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Polyhedral entire solutions in reaction-diffusion equations

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Abstract

This paper studies polyhedral entire solutions to a bistable reaction-diffusion equation in $\mathbb{R}^n$. We consider a pyramidal traveling front solution to the same equation in $\mathbb{R}^{n+1}$. As the speed goes to infinity, its projection converges to an n-dimensional polyhedral entire solution. Conversely, as the time goes to $-\infty$, an n-dimensional polyhedral entire solution gives n-dimensional pyramidal traveling front solutions. The result in this paper suggests a correlation between traveling front solutions and entire solutions in general reaction-diffusion equations or systems.

MSC: 35C07; 35B08; 35K57

Keywords: Traveling front solution; Entire solution; Reaction-diffusion equation

1. Introduction

In this paper we study a reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \widetilde{\Delta} u + f(u), \qquad (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1},\ t > 0, \tag{1.1}$$

$$u(x_1, \ldots, x_{n+1}, 0) = u_0(x_1, \ldots, x_{n+1}), \qquad (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}, \tag{1.2}$$

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where n is a positive integer and

$$\widetilde{\Delta} = \sum_{j=1}^{n+1} \frac{\partial^2}{\partial x_j^2}.$$

Here u_0 is a given function that belongs to a function space of bounded and uniformly continuous functions with

$$\|u_0\| = \sup_{(\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}} |u_0(\boldsymbol{x}, x_{n+1})|,$$

where

$$\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

We write the solution of (1.1)–(1.2) as $u(\boldsymbol{x}, x_{n+1}, t; u_0)$. A given function f is of class C^2 on an open interval including [0, 1]. Now we define

$$W(u) = \int_u^1 f(s)\,\mathrm{d}s$$

and have $W(1) = 0$ and

$$f(u) = -W'(u).$$

The following is the standard assumption on f and W in this paper. We assume

$$W'(0) = 0, \quad W'(1) = 0, \quad W''(0) > 0, \quad W''(1) > 0, \tag{1.3}$$

$$W(1) < W(0) \tag{1.4}$$

$$W'(s) \neq 0 \qquad \text{if} \quad s \in (0, 1) \text{ satisfies } W(s) \leq W(0).$$

We can rewrite (1.3) as

$$f(0) = 0, \quad f(1) = 0, \quad f'(0) < 0, \quad f'(1) < 0,$$

which means that f is a bistable nonlinear term. Then there exist a unique $k \in \mathbb{R}$ and $\Phi \in C^2(\mathbb{R})$ that satisfies

$$\Phi''(x_1) + k\Phi'(x_1) + f(\Phi(x_1)) = 0, \qquad x_1 \in \mathbb{R},$$

$$\Phi(-\infty) = 1, \quad \Phi(\infty) = 0.$$

Now Φ is uniquely determined up to phase shift. See [1,3] and [29, Theorem 4.5] for the proof of existence and uniqueness of Φ. A condition (1.4) gives $k \in (0, \infty)$. This condition implies that a bistable nonlinear term f is imbalanced. A typical example is $f(u) = u(u-a)(u-1)$ for $a \in (0, 1/2)$.

Now we put

$$\widetilde{\boldsymbol{x}} = (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1},$$

$$\nabla = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right), \quad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2},$$

$$\widetilde{\nabla} = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{n+1}}\right), \quad \widetilde{\Delta} = \sum_{j=1}^{n+1} \frac{\partial^2}{\partial x_j^2}.$$

Let $c \in (k, \infty)$ be arbitrarily given. We put $z = x_{n+1} - ct$ and $w(\boldsymbol{x}, z, t) = u(\boldsymbol{x}, x_{n+1}, t)$ for $(\boldsymbol{x}, z) \in \mathbb{R}^{n+1}$ and $t > 0$. Then we have

$$\frac{\partial w}{\partial t} - \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial z^2}\right) w - c\frac{\partial w}{\partial z} - f(w) = 0, \qquad (\boldsymbol{x}, z) \in \mathbb{R}^{n+1},\ t > 0,$$

$$w(\boldsymbol{x}, z, 0) = u_0(\boldsymbol{x}, z), \qquad (\boldsymbol{x}, z) \in \mathbb{R}^{n+1}.$$

We write z simply as x_{n+1}. Then we have

$$\frac{\partial w}{\partial t} - \widetilde{\Delta} w - c\frac{\partial w}{\partial x_{n+1}} - f(w) = 0, \qquad (\boldsymbol{x}, x_n) \in \mathbb{R}^{n+1},\ t > 0, \tag{1.5}$$

$$w(\boldsymbol{x}, x_{n+1}, 0) = u_0(\boldsymbol{x}, x_{n+1}), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}. \tag{1.6}$$

We write the solution of (1.5)–(1.6) as $w(\boldsymbol{x}, x_{n+1}, t; u_0)$. If $(c, v) \in \mathbb{R} \times C^2(\mathbb{R}^{n+1})$ satisfies

$$\widetilde{\Delta} v + c\frac{\partial v}{\partial x_{n+1}} + f(v) = 0, \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}, \tag{1.7}$$

then $u(\boldsymbol{x}, x_{n+1}, t) = v(\boldsymbol{x}, x_{n+1} - ct)$ satisfies (1.1). Then we call v a traveling profile and call c its speed, respectively. Now $u(\boldsymbol{x}, x_{n+1}, t) = v(\boldsymbol{x}, x_{n+1} - ct)$ is called an $(n+1)$-dimensional traveling wave solution. It is also called an $(n+1)$-dimensional traveling front solution if $v(\boldsymbol{x}, x_{n+1})$ is monotone in x_{n+1}. In this paper we always assume

$$0 < v(\boldsymbol{x}) < 1, \qquad \boldsymbol{x} \in \mathbb{R}^n. \tag{1.8}$$

Now $U \in C^2(\mathbb{R}^{n+1})$ is called an n-dimensional entire solution if and only if it satisfies

$$\frac{\partial U}{\partial t}(\boldsymbol{x}, t) = \Delta U(\boldsymbol{x}, t) + f(U(\boldsymbol{x}, t)), \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1}. \tag{1.9}$$

An n-dimensional traveling wave solution is always an n-dimensional entire solution.

Multidimensional traveling front solutions have been studied by [20,21,13,14,16,23,24,17,25,26,29] and so on. Entire solutions have been studied by [31,9,4,12,2,18,19,10,15,11] and so on. Recently [22] shows that an $(n+1)$-dimensional traveling front solution generates an n-dimensional entire solution as the speed goes to infinity, and proves the existence of polyhedral entire solutions under a condition of symmetry, that is, for right polygons.

If $W(0)=W(1)$, f is called balanced. For a reaction-diffusion equation with a bistable balanced reaction term f, multidimensional traveling front solutions have been studies by [5,28,27], and entire solutions have been studied by [6,7,30]. Recently [30] shows that an $(n+1)$-dimensional traveling front solution generates an n-dimensional entire solution as the speed goes to infinity.

In both cases, an $(n+1)$-dimensional traveling front solution generates an n-dimensional entire solution as the speed goes to infinity. Now a problem occurs. Does an entire solution generate a traveling front solution? This problem has been an open problem as far as the author knows. This paper gives an affirmative answer to this question. We show the existence of n-dimensional polyhedral entire solutions for general polygons, and show that an n-dimensional polyhedral entire solution generates n-dimensional pyramidal traveling front solutions as the time goes to $-\infty$ for a reaction-diffusion equation (1.1).

We choose

$$0<s_0<\theta_0<1 \tag{1.10}$$

with

$$\begin{aligned} f(s)<0 \qquad &\text{if} \quad 0<s\le s_0, \\ f(s)>0 \qquad &\text{if} \quad \theta_0\le s<1. \end{aligned} \tag{1.11}$$

By an appropriate translation, we assume that $\Phi(0)=s_0$ without loss of generality. We define

$$\overline{B(\boldsymbol{a};r)}=\{\boldsymbol{x}\in\mathbb{R}^n \mid |\boldsymbol{x}-\boldsymbol{a}|\le r\}, \quad \partial B(\boldsymbol{a};r)=\{\boldsymbol{x}\in\mathbb{R}^n \mid |\boldsymbol{x}-\boldsymbol{a}|=r\}$$

for $\boldsymbol{a}\in\mathbb{R}^n$ and $r>0$.

Pyramidal traveling fronts have been studied by [20,21,23,17,25,26,29] and so on. Let m be a positive integer. Let $\boldsymbol{a}_j\in\mathbb{R}^n$ be a unit vector for $1\le j\le m$, and assume

$$\boldsymbol{a}_i\neq\boldsymbol{a}_j \qquad \text{if} \quad i\neq j.$$

For any given $c\in(k,\infty)$, we define

$$m_*=\frac{\sqrt{c^2-k^2}}{k}\in(0,\infty).$$

For given $\boldsymbol{\xi}\in\mathbb{R}^n$, we put

$$\widetilde{\boldsymbol{\xi}}=\left(\boldsymbol{\xi},m_*\right)\in\mathbb{R}^{n+1}. \tag{1.12}$$

Let $\gamma_j\in\mathbb{R}$ be given for $1\le j\le m$. Now we introduce

$$h(\boldsymbol{x}) = m_* \max_{1\le j\le m} \left((\boldsymbol{a}_j, \boldsymbol{x}) - \gamma_j\right) = m_* \max_{1\le j\le m} \left((\boldsymbol{a}_j, \boldsymbol{x} - \gamma_j \boldsymbol{a}_j)\right), \qquad \boldsymbol{x} \in \mathbb{R}^n \tag{1.13}$$

and

$$v_0(\boldsymbol{x}, x_{n+1}; h) = \Phi\left(\frac{k}{c}(x_{n+1} - h(\boldsymbol{x}))\right), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}.$$

We often write $v_0(\boldsymbol{x}, x_{n+1}; h)$ simply as $v_0(\boldsymbol{x}, x_{n+1})$. With respect to (1.13), we define a pyramidal traveling front by

$$V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h) = \lim_{t\to\infty} w(\boldsymbol{x}, x_{n+1}, t; v_0) \qquad \text{on any compact set in } \mathbb{R}^{n+1}. \tag{1.14}$$

See Section 7 for the details. Then we have

$$v_0(\boldsymbol{x}, x_{n+1}) < V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}.$$

From Theorem 7.1 and (7.4), there exists unique $\zeta_c \in \mathbb{R}$ with $V_{\mathrm{pmd}}(\mathbf{0}, -\zeta_c; h) = \theta_0$. We define

$$v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h) = V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1} - \zeta_c; h), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1} \tag{1.15}$$

and have

$$v_{\mathrm{pmd}}(\mathbf{0}, 0; h) = \theta_0. \tag{1.16}$$

We call $v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h)$ the pyramidal traveling front associated with h. For

$$h(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \mathbb{R}^n,$$

we define

$$v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; 0) = \Phi\left(x_{n+1} + \Phi^{-1}(\theta_0)\right), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}.$$

Now we define

$$\mathcal{A} = \left\{\boldsymbol{\xi} \in \mathbb{R}^n \mid \max_{1\le j\le m} (\boldsymbol{a}_j, \boldsymbol{\xi}) \le 1\right\} \tag{1.17}$$

with the boundary

$$\partial\mathcal{A} = \left\{\boldsymbol{\xi} \in \mathbb{R}^n \mid \max_{1\le j\le m} (\boldsymbol{a}_j, \boldsymbol{\xi}) = 1\right\}.$$

We call $\mathcal{A}$ a *base set* of $v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h)$. Note that $\mathcal{A}$ is not necessarily a compact set in $\mathbb{R}^n$. For every h given by (1.13), we have

$$\overline{B(\mathbf{0}; 1)} \subset \mathcal{A}.$$

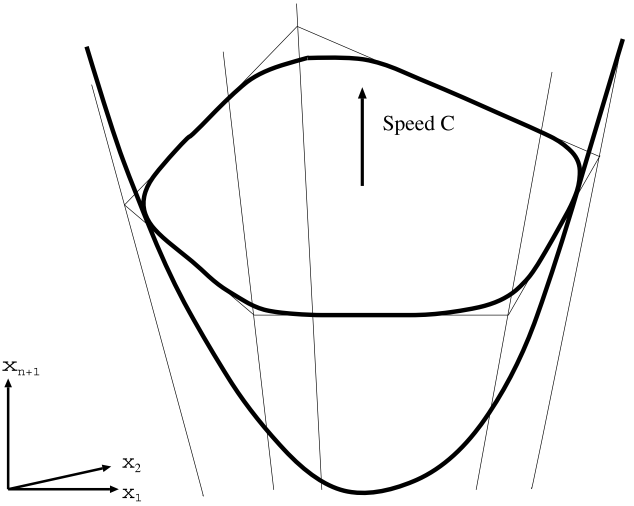

Fig. 1. A pyramidal traveling front in $\mathbb{R}^{n+1}$.

Using Theorem 7.1 and (7.4), we have

$$\left(\widetilde{\nabla} v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h), \widetilde{\boldsymbol{\xi}}\right) < 0, \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}, \ \boldsymbol{\xi} \in \mathcal{A},$$

where $\widetilde{\boldsymbol{\xi}}$ is as in (1.12). We call $v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h)$ the pyramidal traveling front associated with h given by (1.13). If $m = 1$, it is a planar front, that is, a pyramidal traveling front with a single lateral face. We often write $V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h)$ and $v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h)$ as $V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1})$ and $v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1})$, respectively.

The following theorem gives the pyramidal traveling front that converges to a polyhedral entire solution with a base set $\mathcal{A}$. See Fig. 1 for its level set.

Theorem 1.1. *Let $c \in (k, \infty)$ be arbitrarily given. Let $T_{\mathrm{anc}} \in (0, \infty)$ be arbitrarily given and let $z_{\mathrm{anc}} = \sqrt{c^2 - k^2}\, T_{\mathrm{anc}}$. Let h be given by (1.13) and let $v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}) = v^c_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h)$ be the pyramidal traveling front given by (1.15). Let*

$$\mu^c_0 = \max_{j \in \{1,\dots,m\}} \left(\sup\left\{\mu \in [0, \infty) \mid v_{\mathrm{pmd}}(\mu \boldsymbol{a}_j, z_{\mathrm{anc}}; h) = \theta_0\right\}\right) \in (0, \infty). \tag{1.18}$$

Then one can choose

$$(\gamma_1, \dots, \gamma_m) \in [0, \infty)^m$$

such that one has

$$\begin{aligned} &v_{\mathrm{pmd}}(\mathbf{0}, 0; h) = \theta_0, \\ &v_{\mathrm{pmd}}(\mu^c_0 \boldsymbol{a}_j, z_{\mathrm{anc}}; h) = \theta_0 \qquad \textit{for every } j \in \{1, \dots, m\}. \end{aligned} \tag{1.19}$$

Here $\mu^c_0 \in (0, \infty)$ is given by (1.18).

Remark 1.2. If $T_{\rm anc} \in (0, \infty)$ is large enough, $\mu_0^c \in (0, \infty)$ given by (1.18) satisfies

$$0 < \liminf_{c\to\infty} \mu_0^c \leq \limsup_{c\to\infty} \mu_0^c < \infty.$$

See Section 6 for the proof.

The following theorem asserts that a traveling front in $\mathbb{R}^{n+1}$ converges to a polyhedral entire solution in $\mathbb{R}^n$ as the speed c goes to infinity.

Theorem 1.3. *Let $\mathcal{A}$ be given by (1.17). Let $T_{\rm anc} \in (0, \infty)$ be arbitrarily given. For any $c \in (k, \infty)$, define $z_{\rm anc} = \sqrt{c^2 - k^2}\, T_{\rm anc}$ and let h and $v^c_{\rm pmd}(\boldsymbol{x}, x_{n+1}; h)$ be as in Theorem 1.1. Then one can define*

$$U(\boldsymbol{x}, t) = \lim_{c\to\infty} v_{\rm pmd}(\boldsymbol{x}, -\sqrt{c^2 - k^2}\, t; h), \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1}$$

and has (1.9) with

$$\begin{aligned}
&U(\boldsymbol{0}, 0) = \theta_0,\\
&0 < U(\boldsymbol{x}, t) < 1, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1},\\
&\frac{\partial U}{\partial t}(\boldsymbol{x}, t) > 0, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1},\\
&\frac{1}{k}\frac{\partial U}{\partial t}(\boldsymbol{x}, t) - (\boldsymbol{\xi}, \nabla U(\boldsymbol{x}, t)) > 0, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1},\ \boldsymbol{\xi} \in \mathcal{A},\\
&\lim_{t\to\infty} \min_{\boldsymbol{x}\in\Omega_0} U(\boldsymbol{x}, t) = 1, \qquad \lim_{t\to-\infty} \max_{\boldsymbol{x}\in\Omega_0} U(\boldsymbol{x}, t) = 0.
\end{aligned}$$

Here Ω_0 is any compact set in $\mathbb{R}^n$.

Let $\boldsymbol{\zeta} \in \mathbb{S}^{n-1}$ be arbitrarily given, that is, $\boldsymbol{\zeta} \in \mathbb{R}^n$ is any vector with $|\boldsymbol{\zeta}| = 1$. Let $\boldsymbol{p}_j \in \mathbb{R}^n$ $(1 \leq j \leq n-1)$ be chosen such that

$$P = \left(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n-1}, -\boldsymbol{\zeta}\right)$$

is an orthogonal matrix. We introduce a coordinate in $\mathbb{R}^n$ as follows. Let $\boldsymbol{y}$ be defined by

$$P\boldsymbol{y} = \boldsymbol{x}. \tag{1.20}$$

We define

$$\boldsymbol{y}' = (y_1, \ldots, y_{n-1})$$

and have

$$\boldsymbol{y} = (\boldsymbol{y}', y_n) \in \mathbb{R}^n.$$

For $1 \le j \le m$, we put

$$\boldsymbol{a}_j = P\boldsymbol{b}_j, \qquad 1 \le j \le m$$

and have

$$|\boldsymbol{b}_j| = 1, \qquad 1 \le j \le m.$$

Now we define

$$J_{\boldsymbol{\zeta}} = \left\{ j \in \{1, \dots, m\} \;\middle|\; (\boldsymbol{a}_j, \boldsymbol{\zeta}) = \max_{i \in \{1,\dots,m\}} (\boldsymbol{a}_i, \boldsymbol{\zeta}) \right\}. \tag{1.21}$$

Let $m_{\boldsymbol{\zeta}}$ be the number of elements of $J_{\boldsymbol{\zeta}}$. We have $1 \le m_{\boldsymbol{\zeta}} \le m$. Now we have

$$\boldsymbol{b}_j = \big((\boldsymbol{a}_j, \boldsymbol{p}_1), \dots, (\boldsymbol{a}_j, \boldsymbol{p}_{n-1}), (\boldsymbol{a}_j, -\boldsymbol{\zeta})\big), \qquad 1 \le j \le m.$$

For $1 \le j \le m$, we set

$$\boldsymbol{b}'_j = \big((\boldsymbol{a}_j, \boldsymbol{p}_1), \dots, (\boldsymbol{a}_j, \boldsymbol{p}_{n-1})\big) \in \mathbb{R}^{n-1}.$$

For $j \in J_{\boldsymbol{\zeta}}$, $(\boldsymbol{a}_j, \boldsymbol{\zeta})$ and $|\boldsymbol{b}'_j|$ are independent of j, and $|(\boldsymbol{a}_j, \boldsymbol{\zeta})| < 1$ and $|\boldsymbol{b}'_j| < 1$ are equivalent. We write $\rho_{\boldsymbol{\zeta}} = (\boldsymbol{a}_j, \boldsymbol{\zeta})$ for $j \in J_{\boldsymbol{\zeta}}$ and $\rho_{\boldsymbol{\zeta}}$ depends only on $\boldsymbol{\zeta}$. If $\rho_{\boldsymbol{\zeta}} = 1$, we have $\boldsymbol{\zeta} = \boldsymbol{a}_j$ for some $j \in \{1, \dots, m\}$. If $\{\tau\boldsymbol{\zeta} \mid \tau \ge 0\} \cap \partial\mathcal{A} \ne \emptyset$, we see that $\rho_{\boldsymbol{\zeta}}$ has to be positive, and define

$$c_{\boldsymbol{\zeta}} = \frac{k}{\rho_{\boldsymbol{\zeta}}} \in [k, \infty). \tag{1.22}$$

If $\{\tau\boldsymbol{\zeta} \mid \tau \ge 0\} \cap \partial\mathcal{A}$ is not an empty set, it equals to $\{\big(1/\rho_{\boldsymbol{\zeta}}\big)\boldsymbol{\zeta}\}$ with $0 < \rho_{\boldsymbol{\zeta}} \le 1$. If $\{\tau\boldsymbol{\zeta} \mid \tau \ge 0\} \cap \partial\mathcal{A} \ne \emptyset$ holds true, we define, for $\boldsymbol{y}' \in \mathbb{R}^{n-1}$,

$$g(\boldsymbol{y}') = \begin{cases} \dfrac{\sqrt{(c_{\boldsymbol{\zeta}})^2 - k^2}}{k} \max\limits_{j \in J_{\boldsymbol{\zeta}}} \left(\left(\dfrac{\boldsymbol{b}'_j}{|\boldsymbol{b}'_j|}, \boldsymbol{y}' \right) - \mu_j \right), & \text{if} \quad c_{\boldsymbol{\zeta}} \in (k, \infty), \\ 0, & \text{if} \quad c_{\boldsymbol{\zeta}} = k \end{cases} \tag{1.23}$$

with $\mu_j \in \mathbb{R}$ for $j \in J_{\boldsymbol{\zeta}}$. Let $v_{\text{pmd}}(\boldsymbol{y}; g)$ be the n-dimensional pyramidal traveling front associated with $y_n = g(\boldsymbol{y}')$. Its speed toward the y_n-direction is $c_{\boldsymbol{\zeta}}$.

The following theorem asserts that an n-dimensional polyhedral entire solution in Theorem 1.3 gives an n-dimensional traveling front solution as the time goes to $-\infty$.

Theorem 1.4. *In addition to the assumption of Theorem 1.3, assume that $T_{\text{anc}} \in (0, \infty)$ is large enough. Let U be given by Theorem 1.3. Then, for any $\boldsymbol{\zeta} \in \mathbb{S}^{n-1}$ with $\{\tau\boldsymbol{\zeta} \mid \tau \ge 0\} \cap \partial A \ne \emptyset$, one can determine $\sigma(t, \boldsymbol{\zeta}) \in [0, \infty)$ by*

$$U(\sigma(t, \boldsymbol{\zeta})\boldsymbol{\zeta}, t) = \theta_0$$

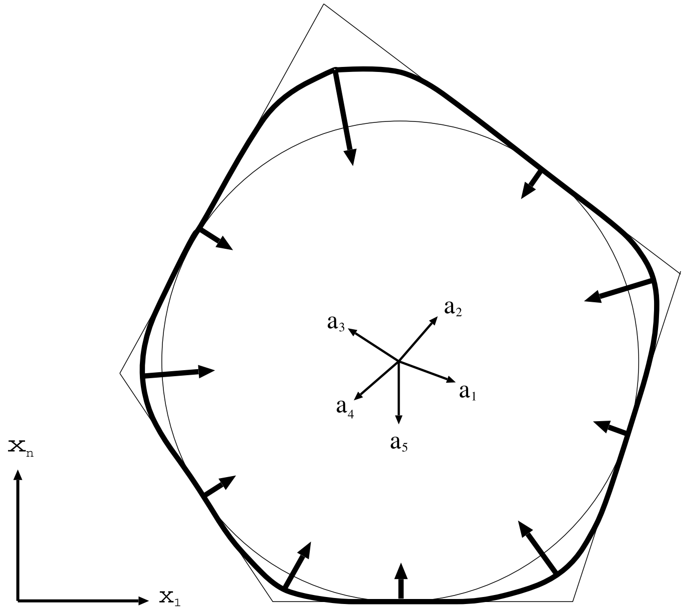

Fig. 2. A level set $\{\boldsymbol{x} \in \mathbb{R}^n \mid U(\boldsymbol{x}, -T_{\text{anc}}) = \theta_0\}$ of a polyhedral entire solution U.

for every $t \in (-\infty, -T_{\text{anc}}]$. Let $J_{\boldsymbol{\zeta}}$ be given by (1.21) and let g be defined by (1.23). Then one can choose $\mu_j \in \mathbb{R}$ for $j \in J_{\boldsymbol{\zeta}}$ such that one has

$$\lim_{t\to-\infty} \sup_{\boldsymbol{x}\in\overline{B(\mathbf{0};R)}} \left| U(\boldsymbol{x} + \sigma(t, \boldsymbol{\zeta})\boldsymbol{\zeta}, t) - v_{\text{pmd}}(\boldsymbol{y}; g) \right| = 0, \tag{1.24}$$

where $\boldsymbol{y}$ is given by (1.20). Here $R \in (0, \infty)$ is any given number. Moreover one has

$$\sigma(t, \boldsymbol{\zeta}) = -c_{\boldsymbol{\zeta}} t + \nu_{\boldsymbol{\zeta}} \qquad as \quad t \to -\infty \tag{1.25}$$

with some $\nu_{\boldsymbol{\zeta}} \in \mathbb{R}$ that depends on $\boldsymbol{\zeta}$. Here $c_{\boldsymbol{\zeta}}$ is given by (1.22).

Theorem 1.4 implies that U generates $\mathcal{A}$ given by (1.17). As a sufficient condition for this generation, $T_{\text{anc}} \in (0, \infty)$ is imposed to be large enough. We call the entire solution in Theorem 1.4 a polyhedral entire solution associated with a polyhedron $\mathcal{A}$. See Fig. 2 for a level set of U. Theorem 1.3 asserts that an $(n+1)$-dimensional pyramidal traveling front solution generates an n-dimensional polyhedral entire solution as the speed goes to infinity for (1.1). Theorem 1.4 asserts that an n-dimensional polyhedral entire solution generates n-dimensional pyramidal traveling front solutions as the time goes to $-\infty$ for (1.1). We conjecture that an $(n+1)$-dimensional traveling front solution generates an n-dimensional entire solution as the speed goes to infinity and an n-dimensional entire solution generates n-dimensional traveling front solutions as the time goes to $-\infty$ in various kinds of reaction-diffusion equations or systems. This problem is left to be open for future studies.

2. Uniform estimate on widths of traveling wave solutions in $\mathbb{R}^n$ for all speed

Let

$$\beta = \frac{1}{2}\min\left\{-f'(0), -f'(1)\right\} > 0.$$

Let $\delta_* \in (0, 1/4)$ be small enough so that one has

$$\min_{|s|\le 2\delta_*}\left(-f'(s)\right) > \beta, \qquad \min_{|s-1|\le 2\delta_*}\left(-f'(s)\right) > \beta.$$

We put

$$M = \max_{-2\delta_* \le s \le 1+2\delta_*} |f'(s)|.$$

Throughout this paper, we assume

$$-\delta_* \le u_0(\boldsymbol{x}) \le 1 + \delta_*, \qquad \boldsymbol{x} \in \mathbb{R}^{n+1}.$$

Then $u(\boldsymbol{x}, t) = u(\boldsymbol{x}, t; u_0)$ satisfies

$$-\delta_* \le u(\boldsymbol{x}, t) \le 1 + \delta_*, \qquad \boldsymbol{x} \in \mathbb{R}^{n+1},\ t > 0.$$

Now the Schauder estimate [29, Proposition 2.9, Lemma 2.6] gives

$$\sup_{\boldsymbol{x}\in\mathbb{R}^{n+1},\, t\ge 1}\left|\frac{\partial u}{\partial x_j}(\boldsymbol{x}, t)\right| < K_*, \tag{2.1}$$

$$\sup_{\boldsymbol{x}\in\mathbb{R}^{n+1},\, t\ge 1}\left|\frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}, t)\right| < K_*, \qquad \sup_{\boldsymbol{x}\in\mathbb{R}^{n+1},\, t\ge 1}\left|\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_\ell}(\boldsymbol{x}, t)\right| < K_*, \tag{2.2}$$

$$\sup_{\boldsymbol{x}\in\mathbb{R}^{n+1},\, t\ge 1}\left|\frac{\partial u}{\partial t}(\boldsymbol{x}, t)\right| < K_*$$

for $1 \le i, j, \ell \le n + 1$. Here $K_* \in (0, \infty)$ is a constant depending only on (f, n, δ_*), and is independent of u_0. We sometimes use

$$\mathrm{D}_t = \frac{\partial}{\partial t}, \quad \mathrm{D}_j = \frac{\partial}{\partial x_j}, \quad \mathrm{D}_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \quad \mathrm{D}_{ij\ell} = \frac{\partial^3}{\partial x_i \partial x_j \partial x_\ell},$$

for $1 \le i, j, \ell \le n + 1$ and write $\mathrm{D}_t u(\boldsymbol{x}, t)$ as $u_t(\boldsymbol{x}, t)$ for simplicity. Now we show the following lemma.

Lemma 2.1. *Assume $c \in \mathbb{R}$ and $v \in C^2(\mathbb{R}^{n+1})$ satisfy (1.7) and (1.8). Then one has*

$$\sup_{\boldsymbol{x}\in\mathbb{R}^{n+1}}\left|\mathrm{D}_j v(\boldsymbol{x})\right| \le K_*, \qquad \sup_{\boldsymbol{x}\in\mathbb{R}^{n+1}}\left|\mathrm{D}_{ij} v(\boldsymbol{x})\right| \le K_*, \qquad \sup_{\boldsymbol{x}\in\mathbb{R}^{n+1}}\left|\mathrm{D}_{ij\ell} v(\boldsymbol{x})\right| \le K_*$$

for $1 \le i, j, \ell \le n+1$, *where* K_* *is independent of* $(c, v) \in \mathbb{R} \times C^2(\mathbb{R}^{n+1})$.

Proof. By putting $u(\boldsymbol{x}, x_{n+1}, t) = v(\boldsymbol{x}, x_{n+1} - ct)$, u satisfies (1.1) with

$$u(\boldsymbol{x}, x_{n+1}, 0) = v(\boldsymbol{x}, x_{n+1}), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}.$$

Then (2.1) and (2.2) give

$$\sup_{(\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1},\, 1 \le t} \left| \mathrm{D}_j v(\boldsymbol{x}, x_{n+1} - ct) \right| \le K_*, \qquad \sup_{(\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1},\, 1 \le t} \left| \mathrm{D}_{ij} v(\boldsymbol{x}, x_{n+1} - ct) \right| \le K_*,$$
$$\sup_{(\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}, 1 \le t} \left| \mathrm{D}_{ij\ell} v(\boldsymbol{x}, x_{n+1} - ct) \right| \le K_*$$

for $1 \le i, j, \ell \le n+1$, which give

$$\sup_{(\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}} \left| \mathrm{D}_j v(\boldsymbol{x}, x_{n+1} - c) \right| \le K_*, \qquad \sup_{(\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}} \left| \mathrm{D}_{ij} v(\boldsymbol{x}, x_{n+1} - c) \right| \le K_*$$
$$\sup_{(\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}} \left| \mathrm{D}_{ij\ell} v(\boldsymbol{x}, x_{n+1} - c) \right| \le K_*$$

and

$$\sup_{(\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}} \left| \mathrm{D}_j v(\boldsymbol{x}, x_{n+1}) \right| \le K_*, \qquad \sup_{(\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}} \left| \mathrm{D}_{ij} v(\boldsymbol{x}, x_{n+1}) \right| \le K_*,$$
$$\sup_{(\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}} \left| \mathrm{D}_{ij\ell} v(\boldsymbol{x}, x_{n+1}) \right| \le K_*$$

for $1 \le i, j, \ell \le n+1$. This completes the proof. □

Let $\theta \in (0, 1)$ be arbitrarily given with $W(0) < W(\theta)$. Then we choose $R = R_\theta \in (0, \infty)$ large enough with

$$(W(\theta) - W(0))\, R > n K_* \theta. \tag{2.3}$$

Taking $s_1 \in (0, \theta)$ small enough with

$$\begin{gathered} W(0) < W(s_1) < W(\theta), \\ (W(\theta) - W(s_1))\, R > n K_* (\theta - s_1). \end{gathered} \tag{2.4}$$

For arbitrarily given $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, we define

$$\mathcal{D}_{\boldsymbol{\xi}} = (\xi_1 - R, \xi_1 + R) \times (\xi_2 - R, \xi_2 + R) \times \cdots \times (\xi_n - R, \xi_n + R) \subset \mathbb{R}^n. \tag{2.5}$$

We write $\mathcal{D}_{\boldsymbol{\xi}}$ simply as $\mathcal{D}$. The volume of $\mathcal{D}$ is given by $(2R)^n$, and the surface area of the boundary of $\mathcal{D}$ is given by $2n(2R)^{n-1}$. Then we have

$$(W(\theta) - W(s_1))\,|\mathcal{D}| > K_*(\theta - s_1)\,|\partial\mathcal{D}|\,,$$

using (2.4). Let $c \in (k, \infty)$ be arbitrarily given and let $V \in C^2(\mathbb{R}^{n+1})$ satisfy

$$\Delta V + c\frac{\partial V}{\partial x_{n+1}} + f(V) = 0, \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}, \tag{2.6}$$

$$0 < V(\boldsymbol{x}, x_{n+1}) < 1, \quad \frac{\partial V}{\partial x_{n+1}}(\boldsymbol{x}, x_{n+1}) < 0, \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}, \tag{2.7}$$

$$\lim_{x_{n+1}\to\infty} \sup_{\boldsymbol{x}\in\Omega_0} V(\boldsymbol{x}, x_{n+1}) = 0, \qquad \lim_{x_{n+1}\to-\infty} \sup_{\boldsymbol{x}\in\Omega_0} (1 - V(\boldsymbol{x}, x_{n+1})) = 0 \tag{2.8}$$

for any given compact set $\Omega_0 \subset \mathbb{R}^n$. Now we define $q_\theta(\boldsymbol{x}) = q_{\theta,c}(\boldsymbol{x})$ by

$$V(\boldsymbol{x}, q_\theta(\boldsymbol{x})) = \theta, \qquad \boldsymbol{x} \in \mathbb{R}^n.$$

The following proposition plays important role in the later discussions.

Proposition 2.2 *([27,22]). Let $\theta \in (0, 1)$ be given with $W(0) < W(\theta)$, and let R be given with (2.3). Assume $c \in (k, \infty)$ is arbitrarily given and $V \in C^2(\mathbb{R}^{n+1})$ satisfies (2.6), (2.7) and (2.8). Then one has*

$$\frac{1}{2}\int_{\mathcal{D}} |\nabla V(\boldsymbol{x}, q_\theta(\boldsymbol{x}))|^2 \,\mathrm{d}\boldsymbol{x} \geq 2(2R)^{n-1}\left[(W(\theta) - W(0))R - nK_*\theta\right] > 0. \tag{2.9}$$

The right-hand side is independent of $c \in (k, \infty)$.

We write the right-hand side of (2.9) as $(1/2)A(\theta, R)^2\,|\mathcal{D}|$ with

$$A(\theta, R) = \sqrt{\frac{2}{R}\left[W(\theta)R - nK_*(1+\theta)\right]} > 0. \tag{2.10}$$

Then (2.9) is written as

$$\int_{\mathcal{D}} |\nabla V(\boldsymbol{x}, q_\theta(\boldsymbol{x}))|^2 \,\mathrm{d}\boldsymbol{x} \geq A(\theta, R)^2|\mathcal{D}| > 0.$$

3. A pyramidal traveling front that convergences to a polyhedral entire solution as the speed goes to infinity

In this section we will prove Theorem 1.1. Let $T_{\mathrm{anc}} \in (0, \infty)$ be given. Let $c \in (k, \infty)$ be given and let h be given by (1.13). Let $V_{\mathrm{pmd}}(\boldsymbol{x}; h)$ and $v_{\mathrm{pmd}}(\boldsymbol{x}; h)$ be given by (1.14) and (1.15), respectively. We have

$$v_{\mathrm{pmd}}(\boldsymbol{0}, 0; h) = \theta_0.$$

Now we set

$$z_{\mathrm{anc}} = \sqrt{c^2 - k^2}\, T_{\mathrm{anc}}.$$

From Theorem 7.1, we have, for $j \in \{1, \ldots, m\}$,

$$\sup\left\{\mu \in [0, \infty) \mid v_{\mathrm{pmd}}(\mu \boldsymbol{a}_j, z_{\mathrm{anc}}; h) = \theta_0\right\} < \infty.$$

Now we define μ_0^c by (1.18). In view of (1.16) and (7.4), we get $\mu_0^c > 0$.

Now we prove Theorem 1.1 as follows.

Proof of Theorem 1.1. We take $\Lambda_2 \in [0, \infty)$ large enough. Then, by Lemma 7.4, for every

$$(\gamma_2, \ldots, \gamma_m) \in [\Lambda_2, \infty)^{m-1},$$

there exists a function $\overline{\gamma}_1(\gamma_2, \ldots, \gamma_m) \in [0, \infty)$ such that h given by (1.13) with

$$(\Gamma_1, \gamma_2, \ldots, \gamma_m),$$

satisfies

$$\sup\left\{\mu \in [0, \infty) \mid v_{\mathrm{pmd}}(\mu \boldsymbol{a}_1, z_{\mathrm{anc}}; h)\right\} = \max_{j \in \{2, \ldots, m\}} \left(\sup\left\{\mu \in [0, \infty) \mid v_{\mathrm{pmd}}(\mu \boldsymbol{a}_j, z_{\mathrm{anc}}; h) = \theta_0\right\}\right),$$

where

$$\Gamma_1 = \overline{\gamma}_1(\gamma_2, \ldots, \gamma_m).$$

We take $\Lambda_3 \in [0, \infty)$ large enough. For every

$$(\gamma_3, \ldots, \gamma_m) \in [\Lambda_3, \infty)^{m-2},$$

there exists a function $\overline{\gamma}_2 = \overline{\gamma}_2(\gamma_3, \ldots, \gamma_m) \in [0, \infty)$ such that h given by (1.13) with

$$(\Gamma_1, \Gamma_2, \gamma_3, \ldots, \gamma_m),$$

satisfies

$$\sup\left\{\mu \in [0, \infty) \mid v_{\mathrm{pmd}}(\mu \boldsymbol{a}_2, z_{\mathrm{anc}}; h)\right\} = \max_{j \in \{3, \ldots, m\}} \left(\sup\left\{\mu \in [0, \infty) \mid v_{\mathrm{pmd}}(\mu \boldsymbol{a}_j, z_{\mathrm{anc}}; h) = \theta_0\right\}\right),$$

where

$$\Gamma_2 = \overline{\gamma}_2(\gamma_3, \ldots, \gamma_m)$$
$$\Gamma_1 = \overline{\gamma}_1(\Gamma_2, \gamma_3, \ldots, \gamma_m).$$

We continue this argument and finally we obtain the following. We take $\Lambda_m \in [0, \infty)$ large enough. For every

$$\gamma_m \in [\Lambda_m, \infty),$$

there exists a function $\overline{\gamma}_{m-1}(\gamma_m) \in [0, \infty)$ such that h given by (1.13) with

$$(\Gamma_1, \dots, \Gamma_{m-2}, \Gamma_{m-1}, \gamma_m)$$

satisfies

$$\sup\left\{\mu \in [0,\infty) \mid v_{\mathrm{pmd}}(\mu \boldsymbol{a}_{m-1}, z_{\mathrm{anc}}; h)\right\} = \left(\sup\left\{\mu \in [0,\infty) \mid v_{\mathrm{pmd}}(\mu \boldsymbol{a}_m, z_{\mathrm{anc}}; h) = \theta_0\right\}\right),$$

where

$$\begin{aligned} \Gamma_{m-1} &= \overline{\gamma}_{m-1}(\gamma_m), \\ \Gamma_{m-2} &= \overline{\gamma}_{m-2}(\Gamma_{m-1}, \gamma_m), \\ &\vdots \\ \Gamma_1 &= \overline{\gamma}_1(\Gamma_2, \Gamma_3, \dots, \Gamma_{m-1}, \gamma_m). \end{aligned}$$

Now we choose $\gamma_m = \Lambda_m$ and have

$$\Gamma_{m-1} \geq \Lambda_{m-1}, \dots, \Gamma_2 \geq \Lambda_2.$$

Then we obtain (1.19) for h given by (1.13) with $(\Gamma_1, \dots, \Gamma_{m-2}, \Gamma_{m-1}, \Lambda_m)$. This completes the proof. □

4. A traveling front in $\mathbb{R}^{n+1}$ converges to an entire solution in $\mathbb{R}^n$ as the speed goes to infinity

Let $\mathcal{A}$ be any closed set in $\mathbb{R}^n$ with $\overline{B(\mathbf{0}; 1)} \subset \mathcal{A}$. For every $c \in (k, \infty)$, let $V_c(\boldsymbol{x}, x_{n+1})$ satisfy (1.7) and (2.9) with

$$\begin{aligned} &0 < V_c(\boldsymbol{x}, x_{n+1}) < 1, \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}, \\ &V_c(\mathbf{0}, 0) = \theta_0, \\ &\left(\widetilde{\nabla} v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h), \widetilde{\boldsymbol{\xi}}\right) < 0, \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1},\ \boldsymbol{\xi} \in \mathcal{A}, \end{aligned}$$

where $\widetilde{\boldsymbol{\xi}}$ is as in (1.12). We have

$$\sup_{\boldsymbol{x} \in \mathbb{R}^n} |f(V_c(\boldsymbol{x}))| \leq \|f\|_{C[0,1]} < \infty,$$

where

$$\|f\|_{C[0,1]} = \max_{0 \leq s \leq 1} |f(s)|.$$

Using Lemma 2.1, we have

$$\sup_{(\boldsymbol{x},x_{n+1})\in\mathbb{R}^{n+1}} \left|\widetilde{\Delta} V_c(\boldsymbol{x},x_{n+1})\right| \le (n+1)K_* < \infty.$$

Using

$$0 < -\frac{\partial V_c}{\partial x_{n+1}}(\boldsymbol{x},x_{n+1}) = \frac{\widetilde{\Delta} V_c(\boldsymbol{x},x_{n+1}) + f(V_c(\boldsymbol{x},x_{n+1}))}{c} \le \frac{(n+1)K_* + \|f\|_{C[0,1]}}{c}$$

for $(\boldsymbol{x},x_{n+1}) \in \mathbb{R}^{n+1}$, we obtain

$$\lim_{c\to\infty} \sup_{(\boldsymbol{x},x_{n+1})\in\mathbb{R}^{n+1}} \left|\frac{\partial V_c}{\partial x_{n+1}}(\boldsymbol{x},x_{n+1})\right| = 0. \tag{4.1}$$

Using Lemma 2.1, we have

$$\max_{1\le j\le n+1} \sup_{(\boldsymbol{x},x_{n+1})\in\mathbb{R}^{n+1}} \left|\frac{\partial^3 V_c}{\partial x_{n+1}^3}(\boldsymbol{x},x_{n+1})\right| < K_*. \tag{4.2}$$

Here a positive constant K_* is as in Lemma 2.1 and is independent of c. Combining (4.1) and (4.2), we obtain

$$\lim_{c\to\infty} \sup_{(\boldsymbol{x},x_{n+1})\in\mathbb{R}^{n+1}} \left|\frac{\partial^2 V_c}{\partial x_{n+1}^2}(\boldsymbol{x},x_{n+1})\right| = 0. \tag{4.3}$$

Now we introduce

$$t = -\frac{x_{n+1}}{\sqrt{c^2-k^2}},$$

that is, $x_{n+1} = -\sqrt{c^2-k^2}\,t$. We define

$$\overline{u}_c(\boldsymbol{x},t) = V_c(\boldsymbol{x}, -\sqrt{c^2-k^2}\,t), \qquad (\boldsymbol{x},t)\in\mathbb{R}^{n+1}. \tag{4.4}$$

Now $\overline{u}_c$ satisfies

$$0 < \overline{u}_c(\boldsymbol{x},t) < 1, \qquad (\boldsymbol{x},t)\in\mathbb{R}^{n+1}, \tag{4.5}$$

$$\frac{\partial \overline{u}_c}{\partial t}(\boldsymbol{x},t) > 0, \qquad (\boldsymbol{x},t)\in\mathbb{R}^{n+1}, \tag{4.6}$$

$$\frac{1}{k}\frac{\partial \overline{u}_c}{\partial t}(\boldsymbol{x},t) - (\boldsymbol{\xi}, \nabla\overline{u}_c(\boldsymbol{x},t)) > 0, \qquad (\boldsymbol{x},t)\in\mathbb{R}^{n+1} \tag{4.7}$$

for every $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n) \in \mathcal{A}$, where $\mathcal{A}$ is given by (1.17). Now we have

$$\frac{\partial V_c}{\partial x_{n+1}}(\boldsymbol{x}, -\sqrt{c^2-k^2}\,t) = -\frac{1}{\sqrt{c^2-k^2}}\frac{\partial \overline{u}_c}{\partial t}(\boldsymbol{x},t), \qquad (\boldsymbol{x},t)\in\mathbb{R}^{n+1}.$$

Then we find

$$\sum_{j=1}^{n} \frac{\partial^2 \overline{u}_c}{\partial x_j^2}(\boldsymbol{x}, t) + \frac{\partial^2 V_c}{\partial x_{n+1}^2}(\boldsymbol{x}, -\sqrt{c^2-k^2}t) - \frac{c}{\sqrt{c^2-k^2}} \frac{\partial \overline{u}_c}{\partial t}(\boldsymbol{x}, t) + f(\overline{u}_c(\boldsymbol{x}, t)) = 0$$

for $(\boldsymbol{x}, t) \in \mathbb{R}^{n+1}$. Thus we obtain

$$\begin{aligned} \frac{\partial \overline{u}_c}{\partial t}(\boldsymbol{x}, t) =& \frac{\sqrt{c^2-k^2}}{c} \left(\sum_{j=1}^{n} \frac{\partial^2 \overline{u}_c}{\partial x_j^2}(\boldsymbol{x}, t) + f(\overline{u}_c(\boldsymbol{x}, t)) \right) \\ &+ \frac{\sqrt{c^2-k^2}}{c} \frac{\partial^2 V_c}{\partial x_{n+1}^2}(\boldsymbol{x}, -\sqrt{c^2-k^2}t), \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1}. \end{aligned} \tag{4.8}$$

Now we introduce

$$U(\boldsymbol{x}, t) = \lim_{c \to \infty} \overline{u}_c(\boldsymbol{x}, t) \tag{4.9}$$

for $(\boldsymbol{x}, t) \in \mathbb{R}^{n+1}$ on any compact set in $\mathbb{R}^{n+1}$.

The heat kernel in $\mathbb{R}^{n-1}$ is given by

$$K(\boldsymbol{x}, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|\boldsymbol{x}|^2}{4t}\right), \qquad \boldsymbol{x} \in \mathbb{R}^n, \ t > 0.$$

Let $t_0 \in \mathbb{R}$ be arbitrarily given. Using (4.8), we get

$$\begin{aligned} \overline{u}_c(\boldsymbol{x}, t) =& \int_{\mathbb{R}^n} K\left(\boldsymbol{x} - \boldsymbol{y}, \frac{\sqrt{c^2-k^2}}{c}(t - t_0)\right) \overline{u}_c(\boldsymbol{y}, t_0)\, \mathrm{d}\boldsymbol{y} \\ &+ \frac{\sqrt{c^2-k^2}}{c} \int_{t_0}^{t} \left(\int_{\mathbb{R}^n} K\left(\boldsymbol{x} - \boldsymbol{y}, \frac{\sqrt{c^2-k^2}}{c}(t - s)\right) \right. \\ &\qquad \left. \times \left(f(\overline{u}_c(\boldsymbol{y}, s)) + \frac{\partial^2 V_c}{\partial x_n^2}(\boldsymbol{y}, -\sqrt{c^2-k^2}s) \right) \mathrm{d}\boldsymbol{y} \right) \mathrm{d}s \end{aligned}$$

for $t > t_0$. Taking the limit of $c \to \infty$ for the both sides and using (4.3), we find

$$U(\boldsymbol{x}, t) = \int_{\mathbb{R}^n} K(\boldsymbol{x} - \boldsymbol{y}, t - t_0) U(\boldsymbol{y}, t_0)\, \mathrm{d}\boldsymbol{y} + \int_{t_0}^{t} \left(\int_{\mathbb{R}^n} K(\boldsymbol{x} - \boldsymbol{y}, t - s) f(U(\boldsymbol{y}, s))\, \mathrm{d}\boldsymbol{y} \right) \mathrm{d}s$$

for $t > t_0$, which gives

$$\frac{\partial U}{\partial t}(\boldsymbol{x}, t) = \sum_{j=1}^{n} \frac{\partial^2 U}{\partial x_j^2}(\boldsymbol{x}, t) + f(U(\boldsymbol{x}, t)), \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1}$$

for $t > t_0$. Since $t_0 \in \mathbb{R}$ is taken arbitrarily, we obtain

$$\frac{\partial U}{\partial t}(\boldsymbol{x}, t) = \Delta U(\boldsymbol{x}, t) + f(U(\boldsymbol{x}, t)), \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1}.$$

Thus the limit of an $(n+1)$-dimensional traveling wave solution V_c gives an n-dimensional entire solution U as $c \to \infty$. Now we have

$$\lim_{c\to\infty} |\nabla \overline{u}_c(\boldsymbol{x}, t) - \nabla U(\boldsymbol{x}, t)| = 0 \tag{4.10}$$

on every compact set in $\mathbb{R}^{n+1}$. Taking the limit of $c \to \infty$ in (4.6) and (4.7), we obtain

$$\frac{\partial U}{\partial t}(\boldsymbol{x}, t) \geq 0, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1}, \tag{4.11}$$

$$\frac{1}{k}\frac{\partial U}{\partial t}(\boldsymbol{x}, t) - (\boldsymbol{\xi}, \nabla U(\boldsymbol{x}, t)) \geq 0, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1} \tag{4.12}$$

for every $\boldsymbol{\xi} \in \mathcal{A}$. Now we have

$$U(\mathbf{0}, 0) = \theta_0, \tag{4.13}$$

$$0 \leq U(\boldsymbol{x}, t) \leq 1, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1}.$$

Since U is not identically 0 or 1, we have

$$0 < U(\boldsymbol{x}, t) < 1, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1} \tag{4.14}$$

by the strong maximum principle. Now we state properties of U as follows.

Lemma 4.1. *Let U be defined by (4.9). Then one has (1.9) with*

$$U(\mathbf{0}, 0) = \theta_0, \tag{4.15}$$

$$0 < U(\boldsymbol{x}, t) < 1, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1}, \tag{4.16}$$

$$\frac{1}{k}\frac{\partial U}{\partial t}(\boldsymbol{x}, t) - (\boldsymbol{\xi}, \nabla U(\boldsymbol{x}, t)) \geq 0, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1} \tag{4.17}$$

for every $\boldsymbol{\xi} \in \overline{B(\mathbf{0}; 1)}$. Let $\theta \in (0, 1)$ satisfy $W(0) < W(\theta)$ and let $R = R_\theta$ be given by (2.3). Let $\mathcal{D}$ be given by (2.5) for arbitrarily given $(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-2}$. Let $A(\theta, R)$ be given by (2.10). Then one has

$$\max\left\{|\nabla U(\boldsymbol{x}, t)| \;\middle|\; \boldsymbol{x} \in \mathcal{D}, t \in \mathbb{R}, U(\boldsymbol{x}, t) = \theta\right\} \geq A(\theta, R) > 0. \tag{4.18}$$

Proof. We already obtained (4.15), (4.16) and (4.17). Using (4.11), we have $\overline{U}, \underline{U} \in C^2(\mathbb{R}^{n-1})$ such that

$$\Delta \underline{U}(\boldsymbol{x}) + f(\underline{U}(\boldsymbol{x})) = 0, \quad \Delta \overline{U}(\boldsymbol{x}) + f(\overline{U}(\boldsymbol{x})) = 0, \qquad \boldsymbol{x} \in \mathbb{R}^n,$$
$$0 \le \underline{U}(\boldsymbol{x}) \le U(\boldsymbol{x}, t) \le \overline{U}(\boldsymbol{x}) \le 1, \qquad \boldsymbol{x} \in \mathbb{R}^n, \ t \in \mathbb{R},$$
$$\lim_{t\to\infty} \big(\overline{U}(\boldsymbol{x}) - U(\boldsymbol{x}, t)\big) = 0, \ \lim_{t\to-\infty} \big(U(\boldsymbol{x}, t) - \underline{U}(\boldsymbol{x})\big) = 0$$

uniformly on every compact set in $\mathbb{R}^n$. Let $\varepsilon \in (0, 1)$ be arbitrarily given. Let $R_\theta \in (0, \infty)$ satisfy (2.3). We can define

$$R_{\max} = \max_{\theta\in[\varepsilon,\theta_0]} R_\theta \in (0, \infty)$$

For arbitrarily given $(\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, we define

$$D_0 = (\xi_1 - R_{\max}, \xi_1 + R_{\max}) \times (\xi_2 - R_{\max}, \xi_2 + R_{\max}) \times \cdots \times (\xi_{n-1} - R_{\max}, \xi_{n-1} + R_{\max}) \subset \mathbb{R}^{n-1}.$$

We will show

$$0 \le \max_{D_0} \underline{U} < \varepsilon, \quad \theta_0 \le \min_{D_0} \overline{U} \le 1. \tag{4.19}$$

Let $\theta \in [\varepsilon, \theta_0]$ be arbitrarily given and let $\mathcal{D}$ be given by (2.5) for R_θ. Combining Proposition 2.2 and (4.1), we have

$$\max_{\boldsymbol{x}\in\mathcal{D}} \{| \nabla V_c(\boldsymbol{x}, x_{n+1})| \,|\, x_{n+1} \in \mathbb{R}, \ V_c(\boldsymbol{x}, x_{n+1}) = \theta\} \ge (1 - \varepsilon)A(\theta, R)$$

if $c \in (0, \infty)$ is large enough, say, if $c \in [C_\varepsilon, \infty)$, where $C(\varepsilon)$ is a positive number that depends on $\varepsilon \in (0, 1)$. Consequently, by (4.4), we get

$$\max_{\boldsymbol{x}\in\mathcal{D}} \{|\nabla \overline{u}_c(\boldsymbol{x}, t)| \,|\, t \in \mathbb{R}, \overline{u}_c(\boldsymbol{x}, t) = \theta\} \ge (1 - \varepsilon)A(\theta, R) \tag{4.20}$$

for every $c \in [C(\varepsilon), \infty)$. Using (4.7), we have

$$\frac{1}{k} \frac{\partial \overline{u}_c}{\partial t}(\boldsymbol{x}, t) \ge |\nabla \overline{u}_c(\boldsymbol{x}, t)| \,.$$

Thus we obtain, for every $\theta \in [\varepsilon, \theta_0]$,

$$\begin{aligned} &\max_{\boldsymbol{x}\in D_0} \left\{ \frac{\partial \overline{u}_c}{\partial t}(\boldsymbol{x}, t) \,\middle|\, t \in \mathbb{R}, \ \overline{u}_c(\boldsymbol{x}, t) = \theta \right\} \\ \ge &\max_{\boldsymbol{x}\in\mathcal{D}} \left\{ \frac{\partial \overline{u}_c}{\partial t}(\boldsymbol{x}, t) \,\middle|\, t \in \mathbb{R}, \ \overline{u}_c(\boldsymbol{x}, t) = \theta \right\} \\ \ge &k(1 - \varepsilon)A(\theta, R) > 0 \end{aligned}$$

for every $c \in [C(\varepsilon), \infty)$. Using this inequality, if $t \in (0, \infty)$ is large enough, we have

$$\max_{\boldsymbol{x}\in D_0} \overline{u}_c(\boldsymbol{x}, t) > \theta_0, \quad \min_{\boldsymbol{x}\in D_0} \overline{u}_c(\boldsymbol{x}, -t) < \varepsilon$$

for every $c \in [C(\varepsilon), \infty)$. Then, using (4.7) and taking $t \in (0, \infty)$ larger if necessary, we have

$$\min_{\boldsymbol{x}' \in D_0} \overline{u}_c(\boldsymbol{x}', t) > \theta_0, \qquad \max_{\boldsymbol{x}' \in D_0} \overline{u}_c(\boldsymbol{x}', -t) < \varepsilon \tag{4.21}$$

for every $c \in [C(\varepsilon), \infty)$. Thus we obtain (4.19). Now we prove (4.18). Let θ satisfy the assumption of Proposition 4.3 with $\varepsilon < \theta$. Let $\mathcal{D}$ be given by (2.5). Taking the limit of $c \to \infty$ in (4.20) and using (4.10) and (4.21), we obtain

$$\max_{\boldsymbol{x} \in \mathcal{D}} \{ |\nabla U(\boldsymbol{x}, t)| \mid t \in \mathbb{R},\ U(\boldsymbol{x}, t) = \theta \} \geq (1 - \varepsilon) A(\theta, R).$$

Since we can take $\varepsilon \in (0, 1)$ to be arbitrarily small, we obtain (4.18). This completes the proof. $\square$

For any initial value $v \in \mathrm{BU}(\mathbb{R}^n)$, we consider (1.1)–(1.2) in $\mathbb{R}^n$ and write the solution as $u(\boldsymbol{x}, t; v)$ for $(\boldsymbol{x}, t) \in \mathbb{R}^n \times (0, \infty)$. We write the following lemma for the later use. For the proof, see [29].

Lemma 4.2 *([29]). Let $T \in (0, \infty)$, $R \in (0, \infty)$ and $\varepsilon > 0$ be given. Let $\boldsymbol{x}_0 \in \mathbb{R}^n$ be given. Assume $v_1, v_2 \in \mathrm{BU}(\mathbb{R}^n)$ satisfy*

$$0 < v_j(\boldsymbol{x}) < 1, \quad j = 1, 2,$$
$$\sup_{\boldsymbol{x} \in B(\boldsymbol{x}_0; 2R)} |v_2(\boldsymbol{x}) - v_1(\boldsymbol{x})| < \varepsilon.$$

Then one has

$$\sup_{B(\boldsymbol{x}_0; R)} |u(\boldsymbol{x}, T; v_2) - u(\boldsymbol{x}, T; v_1)| \leq \varepsilon e^{MT} + K_1 e^{MT} \exp\left(-\frac{R^2}{8T} \right).$$

Here K_1 is a positive constant depending only on (f, n).

Now U defined by (4.9) satisfies the assumption of the following proposition. We will assert properties of entire solutions including U defined by (4.9) in a general setting as follows.

Proposition 4.3. *Let U satisfy (1.9) with (4.15), (4.16), (4.17) and (4.18). Then one has*

$$\frac{\partial U}{\partial t}(\boldsymbol{x}, t) \geq k |\nabla U(\boldsymbol{x}, t)|, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1}, \tag{4.22}$$

$$\frac{\partial U}{\partial t}(\boldsymbol{x}, t) > 0, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1}. \tag{4.23}$$

Proof. We have (4.11) from (4.17). Thus we have (4.22) if $|\nabla U(\boldsymbol{x}_0, t_0)| = 0$. If $|\nabla U(\boldsymbol{x}_0, t_0)| > 0$ at some $(\boldsymbol{x}_0, t_0) \in \mathbb{R}^{n+1}$. Then, putting

$$\boldsymbol{\xi} = \frac{\nabla U(\boldsymbol{x}_0, t_0)}{|\nabla U(\boldsymbol{x}_0, t_0)|},$$

we have

$$\frac{\partial U}{\partial t}(\boldsymbol{x}_0, t_0) \geq k|\nabla U(\boldsymbol{x}_0, t_0)| > 0.$$

Now we proved (4.22). If $U(\boldsymbol{x}, t)$ depends only on $t \in \mathbb{R}$ and is independent of $\boldsymbol{x} \in \mathbb{R}^n$, we have (4.23) from $U(\mathbf{0}, 0) = \theta_0$. Otherwise, we have $(\boldsymbol{x}_0, t_0) \in \mathbb{R}^{n+1}$ with

$$\frac{\partial U}{\partial t}(\boldsymbol{x}_0, t_0) \geq k|\nabla U(\boldsymbol{x}_0, t_0)| > 0.$$

Then we obtain (4.23) from the strong maximum principle. This completes the proof. □

Using the Schauder estimate [29, Proposition 2.9], we have

$$\sup_{(\boldsymbol{x},t)\in\mathbb{R}^{n+1}} \left|\mathrm{D}_j U(\boldsymbol{x}, t)\right| < \infty, \qquad \sup_{(\boldsymbol{x},t)\in\mathbb{R}^{n+1}} \left|\mathrm{D}_{ij} U(\boldsymbol{x}, t)\right| < \infty, \tag{4.24}$$

$$\sup_{(\boldsymbol{x},t)\in\mathbb{R}^{n+1}} |U_t(\boldsymbol{x}, t)| < \infty, \qquad \sup_{(\boldsymbol{x},t)\in\mathbb{R}^{n+1}} \left|\mathrm{D}_j U_t(\boldsymbol{x}, t)\right| < \infty \tag{4.25}$$

for $1 \leq i, j, \ell \leq n+1$.

Now we will further study properties of U.

Proposition 4.4. *Let U satisfy the assumption of Proposition 4.3. Let Ω_0 be any given compact set in $\mathbb{R}^n$. Then one has*

$$\lim_{t\to\infty} \min_{\boldsymbol{x}\in\Omega_0} U(\boldsymbol{x}, t) = 1, \tag{4.26}$$

$$\lim_{t\to-\infty} \max_{\boldsymbol{x}\in\Omega_0} U(\boldsymbol{x}, t) = 0. \tag{4.27}$$

The convergence is uniform for all entire solutions that satisfy the assumption of Proposition 4.3.

Proof. Using (1.9), we have

$$\left(\frac{\partial}{\partial t} - \Delta - f'(U)\right) U_t = 0, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1}.$$

For any given $t_0 \in \mathbb{R}$, let $w(\boldsymbol{x}, s)$ be defined by

$$\left(\frac{\partial}{\partial t} - \Delta + M\right) w = 0, \qquad \boldsymbol{x} \in \mathbb{R}^n,\ s > t_0,$$

$$w(\boldsymbol{x}, t_0) = U_t(\boldsymbol{x}, t_0), \qquad \boldsymbol{x} \in \mathbb{R}^n.$$

Then we have

$$w(\boldsymbol{x}, s) \leq U_t(\boldsymbol{x}, s), \qquad \boldsymbol{x} \in \mathbb{R}^n,\ s \geq t_0,$$

that is,

$$\int_{\mathbb{R}^n} \frac{e^{-M(s-t_0)}}{(4\pi(s-t_0))^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4(s-t_0)}\right) U_t(y,t_0)\,\mathrm{d}y \le U_t(\boldsymbol{x},s), \quad \boldsymbol{x}\in\mathbb{R}^n,\ s\ge t_0.$$

Thus we obtain

$$\int_{\mathbb{R}^n} \frac{e^{-M}}{(4\pi)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4}\right) U_t(y,t-1)\,\mathrm{d}y \le U_t(\boldsymbol{x},t), \quad (\boldsymbol{x},t)\in\mathbb{R}^{n+1}. \tag{4.28}$$

First we prove (4.26). Since U is strictly monotone increasing in t, we have

$$-1 < \min_{\boldsymbol{x}\in\Omega_0} U(\boldsymbol{x},0) \le U(\boldsymbol{x},t) < 1, \qquad \boldsymbol{x}\in\Omega_0, t\ge 0.$$

Then we have

$$\lim_{t\to\infty} \max_{\boldsymbol{x}\in\Omega_0} U(\boldsymbol{x},t) = 1. \tag{4.29}$$

Indeed, otherwise we have

$$-1 < \min_{\boldsymbol{x}\in\Omega_0} U(\boldsymbol{x},0) \le U(\boldsymbol{x},t) \le \lim_{s\to\infty} \max_{\boldsymbol{x}\in\Omega_0} U(\boldsymbol{x},s) < 1, \qquad \boldsymbol{x}\in\Omega_0,\ t\ge 0.$$

Then, from (4.18), (4.17) and (4.28), $U(\boldsymbol{x},t)$ converges to 1 for all $\boldsymbol{x}\in\Omega_0$, which is a contradiction. Thus we obtain (4.29). Since U is strictly monotone increasing in t, we have the limit function

$$\overline{U}(\boldsymbol{x}) = \lim_{t\to\infty} U(\boldsymbol{x},t), \qquad \boldsymbol{x}\in\Omega_0$$

with

$$\max_{\Omega_0} \overline{U} = 1.$$

Then, from (4.22), $\overline{U}$ has to equal 1 on Ω_0. Now we proved (4.26).

Next we prove (4.27). Since U is strictly monotone increasing in t, we have

$$-1 < U(\boldsymbol{x},t) \le \max_{\boldsymbol{x}\in\Omega_0} U(\boldsymbol{x},0) < 1, \qquad \boldsymbol{x}\in\Omega_0,\ t\le 0.$$

We can define

$$\underline{U}(\boldsymbol{x}) = \lim_{t\to-\infty} U(\boldsymbol{x},t), \qquad \boldsymbol{x}\in\Omega_0$$

and have

$$-1 \le \underline{U}(\boldsymbol{x}) \le \max_{\boldsymbol{x}\in\Omega_0} U(\boldsymbol{x},0) < 1, \qquad \boldsymbol{x}\in\Omega_0.$$

If

$$-1 < \inf_{\Omega_0} \underline{U},$$

we have

$$-1 < \lim_{s\to-\infty} \min_{\boldsymbol{x}\in\Omega_0} U(\boldsymbol{x},s) \le U(\boldsymbol{x},t) \le \max_{\boldsymbol{x}\in\Omega_0} U(\boldsymbol{x},0) < 1, \qquad \boldsymbol{x}\in\Omega_0,\ t\le 0.$$

Then, from (4.18), (4.17) and (4.28), $U(\boldsymbol{x},t)$ converges to -1 for all $\boldsymbol{x}\in\Omega_0$ as $t\to-\infty$. This is a contradiction. Thus we obtain

$$-1 = \inf_{\Omega_0} \underline{U}.$$

Then, from (4.22), $\underline{U}$ has to equal -1 on Ω_0. Now we proved (4.27). The convergence of (4.26) and that of (4.27) are uniform with respect to all U, because the arguments of Proposition 4.3 and Proposition 4.4 can be done only under the assumption of (4.15), (4.16) and (4.17). This completes the proof. □

Proof of Theorem 1.3. Theorem 1.3 follows from (4.12), (4.13), (4.14), Proposition 4.3 and Proposition 4.4. □

5. Asymptotic shapes of level sets for entire solutions

In this section, let $\mathcal{A}$ be a closed subset in $\mathbb{R}^n$ with $\overline{B(\mathbf{0};1)}\subset\mathcal{A}$ and let U satisfy (1.9) and (4.18) with

$$\begin{aligned}
&0<U(\boldsymbol{x},t)<1, \qquad (\boldsymbol{x},t)\in\mathbb{R}^{n+1},\\
&U(\mathbf{0},0)=\theta_0,\\
&\frac{1}{k}\frac{\partial U}{\partial t}(\boldsymbol{x},t)-(\boldsymbol{\xi},\nabla U(\boldsymbol{x},t))\ge 0, \qquad (\boldsymbol{x},t)\in\mathbb{R}^{n+1},\ \boldsymbol{\xi}\in\mathcal{A}.
\end{aligned}$$

Now we define

$$\Omega_{s_0}(t)=\left\{\boldsymbol{x}\in\mathbb{R}^n \mid U(\boldsymbol{x},t)\le s_0\right\}, \quad \partial\Omega_{s_0}(t)=\left\{\boldsymbol{x}\in\mathbb{R}^n \mid U(\boldsymbol{x},t)=s_0\right\}$$

for $t\in\mathbb{R}$. We have $\Omega_{s_0}(t)\ne\emptyset$ for $t\in(-\infty,0)$ if $|t|$ is large enough, say, $t\in(-\infty,T_0]$ with $T_0\in(0,\infty)$. As $t\to-\infty$, $\Omega_{s_0}(t)$ expands by Proposition 4.4. We study the asymptotic shape of $\partial\Omega_{s_0}(t)$ as $t\to-\infty$.

The following theorem characterizes the asymptotic shapes of level sets of entire solutions including those of polyhedral entire solutions.

Theorem 5.1. *Let $\mathcal{A}$ be a closed subset in $\mathbb{R}^n$ with $\overline{B(\mathbf{0};1)} \subset \mathcal{A}$. Let U satisfy (1.9) and (4.18) with*

$$0 < U(\boldsymbol{x}, t) < 1, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1}, \tag{5.1}$$

$$U(\mathbf{0}, 0) = \theta_0, \tag{5.2}$$

$$\frac{1}{k}\frac{\partial U}{\partial t}(\boldsymbol{x}, t) - (\boldsymbol{\xi}, \nabla U(\boldsymbol{x}, t)) \geq 0, \qquad (\boldsymbol{x}, t) \in \mathbb{R}^{n+1},\ \boldsymbol{\xi} \in \mathcal{A}. \tag{5.3}$$

Assume $(\tau_i)_{i\in\mathbb{N}}$ and $(R_i)_{i\in\mathbb{N}}$ satisfy

$$0 \leq \tau_i \leq \tau_{i+1}, \quad 0 < R_i \leq R_{i+1}, \qquad i \in \mathbb{N},$$
$$\overline{B(\boldsymbol{y}_i; R_i)} \subset \Omega_{s_0}(-\tau_i), \quad \partial B(\boldsymbol{y}_i; R_i) \cap \partial\Omega_{s_0}(-\tau_i) \neq \emptyset$$

with some $\boldsymbol{y}_i \in \Omega_{s_0}(-\tau_i)$. Let $z_i \in \partial B(\boldsymbol{y}_i; R_i) \cap \partial\Omega_{s_0}(-\tau_i)$ be arbitrarily chosen and let $\boldsymbol{\xi}_i \in \mathcal{A}$ be any vector pointing from $\boldsymbol{y}_i$ to z_i with $|\boldsymbol{\xi}_i| \geq 1$. If one assumes $\lim_{i\to\infty} R_i = \infty$ in addition, one has

$$\lim_{i\to\infty} \left|\boldsymbol{\xi}_i\right| = 1, \tag{5.4}$$

$$\lim_{i\to\infty} \sup_{\boldsymbol{x}\in\overline{B(\mathbf{0};X)}} \left|U(\boldsymbol{x} + z_i, t) - \Phi(-(\boldsymbol{x}, \boldsymbol{\xi}_i))\right| = 0 \tag{5.5}$$

for any given $X \in (0, \infty)$. The convergence is uniform in U's that satisfy the assumption and is also uniform in $(t, \boldsymbol{y})$.

Proof. For every $i \in \mathbb{N}$, we change variables with respect to $\boldsymbol{x} \in \mathbb{R}^n$ if necessary, and we can assume

$$\boldsymbol{\xi}_i = \lambda \boldsymbol{e}_n, \qquad i \in \mathbb{N},$$

where $\boldsymbol{e}_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$ without loss of generality. We have either $\lambda > 1$ or $\lambda = 1$. We will show that only $\lambda = 1$ can happen if we can take the radius of a ball arbitrarily large. Now (4.17) in Proposition 4.3 gives

$$\frac{1}{k}\mathrm{D}_t U(\boldsymbol{x}, -\tau_i) - \lambda \mathrm{D}_n U(\boldsymbol{x}, -\tau_i) \geq 0, \qquad \boldsymbol{x} \in \mathbb{R}^n,\ i \in \mathbb{N}.$$

Then we have

$$z_i = \boldsymbol{y}_i + R_i \boldsymbol{e}_n \in \partial\Omega_{s_0}(-\tau_i),$$

and

$$U(z_i, -\tau_i) = s_0.$$

We set

$$z = -x_n - \lambda k t - \lambda k \tau_i + R_i + (\boldsymbol{y}_i, \boldsymbol{e}_n)$$

and define W by

$$W(\boldsymbol{x}', z, t) = U(\boldsymbol{x} + \boldsymbol{y}_i', x_n, t)$$

for $(\boldsymbol{x}', z, t) \in \mathbb{R}^{n+1}$. Now we put $\boldsymbol{x}' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $\boldsymbol{y}_i'$ is defined by $\boldsymbol{y}_i = (\boldsymbol{y}_i', (\boldsymbol{y}_i, \boldsymbol{e}_n))$. Then we have

$$\mathrm{D}_t W(\boldsymbol{x}', z, -\tau_i) = \mathrm{D}_t U(\boldsymbol{x}', x_n, -\tau_i) - \lambda k \mathrm{D}_n U(\boldsymbol{x}', x_n, -\tau_i) \geq 0, \quad (\boldsymbol{x}', z) \in \mathbb{R}^n \tag{5.6}$$

and

$$W(\boldsymbol{0}, -\tau_i) = s_0. \tag{5.7}$$

Now W satisfies

$$\mathrm{D}_t W(\boldsymbol{x}', z, t) = \left(\sum_{j=1}^{n-1} \mathrm{D}_j^2 + D_z^2 \right) W(\boldsymbol{x}', z, t) + \lambda k \mathrm{D}_z W(\boldsymbol{x}', z, t) + f(W(\boldsymbol{x}', z, t)) \tag{5.8}$$

for $(\boldsymbol{x}', z, t) \in \mathbb{R}^{n+1}$. We set

$$W_i(\boldsymbol{x}', z) = W(\boldsymbol{x}', z, -\tau_i), \qquad (\boldsymbol{x}', z) \in \mathbb{R}^n$$

and have

$$W_i(\boldsymbol{0}) = s_0, \qquad i \in \mathbb{N}.$$

In view of (5.6), W_i is a subsolution of an elliptic equation

$$\left(\sum_{j=1}^{n-1} \mathrm{D}_j^2 + D_z^2 \right) W + \lambda k \mathrm{D}_z W + f(W) = 0, \qquad (\boldsymbol{x}', z) \in \mathbb{R}^n$$

for every $i \in \mathbb{N}$. Taking a subsequence if necessary, we have

$$W_\infty(\boldsymbol{x}', z) = \lim_{i \to \infty} W_i(\boldsymbol{x}', z), \qquad (\boldsymbol{x}', z) \in \mathbb{R}^n$$

with

$$W_\infty(\boldsymbol{0}) = s_0. \tag{5.9}$$

The Schauder estimate (4.24) and (4.25) imply that the convergence is uniform in U's that satisfy the assumption of Theorem 5.1. Now W_∞ is a subsolution of

$$\left(\sum_{j=1}^{n-1} \mathrm{D}_j^2 + D_z^2\right) W + \lambda k \mathrm{D}_z W + f(W) = 0, \qquad (\boldsymbol{x}', z) \in \mathbb{R}^n. \tag{5.10}$$

So far we have not used an assumption $\lim_{i\to\infty} R_i = \infty$. For any $R \in (0,\infty)$, $\tau \in [T_0,\infty)$, $\boldsymbol{y} \in \Omega_{s_0}(-\tau)$ and $\boldsymbol{z} \in \partial B(\boldsymbol{y}; R) \cap \partial\Omega_{s_0}(-\tau)$, we can take $\tau_i = \tau$, $R_i = R$, $\boldsymbol{y}_i = \boldsymbol{y}$ and $\boldsymbol{z}_i = \boldsymbol{z}$ for all $i \in \mathbb{N}$. Because W is a solution of (5.8) with (5.7) and W_∞ is a subsolution of (5.10) with (5.9), the speed of $\partial\Omega_{s_0}(t)$ at the point of contact is no less than $k\lambda$ at $t = -\tau$. Thus the speed of $\partial\Omega_{s_0}(t)$ at the point of contact is no less than $k\lambda$ for all $t \in (-\infty, -T_0]$.

For any bounded and uniformly continuous function $\widetilde{u}_0$ from $\mathbb{R}^n$ to $\mathbb{R}$, we consider

$$\mathrm{D}_t \widetilde{w} = \left(\sum_{j=1}^{n-1} \mathrm{D}_j^2 + D_z^2\right) \widetilde{w} + \lambda k \mathrm{D}_z \widetilde{w} + f(\widetilde{w}), \qquad (\boldsymbol{x}', z) \in \mathbb{R}^n,\ t > 0,$$
$$\widetilde{w}(\boldsymbol{x}', z, 0) = \widetilde{u}_0(\boldsymbol{x}', z), \qquad (\boldsymbol{x}', z) \in \mathbb{R}^n$$

and write the solution as

$$\widetilde{w}(\boldsymbol{x}', z, t; \widetilde{u}_0), \qquad (\boldsymbol{x}', z, t) \in \mathbb{R}^n \times [0,\infty).$$

Then, for every $i \in \mathbb{N}$, we have

$$W_\infty(\boldsymbol{x}', z) \le \widetilde{w}(\boldsymbol{x}', z, t; W_\infty), \qquad (\boldsymbol{x}', z, t) \in \mathbb{R}^n \times [0,\infty).$$

Now we have

$$\widetilde{w}(\boldsymbol{0}', 0, t; W_\infty) \ge s_0, \qquad t \ge 0. \tag{5.11}$$

We write $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}_- = (-\infty, 0)$. Now we have

$$s_0 \chi_{\mathbb{R}_+}(z) + \chi_{\mathbb{R}_-}(z) = \begin{cases} s_0 & \text{if} \quad z > 0, \\ 1 & \text{if} \quad z < 0. \end{cases}$$

Here $\chi_{\mathbb{R}_+}$ and $\chi_{\mathbb{R}_-}$ are the characteristic functions of $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. Since the radius of the ball goes to infinity as $i \to \infty$, we have

$$W_\infty(\boldsymbol{x}', z) \le s_0 \chi_{\mathbb{R}_+}(z) + \chi_{\mathbb{R}_-}(z), \qquad (\boldsymbol{x}', z) \in \mathbb{R}^n.$$

We define

$$W_0(\boldsymbol{x}', z) = s_0 \chi_{\mathbb{R}_+}(z) + \chi_{\mathbb{R}_-}(z), \qquad (\boldsymbol{x}', z) \in \mathbb{R}^n$$

and have

$$W_\infty(\boldsymbol{x}', z) \le W_0(\boldsymbol{x}', z) \qquad (\boldsymbol{x}', z) \in \mathbb{R}^n. \tag{5.12}$$

Note that W_0 depends on z and is independent of $\boldsymbol{x}'$. Now we have

$$u(\boldsymbol{x}', z, t; \widetilde{u}_0) = \widetilde{w}(\boldsymbol{x}', z - \lambda kt, t; \widetilde{u}_0), \qquad (\boldsymbol{x}', z, t) \in \mathbb{R}^n \times (0, \infty), \tag{5.13}$$

where

$$\mathrm{D}_t u = \left(\sum_{j=1}^{n-1} \mathrm{D}_j^2 + D_z^2 \right) u + f(u), \qquad (\boldsymbol{x}', z) \in \mathbb{R}^n,\ t > 0,$$
$$u(\boldsymbol{x}', z, 0) = \widetilde{u}_0(\boldsymbol{x}', z), \qquad (\boldsymbol{x}', z) \in \mathbb{R}^n.$$

For a reaction-diffusion equation (1.1) in the one-dimensional space $\mathbb{R}$, the planar traveling front is asymptotically stable. See [8,3,29] for instance. This implies

$$\lim_{t\to\infty} \sup_{(\boldsymbol{x}', z)\in\mathbb{R}^n} \left| u(\boldsymbol{x}', z, t; W_0) - \Phi(z - kt - z_0) \right| = 0 \tag{5.14}$$

for some $z_0 \in \mathbb{R}$. Here z_0 depends only on (f, s_0). For any given $\varepsilon > 0$, we have

$$\Phi(z - kt - z_0) - \varepsilon < u(\boldsymbol{x}', z, t; W_0) < \Phi(z - kt - z_0) + \varepsilon, \qquad (\boldsymbol{x}', z) \in \mathbb{R}^n$$

if $t \in (0, \infty)$ is large enough. Using (5.11) and (5.12), we find

$$\widetilde{w}(\boldsymbol{x}', z, t; W_\infty) \le \widetilde{w}(\boldsymbol{x}', z, t; W_0), \qquad (\boldsymbol{x}', z, t) \in \mathbb{R}^n \times (0, \infty),$$
$$s_0 \le \widetilde{w}(\mathbf{0}', 0, t; W_\infty) \le \widetilde{w}(\mathbf{0}', 0, t; W_0), \quad t \in (0, \infty).$$

Recalling (5.13), we get

$$s_0 \le \widetilde{w}(\mathbf{0}', 0, t; W_0) = u(\mathbf{0}', \lambda kt, t; W_0) \qquad t \in (0, \infty).$$

Thus we obtain

$$s_0 < \Phi((\lambda - 1)kt - z_0) + \varepsilon, \qquad t \in (0, \infty).$$

Taking $0 < \varepsilon < s_0/2$ and sending $t \to \infty$, we get a contradiction on the definition of s_0 if $\lambda > 1$. Now we obtain $\lambda = 1$.

Using (5.4), for any given $X \in (0, \infty)$, we can approximate $\{\boldsymbol{x} \in \mathbb{R}^n \mid U(\boldsymbol{x}, -\tau_i) = s_0\}$ by a plane on $\overline{B(\boldsymbol{z}; X)}$, if $i \in \mathbb{N}$ is large enough. This fact and (4.18) imply that $\{\boldsymbol{x} \in \mathbb{R}^n \mid U(\boldsymbol{x}, -\tau_i) = \theta\}$ is approximated by a plane for every $\theta \in (0, 1)$, if $i \in \mathbb{N}$ is large enough. That is, $U(\boldsymbol{x}, -\tau_i)$ can be approximated by a function of $-(\boldsymbol{x}, \boldsymbol{e}_n)$ on $\overline{B(\boldsymbol{z}; X)}$, if $i \in \mathbb{N}$ is large enough. Then we obtain (5.5) using $\Phi(0) = s_0$, (4.18), Lemma 4.2 and the asymptotic stability of a one-dimensional traveling front as in (5.14). □

Remark 5.2. As is stated in the proof of Theorem 5.1, one has the following lower estimate of the speed of $\partial\Omega_{s_0}(t)$. Let U be as in Theorem 5.1. For any fixed $t \in (-\infty, -T_0]$, let $R \in (0, \infty)$, $\boldsymbol{y} \in \mathbb{R}^n$ and $\boldsymbol{z} \in \mathbb{R}^n$ satisfy

$$\overline{B(\boldsymbol{y}; R)} \subset \Omega_{s_0}(t), \quad \boldsymbol{z} \in \partial B(\boldsymbol{y}; R) \cap \partial\Omega_{s_0}(t).$$

Let $\boldsymbol{\xi} \in \mathcal{A}$ satisfy $|\boldsymbol{\xi}| \geq 1$ and

$$\frac{\boldsymbol{z} - \boldsymbol{y}}{|\boldsymbol{z} - \boldsymbol{y}|} = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}.$$

Then the speed of $\partial\Omega_{s_0}(t)$ toward the $-\boldsymbol{\xi}$-direction at the point of contact $\boldsymbol{z}$ is no less than $k|\boldsymbol{\xi}|$.

6. Properties of polyhedral entire solutions

Let $c \in (k, \infty)$ be arbitrarily given. In this section we denote $v_{\text{pmd}}(\boldsymbol{x}, x_{n+1})$ in Theorem 1.1 by $V(\boldsymbol{x}, x_{n+1}) = V_c(\boldsymbol{x}, x_{n+1})$ for simplicity. Then we have

$$V_c(\boldsymbol{0}, 0) = \theta_0,$$
$$V_c(\mu_0^c \boldsymbol{a}_j, z_{\text{anc}}) = \theta_0, \qquad j \in \{1, \ldots, m\}.$$

Now U given by (4.9) satisfies the assumption of Theorem 5.1 for a base set $\mathcal{A}$ in (1.17). For any $X \in (0, \infty)$ and $t \in (0, \infty)$, we have $\overline{B(\boldsymbol{0}; X)} \subset \{\boldsymbol{x} \in \mathbb{R}^n \mid U(\boldsymbol{x}, t) < s_0\}$ if $|t|$ is large enough.

Now we define

$$\zeta = \max_{i \neq j} \left(\boldsymbol{a}_i, \boldsymbol{a}_j\right)$$

and have $\zeta < 1$. For $j \in \{1, \ldots, m\}$, we define

$$\Omega_j^0 = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \left(\boldsymbol{a}_j, \boldsymbol{x}\right) \geq \max_{i \in \{1,\ldots,m\} \setminus \{j\}} (\boldsymbol{a}_i, \boldsymbol{x}) + \frac{1-\zeta}{2}|\boldsymbol{x}| \right\}.$$

For $j \in \{1, \ldots, m\}$, we have

$$\{\sigma \boldsymbol{a}_j \mid \sigma \geq 0\} \subset \Omega_j^0,$$
$$\{\sigma \boldsymbol{a}_i \mid \sigma \geq 0\} \cap \Omega_j^0 = \{\boldsymbol{0}\} \qquad \text{if} \quad i \in \{1, \ldots, m\} \setminus \{j\}.$$

Now we prove Theorem 1.4.

Proof of Theorem 1.4. For any $t \in \mathbb{R}$ and any $\boldsymbol{\zeta} \in \mathbb{S}^{n-1}$ with $\{\tau\boldsymbol{\zeta} \mid \tau \geq 0\} \cap \partial A \neq \emptyset$, we choose $s(t, \boldsymbol{\zeta}) \in [0, \infty)$ and $\sigma(t, \boldsymbol{\zeta}) \in [0, \infty)$ with

$$U(s(t, \boldsymbol{\zeta})\boldsymbol{\zeta}, t) = s_0, \quad U(\sigma(t, \boldsymbol{\zeta})\boldsymbol{\zeta}, t) = \theta_0$$

if they exist. Let $\varepsilon > 0$ be arbitrarily given. Let $(X_i)_{i \in \mathbb{N}}$ satisfy

$$0 < X_i < X_{i+1}, \qquad i \in \mathbb{N},$$
$$\lim_{i \to \infty} X_i = \infty.$$

In view of Proposition 4.3, choosing $(\tau_i)_{i \in \mathbb{N}}$ with

$$0 \le \tau_i \le \tau_{i+1}, \qquad i \in \mathbb{N},$$

we have

$$\overline{B(\mathbf{0}; X_i)} \subset \Omega_{s_0}(-\tau_i), \qquad i \in \mathbb{N}.$$

Let $\boldsymbol{y}_i$ be any point in $\Omega_{s_0}(-\tau_i)$ with

$$\partial B(\boldsymbol{y}_i; X_i) \cap \partial\Omega_{s_0}(-\tau_i) \neq \emptyset.$$

By Theorem 5.1 and taking subsequences of $(X_i)_{i\in\mathbb{N}}$ and $(\tau_i)_{i\in\mathbb{N}}$ if necessary, we have $j \in \{1, \dots, m\}$ such that

$$\partial B(\boldsymbol{y}_i; X_i) \cap \partial\Omega_{s_0}(-\tau_i) \cap \Omega_j^0 \neq \emptyset, \qquad i \in \mathbb{N}.$$

Let this j be fixed. We choose

$$\boldsymbol{z}_i \in \partial B(\boldsymbol{y}_i; X_i) \cap \partial\Omega_{s_0}(-\tau_i) \cap \Omega_j^0$$

arbitrarily. Using Theorem 5.1 and taking $i \in \mathbb{N}$ large enough, we obtain

$$\left| U(\boldsymbol{x} + \boldsymbol{z}_i, -\tau_i) - \Phi(-(\boldsymbol{x}, \boldsymbol{a}_j)) \right| < \varepsilon$$

if $\boldsymbol{x} + \boldsymbol{z}_i \in \Omega_j^0$. Since $\varepsilon > 0$ can be taken arbitrarily small, this implies that there exists $s(-\tau_i, \boldsymbol{a}_j)$ that satisfies $U(s(-\tau_i, \boldsymbol{a}_j)\boldsymbol{a}_j, -\tau_i) = s_0$ for sufficiently large i. Now we set $T_{\mathrm{anc}} = \tau_i$ and know that $s(-T_{\mathrm{anc}}, \boldsymbol{a}_j)$ exists. Since we can choose $(\tau_\ell)_{\ell > i}$ arbitrarily, we have

$$\left| U(\boldsymbol{x} + s(t, \boldsymbol{a}_j)\boldsymbol{a}_j, t) - \Phi(-(\boldsymbol{x}, \boldsymbol{a}_j)) \right| < \varepsilon$$

if $t \in (-\infty, -T_{\mathrm{anc}}]$ and $\boldsymbol{x} + s(t, \boldsymbol{a}_j)\boldsymbol{a}_j \in \Omega_j^0$. Then we have

$$\limsup_{\sigma_1 \to \infty} U(\sigma_1 \boldsymbol{a}_j, -T_{\mathrm{anc}}) > \theta_0.$$

Recalling the definition (4.9) of U and the definition (1.18) of μ_0^c, we obtain

$$0 < \liminf_{c \to \infty} \mu_0^c \le \limsup_{c \to \infty} \mu_0^c < \infty.$$

Consequently we find

$$\limsup_{\sigma_1 \to \infty} U(\sigma_1 \boldsymbol{a}_j, -T_{\mathrm{anc}}) > \theta_0 \qquad \text{for every } j \in \{1, \dots, m\}.$$

Thus $s(-T_{\mathrm{anc}}, \boldsymbol{a}_j)$ and $\sigma(-T_{\mathrm{anc}}, \boldsymbol{a}_j)$ exist for every $j \in \{1, \dots, m\}$. By Theorem 5.1, we have, for every $j \in \{1, \dots, m\}$,

$$\left| U(\boldsymbol{x} + s(t, \boldsymbol{a}_j)\boldsymbol{a}_j, t) - \Phi(-(\boldsymbol{x}, \boldsymbol{a}_j)) \right| < \varepsilon \tag{6.1}$$

if $t \in (-\infty, -T_{\rm anc}]$ and $\boldsymbol{x} + s(t, \boldsymbol{a}_j)\boldsymbol{a}_j \in \Omega_j^0$. Then, using Proposition 4.3 and Theorem 5.1, we obtain, for every $\boldsymbol{\zeta} \in \mathbb{S}^{n-1}$ with $\{\tau\boldsymbol{\zeta} \mid \tau \geq 0\} \cap \partial\mathcal{A} \neq \emptyset$,

$$\limsup_{\sigma_1 \to \infty} U(\sigma_1\boldsymbol{\zeta}, -T_{\rm anc}) > \theta_0.$$

Thus we can define $\sigma(-T_{\rm anc}, \boldsymbol{\zeta})$ for every $\boldsymbol{\zeta} \in \mathbb{S}^{n-1}$ with $\{\tau\boldsymbol{\zeta} \mid \tau \geq 0\} \cap \partial\mathcal{A} \neq \emptyset$. Then, using (6.1), we can define $\sigma(t, \boldsymbol{\zeta})$ for every $t \in (-\infty, -T_{\rm anc}]$ and every $\boldsymbol{\zeta} \in \mathbb{S}^{n-1}$ with $\{\tau\boldsymbol{\zeta} \mid \tau \geq 0\} \cap \partial\mathcal{A} \neq \emptyset$. Now we have

$$\left| U(\boldsymbol{x} + \sigma(t, \boldsymbol{a}_j)\boldsymbol{a}_j, t) - \Phi(-(\boldsymbol{x}, \boldsymbol{a}_j) + \Phi^{-1}(\theta_0)) \right| < \varepsilon \tag{6.2}$$

if $t \in (-\infty, -T_{\rm anc}]$ and $\boldsymbol{x} + \sigma(t, \boldsymbol{a}_j)\boldsymbol{a}_j \in \Omega_j^0$. Since the speed of a planar traveling front is k, we have

$$\lim_{t \to -\infty} \frac{\sigma(t, \boldsymbol{a}_j)}{-t} = k$$

for every $j \in \{1, \ldots, m\}$. The influence by $\mathbb{R}^n \backslash \Omega_j^0$ decays exponentially as $t \to -\infty$ by Lemma 4.2. Then, for every $j \in \{1, \ldots, m\}$, we obtain

$$\lim_{t \to -\infty} \left| \sigma(t, \boldsymbol{a}_j) + kt - \nu_j \right| = 0 \tag{6.3}$$

with some $\nu_j \in \mathbb{R}$ for $j \in \{1, \ldots, m\}$. Let $\boldsymbol{\zeta} \in \mathbb{S}^{n-1}$ satisfy $\{\tau\boldsymbol{\zeta} \mid \tau \geq 0\} \cap \partial\mathcal{A} \neq \emptyset$. Then we have $m_{\boldsymbol{\zeta}} \geq 2$ or $m_{\boldsymbol{\zeta}} = 1$. If $m_{\boldsymbol{\zeta}} = 1$, U converges to a planar traveling front on every compact neighborhood of $s(t, \boldsymbol{\zeta})$ as $t \to -\infty$ by Theorem 5.1. Let $J_{\boldsymbol{\eta}} = \{j\}$ with $j \in \{1, \ldots, m\}$. The speed of this planar front to a direction $-\boldsymbol{\zeta}$ equals $c_{\boldsymbol{\zeta}}$ given by (1.22). Choosing $\mu_j \in \mathbb{R}$ with $|\mu_j| \leq \sqrt{1 - (\rho_{\boldsymbol{\zeta}})^2}|\nu_j|$, we obtain (1.24). Assume $m_{\boldsymbol{\zeta}} \geq 2$. For every $j \in J_{\boldsymbol{\zeta}}$, let $\mu_j \in [0, \infty)$ be arbitrarily given and let g be given by (1.23). We consider a pyramid $\{(\boldsymbol{y}', g(\boldsymbol{y}')) \mid \boldsymbol{y}' \in \mathbb{R}^{n-1}\}$ with its edge E_g. See (7.1) for the edge of a pyramid. Let $v_{\rm pmd}(\boldsymbol{y}; g)$ be the pyramidal traveling front associated with a pyramid $\{(\boldsymbol{y}', g(\boldsymbol{y}')) \mid \boldsymbol{y}' \in \mathbb{R}^{n-1}\}$. The speed of $v_{\rm pmd}(\boldsymbol{y}; g)$ equals $c_{\boldsymbol{\eta}}$. Then Theorem 5.1 implies that U converges to planar traveling fronts at any points that are away from the edge of the pyramid. Using (6.2) and (6.3), we can choose $\mu_j \in \mathbb{R}$ for every $j \in J_{\boldsymbol{\zeta}}$ such that we have, for any given $R \in (0, \infty)$,

$$\sup_{\boldsymbol{y} \in D(\gamma) \cap \overline{B(\boldsymbol{0}; R)}} \left| U(\boldsymbol{x} + \sigma(t, \boldsymbol{\zeta})\boldsymbol{\zeta}, t) - v_{\rm pmd}(\boldsymbol{y}; g) \right| < \varepsilon$$

if $\min\{\gamma, |t|\}$ is large enough, where $\boldsymbol{y}$ is given by (1.20). Here $D(\gamma)$ is given by (7.2) for E_g. Now an n-dimensional pyramidal traveling front is asymptotically stable for perturbations around the edge as in Theorem 7.1 and Remark 7.2. Then, using Theorem 7.1 and Lemma 4.2, we obtain (1.24). Moreover we obtain (1.25) with some $\nu_{\boldsymbol{\zeta}} \in \mathbb{R}$ that depends on $\boldsymbol{\zeta}$. This completes the proof of Theorem 1.4. □

Declaration of competing interest

The author states that there is no conflict of interest.

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Appendix 7

Let $c \in (k, \infty)$ be arbitrarily given. Let m be a positive integer. Let $\boldsymbol{a}_j \in \mathbb{R}^n$ be a unit vector for $1 \le j \le m$. We assume $\boldsymbol{a}_i \ne \boldsymbol{a}_j$ if $i \ne j$. Let $\gamma_j \in \mathbb{R}$ be given for $1 \le j \le m$. For $j \in \{1, \dots, m\}$, we set

$$\Omega_j = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \left(\boldsymbol{a}_j, \boldsymbol{x}\right) - \gamma_j = \max_{1 \le i \le m} ((\boldsymbol{a}_i, \boldsymbol{x}) - \gamma_i) \right\}$$

and have

$$\mathbb{R}^n = \bigcup_{1 \le j \le m} \Omega_j .$$

We put

$$h_j(\boldsymbol{x}) = m_* \left((\boldsymbol{a}_j, \boldsymbol{x}) - \gamma_j \right) = m_*(\boldsymbol{a}_j, \boldsymbol{x} - \gamma_j \boldsymbol{a}_j), \qquad \boldsymbol{x} \in \mathbb{R}^n$$

for $1 \le j \le m$ and

$$h(\boldsymbol{x}) = \max_{1 \le j \le m} h_j(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^n .$$

Then the graph of $x_{n+1} = h(\boldsymbol{x})$ is a pyramid in $\mathbb{R}^{n+1}$. For $j \in \{1, \dots, m\}$,

$$\{(\boldsymbol{x}, h(\boldsymbol{x})) \mid \boldsymbol{x} \in \Omega_j\}$$

is called a lateral face of the pyramid. We call

$$E = \{(\boldsymbol{x}, h(\boldsymbol{x})) \mid \boldsymbol{x} \in \Omega_i \cap \Omega_j \quad \text{for } i \ne j\} \tag{7.1}$$

the edge of the pyramid. For given $\gamma \in (0, \infty)$, we define

$$D(\gamma) = \left\{ (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1} \mid \operatorname{dist}((\boldsymbol{x}, x_{n+1}), E) > \gamma \right\} . \tag{7.2}$$

Now we set

$$\mathcal{A} = \left\{ \boldsymbol{\xi} \in \mathbb{R}^n \mid \max_{1 \le j \le m} (\boldsymbol{a}_j, \boldsymbol{\xi}) \le 1 \right\} .$$

In this appendix, we show that a pyramidal traveling front $v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h)$ to (1.7) associated with $x_{n+1} = h(\boldsymbol{x})$ satisfies

$$\left(\nabla v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h), \widetilde{\boldsymbol{\xi}}\right) < 0, \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}, \ \boldsymbol{\xi} \in \mathcal{A}$$

if

$$\widetilde{\boldsymbol{\xi}} = (\boldsymbol{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}, \quad \xi_{n+1} \geq m_*.$$

Now we recall the pyramidal traveling fronts in [20,21,23,17,29]. We define

$$v_0(\boldsymbol{x}, x_{n+1}; h) = \Phi\left(\frac{k}{c}(x_{n+1} - h(\boldsymbol{x}))\right) = \max_{1 \leq j \leq m} \Phi\left(\frac{k}{c}\left(x_{n+1} - h_j(\boldsymbol{x})\right)\right), \quad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}.$$

We often write $v_0(\boldsymbol{x}, x_{n+1}; h)$ simply as $v_0(\boldsymbol{x}, x_{n+1})$. Now we have

$$v_0(\boldsymbol{x}, x_{n+1}; h) \leq w(\boldsymbol{x}, x_{n+1}, t; v_0(\,\cdot\,, h)), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}, \ t > 0.$$

First we consider a case

$$\gamma_j = 0, \qquad 1 \leq j \leq m.$$

Then we have

$$h_{\mathrm{zero}}(\boldsymbol{x}) = m_* \max_{1 \leq j \leq m} (\boldsymbol{a}_j, \boldsymbol{x}).$$

For $x_{n+1} = h_{\mathrm{zero}}(\boldsymbol{x})$, we define a pyramidal traveling front by

$$V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h_{\mathrm{zero}}) = \lim_{t \to \infty} w(\boldsymbol{x}, x_{n+1}, t; v_0(\,\cdot\,, h_{\mathrm{zero}}))$$

on every compact set in $\mathbb{R}^{n+1}$ as in [17,29]. Next, for h given by (1.13), we define

$$\overline{v}(\boldsymbol{x}, x_{n+1}; h) = \min_{1 \leq j \leq m} V_{\mathrm{pmd}}(\boldsymbol{x} - \gamma_j \boldsymbol{a}_j, x_{n+1}; h_{\mathrm{zero}}), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}.$$

Now $v_0(\,\cdot\,; h)$ is a weak subsolution and $\overline{v}(\,\cdot\,; h)$ is a weak supersolution to (1.5) with

$$v_0(\boldsymbol{x}, x_{n+1}; h) < \min\{1, \overline{v}(\boldsymbol{x}, x_{n+1}; h)\}, \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}.$$

Then we have

$$v_0(\boldsymbol{x}, x_{n+1}; h) < w(\boldsymbol{x}, x_{n+1}, t; v_0(\,\cdot\,, h)) < w(\boldsymbol{x}, x_{n+1}, t; \overline{v}(\,\cdot\,; h)) < \overline{v}(\boldsymbol{x}, x_{n+1}; h)$$

for $(\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}$ and $t > 0$. With respect to a pyramid $x_{n+1} = h(\boldsymbol{x})$, we define a pyramidal traveling front by

$$V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h) = \lim_{t \to \infty} w(\boldsymbol{x}, x_{n+1}, t; v_0(\,\cdot\,, h)) \tag{7.3}$$

on every compact set in $\mathbb{R}^{n+1}$. The properties of $V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h)$ are as follows.

Theorem 7.1 *(Uniqueness and asymptotic stability of pyramidal traveling fronts). Let h be given by (1.13) and let $D(\gamma)$ be given by (7.2) for $\gamma > 0$. Let $V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h)$ be defined by (7.3). Let a bounded and continuous function u_0 in $\mathbb{R}^{n+1}$ satisfy*

$$-1 - \delta_* \le u_0(\boldsymbol{x}, x_{n+1}) \le 1 + \delta_*, \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1},$$
$$\lim_{\gamma \to \infty} \sup_{(\boldsymbol{x}, x_{n+1}) \in D(\gamma)} \left| u_0(\boldsymbol{x}, x_{n+1}) - V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h) \right| = 0.$$

Then one has

$$\lim_{t \to \infty} \sup_{(\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}} \left| w(\boldsymbol{x}, x_{n+1}, t; u_0) - V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h) \right| = 0.$$

Proof. If $\gamma_j = 0$ for $1 \le j \le m$, the assertion follows from [29, Theorem 7.20]. Its proof is valid even for $\gamma_j \in \mathbb{R}$, $1 \le j \le m$. See the proof of [29, Theorem 7.20] for the details. □

We often write $V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h)$ simply as $V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1})$. Let $\widetilde{\boldsymbol{\xi}} = (\boldsymbol{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}$ be given. Now $v_0(\tau\widetilde{\boldsymbol{\xi}}; h)$ is monotone non-increasing in $\tau \in (0, \infty)$ if

$$\boldsymbol{\xi} \in \mathcal{A}, \quad \xi_{n+1} \ge m_*,$$

and $v_0(\tau\widetilde{\boldsymbol{\xi}}; h)$ converges to 1 as $\tau \to \infty$ if

$$\boldsymbol{\xi} \in \mathcal{A}, \quad \xi_{n+1} < m_*.$$

Using this fact and Theorem 7.1, we get

$$\frac{\partial V_{\mathrm{pmd}}}{\partial \widetilde{\boldsymbol{\xi}}}(\boldsymbol{x}, x_{n+1}; h) < 0, \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1} \tag{7.4}$$

if

$$\boldsymbol{\xi} \in \mathcal{A}, \quad \xi_{n+1} \ge m_*,$$

and $V_{\mathrm{pmd}}(\tau\widetilde{\boldsymbol{\xi}}; h)$ converges to 1 as $\tau \to \infty$ if

$$\boldsymbol{\xi} \in \mathcal{A}, \quad \xi_{n+1} < m_*.$$

By (7.4), there exists a unique $\zeta_c \in \mathbb{R}$ with $V_{\mathrm{pmd}}(\mathbf{0}, -\zeta_c; h) = \theta_0$. Now we define

$$v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h) = V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1} - \zeta_c; h), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}$$

and call it the pyramidal traveling front associated with h given by (1.13).

Remark 7.2. Theorem 7.1 holds true if one replaces $V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h)$ by $v_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h)$.

The following lemma studies the dependency of $V_{\rm pmd}(\boldsymbol{x}, x_{n+1}; h)$ on $(\gamma_1, \dots, \gamma_m)$.

Lemma 7.3. *Let h be given by (1.13) and let $V_{\rm pmd}(\boldsymbol{x}, x_{n+1}; h)$ be defined by (1.14). Then $V_{\rm pmd}(\boldsymbol{x}, x_{n+1}; h)$ depends continuously on $(\gamma_1, \dots, \gamma_m) \in [0, \infty)^m$. Moreover $V_{\rm pmd}(\boldsymbol{x}, x_{n+1}; h)$ is monotone decreasing in γ_j for any fixed $j \in \{1, \dots, m\}$.*

Proof. For any fixed $T \in (0, \infty)$, $w(\boldsymbol{x}, x_{n+1}, T; \cdot)$ is a continuous function in $\mathrm{BU}(\mathbb{R}^{n+1})$. By this fact and the definition (1.14), $V_{\rm pmd}(\boldsymbol{x}, x_{n+1}; h)$ depends continuously on $(\gamma_1, \dots, \gamma_m) \in [0, \infty)^m$. Since $v_0(\boldsymbol{x}, x_{n+1}; h)$ is monotone decreasing in γ_j for any fixed $j \in \{1, \dots, m\}$, $V_{\rm pmd}(\boldsymbol{x}, x_{n+1}; h)$ is monotone decreasing in γ_j for any fixed $j \in \{1, \dots, m\}$. □

For

$$h_{\rm zero}(\boldsymbol{x}) = m_* \max_{1 \le j \le m} (\boldsymbol{a}_j, \boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^n,$$

we have

$$\Omega_j^0 = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \left(\boldsymbol{a}_j, \boldsymbol{x}\right) = \max_{1 \le i \le m} (\boldsymbol{a}_i, \boldsymbol{x}) \right\}$$

and

$$\mathbb{R}^n = \bigcup_{1 \le j \le m} \Omega_j^0. \tag{7.5}$$

We define

$$\Lambda = \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) \le 0 \right\}, \quad \partial\Lambda = \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) = 0 \right\}.$$

Let $\boldsymbol{\eta} \in \Lambda$ be arbitrarily given. Then we have

$$\left(\boldsymbol{a}_j, \boldsymbol{\eta}\right) \le \gamma_j, \qquad 1 \le j \le m$$

and

$$(\boldsymbol{a}_i, \boldsymbol{x}) - \gamma_i \le \max_{1 \le j \le m} \left(\left(\boldsymbol{a}_j, \boldsymbol{x}\right) - \gamma_j \right) \le \max_{1 \le j \le m} \left(\boldsymbol{a}_j, \boldsymbol{x} - \boldsymbol{\eta}\right) = h_{\rm zero}(\boldsymbol{x} - \boldsymbol{\eta}), \qquad 1 \le i \le m.$$

Thus we have

$$h(\boldsymbol{x}) \le \min_{\boldsymbol{\eta} \in \boldsymbol{\Lambda}} h_{\rm zero}(\boldsymbol{x} - \boldsymbol{\eta}), \qquad \boldsymbol{x} \in \mathbb{R}^n. \tag{7.6}$$

Now we prove

$$h(\boldsymbol{x}) = \min_{\boldsymbol{\eta} \in \partial\Lambda} h_{\rm zero}(\boldsymbol{x} - \boldsymbol{\eta}), \qquad \boldsymbol{x} \in \mathbb{R}^n \backslash \Lambda. \tag{7.7}$$

Indeed, we take $i \in \{1, \dots m\}$ arbitrarily and take any $\boldsymbol{\eta} \in \partial\Lambda \cap \Omega_i^0$ with $(\boldsymbol{a}_i, \boldsymbol{\eta}) = \gamma_i$. Then, for any $\boldsymbol{x} \in \mathbb{R}^n \backslash \Lambda$, we have

$$h_{\text{zero}}(\boldsymbol{x} - \boldsymbol{\eta}) = m_* \max_{1 \le j \le m} \left(\boldsymbol{a}_j, \boldsymbol{x} - \boldsymbol{\eta}\right) = m_* \left((\boldsymbol{a}_i, \boldsymbol{x}) - (\boldsymbol{a}_i, \boldsymbol{\eta})\right) \qquad \text{if} \quad \boldsymbol{x} - \boldsymbol{\eta} \in \Omega_i^0.$$

Then we get

$$h_{\text{zero}}(\boldsymbol{x} - \boldsymbol{\eta}) = m_* \left((\boldsymbol{a}_i, \boldsymbol{x}) - \gamma_i\right).$$

Using

$$h(\boldsymbol{x}) \ge m_* \left((\boldsymbol{a}_i, \boldsymbol{x}) - \gamma_i\right), \qquad \boldsymbol{x} \in \mathbb{R}^n,$$

we get

$$h(\boldsymbol{x}) \ge h_{\text{zero}}(\boldsymbol{x} - \boldsymbol{\eta}) \qquad \text{if } \boldsymbol{\eta} \in \partial\Lambda \cap \Omega_i^0 \text{ and } \boldsymbol{x} - \boldsymbol{\eta} \in \Omega_i^0.$$

Combining this inequality, (7.6) and (7.5), we obtain (7.7).

Using (7.7), we find

$$\Phi\left(\frac{k}{c}\left(x_{n+1} - h(\boldsymbol{x})\right)\right) \le \min_{\eta \in \partial\Lambda} \Phi\left(\frac{k}{c}\left(x_{n+1} - h_{\text{zero}}(\boldsymbol{x} - \boldsymbol{\eta})\right)\right)$$

for $(\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}$. Thus we get

$$v_0(\boldsymbol{x}, x_{n+1}; h) \le v_0\left(\boldsymbol{x} - \boldsymbol{\eta}, x_{n+1}; h_{\text{zero}}\right), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}$$

for every $\boldsymbol{\eta} \in \partial\Lambda$. Taking both sides as initial values of (1.5) and taking the limit of $t \to \infty$, we get

$$v_0(\boldsymbol{x}, x_{n+1}; h) \le V_{\text{pmd}}(\boldsymbol{x}, x_{n+1}; h) \le V_{\text{pmd}}\left(\boldsymbol{x} - \boldsymbol{\eta}, x_{n+1}; h_{\text{zero}}\right), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}$$

for every $\boldsymbol{\eta} \in \partial\Lambda$. Thus we obtain

$$v_0(\boldsymbol{x}, x_{n+1}; h) \le V_{\text{pmd}}(\boldsymbol{x}, x_{n+1}; h) \le \min_{\boldsymbol{\eta} \in \partial\Lambda} V_{\text{pmd}}\left(\boldsymbol{x} - \boldsymbol{\eta}, x_{n+1}; h_{\text{zero}}\right)$$

Now we define

$$v_1(\boldsymbol{x}, x_{n+1}) = \min_{\boldsymbol{\eta} \in \Lambda} V_{\text{pmd}}\left(\boldsymbol{x} - \boldsymbol{\eta}, x_{n+1}; h_{\text{zero}}\right).$$

It is a weak supersolution to (1.5), that is, it satisfies

$$w(\boldsymbol{x}, x_{n+1}, t; v_1) \le v_1(\boldsymbol{x}, x_{n+1}), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1},\ t > 0.$$

Using Theorem 7.1, we have

$$V_{\text{pmd}}(\boldsymbol{x}, x_{n+1}; h) = \lim_{t\to\infty} w\left(\boldsymbol{x}, x_{n+1}, t; v_1\right), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}.$$

Let $J_0 \subset \{1, \dots, m\}$ be an arbitrarily given nonempty subset with $J_0 \neq \{1, \dots, m\}$. We define

$$h_{J_0}(\boldsymbol{x}) = m_* \max_{j\in\{1,\dots,m\}\setminus J_0} \left((\boldsymbol{a}_j, \boldsymbol{x}) - \gamma_j\right), \qquad \boldsymbol{x} \in \mathbb{R}^n.$$

Then we have

$$h_{J_0}(\boldsymbol{x}) \le h(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^n.$$

Now we set

$$\Lambda_{J_0} = \left\{\boldsymbol{x} \in \mathbb{R}^n \mid h_{J_0}(\boldsymbol{x}) \le 0\right\}$$

and define

$$R_0 = \min_{j\in J_0} \gamma_j \in [0, \infty). \tag{7.8}$$

The following lemma shows that $V_{\text{pmd}}(\boldsymbol{x}, x_{n+1}; h)$ converges to $V_{\text{pmd}}(\boldsymbol{x}, x_{n+1}; h_{J_0})$ as R_0 goes to infinity.

Lemma 7.4. *Let h be given by (1.13). Let J_0 be a non-empty subset of $\{1, \dots, m\}$ with $J_0 \not\subset \{1, \dots, m\}$. Let R_0 be given by (7.8). Let $V_{\text{pmd}}(\boldsymbol{x}, x_{n+1}; h)$ and $V_{\text{pmd}}(\boldsymbol{x}, x_{n+1}, ; h_{J_0})$ be defined by (1.14) for h and h_{J_0}, respectively. Then, for every compact set $\Omega \subset \mathbb{R}^{n+1}$, one has*

$$\lim_{R_0\to\infty} \sup_{(\boldsymbol{x},x_{n+1})\in\Omega} \left|V_{\text{pmd}}(\boldsymbol{x}, x_{n+1}; h) - V_{\text{pmd}}(\boldsymbol{x}, x_{n+1}; h_{J_0})\right| = 0. \tag{7.9}$$

Proof. Now we have

$$h_{J_0}(\boldsymbol{x}) \le h(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^n$$

and

$$v_0(\boldsymbol{x}, x_{n+1}; h_{J_0}) \le v_0(\boldsymbol{x}, x_{n+1}; h), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}.$$

For every compact set D in $\mathbb{R}^n$, we have

$$\lim_{R_0\to\infty} \max_{\boldsymbol{x}\in D} \left(h(\boldsymbol{x}) - h_{J_0}(\boldsymbol{x})\right) = 0.$$

Then we find

$$\lim_{R_0\to\infty} \max_{\boldsymbol{x}\in D} \left(v_0(\boldsymbol{x}, x_{n+1}; h) - v_0(\boldsymbol{x}, x_{n+1}; h_{J_0})\right) = 0. \tag{7.10}$$

Sending $t \to \infty$ for

$$w(\boldsymbol{x}, x_{n+1}, t; v_0(\,\cdot\,; h_{J_0})) \le w(\boldsymbol{x}, x_{n+1}, t; v_0(\,\cdot\,; h)), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1},\ t > 0,$$

we find

$$V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h_{J_0}) \le V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h), \qquad (\boldsymbol{x}, x_{n+1}) \in \mathbb{R}^{n+1}.$$

Let $\varepsilon > 0$ be arbitrarily given. Using (7.10) and applying Lemma 4.2 to

$$w(\boldsymbol{x}, x_{n+1}, t; v_0(\,\cdot\,; h)) - w(\boldsymbol{x}, x_{n+1}, t; v_0(\,\cdot\,; h_{J_0})),$$

we obtain

$$V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h) \le V_{\mathrm{pmd}}(\boldsymbol{x}, x_{n+1}; h_{J_0}) + \varepsilon$$

on every compact set in $\mathbb{R}^{n+1}$ if R_0 is large enough. Since we can take ε arbitrarily small, we obtain (7.9). This completes the proof. □

Data availability

No data was used for the research described in the article.

References

[1] D.G. Aronson, H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in: Partial Differential Equations and Related Topics, Program, Tulane Univ., New Orleans, LA, 1974, in: Lecture Notes in Math., vol. 446, Springer-Verlag, Berlin-New York, 1975, pp. 5–49.

[2] H. Berestycki, F. Hamel, H. Matano, Bistable traveling waves around an obstacle, Commun. Pure Appl. Math. 62 (6) (2009) 729–788.

[3] X. Chen, Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations, Adv. Differ. Equ. 2 (1) (1997) 125–160.

[4] X. Chen, J.-S. Guo, Existence and uniqueness of entire solutions for a reaction-diffusion equation, J. Differ. Equ. 212 (1) (2005) 62–84.

[5] X. Chen, J.-S. Guo, F. Hamel, H. Ninomiya, J.-M. Roquejoffre, Traveling waves with paraboloid like interfaces for balanced bistable dynamics, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 24 (3) (2007) 369–393.

[6] X. Chen, J.-S. Guo, H. Ninomiya, Entire solutions of reaction-diffusion equations with balanced bistable nonlinearities, Proc. R. Soc. Edinb., Sect. A 136 (6) (2006) 1207–1237.

[7] M. del Pino, K.T. Gkikas, Ancient multiple-layer solutions to the Allen-Cahn equation, Proc. R. Soc. Edinb., Sect. A 148 (6) (2018) 1165–1199.

[8] P.C. Fife, J.B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions, Arch. Ration. Mech. Anal. 65 (4) (1977) 335–361.

[9] Y. Fukao, Y. Morita, H. Ninomiya, Some entire solutions of Allen-Cahn equation, Taiwan. J. Math. 8 (1) (2004) 15–32.

[10] H. Guo, H. Monobe, V-shaped fronts around an obstacle, Math. Ann. 379 (1–2) (2021) 661–689.

[11] H. Guo, K. Wang, Some new bistable transition fronts with changing shape, preprint.

[12] J.-S. Guo, Y. Morita, Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations, Discrete Contin. Dyn. Syst. 12 (2) (2005) 193–212.

[13] F. Hamel, R. Monneau, J.-M. Roquejoffre, Existence and qualitative properties of multidimensional conical bistable fronts, Discrete Contin. Dyn. Syst. 13 (4) (2005) 1069–1096.

[14] F. Hamel, R. Monneau, J.-M. Roquejoffre, Asymptotic properties and classification of bistable fronts with Lipschitz level sets, Discrete Contin. Dyn. Syst. 14 (1) (2006) 75–92.

[15] F. Hamel, H. Ninomiya, Localized and expanding entire solutions of reaction-diffusion equations, J. Dyn. Differ. Equ. 34 (4) (2022) 2937–2974.

[16] M. Haragus, A. Scheel, Corner defects in almost planar interface propagation, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 23 (3) (2006) 283–329.
[17] Y. Kurokawa, M. Taniguchi, Multi-dimensional pyramidal traveling fronts in the Allen–Cahn equations, Proc. R. Soc. Edinb., Sect. A 141 (5) (2011) 1031–1054.
[18] H. Ninomiya, Entire solutions and traveling wave solutions of the Allen-Cahn-Nagumo equation, Discrete Contin. Dyn. Syst. 39 (4) (2019) 2001–2019.
[19] H. Ninomiya, Entire solutions of the Allen-Cahn-Nagumo equation in a multi-dimensional space, Discrete Contin. Dyn. Syst. 41 (1) (2021) 395–412.
[20] H. Ninomiya, M. Taniguchi, Existence and global stability of traveling curved fronts in the Allen–Cahn equations, J. Differ. Equ. 213 (1) (2005) 204–233.
[21] H. Ninomiya, M. Taniguchi, Global stability of traveling curved fronts in the Allen–Cahn equations, Discrete Contin. Dyn. Syst. 15 (3) (2006) 819–832.
[22] H. Ninomiya, M. Taniguchi, Traveling front solutions of dimension n generate entire solutions of dimension $(n-1)$ in reaction-diffusion equations as the speeds go to infinity, Arch. Ration. Mech. Anal. 249 (1) (2025) 13.
[23] M. Taniguchi, Traveling fronts of pyramidal shapes in the Allen–Cahn equations, SIAM J. Math. Anal. 39 (1) (2007) 319–344.
[24] M. Taniguchi, The uniqueness and asymptotic stability of pyramidal traveling fronts in the Allen–Cahn equations, J. Differ. Equ. 246 (5) (2009) 2103–2130.
[25] M. Taniguchi, Multi-dimensional traveling fronts in bistable reaction-diffusion equations, Discrete Contin. Dyn. Syst. 32 (3) (2012) 1011–1046.
[26] M. Taniguchi, An $(N-1)$-dimensional convex compact set gives an N-dimensional traveling front in the Allen–Cahn equation, SIAM J. Math. Anal. 47 (1) (2015) 455–476.
[27] M. Taniguchi, Axially asymmetric traveling fronts in balanced bistable reaction-diffusion equations, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 36 (7) (2019) 1791–1816.
[28] M. Taniguchi, Axisymmetric traveling fronts in balanced bistable reaction-diffusion equations, Discrete Contin. Dyn. Syst. 40 (6) (2020) 3981–3995.
[29] M. Taniguchi, Traveling Front Solutions in Reaction-Diffusion Equations, MSJ Memoirs, vol. 39, Mathematical Society of Japan, Tokyo, 2021.
[30] M. Taniguchi, Entire solutions with and without radial symmetry in balanced bistable reaction-diffusion equations, Math. Ann. 390 (3) (2024) 3931–3967.
[31] H. Yagisita, Backward global solutions characterizing annihilation dynamics of travelling fronts, Publ. Res. Inst. Math. Sci. 39 (1) (2003) 117–164.