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NOTES ON THE FILTRATION OF THE K-THEORY FOR ABELIAN p-GROUPS

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Abstract. Let p be a prime number. For a given finite group G, let $gr_\gamma^*(BG)$ be the associated ring of the gamma filtration of the topological K-theory for the classifying space BG. In this paper, we study $gr_\gamma^*(BG)$ when G are abelian p-groups which are not elementary. In particular, we extend related Chetard's results for such 2-groups to p-groups for odd p.

1. Introduction

Let p be a prime number. For a given finite group G, let $gr_{top}^*(BG)$ (resp. $gr_\gamma^*(BG)$) be the associated graded ring of the topological (resp. gamma) filtration of the K-theory $K^0(BG)$ for the classifying space BG.

In Theorem 4.1 in [5], I wrote that for $q = p^r$ and $G = \oplus^n \mathbb{Z}/q$, we had

$$(*) \quad gr_{top}^*(BG) \cong \mathbb{Z}[y_1, ..., y_n]/(qy_i, y_i^q y_j - y_i y_j^q | 1 \le i, j \le n), \quad |y_i| = 2.$$

But $(*)$ is not correct for $r \ge 2$, indeed, arguments for the higher Bokstein Q_0' in its proof were errors. However the statement $(*)$ holds still (without changing any arguments) for $r = 1$, i.e., for an elementary abelian p-group G. (The fact $gr_{top}^*(BG) \cong gr_\gamma^*(BG)$ holds for all abelian p-groups [1].)

Beatrice Chetard pointed out this fact [2]. She also gives another proof of $(*)$ for $r = 1$, and shows the following isomorphism by using the definition of the gamma filtration of the representation ring

$$gr_\gamma^*(B(\mathbb{Z}/4 \times \mathbb{Z}/4)) \cong \mathbb{Z}[y_1, y_2]/(4y_1, 4y_2, 2y_1^2 y_2 + 2y_1 y_2^2, y_1^4 y_2^2 - y_1^2 y_2^4).$$

She also computes $gr_\gamma^*(B(\mathbb{Z}/4 \times \mathbb{Z}/2))$, and conjectured

$$gr_\gamma^*(B(\mathbb{Z}/2^r \times \mathbb{Z}/2)) \cong \mathbb{Z}[y_1, y_2]/(2^r y_1, 2y_2, y_1 y_2^{r+1} + y_1^2 y_2^r).$$

In this note, we will prove her conjecture and see that the above Chetard results can be extended to odd prime cases. Let us write $y(1) = y_1^p y_2 - y_1 y_2^p$. Then we have

Theorem 1.1. *For each prime p, let $G = \mathbb{Z}/p^2 \times \mathbb{Z}/p^2$. Then*

$$gr_{top}^*(BG) \cong \mathbb{Z}[y_1, y_2]/(p^2 y_1, p^2 y_2, py(1), y(1)^p).$$

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Theorem 1.2. *For each prime p, let $G = (\mathbb{Z}/p^r \times \mathbb{Z}/p)$, $r \geq 1$. Then*

$$gr_{top}^*(BG) \cong \mathbb{Z}[y_1, y_2]/(p^r y_1, py_2, s_r)$$

where $s_r = y_1 y_2^{r(p-1)+1} - y_1^p y_2^{(r-1)(p-1)+1} = y(1) y_2^{(r-1)(p-1)}$.

Here we note that $gr_\gamma^*(BG)$ are known for many nonabelian p-groups G by using $gr_{top}^*(BG)$ and the Atiyah-Hirzebruch spectral sequence, while the direct computations of $gr_\gamma^*(BG)$ by using representations theory are not so many.

For example, when $|G| = p^3$ and nonabelian, we know [5]

$$gr_\gamma^*(BG) \cong gr_{top}^*(BG) \cong H^{even}(BG)/(Q_1 H^{odd}(BG)).$$

Here $H^{odd}(BG)$ is just p-torsion and we can define the Milnor Q_1-operation on $H^{odd}(BG)$ (see the proof of Theorem 4.2 in [5]). In particular, when $G = Q_8$ the quaternion group of the order 8, it is known $H^{odd}(BG) = 0$, which implies $H^{even}(BG) \cong gr_\gamma^*(BG)$. Using representation arguments and the ring structure of $gr_\gamma^*(BG)$, Atiyah [1] gets the ring structure of $H^*(BQ_8)$.

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2. $H^*(B(\mathbb{Z}/q \times \mathbb{Z}/q))$ AND $H^*(B\mathbb{Z}/q \times B\mathbb{Z}/p)$

Let $X = B\mathbb{Z}/q$ with $q = p^r$, $r \geq 2$. Its integral cohomology is $H^*(X) \cong \mathbb{Z}[y]/(qy)$ with the degree $|y| = 2$. Considering the long exact sequence for $q' = q$ or $q' = p$

$$... \to H^{*-1}(X; \mathbb{Z}/q') \xrightarrow{\delta} H^*(X) \xrightarrow{q'} H^*(X) \to H^*(X; \mathbb{Z}/q') \to ...,$$

we have

$$(2.1) \quad H^*(X; \mathbb{Z}/q) \cong H^*(X)/q\{1, x\}, \quad x = \delta^{-1} y$$

$$(2.2) \quad H^*(X; \mathbb{Z}/p) \cong H^*(X)/p\{1, x'\} \quad x' = \delta^{-1}(p^{r-1} y).$$

Here the notation $H\{x, ..., z\}$ means the H-free module generated by $x, ..., z$.

We consider the Serre spectral sequence for $X = X_1 = X_2$

$$E_2^{*,*'} \cong H^*(X_1; H^{*'}(X_2)) \Longrightarrow H^*(X_1 \times X_2)$$

$$\textit{with} \qquad E_2^{*,*'} \cong \begin{cases} \mathbb{Z}[y_1]/(qy_1) & *' = 0 \\ \mathbb{Z}/(q)[y_1]\{1, x_1\} \otimes y_2^{*'} & *' > 0. \end{cases}$$

Here we identify $y_1 \in E_2^{2,0} \cong H^2(X_1)$, and $y_2 \in E_2^{0,2} \cong H^2(X_2)$. Moreover $x_1 y_2 \in E_2^{1,2} \cong H^1(X_1, H^2(X_2))$ with $H^2(X_2) \cong \mathbb{Z}/q$ from (2.1).

Since $H^*(X_i) \subset H^*(X_1 \times X_2)$, elements y_1, y_2 are permanent cycles, and so is x_1y_2. The $E_2^{*,*'}$-term is multiplicatively generated by these elements. Hence we have

$$E_\infty^{*,*'} \cong \mathbb{Z}/q[y_1, y_2]\{1, y_2x_1\} \quad for\ (*, *') \neq (0,0).$$

Writing by $\alpha \in H^3(X \times X)$ which represents $y_2x_1 \in E_\infty^{1,2}$, we have

Lemma 2.1. *For $X = B\mathbb{Z}/q$, $q = p^r$, we have*

$$H^*(X \times X) \cong \mathbb{Z}[y_1, y_2]\{1, \alpha\}/(qy_1, qy_2, q\alpha), \quad |\alpha| = 3.$$

Next, we compute the spectral sequence for $X \times B\mathbb{Z}/p$ by using (2.2)

$$E_2^{*,*'} \cong H^*(X; H^{*'}(B\mathbb{Z}/p)) \Longrightarrow H^*(X \times B\mathbb{Z}/p)$$

$$with \qquad E_2^{*,*'} \cong \begin{cases} \mathbb{Z}[y_1]/(qy_1) & *' = 0 \\ \mathbb{Z}/p[y_1]\{1, x_1'\} \otimes y_2^{*'} & *' > 0. \end{cases}$$

Lemma 2.2. *For $X = B\mathbb{Z}/q$, $q = p^r$, (identifying $\alpha' = x_1'y_2$), we have*

$$H^*(X \times B\mathbb{Z}/p) \cong \mathbb{Z}[y_1, y_2]\{1, \alpha'\}/(qy_1, py_2, p\alpha'), \quad |\alpha'| = 3.$$

3. $gr_{top}^*(X \times B\mathbb{Z}/p)$

In this note we study $gr_{top}^*(BG)$ only for a p-group G. Then $gr_{top}^*(BG) \cong E_\infty^{*,0}$ for the infinite term of the Atiyah-Hirzebruch spectral sequence converging to the integral Morava K-theory $\tilde{K}(1)^*(BG)$ with the coefficient $\tilde{K}(1)^* = \mathbb{Z}_{(p)}[v_1, v_1^{-1}]$. In this note, we use this Morava K-theory, instead of the usual complex K-theory. So hereafter this note, let $K^*(BG)$ mean the Morava K-theory $\tilde{K}(1)^*(BG)$.

Also hereafter this section, we assume $G = (\mathbb{Z}/q \times \mathbb{Z}/p)$ and $X = B\mathbb{Z}/q$. We will prove

Theorem 3.1. *Let $G = \mathbb{Z}/p^r \times \mathbb{Z}/p$. Then we have*

$$gr_{top}^*(BG) \cong \mathbb{Z}[y_1, y_2]/(p^r y_1, py_2, s_r)$$

where $s_r = y_1 y_2^{r(p-1)+1} - y_1^p y_2^{(r-1)(p-1)+1}$.

At first, we study relations in $K^*(BG)$. Recall that $[p](y)$ is the p-th product of the formal group law of the Morava K-theory ([3], [4]) so that

$$K^*(B\mathbb{Z}/p) \cong K^*[[y]]/([p](y)) \quad |y| = 2.$$

We can identify $K^* = \mathbb{Z}_{(p)}[v_1, v_1^{-1}]$, with $|v_1| = -2(p-1)$, and write

$$[p](y) = py + v_1 y^p.$$

Similarly we have $K^*(B\mathbb{Z}/q) \cong K^*[[y]]/([q](y))$. The K-theory of BG has the Kunneth formula, and we have

$$K^*(BG) \cong K^*(X) \otimes_{K^*} K^*(B\mathbb{Z}/p) \cong K^*[[y_1, y_2]]/([p^r](y_1), [p](y_2)).$$

The equation $[p](y_2) = py_2 + v_1 y_2^p$ implies

$$(*) \quad p^r y_2 = -p^{r-1} v_1 y_2^p = p^{r-2} v_1^2 y_2^{2p-1} = ... = (-1)^r v_1^r y_2^{r(p-1)+1}$$

in $K^*(B\mathbb{Z}/p)$.

To study $[p^r](y_1)$, at first, we consider it in $C = \mathbb{Z}_{(p)}[v_1, y_1, y_2]$. Let $I = (p, v_1)$ be the ideal in $\mathbb{Z}_{(p)}[v_1]$ generated by p, v_1, and let $I^k(y_1)$ be the ideal in C generated by the product of I^k and y_1 for $k = 1, 2, ...$

Then we easily see by induction

$$[p^r](y_1) = [p]([p^{r-1}](y_1)) = p^r y_1 + p^{r-1} v_1 y_1^p \quad mod(I^{r+1}).$$

We compute $y_2[p^r](y_1)$ in $C' = C/([p](y_2))$, (which is zero in $K^*(BG)(y_2)$). Let us write $f \equiv g \ mod(A)$ for $f, g \in C$ if there is $x \in A \subset C$ such that $f = g + x \in C'$. Then modulo $I^{r+1}(y_1, y_2)$, we can write

$$y_2[p^r](y_1) \equiv p^r y_1 y_2 + p^{r-1} v_1 y_1^p y_2$$
$$\equiv (-1)^r v_1^r y_1 y_2^{r(p-1)+1} + (-1)^{r-1} v_1^r y_1^p y_2^{(r-1)(p-1)+1} \quad (from(*))$$
$$\equiv (-1)^r v_1^r s_r \quad (by\ definition).$$

Take $x \in I^{r+1}(y_1, y_2)$ such that

$$y_2[p^r](y_1) = v_1^r s_r + x \quad in\ C'.$$

Moreover, by using $py_2 = -v_1 y_2^p$, we can take $x = v_1^{r+1} x'$. Recall that the filtration for $gr_{top}^*(BG)$ is defined by the degree of $H^*(BG)$. Since $|v_1| < 0$, we have

Lemma 3.2. *There is $x' \in \mathbb{Z}/p[v_1, y_1, y_2]$ such that*

$$K^*(BG)(y_2) \cong K^*[[y_1, y_2]]\{y_2\}/([p](y_2), s_r + v_1 x').$$

Hence $s_r = 0$ in $gr_{top}^*(BG)$.

Now we study the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*'} \cong H^*(BG) \otimes K^{*'} \Longrightarrow K^*(BG).$$

By Atiyah [1], we know $E_\infty^{*,0} \cong gr_{top}^*(BG)$. Moreover if $E_\infty^{*,0}$ is multiplicatively generated by Chern classes in $H^*(BG)$, then $gr_{top}^*(BG) \cong gr_\gamma^*(BG)$. Note y_1, y_2 are the first Chern classes for $\mathbb{Z}/q$ and $\mathbb{Z}/p$ (and so for G).

Here we recall from Lemma 2.2, $H^{odd}(BG) \cong \mathbb{Z}/p[y_1, y_2]\{\alpha'\}$ with $|\alpha'| = 3$.

Since $K^*(BG)$ is generated by even dimensional elements, there are $t, t' > 1$ and $s' \neq 0$ in $H^{even}(BG)$ such that $d_t(\alpha') = v_1^{t'} \otimes s'$. Here note $s' \in$

$H^{even}(BG)/p\{y_2\} \cong \mathbb{Z}/p[y_1, y_2]\{y_2\}$ since elements in $\mathbb{Z}[y_1]/(q)$ are permanent from $K^*(X) \subset K^*(BG)$.

Hence the map

$$v_1^{-t'} \otimes d_t : H^{odd}(BG) \cong \mathbb{Z}/p[y_1, y_2]\{\alpha'\} \to \mathbb{Z}/p[y_1, y_2]\{s'\} \subset Z/p[y_1, y_2]\{y_2\}$$

(via $\alpha' \mapsto s'$) is injective (since $s' \neq 0$). Hence we get

$$E_{t+1}^{*,0} \cong \mathbb{Z}[y_1]/(qy_1) \oplus \mathbb{Z}/p[y_1, y_2]\{y_2\}/(s').$$

This term is generated by even dimensional elements, and is isomorphic to

$$E_{t+1}^{*,0} \cong E_{\infty}^{*,0} \cong gr_{top}^*(BG).$$

From the preceding lemma, we have the graded ring, by the filtration (v_1)

$$grK^*(BG)(y_2) \cong \mathbb{Z}/p[y_1, y_2]\{y_2\}/(s_r).$$

Hence we can take $s' = s_r$. Thus we have $E_{\infty}^{*,0} \cong gr_{top}^*(BG)$, and Theorem 3.1.

4. $gr_{top}^*(B\mathbb{Z}/p^2 \times B\mathbb{Z}/p^2)$

Throughout this section let $G = \mathbb{Z}/p^2 \times \mathbb{Z}/p^2$ and $X = B\mathbb{Z}/p^2$. We study the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*'} \cong H^*(BG) \otimes K^{*'} \Longrightarrow K^*(BG).$$

Here we recall $H^*(BG) \cong \mathbb{Z}[y_1, y_2]\{1, \alpha\}/(p^2y_1, p^2y_2, p^2\alpha)$ with $|\alpha| = 3$. We will prove

$$d_{2p-1}(\alpha) = v_1 \otimes py(1), \quad d_{2p^2+2p-3}(p\alpha) = v_1^{p+2} \otimes y(1)^p$$

for $y(1) = y_1^p y_2 - y_1 y_2^p$. Then we see that

Theorem 4.1. *Let $G = \mathbb{Z}/p^2 \times \mathbb{Z}/p^2$. Then we have the isomorphism*

$$gr_{top}^*(BG) \cong \mathbb{Z}[y_1, y_2]/(p^2y_1, p^2y_2, py(1), y(1)^p).$$

Recall that the p-product of the formal group law for K^*-theory is given by $[p](y) = py + v_1y^p$. Recall $f \equiv g \ mod(A)$ for $f, g \in C = \mathbb{Z}_{(p)}[v_1, y_1, y_2]$ if there is $x \in A \subset C$ such that $f = g + x \in C'$. Hereafter, we take $C' = C/([p^2](y_1), [p^2](y_2))$.

We note in C

$$\begin{aligned}[p^2](y_1) &= p(py_1 + v_1y_1^p) + v_1(py_1 + v_1y_1^p)^p \\ &= p^2y_1 + pv_1y_1^p + p^pv_1y_1^p + B + v_1^py_1^{p^2} \\ \textit{where} \quad B &= v_1 \sum_{k=1}^{p-1} \binom{p}{k} p^k v_1^{p-k} y_1^{k+p(p-k)}.\end{aligned}$$

Since $p^2y_1 \equiv -pv_1y_1^p\ mod(I^{p+1}(y_1))$ and $p^kv_1^{p-k} \in I^p$, we have in C'

$$B \equiv (\sum_{k=1}^{p-1}\binom{p}{k}(-1)^k)v_1^{p+1}y_1^{p^2} \equiv 0 \quad mod(I^{2p+1}(y_1)).$$

Hence we have

$$(*) \quad [p^2](y_1) \equiv p^2y_1 + (pv_1 + p^pv_1)y_1^p + v_1^{p+1}y_1^{p^2} \quad mod(I^{2p+1}(y_1)).$$

Similar equation holds for y_2.

We consider the following elements a_1, a_2 in C (which are zero in $K^*(BG)$)

$$a_1 = y_2[p^2](y_1) - y_1[p^2](y_2),$$
$$a_2 = y_2^p[p^2](y_1) - y_1^p[p^2](y_2).$$

Then from $(*)$, we have

$$a_1 \equiv (pv_1 + p^pv_1)y(1) + v_1^{p+1}y(2) \quad mod(I^{2p+1}(y_1,y_2)),$$
$$a_2 \equiv -p^2y(1) + v_1^{p+1}y(1)' \quad mod(I^{2p+1}(y_1,y_2))$$

where $y(2) = y_1^{p^2}y_2 - y_1y_2^{p^2}$ and $y(1)' = y_1^{p^2}y_2^p - y_1^py_2^{p^2}$. (Hence $y(1)' = y(1)^p\ mod(p)$.)

Here we note if $x \in I^{k+1}(y_1,y_2)$, then there is $x' \in (v_1)^k(y_1,y_2)$ such that $x = x'$ in C' by using $[p^2](y_1) = 0 \in C'$. Since $a_1 \equiv 0$, we have in C

$$(**) \quad p(1+p^{p-1})y(1) \equiv -v_1^py(2) \quad mod((v_1)^{2p-1}(y_1,y_2)).$$

In particular, we have $py(1) = 0 \in gr_{top}^*(BG)$.

Next, we will see $y(1)^p = 0 \in gr_{top}^*(BG)$. Delete $y(1)$ from the equations for a_1, a_2. Modulo $I^{2p+2}(y_1,y_2)$, we have

$$(1+p^{p-1})pa_1 + (1+p^{p-1})^2v_1a_2 \equiv p(1+p^{p-1})v_1^{p+1}y(2) + (1+p^{p-1})^2v_1^{p+2}y(1)'$$
$$\equiv -v_1^{2p+1}y(2)^2/y(1) + (1+p^{p-1})^2v_1^{p+2}y(1)' \quad from\ (**).$$

Since a_1, a_2 are zero in C', there is $x \in C$ such that

$$(1+p^{p-1})^2y(1)' - v_1^{p-1}(y(2)^2/y(1) + x) = 0 \quad in\ K^*(BG).$$

Therefore $y(1)' = 0$ in $gr_{top}^*(BG)$.

Now we sudy the Atiyah-Hirzebruch spectral sequence. Recall $py(1)$ is zero in $gr_{top}^*(BG) \cong E_\infty^{*,0}$ (but it is nonzero in $K^*(BG)$, hence $y(2) \neq 0$, since $K^*(BG)$ is torsion free). Therefore $py(1)$ is not permanent cycle in the spectral sequence for $K^*(G)$.

It is known that the first possible nonzero differential is d_{2p-1} since $|v_1| = -2p+2$. For dimensional reasons, we see

$$d_{2p-1}(\alpha) = v_1 \otimes py(1), \quad and \quad E_{2p}^{*,0} \cong \mathbb{Z}[y_1,y_2]\{1,p\alpha\}/(p^2y_1, p^2y_2, p^2\alpha, py(1)).$$

Since $K^*(BG)$ is generated by even dimensional elements, we see $\alpha'' = p\alpha$ is not a permanent cycle, i.e. there are $r > 2p, t'' > 1$, $d \in E_r^{*,0}$ such that $d_r(\alpha'') = v_1^{t''} \otimes d \neq 0$.

We study this d. At first d is invariant $mod(p)$ under the action of $SL_2(\mathbb{Z}/p)$, (since so is α'') namely d is written as b or pb for $b \in \mathbb{Z}/p[y_1, y_2]^{SL_2(\mathbb{Z}/p)}$. The invariant ring is known as the Dickson algebra

$$\mathbb{Z}/p[y_1, y_2]^{SL_2(\mathbb{Z}/p)} \cong \mathbb{Z}/p[y(1), y(2)/y(1)],$$

$$where \quad y(2)/y(1) = y_1^{p(p-1)} + y_1^{(p-1)(p-1)} y_2^{p-1} + ... + y_2^{p(p-1)}.$$

Consider the restriction to $K^*(X)$

$$res(y(2)/y(1)) = y_1^{p(p-1)} \neq 0 \in K^*(X) \cong K^*[y_1]/([p^2](y_1)).$$

Hence we can not take $d = y(2)/y(1)$ neither $d = py(2)/y(1)$.

Moreover we still see that $y(2)$ is nonzero.

Therefore if $|d| \leq 2(p^2 + p)$, then we see $d = y(1)^i$ for $i \leq p$. Here we consider the restriction to the $mod(p)$ K-theory

$$K^*(BG; \mathbb{Z}/p) \cong K^*/p[y_1, y_2]/(y_1^{p^2}, y_2^{p^2}).$$

Hence d is in the $Ideal(y_1^{p^2}, y_2^{p^2})$. Thus we see that the possibility of the smallest degree element for d is $y(1)^p$.

We still see $y(1)' = y(1)^p = 0$ in $gr_{top}^*(BG)$. Thus we can take $d = y(1)^p$.

We see that the map

$$\mathbb{Z}/p[y_1, y_2]\{\alpha'\} \to \mathbb{Z}/p[y_1, y_2]\{y(1)\}$$

by $\alpha' \mapsto y(1)^p$ is injective. Hence $E_{2p^2+2p-3}^{*,*'}$ is generated by even dimensional elements, and is isomorphic to the infinite term $E_\infty^{*,*'}$. Thus we have Theorem 4.1.

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