Traveling Front Solutions to Reaction-Diffusion Equations and Their Robustness for Perturbation on Reaction Terms

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Abstract

This thesis is concerned with the existence of traveling front solutions to nonlinear reaction-diffusion equations under perturbation. Traveling front solutions have been studied for reaction-diffusion equations with various kinds of nonlinear terms. The reaction terms influence and evaluates the rate of changes in the processes such as chemical reactions, biological population, and transition phenomena. One of the interesting subjects is their existence and non-existence of them. In this thesis, we proved that, if a traveling front solution exists for a reaction-diffusion equation with a nonlinear term, it also exists for a reaction-diffusion equation with a perturbed nonlinear term. In other words, a traveling front is robust under perturbation on a nonlinear term. There are three main results in this thesis. The first assertation is a traveling front of a nonlinear reaction-diffusion equation is robust under perturbation by assuming the derivative of the reaction term that is negative at stable rest state 1. Secondly, a traveling front of a nonlinear reaction-diffusion equation is robust under perturbation by assuming the derivative of the reaction term that is negative at stable rest state 0. The last one is the traveling fronts to nonlinear reaction-diffusion equations for bistable or multistable nonlinear terms are robust under $C^1[0,1]$ perturbation. More precisely, the robustness of traveling fronts is illustrated by the graphs based on the phase plane analysis.

Contents

Chapter 1

Introduction to traveling front solutions and reaction-diffusion equations

In this chapter, we study the reaction-diffusion equation, traveling front solution, and their qualitative properties. Further, a sufficient condition for the existence and non-existence of a traveling front solution is stated with an example.

1.1 Introduction

In a variety of scientific fields wave propagating phenomena are modeled by reaction-diffusion equations and such wave propagations notably exist in different forms of traveling fronts. A traveling front is a monotone wave that progresses in a particular direction with a constant speed and keeps its shape along the propagation. Traveling front solutions can solve physical phenomena and they play a significant role in scientific research. Their applications are extensively in mathematical biology, physical chemistry, and chemical reactions.

In this thesis, we study a traveling front solution of a reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \mathbb{R},\, t > 0, \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where initial function u_0 is a given bounded and uniformly continuous function from $\mathbb{R}$ to $\mathbb{R}$. Here nonlinear function f is of class C^1 in an open interval including $[0, 1]$. Let $u(x, t; u_0)$ be the desired solution of (1.1). Throughout this work we assume the nonlinear f satisfies

$$f(0) = 0, \qquad f(1) = 0.$$

Many scientists have been studying various perspectives on the dynamics of traveling waves since the early twentieth century. Their interests create useful research projects and then famous research papers come out. Especially, they have focused on traveling front solutions with various kinds of nonlinear reaction terms. Nonlinear reaction waves are covered a broad spectrum of many problems in science. The most interesting subject is the existence and non-existence of traveling front solutions to the concerning problems with nonlinear reaction terms.

Equation (1.1) with such a nonlinear term f appears in many models, and it has often a traveling front solution. See [1, 2, 7, 8, 21, 16, 20] for a general theory of traveling front solutions. Equation (1.1) is called a monostable type if we assume with

$$f'(1) < 0. \tag{1.3}$$

It is well-known the Fisher–KPP equations and a typical nonlinear term is $f(u) = u(1-u)$. See [9, 12, 14, 4, 21] for traveling fronts of (1.1) for the Fisher–KPP equations. It is applicable to population genetics, the spread of epidemic problems, flame propagation, the Brownian motion process, the nuclear reactor theory, and combustion in chemical reactions.

Equation (1.1) is called bistable or multistable type if we assume $f'(0) < 0$ in addition. The nonlinear term of this case is $f(u) = -u(u-a)(u-1)$ for $a \in (0,1)$. (1.1) with this reaction term is called the Nagumo equation in an excitable system of nerve cells while the Allen-Cahn equation in general phase transition problems. See [15, 1, 2, 5, 7, 19, 6, 18, 20] for traveling fronts of (1.1) for bistable or multistable nonlinear case.

For traveling fronts of (1.1) for combustion models, see [10, 11, 3, 17] for instance. For traveling fronts of (1.1) for degenerate monostable nonlinear terms, see [13, 23, 24]. Many scientists are interested in the existence and non-existence of such solutions and their robustness. For the robustness of traveling fronts, one can see [7, 8, 1, 2, 19] for instance. They have approached and searched them by using various kinds of methods. In this thesis, it is focused on their robustness for perturbation on nonlinear reaction terms by using the phase plane technique.

In this Ph.D. thesis I emphasize the existence of traveling front solutions to reaction-diffusion equations with nonlinearity under perturbation and how the speed and the profile solution relate with the nonlinear reaction term. Further, the necessary conditions for the existence of such solutions are analyzed. Especially, It is examined whether a traveling front is robust or not under perturbation on a nonlinear term.

1.2 Background materials

1.2.1 Profile solution of the reaction-diffusion equation

We know how the profile solution of (1.1) provides for the existence of a traveling front solution to such an equation. See [7] for instance. Now we construct the profile solution of (1.1).
If the profile $U \in C^2(\mathbb{R})$ and the speed $c \in \mathbb{R}$ satisfy

$$\begin{cases} U''(y) + cU'(y) + f(U(y)) = 0, & y \in \mathbb{R}, \\ U(-\infty) = 1, \quad U(\infty) = 0, \end{cases} \tag{1.4}$$

then we have $U(x - ct)$ and it becomes a traveling front solution to (1.1). We call (1.4) the profile equation of (c, U), if it exists. In this case, we necessarily have

$$U'(y) < 0, \qquad y \in \mathbb{R}$$

by using [7, Lemma 2.1].

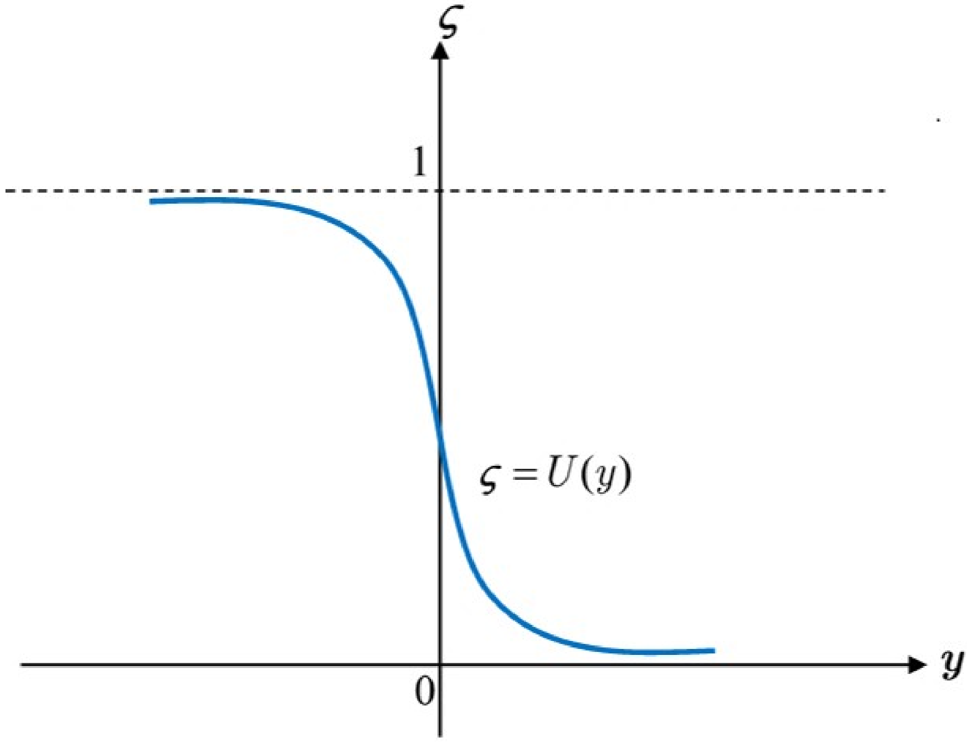

Figure 1.1: Traveling Profile $U(y)$

We will use the standing assumption for comparison. Continuously, we assume that the standing assumption nonlinear f_0 is of class C^1 in an open interval including $[0,1]$ with

$$f_0(0) = 0, \qquad f_0(1) = 0,$$

and

$$f_0'(1) < 0. \tag{1.5}$$

Additionally, we assume that there exist $U_0 \in C^2(\mathbb{R})$ and $c_0 \in \mathbb{R}$ that satisfy

$$\begin{cases} U_0''(y) + c_0 U_0'(y) + f_0(U_0(y)) = 0, \qquad y \in \mathbb{R}, \\ U_0(-\infty) = 1, \quad U_0(\infty) = 0. \end{cases} \tag{1.6}$$

Then we necessarily have

$$U_0'(y) < 0, \qquad y \in \mathbb{R}. \tag{1.7}$$

Assume that $f - f_0 \in C_0^1(0,1]$. Here $C_0^1(0,1]$ is the set of functions in $C^1(0,1]$ whose supports lie in $(0,1]$.

Our approach to the traveling front solution is based on the phase plane analysis. To prove the existence of wave fronts, a trajectory is drawn off from the stationary point to another stationary point with the constant speed $c \in \mathbb{R}$, we reduce the system of profile equation (1.4) from second order to a system of first order ordinary differential equation (1.8).

In view of (1.4), we search (c, U) that satisfies

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix} U \\ U' \end{pmatrix} = \begin{pmatrix} U' \\ -cU' - f(U) \end{pmatrix}, \qquad y \in \mathbb{R}, \\ U'(y) < 0, \qquad y \in \mathbb{R}, \\ U(-\infty) = 1, \quad U(\infty) = 0. \end{gathered} \tag{1.8}$$

Equations (1.4) and (1.8) are equivalent. Using (1.6), we have (c_0, U_0) that satisfies

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix} U_0 \\ U_0' \end{pmatrix} = \begin{pmatrix} U_0' \\ -c_0U_0' - f_0(U_0) \end{pmatrix}, \qquad y \in \mathbb{R}, \\ U_0'(y) < 0, \qquad y \in \mathbb{R}, \\ U_0(-\infty) = 1, \quad U_0(\infty) = 0. \end{gathered} \tag{1.9}$$

1.2.2 Characteristics of solution equations

In general, it is difficult to evaluate the analytical solution of nonlinear reaction-diffusion equations. The phase plane analysis provides to study of the qualitative behaviors of such a dynamic system of equations. We will employ this technique to obtain information based on the existence of solutions.
We now study the following ordinary differential equations or solution equations:

$$\begin{cases} p'(z) = -c - \dfrac{f(z)}{p(z)}, \qquad 0 < z < 1, \\ p(z) < 0, \qquad 0 < z < 1, \\ p(0) = 0, \quad p(1) = 0. \end{cases} \tag{1.10}$$

We write the solution of (1.10) as $p(z; c, f)$ if it exists. There exists a solution (c, U) to (1.8) if and only if $p(z; c, f)$ exists. Indeed, if (c, U) satisfies (1.8), we define p by $p(U(y)) = U'(y)$ for $y \in \mathbb{R}$, and have (1.10).
If $p(z; c, f)$ satisfies (1.10), we define

$$y = \int_a^U \frac{\mathrm{d}z}{p(z)}, \qquad 0 < z < 1, \tag{1.11}$$

and have (1.8). Here a is an arbitrarily given number.

Similarly, there exists a solution (c_0, U_0) to (1.9) if and only if $p(z; c_0, f_0)$ exists. By the standing assumption, we have $p(z; c_0, f_0)$ that satisfies

$$\begin{cases} p_z(z; c_0, f_0) = -c_0 - \dfrac{f_0(z)}{p(z; c_0, f_0)}, \qquad 0 < z < 1, \\ p(z; c_0, f_0) < 0, \qquad 0 < z < 1, \\ p(0; c_0, f_0) = 0, \quad p(1; c_0, f_0) = 0. \end{cases} \tag{1.12}$$

Now we choose $\alpha_0 \in (0, 1)$ such that we have

$$f_0(u) > 0 \qquad \text{if} \quad u \in [\alpha_0, 1).$$

Also, we choose $\alpha \in (0, 1)$ such that we have

$$f(u) > 0 \qquad \text{if} \quad u \in [\alpha, 1).$$

Now we can have $|\alpha - \alpha_0| \to 0$ as $\|f - f_0\|_{C^1[0,1]} \to 0$. We set

$$\alpha_* = \frac{1 + \alpha_0}{2}. \tag{1.13}$$

It suffices to assume that $\|f - f_0\|_{C^1[0,1]}$ is small enough and we always have

$$\alpha < \alpha_*.$$

Next, we will study the characteristics of the solution curve.

1.2.3 Characteristics of solution curve

As stated above to search (c, U) that satisfies (1.8) we will study the following solution equations:

$$\begin{cases} p'(z) = -s - \dfrac{f(z)}{p(z)}, \qquad 1-\alpha < z < 1, \\ p(z) < 0, \qquad 1-\alpha < z < 1, \\ p(1) = 0 \end{cases} \tag{1.14}$$

for every speed $s \in \mathbb{R}$, and $\alpha \in (0,1)$. If $p(z)$ satisfies (1.14), there exists a traveling profile (c, U) that satisfies (1.8) by defining (1.11).
We assume $\|f - f_0\|_{C^1[0,1]}$ and $|s - c_0|$ small enough, say,

$$\sqrt{\|f - f_0\|^2_{C^1[0,1]} + |s - c_0|^2} < \delta_1$$

for every $s \in \mathbb{R}$, and for any $c_0 \in \mathbb{R}$. Here δ_1 is a positive number.
We define

$$\Omega = \left\{(f, s) \in C^1[0,1] \times \mathbb{R} \Big| \sqrt{\|f - f_0\|^2_{C^1[0,1]} + |s - c_0|^2} < \delta_1\right\}.$$

Again, we define

$$\gamma = \frac{-s + \sqrt{s^2 - 4f'(1)}}{2} > 0$$

for every $s \in \mathbb{R}$. Now we choose $\varepsilon_0 \in (0, \gamma)$ to be small enough such that we have

$$(\gamma + \varepsilon_0)^2 + s(\gamma + \varepsilon_0) + f'(1) > 0,$$

and

$$(\gamma - \varepsilon_0)^2 + s(\gamma - \varepsilon_0) + f'(1) < 0$$

for all $(f, s) \in \Omega$.
For every $(f, s) \in \Omega$ we have

$$-s - \frac{f(z)}{p}\bigg|_{p=(\gamma+\varepsilon_0)(z-1)} < \gamma + \varepsilon_0,$$

and

$$-s - \frac{f(z)}{p}\bigg|_{p=(\gamma-\varepsilon_0)(z-1)} > \gamma - \varepsilon_0,$$

if $z \in (0,1)$ and $|z-1|$ is small enough.
Now we have

$$(\gamma + \varepsilon_0)(z-1) < p(z; c_0, f_0) < (\gamma - \varepsilon_0)(z-1),$$

and

$$(\gamma+\varepsilon_0)(z-1) < p_+(z;s,f) < (\gamma-\varepsilon_0)(z-1),$$

if $z \in (0,1)$ and $|z-1|$ is small enough, say, $|z-1| < \delta_1$ with $\delta_1 \in (0, \min\{\frac{1}{8}, 1-\alpha\})$.

By taking δ_1 small enough, Ω is negatively invariant for (1.14) with respect to $1-\alpha \le z \le 1$. If $\|f-f_0\|_{C^1[0,1]}$ goes to zero and $|s-c_0|$ goes to zero, we can assume ε_0 goes to zero. Thus we have

$$\max_{1-\delta_1 \le z \le 1} |p_+(z;s,f) - p(z;c_0,f_0)| = 0.$$

We note that the trajectories move along inside the cone and they never leave from the cone if the traveling front solutions exist. It means that if there exists $p_+(z;c_0,f_0)$ for some $1-\alpha \le z \le 1$ we have $p_+(z;s,f)$ for all $1-\alpha \le z \le 1$.

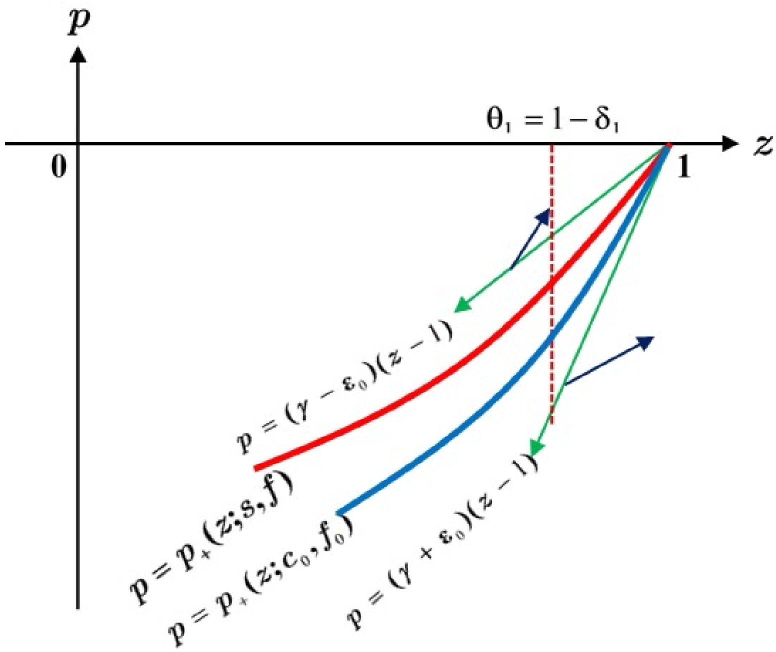

Figure 1.2: Characteristics of $p = p_+(z;s,f)$.

Figure 1.2 shows the characteristics of $p_+(z;s,f)$. We will use this property of $p_+(z;s,f)$ in the proof of main result 1.

1.3 A sufficient condition for the existence of traveling front solutions in balanced nonlinearity

We first study one of the essential features for the existence and non-existence of traveling front solutions with balanced nonlinearity. The following facts are provided to the sufficient condition for the existence of traveling front solutions.

Lemma 1.1. *Let $W \in C^2(\mathbb{R})$ satisfy*

$$W(0) = 0, \quad W(1) = 0, \quad W'(0) = 0, \quad W'(1) = 0,$$

$$W''(0) > 0, \quad W''(1) > 0, \quad W''(s) > 0 \quad \textit{if} \quad 0 < s < 1,$$

$$W(s) > 0 \quad \textit{if} \quad 0 < s < 1.$$

Then $\Phi(x)$ given by

$$x = -\int_{\frac{1}{2}}^{\Phi} \frac{\mathrm{d}v}{\sqrt{2W(v)}}, \qquad 0 < \Phi < 1 \tag{1.15}$$

satisfies

$$\begin{cases} \Phi''(x) - W'(\Phi(x)) = 0, & x \in \mathbb{R}, \\ \Phi(-\infty) = 1, \quad \Phi(0) = \frac{1}{2}, \quad \Phi(\infty) = 0. \end{cases} \tag{1.16}$$

Moreover, for the speed c with $-\sqrt{W''(1)} < c < \sqrt{W''(0)}$ and the reaction term

$$f(u) = -W'(u) + c\sqrt{2W(u)}, \qquad 0 < u < 1,$$

one has there exists (c, Φ) satisfies

$$\begin{cases} \Phi''(x) + c\,\Phi'(x) + f(\Phi(x)) = 0, & x \in \mathbb{R}, \\ \Phi(-\infty) = 1, \quad \Phi(0) = \frac{1}{2}, \quad \Phi(\infty) = 0. \end{cases} \tag{1.17}$$

Proof. From (1.15) we have

$$-\Phi'(x) = \sqrt{2W(\Phi(x))}, \qquad x \in \mathbb{R}, \tag{1.18}$$

and

$$-\Phi''(x) = -W'(\Phi(x)), \qquad x \in \mathbb{R}.$$

Thus we obtain (1.16). Again, we observe that

$$\begin{aligned} &\Phi''(x) + c\Phi'(x) + f(\Phi(x)) \\ &= W'(\Phi(x)) + c\,\Phi'(x) - W'(\Phi(x)) + c\sqrt{2W(\Phi(x))} = 0, \qquad 0 < \Phi < 1, \end{aligned}$$

and so $\Phi(x)$ can solve (1.17). This completes the proof. $\square$

Now we study the solution $p(\Phi(x); c, f)$ that related to the solution $\Phi(x)$. Setting

$$\Phi'(x) = p(\Phi(x); c, f)$$

and using (1.18) we have

$$p(\Phi(x); c, f) = -\sqrt{2W(\Phi(x))}, \qquad 0 < \Phi < 1.$$

For $0 < z < 1$ we have

$$p(z; c, f) = -\sqrt{2W(z)}.$$

It can be checked that

$$p_z(z; c, f) = -\frac{W'(z)}{\sqrt{2W(z)}} = \frac{-c\sqrt{2W(z)} + f(z)}{\sqrt{2W(z)}} = -c - \frac{f(z)}{p(z)}.$$

Now we obtain a solution $p(z; c, f)$ that satisfies

$$p_z(z; c, f) = -c - \frac{f(z)}{p(z; c, f)}, \qquad 0 < z < 1,$$

$$p(0; c, f) = 0, \qquad p(1; c, f) = 0.$$

Using $f'(0) < 0$, and $f'(1) < 0$ we have the speed c with

$$-\sqrt{W''(1)} < c < \sqrt{W''(0)}.$$

We note that for $f(u) = -W'(u) + c\sqrt{2W(u)}$ and speed c with above condition there exists a solution (c, Φ) of (1.17) if and only if a solution $p(\Phi(x); c, f)$ exists for bistable or multistable nonlinearity. Moreover, nonzero speed is a necessary and sufficient condition for the existence of traveling front to such a problem.

We will show an example of the existence of solutions for some reaction terms with zero speed. The following example gives the explicit form of the solution.

Example 1.1. *We can search explicit solution of $\Phi(x)$ for given reaction term*

$$f(u) = -u(u-1)\left(u - \frac{1}{2} - \frac{c}{\sqrt{2}}\right)$$

if the speed $c = 0$.

We note that

$$f(u) = -W'(u) \qquad \text{if} \quad c = 0.$$

Then we have

$$\begin{cases} W'(u) = u(u-\frac{1}{2})(u-1), & 0 < u < 1, \\ \\ W(u) = \dfrac{u^2(1-u)^2}{4}, & 0 < u < 1, \end{cases} \tag{1.19}$$

We know that

$$p(\Phi(x); c, f) = \Phi'(x),$$

and we obtain

$$p(z; c, f) = -\sqrt{2W(z)}, \qquad 0 < z < 1. \tag{1.20}$$

From this equality we have

$$p(z; c, f) = -\frac{z}{\sqrt{2}}(1-z),$$

and

$$-\Phi'(x) = \frac{1}{\sqrt{2}}\Phi(1-\Phi).$$

It follows that we have

$$\frac{1}{\Phi} - 1 = \exp\left(\frac{x}{\sqrt{2}}\right).$$

From the equation

$$\Phi(x) = \frac{1}{1+\exp(\frac{x}{\sqrt{2}})}$$

we have the explicit solution

$$\Phi(x) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right).$$

Putting $c = 0$, we get the explicit form of $\Phi(x)$ and that (c, Φ) satisfies equation (1.17) for the multistable nonlinear reaction term

$$f(u) = -u(u-\frac{1}{2})(u-1).$$

We will show that there is a sufficient condition for the existence of a traveling front solution in balanced nonlinearity. We now assume the nonliner $f \in C^1(\mathbb{R})$ that satisfies

$$f(0) = 0, \qquad f(1) = 0,$$

$$f'(0) < 0, \qquad f'(1) < 0, \qquad \int_0^1 f(u)\, du = 0.$$

In addition, we put

$$W(u) = \int_u^1 f(s)\,ds \qquad \text{for all} \quad u \in (0,1),$$

and

$$W'(u) = -f(u), \qquad W(0) = 0, \qquad W(1) = 0.$$

Proposition 1.2. *There exists* (c, Φ) *that satisfies*

$$\Phi''(x) + c\,\Phi'(x) + f(\Phi(x)) = 0, \qquad 0 < \Phi < 1, \qquad x \in \mathbb{R},$$

$$\Phi(-\infty) = 1, \qquad \Phi(\infty) = 0$$

if and only if

$$W(u) > 0 \qquad \textit{for all} \quad u \in (0,1).$$

Proof. Multiplying (1.17) by $2\Phi'(x)$ and integrating with respect to x over $\mathbb{R}$ we have

$$2\int_{-\infty}^{\infty} \Phi'(x)\Phi''(x)\,dx + 2c\int_{-\infty}^{\infty} \Phi'(x)^2\,dx + 2\int_{-\infty}^{\infty} \Phi'(x)f(\Phi(x))\,dx = 0.$$

It becomes

$$2c\int_{-\infty}^{\infty} \Phi'(x)^2\,dx = \left[-\Phi'(x)^2\right]_{-\infty}^{\infty} - 2\int_{-\infty}^{\infty} f(\Phi(x))\,d\,\Phi(x) \quad \text{for all} \quad x \in \mathbb{R}.$$

Since $-\Phi'(x) > 0$ follows from [7, Lemma 2.1] and $\int_0^1 f(u)\,du = 0$ we have $c = 0$ from the first part. Then from the latter, we obtain

$$\Phi'(x)^2 = 2\int_u^1 f(s)\,ds > 0 \quad \text{for} \quad s \in (0,1).$$

It follows that

$$\int_u^1 f(s)\,ds = W(u) > 0 \quad \text{for} \quad u \in (0,1).$$

Conversely, we have

$$W'(u) = -f(u) \qquad \text{for} \quad 0 < u < 1.$$

Using definition (1.15), we obtain

$$\Phi''(x) - W'(\Phi(x)) = 0.$$

Then we have

$$\Phi''(x) + f(\Phi(x)) = 0 \quad \text{for} \quad x \in \mathbb{R}.$$

For $c = 0$ we have the desired profile equation and it completes the proof. $\square$

As mentioned above we can say that (c, Φ) exists for some f if the function $W(u) > 0$ in balanced nonlinearity. Otherwise, there is no traveling front for some f if $W(u) \leq 0$ for such case. It shows that the condition $\int_u^1 f(s)\,ds > 0$ is necessary for existence of solution in balanced nonlinearity. Furthermore, zero speed is a necessary and sufficient condition for the existence of traveling fronts in such case. It will be an open problem for nonzero speed.

Chapter 2

Traveling front solutions for monostable nonlinear perturbed reaction-diffusion equations

In this chapter, we search for a traveling front solution for the monostable nonlinear perturbed reaction-diffusion equations. We state the detailed proof of Theorem 2.1 in this chapter and it is also pointed out that an idea of proof by Figure 2.1. Furthermore, the auxiliary result is stated as Corollary 2.9 in this chapter.

2.1 Main result 1

The first main result is stated as the following theorem:

Theorem 2.1. *Let f_0 be a function of class C^1 in an open interval including $[0,1]$ with*

$$f_0(0) = 0, \qquad f_0(1) = 0, \qquad f_0'(1) < 0.$$

Assume that there exists (c_0, U_0) that satisfies (1.6). Assume that $f - f_0 \in C_0^1(0,1]$ and let $\|f - f_0\|_{C^1[0,1]}$ be small enough. Then there exists (c, U) that satisfies (1.4). If $\|f - f_0\|_{C^1[0,1]}$ goes to zero, c converges to c_0 and $\|U - U_0\|_{C^2(\mathbb{R})}$ goes to zero.

2.2 Construction for a traveling front solution

2.2.1 Background assertion

We will use [20, Theorem 1.1] for the proof of Theorem (**??**). Here, the background theorem from [20] is mentioned for convenience.

Assume that f is of class $C^1[b,1]$ for $b \in \mathbb{R}$ with $0 < b < 1$. For any $c \in \mathbb{R}$, there exists unique $p_+(z;c,f)$ that satisfies

$$\begin{aligned}
&(p_+)_z\,(z;c,f) = -c - \frac{f(z)}{p_+(z;c,f)}, \qquad z \in (b,1),\\
&p_+(z;c,f) < 0, \qquad z \in [b,1),\\
&p_+(1;c,f) = 0,\\
&(p_+)_z\,(1;c,f) = \frac{-c + \sqrt{c^2 - 4f'(1)}}{2} > 0.
\end{aligned}$$

Moreover, $p_+(z;c,f)$ is strictly monotone increasing in c, that is, if $c_1 < c_2$, one has

$$p_+(z;c_1,f) < p_+(z;c_2,f) \qquad b < z < 1.$$

Furthermore, one has $p_+(b;\infty,f) = 0$ and $p_+(b;-\infty,f) = -\infty$. If $p(z)$ satisfies

$$\begin{cases} p'(z) = -c - \dfrac{f(z)}{p(z)}, \qquad b < z < 1,\\ p(z) < 0, \qquad b < z < 1,\\ p(1) = 0, \end{cases} \tag{2.1}$$

one has $p(z) = p_+(z;c,f)$ for all $b < z < 1$.

Now we start with the existence of the solution for the first interval $z \in [\alpha, 1]$.

2.2.2 Existence of first solution and extended solution

Lemma 2.2 ([20]). *For every $s \in \mathbb{R}$ there exists $p_+(z;s,f)$ defined for $z \in [\alpha,1]$, such that one has*

$$(p_+)_z\,(z;s,f) = -s - \frac{f(z)}{p_+(z;s,f)}, \qquad z \in (\alpha,1), \tag{2.2}$$

$$p_+(z;s,f) < 0, \qquad z \in [\alpha,1), \tag{2.3}$$

$$p_+(1;s,f) = 0, \tag{2.4}$$

$$(p_+)_z\,(1;s,f) = \frac{-s + \sqrt{s^2 - 4f'(1)}}{2} > 0. \tag{2.5}$$

If $s_1 < s_2$, one has

$$p_+(z;s_1,f) < p_+(z;s_2,f), \qquad z \in [\alpha,1).$$

Proof. This assertion follows from [20, Theorem 1.1] and its proof. ☐

Since $f - f_0 \in C_0^1(0,1]$, we can choose $z_* \in (0,1)$ with

$$f(z) = f_0(z) \qquad \text{if} \quad 0 \le z \le z_*. \tag{2.6}$$

Let $s \in \mathbb{R}$ be arbitrarily given and let $p_+(z; s, f)$ be given by Lemma 2.2. We choose $M \ge 1$ large enough such that we have

$$|s| + \frac{\|f\|_{C[0,1]}}{M} \le M. \tag{2.7}$$

In Lemma 2.2, $p_+(z; s, f)$ is defined only on $[\alpha, 1]$. We extend $p_+(z; s, f)$ for all possible z, say $z \in (\zeta_0(s, f), 1)$. Then we have

$$\zeta_0(s, f) \le \alpha < \alpha_*.$$

Since f is defined in an open interval including $[0,1]$, $\zeta_0(s, f)$ can be a negative value. Now we have

$$\begin{aligned}
&(p_+)_z\,(z; s, f) = -s - \frac{f(z)}{p_+(z; s, f)}, \qquad z \in (\zeta_0(s, f), 1), \\
&p_+(z; s, f) < 0, \qquad z \in (\zeta_0(s, f), 1), \\
&p_+(1; s, f) = 0, \\
&(p_+)_z\,(1; s, f) = \frac{-s + \sqrt{s^2 - 4f'(1)}}{2} > 0.
\end{aligned} \tag{2.8}$$

Now we assert the following lemma.

Lemma 2.3. *Let $s \in \mathbb{R}$ be arbitrarily given and let $M \ge 1$ satisfy (2.7). Let $p_+(z; s, f)$ be given by Lemma 2.2 and one extends $p_+(z; s, f)$ for all possible z, say $z \in (\zeta_0(s, f), 1)$. Then one has*

$$0 < -p_+(z; s, f) < 2M, \qquad \zeta_0(s, f) < z < 1. \tag{2.9}$$

One has

$$p_+(0; s, f) < 0, \quad \zeta_0(s, f) < 0,$$

or one has

$$\zeta_0(s, f) \in [0, \alpha), \quad p_+(\zeta_0(s, f); s, f) = 0. \tag{2.10}$$

Proof. Assume that there exists $\eta_0 \in (0,1)$ with

$$-p_+(\eta_0; s, f) \ge 2M.$$

Then we can define $\eta_1 \in (\eta_0, 1]$ by

$$\eta_1 = \sup\{\eta \in (\eta_0, 1) \mid -p_+(z; s, f) \geq M \quad \text{for all } z \in [\eta_0, \eta]\}.$$

Using $p_+(1; s, f) = 0$, we have $0 < \eta_0 < \eta_1 < 1$. Using (2.7) and (2.8), we obtain

$$\begin{aligned}
& -p_+(\eta_1; s, f) \\
=& -p_+(\eta_0; s, f) - \int_0^1 (p_+)_z\,(\theta\eta_1 + (1-\theta)\eta_0; s, f)\,\mathrm{d}\theta\,(\eta_1 - \eta_0) \\
\geq& 2M - M(\eta_1 - \eta_0) > M.
\end{aligned}$$

This contradicts the definition of η_1. Now we obtain (2.9).

If $\zeta_0(s, f) < 0$, we have $p_+(0; s, f) < 0$. It suffices to prove (2.10) by assuming $\zeta_0(s, f) \geq 0$. Then necessarily we have $\zeta_0(s, f) \in [0, \alpha)$. Assume that (2.10) does not hold true. Then we have

$$\beta = \limsup_{z \to \zeta_0(s,f)} (-p_+(z; s, f)) \in (0, 2M].$$

Using (2.8), we obtain

$$(p_+)_z\,(\zeta_0(s, f); s, f) = -s + \frac{f(0)}{\beta}.$$

Since the right-hand side is bounded, it is bounded on a neighborhood of $(\zeta_0(s, f), -\beta)$ and we can extend $p_+(z; s, f)$ for $z \in (\zeta_0(s, f) - \delta, \zeta_0(s, f))$ with some $\delta > 0$ that is small enough. This contradicts the definition of $\zeta_0(s, f)$. Thus we obtain (2.10) and complete the proof. □

Now we have

$$\begin{aligned}
&\zeta_0(c_0, f_0) = 0, \\
&p_+(z; c_0, f_0) = p(z; c_0, f_0), \qquad 0 < z < 1.
\end{aligned} \tag{2.11}$$

2.3 Convergence of a traveling front solution

In this section we find the difference value of $p_+(z; s, f)$ and $p_+(z; c_0, f_0)$ in order to establish the convergence of a traveling front solution. Now we assert the following proposition. This assertion provides the convergence result of our desired solution.

Proposition 2.4. *Let $s \in \mathbb{R}$ be arbitrarily given. Then one has*

$$\begin{aligned}
&p_+(z; s, f) - p_+(z; c_0, f_0) \\
=& \int_z^1 \left(s - c_0 + \frac{f(z') - f_0(z')}{p_+(z'; c_0, f_0)} \right) \exp \left(- \int_z^{z'} \frac{f(\zeta)}{p_+(\zeta; s, f) p_+(\zeta; c_0, f_0)} \, \mathrm{d}\zeta \right) \mathrm{d}z'
\end{aligned}$$

for $\zeta_0(s, f) < z < 1$.

Proof. We put

$$w(z) = p_+(z; s, f) - p_+(z; c_0, f_0)$$

and have

$$w'(z) = -s + c_0 - \frac{f(z)}{p_+(z; s, f)} + \frac{f_0(z)}{p_+(z; c_0, f_0)}$$

for $\zeta_0(s, f) < z < 1$. Now we have

$$-\frac{f(z)}{p_+(z; s, f)} + \frac{f_0(z)}{p_+(z; c_0, f_0)} = \frac{-f(z) p_+(z; c_0, f_0) + f_0(z) p_+(z; s, f)}{p_+(z; s, f) p_+(z; c_0, f_0)}$$

and

$$\begin{aligned}
&- f(z) p_+(z; c_0, f_0) + f_0(z) p_+(z; s, f) \\
=& - f(z) \left(p_+(z; c_0, f_0) - p_+(z; s, f) \right) - f(z) p_+(z; s, f) + f_0(z) p_+(z; s, f) \\
=& f(z) w(z) - \left(f(z) - f_0(z) \right) p_+(z; s, f).
\end{aligned}$$

Then we obtain

$$w'(z) - \frac{f(z)}{p_+(z; s, f) p_+(z; c_0, f_0)} w(z) = -s + c_0 - \frac{f(z) - f_0(z)}{p_+(z; c_0, f_0)}$$

for $\zeta_0(s, f) < z < 1$. Then we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z} \left(w(z) \exp \left(\int_z^1 \frac{f(\zeta)}{p_+(\zeta; s, f) p_+(\zeta; c_0, f_0)} \, \mathrm{d}\zeta \right) \right) \\
=& \left(w'(z) - \frac{f(z)}{p_+(z; s, f) p_+(z; c_0, f_0)} w(z) \right) \\
&\qquad\qquad\qquad \times \exp \left(\int_z^1 \frac{f(\zeta)}{p_+(\zeta; s, f) p_+(\zeta; c_0, f_0)} \, \mathrm{d}\zeta \right) \\
=& \left(-s + c_0 - \frac{f(z) - f_0(z)}{p_+(z; c_0, f_0)} \right) \exp \left(\int_z^1 \frac{f(\zeta)}{p_+(\zeta; s, f) p_+(\zeta; c_0, f_0)} \, \mathrm{d}\zeta \right).
\end{aligned}$$

Let $\theta' \in (z,1)$ be arbitrarily given. Integrating both sides of the equality stated above over (z,θ'), we have

$$\begin{aligned}
&- w(z)\exp\left(\int_z^1 \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\\
&+ w(\theta')\exp\left(\int_{\theta'}^1 \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\\
=& -\int_z^{\theta'}\left(s-c_0+\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right)\exp\left(\int_{z'}^1 \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\mathrm{d}z'
\end{aligned}$$

for $\zeta_0(s,f)<z<\theta'$. Now we find

$$\begin{aligned}
w(z) = w(\theta')&\exp\left(-\int_z^{\theta'} \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\\
&+\int_z^{\theta'}\left(s-c_0+\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right)\\
&\qquad\times\exp\left(-\int_z^{z'} \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\mathrm{d}z'
\end{aligned} \tag{2.12}$$

for $\zeta_0(s,f)<z<\theta'$. Using

$$\begin{aligned}
&f(\zeta)>0 \qquad \text{if} \quad \zeta\in(\alpha_*,1),\\
&p_+(\zeta;s,f)<0,\quad p_+(\zeta;c_0,f_0)<0,\qquad \zeta_0(s,f)<\zeta<1,
\end{aligned}$$

we have

$$\lim_{\theta'\to 1} w(\theta')\exp\left(-\int_z^{\theta'} \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)=0$$

and

$$\begin{aligned}
&\lim_{\theta'\to 1}\int_z^{\theta'}\left(s-c_0+\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right)\\
&\qquad\qquad\times\exp\left(-\int_z^{z'} \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\mathrm{d}z'\\
&=\int_z^{1}\left(s-c_0+\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right)\\
&\qquad\qquad\times\exp\left(-\int_z^{z'} \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\mathrm{d}z'
\end{aligned}$$

for $\zeta_0(s,f)<z<1$. Passing to the limit of $\theta'\to 1$ in (2.12), we obtain

$$\begin{aligned}
&w(z)=\\
&\int_z^{1}\left(s-c_0+\frac{f(z')-f_0(z')}{p_+(z';c_0,f_0)}\right)\exp\left(-\int_z^{z'} \frac{f(\zeta)}{p_+(\zeta;s,f)p_+(\zeta;c_0,f_0)}\,\mathrm{d}\zeta\right)\mathrm{d}z'
\end{aligned}$$

for $\zeta_0(s,f) < z < 1$. This completes the proof. $\square$

Now we take $\varepsilon_0 \in (0, 1-\alpha_*)$ small enough such that we have

$$(p_+)_z\,(z; c_0, f_0) > \frac{1}{2}\,(p_+)_z\,(1; c_0, f_0) > 0 \qquad \text{if} \quad z \in (1-\varepsilon_0, 1). \tag{2.13}$$

We show that $|p_+(\alpha_*; s, f) - p_+(\alpha_*; c_0, f_0)|$ converges to 0 as $|s - c_0| + \|f - f_0\|_{C^1[0,1]}$ goes to 0 in the following lemma.

Lemma 2.5. *Let $\alpha_* \in (0,1)$ be as in (1.13) and let $\varepsilon_0 \in (0, 1-\alpha_*)$ satisfy (2.13). Then one has*

$$\begin{aligned}
&\sup_{z\in[\alpha_*,1]} |p_+(z; s, f) - p_+(z; c_0, f_0)| \\
\le &(1-\alpha_*)|s - c_0| + \frac{(1-\varepsilon_0-\alpha_*)\|f - f_0\|_{C[0,1]}}{\min_{z'\in[\alpha_*,1-\varepsilon_0]}\left(-p_+(z'; c_0, f_0)\right)} \\
&\qquad\qquad\qquad\qquad + \frac{\varepsilon_0\|f - f_0\|_{C^1[0,1]}}{\min_{\zeta'\in[1-\varepsilon_0,1]}|(p_+)_z(\zeta'; c_0, f_0)|}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&f(z) > 0 \qquad \text{if} \quad z \in [\alpha_*, 1), \\
&p_+(z; s, f) < 0 \qquad \text{if} \quad z \in [\alpha_*, 1), \\
&p_+(z; c_0, f_0) < 0 \qquad \text{if} \quad z \in (0, 1).
\end{aligned}$$

Then, using Proposition 2.4, we have

$$\max_{z\in[\alpha_*,1]} |p_+(z; s, f) - p_+(z; c_0, f_0)| \le \int_{\alpha_*}^1 \left(|s - c_0| + \left|\frac{f(z') - f_0(z')}{p_+(z'; c_0, f_0)}\right|\right) \mathrm{d}z'.$$

Now we find

$$\begin{aligned}
&\int_{\alpha_*}^1 \left(|s - c_0| + \left|\frac{f(z') - f_0(z')}{p_+(z'; c_0, f_0)}\right|\right) \mathrm{d}z' \\
&\le (1-\alpha_*)\,|s - c_0| + \int_{\alpha_*}^1 \left|\frac{f(z') - f_0(z')}{p_+(z'; c_0, f_0)}\right| \mathrm{d}z'.
\end{aligned} \tag{2.14}$$

If $z' \in (\alpha_*, 1-\varepsilon_0]$, we have

$$\left|\frac{f(z') - f_0(z')}{p_+(z'; c_0, f_0)}\right| \le \frac{\|f - f_0\|_{C[0,1]}}{\min_{z'\in[\alpha_*,1-\varepsilon_0]}\left(-p_+(z'; c_0, f_0)\right)}$$

and thus

$$\int_{\alpha_*}^{1-\varepsilon_0} \left| \frac{f(z') - f_0(z')}{p_+(z'; c_0, f_0)} \right| \mathrm{d}z' \leq \frac{(1 - \varepsilon_0 - \alpha_*) \|f - f_0\|_{C[0,1]}}{\min_{z' \in [\alpha_*, 1-\varepsilon_0]} (-p_+(z'; c_0, f_0))}.$$

If $z' \in (1 - \varepsilon_0, 1)$, we have

$$\frac{f(z') - f_0(z')}{p_+(z'; c_0, f_0)} = \frac{f'(\zeta') - f_0'(\zeta')}{(p_+)_z(\zeta'; c_0, f_0)}$$

for some $\zeta' \in (z', 1)$. Thus, if $z' \in (1 - \varepsilon_0, 1)$, we find

$$\left| \frac{f(z') - f_0(z')}{p_+(z'; c_0, f_0)} \right| \leq \frac{\|f - f_0\|_{C^1[0,1]}}{\min_{\zeta' \in [1-\varepsilon_0, 1]} |(p_+)_z(\zeta'; c_0, f_0)|}$$

and

$$\int_{1-\varepsilon_0}^{1} \left| \frac{f(z') - f_0(z')}{p_+(z'; c_0, f_0)} \right| \mathrm{d}z' \leq \frac{\varepsilon_0 \|f - f_0\|_{C^1[0,1]}}{\min_{\zeta' \in [1-\varepsilon_0, 1]} |(p_+)_z(\zeta'; c_0, f_0)|}.$$

Then we obtain

$$\begin{aligned} &\int_{\alpha_*}^{1} \left| \frac{f(z') - f_0(z')}{p_+(z'; c_0, f_0)} \right| \mathrm{d}z' \\ &\leq \frac{(1 - \varepsilon_0 - \alpha_*) \|f - f_0\|_{C[0,1]}}{\min_{z' \in [\alpha_*, 1-\varepsilon_0]} (-p_+(z'; c_0, f_0))} + \frac{\varepsilon_0 \|f - f_0\|_{C^1[0,1]}}{\min_{\zeta' \in [1-\varepsilon_0, 1]} |(p_+)_z(\zeta'; c_0, f_0)|}. \end{aligned}$$

Combining this inequality and (2.14), we complete the proof. □

Lemma 2.5 asserts that $|p_+(z; s, f) - p_+(z; c_0, f_0)|$ converges to 0 on an interval $[\alpha_*, 1]$ as $|s - c_0| + \|f - f_0\|_{C^1[0,1]}$ goes to 0. Does this convergence hold true for every compact interval in $(0, 1]$? To answer this question, we assert the following lemma.

Lemma 2.6. *Let $s \in \mathbb{R}$. Let $z_* \in (0, 1)$ satisfy (2.6) and let $z_1 \in (0, z_*)$ be arbitrarily given. As $|s - c_0| + \|f - f_0\|_{C^1[0,1]}$ goes to zero, $\zeta_0(s, f)$ converges to zero and*

$$\sup_{z \in [z_1, 1]} |p_+(z; s, f) - p_+(z; c_0, f_0)|$$

converges to zero.

Proof. We will prove $\zeta_0(s, f) < z_1$ if $|s - c_0| + \|f - f_0\|_{C^1[0,1]}$ is small enough. Let (c_0, U_0) satisfy (1.9). There exists $-\infty < y_0 < y_1 < \infty$ such that we have

$$U_0(y_0) = \alpha_*, \quad U_0(y_1) = \frac{z_1}{2}.$$

For $s \in \mathbb{R}$, let $V = V(y)$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}y} \begin{pmatrix} V \\ V' \end{pmatrix} = \begin{pmatrix} V' \\ -sV' - f(V) \end{pmatrix}, \qquad y \in \mathbb{R} \tag{2.15}$$

with

$$V(y_0) = \alpha_*, \quad V'(y_0) = p_+(\alpha_*; s, f).$$

Now we define

$$w(y) = \begin{pmatrix} w_1(y) \\ w_2(y) \end{pmatrix} = \begin{pmatrix} V(y) - U_0(y) \\ V'(y) - U_0'(y) \end{pmatrix}, \qquad y \in \mathbb{R}.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}y} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_2 \\ -sV' + c_0 U_0' - f(V) + f_0(U_0) \end{pmatrix}, \qquad y \in \mathbb{R}.$$

Now we have

$$f(V) - f(U_0) = [f(\theta V + (1-\theta)U_0)]_{\theta=0}^{\theta=1} = \int_0^1 f'(\theta V + (1-\theta)U_0)\, \mathrm{d}\theta\, (V - U_0)$$

for $y \in \mathbb{R}$. Now we define

$$h(y) = \int_0^1 f'(\theta V(y) + (1-\theta)U_0(y))\, \mathrm{d}\theta, \qquad y \in \mathbb{R},$$

$$A(y) = \begin{pmatrix} 0 & -1 \\ h(y) & s \end{pmatrix}, \qquad y \in \mathbb{R},$$

$$g(y) = -\begin{pmatrix} 0 \\ (s - c_0)U_0'(y) + f(U_0(y)) - f_0(U_0(y)) \end{pmatrix}, \qquad y \in \mathbb{R}.$$

Now we have

$$\sup_{y \in \mathbb{R}} |A(y)| \le \sqrt{1 + s^2 + \|f\|_{C^1[0,1]}^2}.$$

Here

$$|A| = \sup_{x_1^2 + x_2^2 = 1} \left| A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right|$$

for a 2×2 real matrix A. Then, we obtain

$$w'(y) + A(y)w(y) = g(y), \qquad y \in \mathbb{R}$$

and

$$w(y) = w(y_0) \exp\left(-\int_{y_0}^y A(y') \mathrm{d}y' \right) + \int_{y_0}^y \exp\left(-\int_{y'}^y A(y'')\, \mathrm{d}y'' \right) g(y')\, \mathrm{d}y'$$

for $y \in \mathbb{R}$. Now we have

$$\sup_{y \in \mathbb{R}} |g(y)| \le |s - c_0| \max_{\eta \in \mathbb{R}} |U_0'(\eta)| + \|f - f_0\|_{C[0,1]}.$$

Thus, as $|s - c_0| + \|f - f_0\|_{C[0,1]}$ goes to zero,

$$\max_{y \in [y_0, y_1]} |w(y)|$$

converges to zero. Taking $|s - c_0| + \|f - f_0\|_{C[0,1]}$ small enough, we have

$$\begin{aligned} &|w(y_1)| < \frac{z_1}{4}, \\ &\max_{y \in [y_0, y_1]} |w(y)| < \frac{1}{2} \min_{y \in [y_0, y_1]} (-U_0'(y)) . \end{aligned}$$

We define $p(\,\cdot\,; s, f)$ by

$$p(V(y); s, f) = V'(y), \qquad y_0 \le y < y_1.$$

Then we have

$$V(y_1) < \frac{z_1}{2} + \frac{z_1}{4} = \frac{3}{4} z_1$$

and

$$\begin{aligned} &p_z(z; s, f) = -s - \frac{f(z)}{p(z; s, f)}, \qquad \frac{3}{4} z_1 < z \le \alpha_*, \\ &p(z; s, f) < 0, \qquad \frac{3}{4} z_1 < z \le \alpha_*, \\ &p(\alpha_*; s, f) = p_+(\alpha_*; s, f) < 0. \end{aligned}$$

This $p(z; s, f)$ is an extension of $p_+(z; s, f)$ given by Lemma 2.2. Thus we obtain $\zeta_0(s, f) < z_1$. Combining Lemma 2.5 and the argument stated above, we have

$$\sup_{z \in [z_1, 1]} |p_+(z; s, f) - p_+(z; c_0, f_0)| \to 0$$

as $|s - c_0| + \|f - f_0\|_{C^1[0,1]}$ goes to zero. This completes the proof. □

2.4 Monotonicity of a traveling front solution

Lemma 2.2 asserts that $p_+(z; s, f)$ is strictly monotone increasing in s on $[\alpha_*, 1)$. In the following lemma, we assert that $p_+(z; s, f)$ is strictly monotone increasing in s on the whole interval $(0, 1)$.

Lemma 2.7. *Let $-\infty < s_1 < s_2 < \infty$ be arbitrarily given. Let $z_{\mathrm{init}} \in (0, 1)$ be arbitrarily given. Assume that $p_+(z_{\mathrm{init}}; s_1, f)$ and $p_+(z_{\mathrm{init}}; s_2, f)$ exist and satisfy*

$$p_+(z_{\mathrm{init}}; s_1, f) < p_+(z_{\mathrm{init}}; s_2, f) < 0.$$

Then one has

$$\zeta_0(s_1, f) \le \zeta_0(s_2, f) < z_{\mathrm{init}}$$

and

$$p_+(z; s_1, f) < p_+(z; s_2, f) < 0 \qquad \textit{for all} \quad z \in (\zeta_0(s_2, f), z_{\mathrm{init}}].$$

Proof. We put

$$q(z) = p_+(z; s_2, f) - p_+(z; s_1, f), \qquad \max\{\zeta_0(s_2, f), \zeta_0(s_1, f)\} \le z \le z_{\mathrm{init}}.$$

Then we have

$$\begin{aligned} &q'(z) = -(s_2 - s_1) - \frac{f(z)}{p_+(z; s_2, f)} + \frac{f(z)}{p_+(z; s_1, f)}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \max\{\zeta_0(s_2, f), \zeta_0(s_1, f)\} < z < z_{\mathrm{init}}, \\ &q(z_{\mathrm{init}}) > 0. \end{aligned}$$

Consequently we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z}\left(q(z) \exp\left(-\int_z^{z_{\mathrm{init}}} \frac{f(\zeta)}{p_+(\zeta; s_1, f) p_+(\zeta; s_2, f)}\,\mathrm{d}\zeta\right)\right) \\ &= -\,(s_2 - s_1) \exp\left(-\int_z^{z_{\mathrm{init}}} \frac{f(\zeta)}{p_+(\zeta; s_1, f) p_+(\zeta; s_2, f)}\,\mathrm{d}\zeta\right) < 0 \end{aligned}$$

for

$$\max\{\zeta_0(s_2, f), \zeta_0(s_1, f)\} < z < z_{\mathrm{init}}.$$

Then we find

$$\begin{aligned} &q(z) \exp\left(-\int_z^{z_{\mathrm{init}}} \frac{f(\zeta)}{p_+(\zeta; s_1, f) p_+(\zeta; s_2, f)}\,\mathrm{d}\zeta\right) > 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \max\{\zeta_0(s_2, f), \zeta_0(s_1, f)\} < z < z_{\mathrm{init}}. \end{aligned}$$

Thus we obtain

$$q(z) > 0, \qquad \max\{\zeta_0(s_2, f), \zeta_0(s_1, f)\} < z < z_{\mathrm{init}}.$$

Then, using $q(z_{\mathrm{init}}) > 0$, we obtain

$$q(z) = p_+(z; s_2, f) - p_+(z; s_1, f) > 0, \quad \max\{\zeta_0(s_2, f), \zeta_0(s_1, f)\} < z < z_{\mathrm{init}}.$$

Now we obtain $\zeta_0(s_1, f) \le \zeta_0(s_2, f)$. This completes the proof. $\square$

2.5 Searching speed c

Let $\delta_0 \in (0,1)$ be arbitrarily given. We have $\zeta_0(c_0+\delta_0, f_0) \in [0,1)$ with

$$\begin{aligned}
&p_+(\zeta_0(c_0+\delta_0, f_0); c_0+\delta_0, f_0) = 0,\\
&p_+(z; c_0-\delta_0, f_0) < p_+(z; c_0, f_0) < p_+(z; c_0+\delta_0, f_0) < 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad z \in (\zeta_0(c_0+\delta_0, f_0), 1),\\
&p_+(z; c_0-\delta_0, f_0) < 0, \qquad z \in (0,1).
\end{aligned}$$

Taking $\delta_0 \in (0,1)$ small enough and applying Lemma 2.6, we have

$$0 \le \zeta_0(c_0+\delta_0, f_0) < z_*.$$

Taking $\delta_0 \in (0,1)$ smaller if necessary and taking $\|f-f_0\|_{C^1[0,1]}$ small enough, we also have

$$0 \le \zeta_0(c_0+\delta_0, f) < z_* \tag{2.16}$$

by Lemma 2.6.

Now we have

$$p_+(z_*; c_0-\delta_0, f_0) < p_+(z_*; c_0, f_0) < p_+(z_*; c_0+\delta_0, f_0) < 0.$$

Taking $\|f-f_0\|_{C^1[0,1]}$ small enough and applying Lemma 2.6, we have

$$p_+(z_*; c_0-\delta_0, f) < p_+(z_*; c_0, f_0) < p_+(z_*; c_0+\delta_0, f) < 0.$$

Recalling (2.6) and applying Lemma 2.7, we obtain

$$\begin{aligned}
&p_+(z; c_0-\delta_0, f) < p_+(z; c_0, f_0), \qquad z \in (0, z_*], \\
&p_+(z; c_0-\delta_0, f) < p_+(z; c_0, f_0) < p_+(z; c_0+\delta_0, f) < 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad z \in (\zeta_0(c_0+\delta_0, f), z_*]
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
p_+(\zeta_0(c_0+\delta_0, f); c_0-\delta_0, f) &< p_+(\zeta_0(c_0+\delta_0, f); c_0, f_0)\\
&< p_+(\zeta_0(c_0+\delta_0, f); c_0+\delta_0, f) = 0.
\end{aligned}$$

Using (2.17) and $p_+(0; c_0, f_0) = 0$, we have

$$\zeta_0(c_0-\delta_0) \le 0$$

and

$$(p_+)_z\,(z; c_0-\delta_0, f) = -(c_0-\delta_0) - \frac{f(z)}{p_+(z; c_0-\delta_0, f)}, \qquad 0<z<1, \tag{2.18}$$

$$p_+(z; c_0-\delta_0, f) < 0, \qquad 0<z<1, \tag{2.19}$$

$$p_+(1; c_0-\delta_0, f) = 0. \tag{2.20}$$

To prove Theorem 2.1 we have $\zeta = p_+(z; c_0+\delta_0, f)$ in the (z,ζ) plane in Figure 2.1. We study $\zeta = p_+(z; c_0-\delta_0, f)$ in the following lemma and will show the existence of $\zeta = p_+(z; c, f)$ with $p_+(0; c, f) = 0$ for some $c \in [c_0-\delta_0, c_0+\delta_0]$.

Lemma 2.8. *Assume* $|s-c_0| \le 1$ *and*

$$\|f-f_0\|_{C^1[0,1]} \le 1. \tag{2.21}$$

Take $M \ge 1$ *large enough such that one has (2.7) for all* $s \in [c_0-1, c_0+1]$ *and for all* f *with (2.21). Assume that* $|s-c_0| + \|f-f_0\|_{C^1[0,1]}$ *is small enough such that one has (2.16). Then there exists* $\gamma \in [0, 2M]$ *such that one has*

$$\gamma = \lim_{z\to 0}\left(-p_+(z; c_0-\delta_0, f)\right).$$

Proof. We define $W = W(y)$ by

$$\frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix} W \\ W' \end{pmatrix} = \begin{pmatrix} W' \\ -(c_0-\delta_0)W' - f(W) \end{pmatrix}, \qquad y \in \mathbb{R},$$
$$W(0) = \alpha_*, \quad W'(0) = p_+(\alpha_*; c_0-\delta_0, f) < 0.$$

Now we have

$$W'(y) = p_+(W(y); c_0-\delta_0, f), \qquad 0 \le y < \infty.$$

Using (2.17), $p_+(0; c_0, f_0) = 0$ and Lemma 2.3, we have one of the following (i) or (ii).

(i) One has

$$W'(y) < 0, \qquad y \in [0,\infty)$$

and

$$\lim_{y\to\infty}\begin{pmatrix} W(y) \\ W'(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(ii) There exists $y_0 \in (0,\infty)$ such that one has

$$W(y_0) = 0, \quad W'(y_0) < 0.$$

In Case (i), we can extend $p_+(z; c_0 - \delta_0, f)$ by

$$p_+(W(y); c_0 - \delta_0, f) = W'(y), \qquad y \in [0, \infty)$$

and obtain

$$\gamma = \lim_{z \to 0} (-p_+(z; c_0 - \delta_0, f)) = 0.$$

In Case (ii), we can extend $p_+(z; c_0 - \delta_0, f)$ by

$$p_+(W(y); c_0 - \delta_0, f) = W'(y), \qquad y \in [0, y_0)$$

and obtain

$$\gamma = \lim_{z \to 0} (-p_+(z; c_0 - \delta_0, f)) = -W'(y_0) \in (0, 2M].$$

This completes the proof. □

2.6 Proof of Theorem 2.1

Now we are ready to prove the main theorem.

Proof of Theorem 2.1. By the assumption we have (2.16). By the definition of $\zeta_0(c_0 + \delta_0, f) \in [0, z_*)$, we have

$$\begin{aligned} &p_+(\zeta_0(c_0 + \delta_0, f); c_0 + \delta_0, f) = 0. \\ &p_+(z; c_0 + \delta_0, f) < 0, \qquad \zeta_0(c_0 + \delta_0, f) < z < 1. \end{aligned}$$

By Lemma 2.8, we have

$$\lim_{z \to 0} p_+(z; c_0 - \delta_0, f) = -\gamma \in (-\infty, 0].$$

Recalling (2.6) and applying Lemma 2.7, we obtain $c \in [c_0 - \delta_0, c_0 + \delta_0]$ with

$$\begin{aligned} &\lim_{z \to 0} p_+(z; c, f) = 0, \\ &p_+(z; c, f) < 0, \qquad 0 < z < 1. \end{aligned}$$

See Figure 2.1. Thus $p_+(z; c, f)$ satisfies (1.10). Defining U by (1.11), we find that (c, U) satisfies the profile equation (1.4). As $\|f - f_0\|_{C^1[0,1]}$ goes to zero, we can take $\delta_0 \in (0, 1)$ arbitrarily small. Then c converges to c_0. From (1.11) and Lemma 2.6, $\|U - U_0\|_{C(\mathbb{R})}$ converges to zero as $\|f - f_0\|_{C^1[0,1]}$ goes to zero. By

$$U'(y) = p_+(U(y); s, f), \qquad y \in \mathbb{R}$$

and Lemma 2.6, $\|U - U_0\|_{C^1(\mathbb{R})}$ converges to zero. Then $\|U - U_0\|_{C^2(\mathbb{R})}$ converges to zero as $\|f - f_0\|_{C^1[0,1]}$ goes to zero. This completes the proof. □

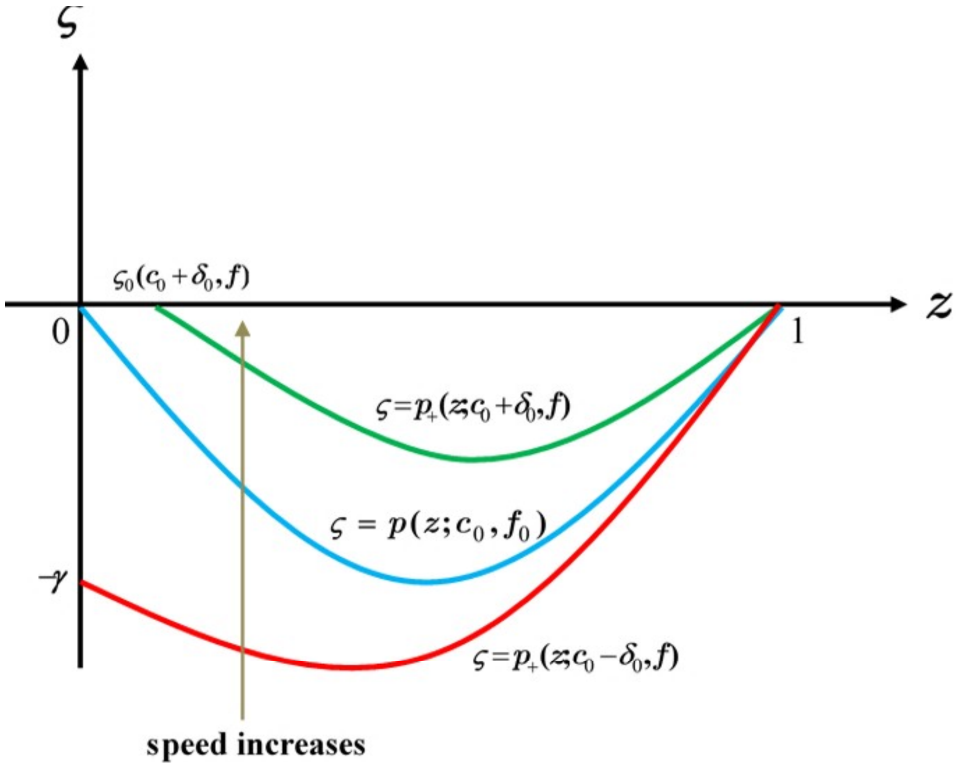

Figure 2.1: Search $c \in [c_0 - \delta_0, c_0 + \delta_0]$ with $p_+(0; c, f) = 0$.

2.7 Auxiliary result

In this section, we assume

$$f_0'(0) < 0 \tag{2.22}$$

instead of (1.5). We assume that f_0 is of class C^1 in an open interval including $[0, 1]$ with $f_0(0) = 0$, $f_0(1) = 0$ and (2.22), and assume that there exist $U_0 \in C^2(\mathbb{R})$ and $c_0 \in \mathbb{R}$ that satisfy (1.6). We define

$$g_0(u) = -f_0(1-u)$$

in an open interval including $[0, 1]$. Then we have

$$g_0(0) = 0, \quad g_0(1) = 0, \quad g_0'(1) < 0.$$

Defining

$$s_0 = -c_0,$$
$$V_0(y) = 1 - U_0(-y), \qquad y \in \mathbb{R},$$

we have

$$
\begin{aligned}
&V_0''(y) + s_0 V_0'(y) + g_0(V_0(y)) = 0, \qquad y \in \mathbb{R}, \\
&V_0'(y) < 0, \qquad y \in \mathbb{R}, \\
&V_0(-\infty) = 1, \quad V_0(\infty) = 0.
\end{aligned}
$$

Let $C_0^1[0,1)$ be the set of functions in $C^1[0,1)$ whose supports lie in $[0,1)$.

Corollary 2.9. *Let f_0 be of class C^1 in an open interval including $[0,1]$ with*

$$
f_0(0) = 0, \quad f_0(1) = 0, \quad f_0'(0) < 0.
$$

Let f be of class C^1 in an open interval including $[0,1]$ with

$$
f(0) = 0, \quad f(1) = 0.
$$

Assume that there exists (c_0, U_0) that satisfies (1.6). Assume that $f - f_0 \in C_0^1[0,1)$ and let $\|f - f_0\|_{C^1[0,1]}$ be small enough. Then there exists (c, U) that satisfies (1.4). If $\|f - f_0\|_{C^1[0,1]}$ goes to zero, c converges to c_0 and $\|U - U_0\|_{C^2(\mathbb{R})}$ goes to zero.

Proof. Combining Theorem 2.1 and the argument stated above, we have this corollary. ∎

Chapter 3

Traveling front solutions for bistable or multistable nonlinear perturbed reaction-diffusion equations

In this chapter we prove the existence of a traveling front solution to (1.1) for some speed $c \in \mathbb{R}$ and bistable or multistable nonlinear function f under perturbation. That is, we have to find some speed $c \in \mathbb{R}$ for which a trajectory joins the two saddle points. Such a trajectory will correspond to a traveling front from stationary points $u = 0$ and $u = 1$. The following theorem is prescribed for this argument.

3.1 Main result 2

Theorem 3.1. *Let f_0 be of class C^1 in an open interval including $[0, 1]$ with*

$$f_0(0) = 0, \qquad f_0(1) = 0, \qquad f_0'(0) < 0, \qquad f_0'(1) < 0.$$

Let f be of class C^1 in an open interval including $[0, 1]$ with

$$f(0) = 0, \qquad f(1) = 0.$$

Assume that there exists (c_0, U_0) that satisfies (1.6). Assume that $f - f_0 \in C^1[0, 1]$ and let $\|f - f_0\|_{C^1[0,1]}$ be small enough. Then there exists (c, U) that satisfies (1.4). If $\|f - f_0\|_{C^1[0,1]}$ goes to zero, c converges to c_0, and $\|U - U_0\|_{C^2(\mathbb{R})}$ goes to zero.

3.2 Background assertion

In Corollary (2.9) we assume that f_0 is of class C^1 in an open interval including [0,1] with

$$f_0(0) = 0, \qquad f_0(1) = 0, \qquad f_0'(0) < 0.$$

and there exist traveling profile $U_0 \in C^2(\mathbb{R})$ and $c_0 \in \mathbb{R}$ that satisfy (1.6). Then we have a traveling profile (c, U) that satisfy (1.4). From Corollary (2.9) we obtain the following solution equations:
For some $c \in \mathbb{R}$ there exists $p_-(z; c, f)$ defined for $z \in (0,1)$ such that one has

$$(p_-)_z\,(z; c, f) = -c - \frac{f(z)}{p_-(z; c, f)}, \qquad z \in (0,1), \tag{3.1}$$

$$p_-(z; c, f) < 0, \qquad z \in (0,1), \tag{3.2}$$

$$p_-(0; c, f) = 0, \tag{3.3}$$

$$(p_-)_z\,(0; c, f) = \frac{-c - \sqrt{c^2 - 4f'(0)}}{2} < 0. \tag{3.4}$$

We will apply these solution equations to the proof of Theorem 3.1.

3.3 Proof of Theorem

Proof. Let $\delta_0 \in (0,1)$ be arbitrarily given. Assume $(f, s) \in \Omega$ and $\|f - f_0\|_{C^1[0,1]}$ is small enough. Taking $\delta_0 \in (0,1)$ small enough if necessary and using Theorem (2.1), we have

$$p_+(z; c_0 - \delta_0, f) \; < \; p_+(z; c_0, f_0) \; < \; p_+(z; c_0 + \delta_0, f)$$

for all $z \in [\frac{1}{2}, 1]$.

By applying Corollary 2.9, we have

$$p_-(z; c_0 + \delta_0, f) \; < \; p_-(z; c_0, f_0) \; < \; p_-(z; c_0 - \delta_0, f)$$

for all $z \in [0, \frac{1}{2})$.

By using [20, Theorem 1.4] we have

$$p_+(\frac{1}{2}; c_0, f_0) = p_-(\frac{1}{2}; c_0, f_0),$$

and it follows that there exists $c \in (c_0 - \delta_0, c_0 + \delta_0,)$ such that one has

$$p_-(\tfrac{1}{2}; c, f) = p_+(\tfrac{1}{2}; c, f).$$

We set

$$p(z; c, f) = p_-(z; c, f) \quad \text{if} \quad z \in (0, \tfrac{1}{2}],$$

and

$$p(z; c, f) = p_+(z; c, f) \quad \text{if} \quad z \in (\tfrac{1}{2}, 1),$$

and have the solution $p(z; c, f)$ that satisfies (1.10). Then we have profile solution (c, U) by defining (1.11). This completes the proof. □

To be more precise we describe the existence of such a traveling front solution by a geometric approach. See Figure 3.1. The following figure shows the combination of two solution curves or the matching condition of solution $p_+(z; c, f)$ and $p_-(z; c, f)$.

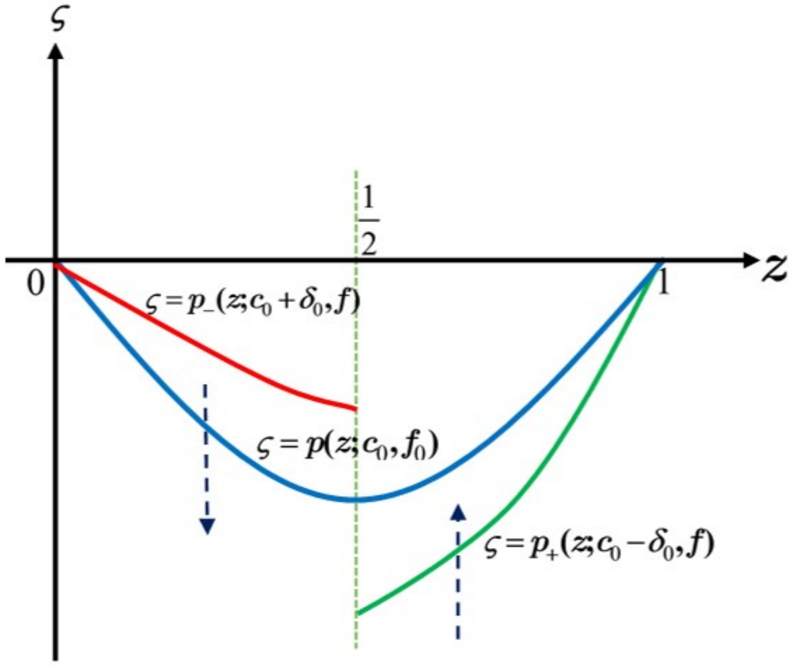

Figure 3.1: Combination of orbit for $p_+(z; c, f)$ and $p_-(z; c, f)$.

Theorem 3.1 asserts that a traveling front to (1.4) for a perturbed bistable or multistable nonlinearity is robust under $C^1[0, 1]$ perturbation.

3.4 Summary and Discussion

Theorem 2.1 asserts that a traveling front is robust under perturbation on a nonlinear term by assuming (1.5). If we assume $f_0'(0) < 0$ in addition, Theorem 3.1 shows that traveling fronts for bistable or multistable nonlinear terms are robust under perturbation. It can be proved by using Theorem 2.1 and auxiliary result or corollary 2.9. So we observe that the assumption $\|f - f_0\|_{C^1[0,1]}$ is small enough to be one of the necessary and sufficient conditions for the existence of traveling front solutions to nonlinear reaction-diffusion equations under perturbation.

The existence of (c, U) to (1.4) is an open problem if one assumes the existence of (c_0, U_0) to (1.6) without assuming (1.5) and just assumes that $\|f - f_0\|_{C^1[0,1]}$ is small enough. Furthermore, the assumption $f - f_0 \in C_0^1(0, 1]$ is necessarily in Theorem 2.1. The main results might be the new steps to attack this general robustness problem of traveling fronts.

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