

ON H -EPIMORPHISMS AND CO- H -SEQUENCES IN TWO-SIDED HARADA RINGS

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ABSTRACT. In [8] M. Harada studied a left artinian ring R such that every non-small left R -module contains a non-zero injective submodule. And in [13] K. Oshiro called the ring a left Harada ring (abbreviated left H -ring). We can see many results on left Harada rings in [6] and many equivalent conditions in [4, Theorem B]. In this paper, to characterize two-sided Harada rings, we introduce new concepts “co- H -sequence” and “ H -epimorphism” and study them.

In §1 we define a H -epimorphism, a co- H -sequence and a w - H -epimorphism. In §2 we characterize H -epimorphisms and left co- H -sequences of two-sided Harada rings using well-indexed set of left Harada ring. In Lemma 2.1 we show that a left (resp. right) H -epimorphism induces the inverse right (resp. left) H -epimorphism. In Theorem 2.2, using Lemma 2.1, we characterize left (right) H -epimorphisms. In Corollary 2.3 we characterize left co- H -sequences by a well-indexed set of left Harada ring. And in Example 2.4 we give a simple example of left co- H -sequences in a Nakayama ring. In §3 we consider a w - H -epimorphism. In Lemma 3.1 we consider a left H -epimorphism with the codomain the Jacobson radical of the first term of some left co- H -sequence. And in Lemma 3.2 we further consider the left (right) H -epimorphism in Lemma 3.1. In Lemma 3.3 we consider colocal pairs in two-sided Harada rings. And in Proposition 3.4 we consider w - H -epimorphisms.

1. DEFINITIONS

Let R be a basic artinian ring. A ring R is called a *left Harada ring* or a *left H -ring* if, for any primitive idempotent e of R , there exists a primitive idempotent f_e of R with $E(T(RRe)) \cong RRf_e/S_{n_e}(RRf_e)$ for some $n_e \in \mathbb{N}$.

By, for instance, [4, Theorem B (5),(6),(14) and the proof of (6) \Rightarrow (5)], the following are equivalent:

- (a) R is a left Harada ring.
- (b) There exist a basic set $\{e_{i,j}\}_{i=1,j=1}^{m,n(i)}$ of orthogonal primitive idempotents of R and a set $\{f_i\}_{i=1}^m$ of primitive idempotents of R such

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that $E(T({}_R R e_{i,j})) \cong {}_R R f_i / S_{j-1}({}_R R f_i)$ for each $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n(i)$.

- (c) There exists a basic set $\{e_{i,j}\}_{i=1,j=1}^{m, n(i)}$ of orthogonal primitive idempotents of R such that $e_{i,1}R_R$ is injective and $e_{i,j}R_R \cong e_{i,1}J_R^{j-1}$ for each $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n(i)$.

We may consider the sets

$$\{e_{i,j}\}_{i=1,j=1}^{m, n(i)}$$

in (b), (c) coincide and call it a *well-indexed set of left Harada ring* or a *left well-indexed set*.

Further, for primitive idempotents e, f of R , we call

$$(eR, Rf)$$

is an *i-pair* if both $S(eR_R) \cong T(fR_R)$ and $S({}_R R f) \cong T({}_R R e)$ hold. And, since $\{e_{i,1}R\}_{i=1}^m$ is a basic set of indecomposable projective injective right R -modules, for each $i = 1, 2, \dots, m$, there exists $e_{\sigma(i), \rho(i)} \in \{e_{i,j}\}_{i=1,j=1}^{m, n(i)}$ such that $(e_{i,1}R, R e_{\sigma(i), \rho(i)})$ is an *i-pair* by [7, Theorem 3.1], where $\sigma, \rho : \{1, 2, \dots, m\} \rightarrow \mathbb{N}$ are mappings.

Unless otherwise stated, throughout this paper, we let R be a basic two-sided Harada ring, let $\{e_{i,j}\}_{i=1,j=1}^{m, n(i)}$ be its well-indexed set of left Harada ring, let σ, ρ be mappings above, and, for each $i = 1, 2, \dots, m$ and each $j = 2, 3, \dots, n(i)$, let

$$\theta_{i,j} : e_{i,j}R_R \rightarrow e_{i,j-1}J_R$$

be an R -isomorphism.

Let R be an artinian ring, let $\{e_i\}_{i=1}^n$ be a complete set of orthogonal primitive idempotents of R and let $\{f_i\}_{i=1}^k \subseteq \{e_i\}_{i=1}^n$. A sequence f_1R, f_2R, \dots, f_kR is called a *right co-H-sequence* of R if the following (CHS1), (CHS2), (CHS3) hold.

- (CHS1) For each $i = 1, 2, \dots, k-1$, there exists an R -isomorphism $\xi_i : f_iR_R \rightarrow f_{i+1}J_R$.
- (CHS2) The last term f_kR_R is injective.
- (CHS3) f_1R, f_2R, \dots, f_kR is the longest sequence among the sequences which satisfy both (CHS1) and (CHS2), i.e., there does not exist an R -isomorphism: $fR_R \rightarrow f_1J_R$, where $f \in \{e_i\}_{i=1}^n$.

Similarly, we define a *left co-H-sequence* Rf_1, Rf_2, \dots, Rf_k of R .

Obviously, for each $i = 1, 2, \dots, m$

$$e_{i,n(i)}R_R, e_{i,n(i)-1}R_R, \dots, e_{i,1}R_R$$

is a right co- H -sequence of R . And, for an artinian ring R' , it is a left Harada ring if and only if there exists a basic set $\{e_{i,j}\}_{i=1,j=1}^{m,n(i)}$ of orthogonal primitive idempotents of R' such that $e_{i,n(i)}R', e_{i,n(i)-1}R', \dots, e_{i,1}R'$ is a right co- H -sequence of R' for all $i = 1, 2, \dots, m$.

From the definition of a left Harada ring, the following lemma holds:

Lemma 1.1. *For a left Harada ring R' and primitive idempotents f_1, f_2, \dots, f_k of R' , the following are equivalent.*

- (a) $f_1R', f_2R', \dots, f_kR'$ is a right co- H -sequence.
- (b) $f_1R', f_2R', \dots, f_kR'$ satisfies (CHS1) and the following (CHS3'):
 (CHS3') $f_1R', f_2R', \dots, f_kR'$ is the longest sequence among sequences which satisfy (CHS1).

In this paper, using a well-indexed set $\{e_{i,j}\}_{i=1,j=1}^{m,n(i)}$ of left Harada ring, we characterize left co- H -sequences of R , i.e., we give the structure of R as a right Harada ring.

Let $\{e_i\}_{i=1}^n$ be a complete set of orthogonal primitive idempotents of R and let $\{f_i\}_{i=1}^{j+1} \subseteq \{e_i\}_{i=1}^n$, where f_1, f_2, \dots, f_{j+1} are mutually distinct. Then we call $\varphi : f_1R_R \rightarrow f_2J_R$ (resp. ${}_R R f_1 \rightarrow {}_R J f_2$) a *right* (resp. *left*) H -epimorphism if φ is a non-zero R -epimorphism with $J \cdot \text{Ker } \varphi = 0$ (resp. $\text{Ker } \varphi \cdot J = 0$). And we call $\varphi : f_1R_R \rightarrow f_{j+1}J_R^j$ (resp. ${}_R R f_1 \rightarrow {}_R J^j f_{j+1}$) a *right* (resp. *left*) *weak H -epimorphism* (or simply a *right* (resp. *left*) *w- H -epimorphism*) if there exist right (resp. left) H -epimorphisms $\varphi_i : f_iR_R \rightarrow f_{i+1}J_R$ ($i = 1, 2, \dots, j$) with $\varphi = \varphi_j \varphi_{j-1} \cdots \varphi_1$ (resp. $\varphi_i : {}_R R f_i \rightarrow {}_R J f_{i+1}$ ($i = 1, 2, \dots, j$) with $\varphi = \varphi_1 \varphi_2 \cdots \varphi_j$).

For $a \in R$, we write the left (resp. right) multiplication map by a

$$(a)_L \quad (\text{resp. } (a)_R).$$

And, for primitive idempotents e, f and g , we use the following terminologies.

- If both $S(eReeRf)$ and $S(eRf_fRf)$ are simple, we call (eR, Rf) is a *colocal pair* following [12] and [11]. And then $S(eReeRf) = S(eRf_fRf)$ holds. We abbreviate it to

$$S(eRf).$$

- We put

$$R(e) \stackrel{\text{put}}{:=} eRe.$$

2. H -EPIMORPHISMS AND LEFT CO- H -SEQUENCES OF TWO-SIDED
HARADA RINGS

First we give basic results.

Lemma A.

(I) For any $i = 1, 2, \dots, m$ and any $j = 1, 2, \dots, n(i)$, the following hold.

- (1) ${}_R R e_{\sigma(i), \rho(i)} / S_{j-1}({}_R R e_{\sigma(i), \rho(i)}) \cong E(T({}_R R e_{i,j}))$. So $S_{n(i)}({}_R R e_{\sigma(i), \rho(i)})$ is uniserial.
- (2) $S_j({}_R R e_{\sigma(i), \rho(i)}) = \bigoplus_{k=1}^j S(e_{i,k} R_R)$.
- (3) $S(e_{ij} R_R) \cong T(e_{\sigma(i), \rho(i)} R_R)$ and $S(e_{i,j} R_R) = S(e_{i,j} R_R) e_{\sigma(i), \rho(i)} = S(e_{i,j} R e_{\sigma(i), \rho(i)})$.
- (4) Suppose that $T(e_{i,1} J_R^{n(i)}) \oplus > T(e_{k,l} R_R)$. Then $l = 1$.
- (5) $S(R_R) \cong \bigoplus_{i=1}^m T(e_{\sigma(i), \rho(i)} R_R)^{n(i)}$.

(II) Let

$$Rf_1, Rf_2, \dots, Rf_{n'}$$

be a left co- H -sequence of R and let $(e_{i,1} R, Rf_{n'})$ be an i -pair, where $f_1, f_2, \dots, f_{n'}$ are primitive idempotents of R . Then, for any $p = 1, 2, \dots, n'$, the following hold.

- (1) $e_{i,1} R_R / S_{p-1}(e_{i,1} R_R) \cong E(T(f_{n'-p+1} R_R))$. So $S_{n'}(e_{i,1} R_R)$ is uniserial.
- (2) $S_p(e_{i,1} R_R) = \bigoplus_{k=n'-p+1}^{n'} S({}_R R f_k)$.
- (3) $S({}_R R f_p) \cong T({}_R R e_{i,1})$ and $S({}_R R f_p) = e_{i,1} S({}_R R f_p) = S(e_{i,1} R f_p)$.
So, in particular, if $S({}_R R e_{k,l}) \cong T({}_R R e_{s,t})$ for some $e_{k,l}$ and $e_{s,t}$, then $t = 1$ and $S({}_R R e_{k,l}) = e_{s,1} S({}_R R e_{k,l}) = S(e_{s,1} R e_{k,l})$.
- (4) Suppose that $T({}_R J^{n'} f_{n'}) \oplus > T({}_R R g)$, where g is a primitive idempotent of R . Then ${}_R R g$ is injective.
- (5) For each $i = 1, 2, \dots, m$, we let $Rf_{i,1}, Rf_{i,2}, \dots, Rf_{i,n'_i} = R e_{\sigma(i), \rho(i)}$ be a left co- H -sequence with the last term $R e_{\sigma(i), \rho(i)}$. Then $S({}_R R) = \bigoplus_{i=1}^m \bigoplus_{j=1}^{n'_i} S({}_R R f_{i,j})$. So $S({}_R R) \cong \bigoplus_{i=1}^m T({}_R R e_{i,1})^{n'_i}$.

Proof.

- (I) (1) By [6, Lemma 3.3.1].
- (2) By [14, Proposition 3.5 (1)].
- (3) By [3, Theorem 1], $S(e_{i,j} R_R) \cong T(e_{\sigma(i), \rho(i)} R_R)$ and $S({}_{R(e_{i,j})} e_{i,j} R e_{\sigma(i), \rho(i)})$ is simple. So the statements follow from, for instance, [3, Lemma 1 (2),(3)] since R is a basic ring.

- (4) There exists $\varphi : e_{k,l}R_R \rightarrow e_{i,1}J_R^{n(i)}$ with $\text{Im } \varphi \not\subseteq e_{i,1}J^{n(i)+1}$ by the assumption. Suppose that $l \neq 1$. There exists $\tilde{\varphi} : e_{k,l-1}R_R \rightarrow e_{i,1}R_R$ with $\tilde{\varphi}\theta_{k,l} = \varphi$ since $e_{i,1}R_R$ is injective. Then, because $\text{Im } \varphi \subseteq e_{i,1}J^{n(i)}$ and $\text{Im } \varphi \not\subseteq e_{i,1}J^{n(i)+1}$, $\text{Im } \tilde{\varphi} \subseteq e_{i,1}J^{n(i)-1}$ and $\text{Im } \tilde{\varphi} \not\subseteq e_{i,1}J^{n(i)}$. So $T(e_{k,l-1}R_R) \cong T(e_{i,1}J_R^{n(i)-1})$ since $e_{i,1}R_R/e_{i,1}J^{n(i)}$ is uniserial by the definition of $\{e_{i,j}\}_{i=1,j=1}^{m \ n(i)}$. On the other hand, $T(e_{i,1}J_R^{n(i)-1}) \cong T(e_{i,n(i)}R_R)$ by the definition of $\{e_{i,j}\}_{i=1,j=1}^{m \ n(i)}$. Therefore $(k, l) = (i, n(i) + 1)$. But $e_{i,n(i)+1}$ does not exist, a contradiction.
- (5) By the equivalent condition (c) given in the definition of a left Harada ring.

(II) By left-right symmetric argument of (I), we see that (1),(2),(4),(5) and the first half of (3) hold. We only show the second half of (3). We assume that $S({}_RRe_{k,l}) \cong T({}_RRe_{s,t})$ for some $e_{k,l}$ and $e_{s,t}$. Then $t = 1$, i.e., $e_{s,t}R_R$ is injective, by (II)(5). And $S({}_RRe_{k,l}) \cong S({}_RRe_{\sigma(s),\rho(s)})$. Now we consider a left co- H -sequence with the last term $Re_{\sigma(s),\rho(s)}$. And we may assume that $Re_{k,l}$ is a term of the left co- H -sequence. Hence $S({}_RRe_{k,l}) = e_{s,1}S({}_RRe_{k,l}) = S(e_{s,1}Re_{k,l})$ by the first half.

□

Now we give an important lemma which gives the relationship between left H -epimorphisms and right H -epimorphisms.

Lemma 2.1.

- (I) Suppose that $\zeta : {}_RRe_{i,j} \rightarrow {}_RJe_{k,l}$ is a left H -epimorphism. Then the left multiplication map

$$\xi \stackrel{\text{put}}{:=} ((e_{i,j})\zeta)_L : e_{k,l}R_R \rightarrow e_{i,j}J_R$$

by $(e_{i,j})\zeta$ is a right H -epimorphism.

Further, we put $I_k \stackrel{\text{put}}{:=} \{ (p, q) \mid S({}_RRe_{p,q}) \cong T({}_RRe_{k,1}) \}$ and let $Rf_1, Rf_2, \dots, Rf_{n'} = Re_{\sigma(k),\rho(k)}$ be a left co- H -sequence with the last term $Re_{\sigma(k),\rho(k)}$, where $\{f_1, f_2, \dots, f_{n'}\} \subseteq \{e_{i,j}\}_{i=1,j=1}^{m \ n(i)}$ (then, since R is a right Harada ring, $\{f_1, f_2, \dots, f_{n'}\} = \{e_{p,q}\}_{(p,q) \in I_k}$). ξ satisfies the following conditions (1),(2),(3).

$$(1) \quad \text{Ker } \xi = e_{k,l}S({}_RRe_{k,l}) = \begin{cases} \bigoplus_{(p,q) \in I_k} S({}_RRe_{p,q}) \neq 0 & (\text{if } l = 1) \\ 0 & (\text{if } l \neq 1) \end{cases}$$

(2) (i) Suppose that $e_{k,l}R_R$ is injective, i.e., $l = 1$. Then the following $(x), (y), (z)$ hold.

(x) For each $(p, q) \in I_k$,

$$S({}_R R e_{p,q}) = e_{k,1} S({}_R R e_{p,q}) = S(e_{k,1} R e_{p,q}).$$

(y) $\text{Ker } \xi = \bigoplus_{i=1}^{n'} S({}_R R f_i) = S_{n'}(e_{k,1} R_R)$ and it is uniserial as a right R -module.

(z) $j = n(i)$, i.e., $\xi : e_{k,1} R_R \rightarrow e_{i,n(i)} J_R$.

(ii) Suppose that $e_{k,l}R_R$ is not injective, i.e., $l \neq 1$. Then $(k, l) = (i, j + 1)$ and $j < n(i)$, i.e., $\xi : e_{i,j+1} R_R \rightarrow e_{i,j} J_R$.

(3) For any $e_{s,t}$, the restriction map of ξ

$$\xi_{s,t} \stackrel{\text{put}}{:=} \xi|_{e_{k,l} R e_{s,t}} : e_{k,l} R e_{s,t} \rightarrow e_{i,j} J e_{s,t}$$

is a right $R(e_{s,t})$ -epimorphism with

$\text{Ker } \xi_{s,t} =$

$$\begin{cases} S(e_{k,1} R e_{s,t}) & \left(\begin{array}{l} \text{if } S({}_R R e_{s,t}) \cong T({}_R R e_{k,l}), \\ \text{i.e., } l = 1 \text{ and } (e_{k,1} R, E({}_R R e_{s,t})) \text{ is an } i\text{-pair} \end{array} \right) \\ 0 & \left(\begin{array}{l} \text{if } S({}_R R e_{s,t}) \not\cong T({}_R R e_{k,l}), \\ \text{i.e., either } l \neq 1 \text{ or } (e_{k,1} R, E({}_R R e_{s,t})) \text{ is not an } i\text{-pair} \end{array} \right) \end{cases}$$

(II) Suppose that $\xi : e_{i,j} R_R \rightarrow e_{k,l} J_R$ is a right H -epimorphism. Then the right multiplication map

$$\zeta \stackrel{\text{put}}{:=} (\xi(e_{i,j}))_R : {}_R R e_{k,l} \rightarrow {}_R J e_{i,j},$$

by $\xi(e_{i,j})$ is a left H -epimorphism. Further, if ${}_R R e_{k,l}$ is injective, we let $(e_{p,1} R, {}_R R e_{k,l})$ be an i -pair. ζ satisfies the following conditions (1), (2), (3), (4).

$$(1) \text{ Ker } \zeta = S({}_R R) e_{k,l} = \begin{cases} \bigoplus_{q=1}^{n(p)} S(e_{p,q} R_R) \neq 0 & (\text{if } {}_R R e_{k,l} \text{ is injective}) \\ 0 & (\text{if } {}_R R e_{k,l} \text{ is not injective}) \end{cases}$$

(2) Suppose that ${}_R R e_{k,l}$ is injective. Then the following $(x), (y)$ hold.

(x) For each $q = 1, 2, \dots, n(p)$,

$$S(e_{p,q} R_R) = S(e_{p,q} R_R) e_{k,l} = S(e_{p,q} R e_{k,l}).$$

(y) $\text{Ker } \zeta = S_{n(p)}({}_R R e_{k,l})$ and it is uniserial as a left R -module.

(3) For any $e_{s,t}$, the restriction map of ζ

$$\zeta_{s,t} \stackrel{\text{put}}{:=} \zeta|_{e_{s,t} R e_{k,l}} : e_{s,t} R e_{k,l} \rightarrow e_{s,t} J e_{i,j}$$

is a left $R(e_{s,t})$ -epimorphism with

$$\text{Ker } \zeta_{s,t} = \begin{cases} S(e_{s,t}Re_{k,l}) & (\text{if } S(e_{s,1}R_R) \cong T(e_{k,l}R_R), \text{ i.e., } (e_{s,1}R, Re_{k,l}) \\ & \text{is an } i\text{-pair}) \\ 0 & (\text{if } S(e_{s,1}R_R) \not\cong T(e_{k,l}R_R)) \end{cases}$$

(4) For any $i = 1, 2, \dots, m$, ${}_RRe_{i,n(i)}/J^{n(i)}e_{i,n(i)}$ is uniserial.

Proof. (I) First we show that ξ is surjective. Take any $e_{s,t}$. Since R is a right Harada ring, $E({}_RRe_{s,t}) \cong Re_{u,v}$ for some $e_{u,v}$, there exists an R -isomorphism $\iota : {}_RRe_{s,t} \rightarrow {}_RJ^n e_{u,v}$, where $n \in \mathbb{N}$, and ${}_RRe_{u,v}/J^{n+1}e_{u,v}$ is uniserial. Then, for any $\gamma \in e_{i,j}J e_{s,t}$, we can define the right multiplication map

$$(\gamma)_R : Re_{i,j}/\text{Ker } \zeta \rightarrow Re_{s,t}$$

by γ since ζ is a left H -epimorphism. Therefore we consider the diagram

$$\begin{array}{ccc} 0 & \rightarrow & Re_{i,j}/\text{Ker } \zeta \xrightarrow{\bar{\zeta}} Re_{k,l}, \\ & & (\gamma)_R \downarrow \\ & & Re_{s,t} \\ & & \iota \downarrow \cong \\ & & J^n e_{u,v} \\ & & \cap \\ & & Re_{u,v} \end{array}$$

where $\bar{\zeta}$ is an R -monomorphism naturally induced from ζ . And we have $\psi : Re_{k,l} \rightarrow Re_{u,v}$ with $\bar{\zeta}\psi = (\gamma)_R \iota$ since ${}_RRe_{u,v}$ is injective. Then $J \text{Im } \psi = (Je_{k,l})\psi = (\text{Im } \bar{\zeta})\psi = \text{Im}(\bar{\zeta}\psi) = \text{Im}((\gamma)_R \iota) \subseteq J^{n+1}e_{u,v}$. So $\text{Im } \psi \subseteq J^n e_{u,v}$ because ${}_RRe_{u,v}/J^{n+1}e_{u,v}$ is uniserial. Therefore there exists $\beta \in e_{k,l}Re_{s,t}$ with $\bar{\zeta}(\beta)_R = (\gamma)_R$. So

$$\xi(\beta) = ((e_{i,j})\zeta)_L(\beta) = (\overline{e_{i,j}})\bar{\zeta}(\beta)_R = (\overline{e_{i,j}})(\gamma)_R = \gamma.$$

Hence $\text{Im } \xi \supseteq e_{i,j}J e_{s,t}$ for any $e_{s,t}$. We see that $\text{Im } \xi = e_{i,j}J$.

Next we show that (1),(2),(3). From (1), we see that ξ is a right H -epimorphism.

(1) For any $a \in \text{Ker } \xi (\subset e_{k,l}R)$, $0 = R(e_{i,j})\zeta a = Je_{k,l}a = Ja$ since ζ is surjective. So $a \in S({}_RR)$. Hence $\text{Ker } \xi = e_{k,l}S({}_RR)$. And, by Lemma A (II)(5),

$$e_{k,l}S({}_RR) = \begin{cases} \bigoplus_{i=1}^{n'} S({}_RRf_i) & (\text{if } l = 1) \\ 0 & (\text{if } l \neq 1) \end{cases}$$

since $Rf_1, Rf_2, \dots, Rf_{n'} = Re_{\sigma(k), \rho(k)}$ is a left co- H -sequence with the last term $Re_{\sigma(k), \rho(k)}$. So the statement follows since $\{f_1, f_2, \dots, f_{n'}\} = \{e_{p,q}\}_{(p,q) \in I_k}$.

- (2) (i) (x) Since $(p, q) \in I_k$, $S(RRe_{p,q}) \cong T(RRe_{k,1})$. So the statement holds by the second half of Lemma A (II)(3).
 (y) $\text{Ker } \xi = \bigoplus_{i=1}^{n'} S(RRf_i)$ by the proof of (1). The remainder follows from Lemma A (II)(1),(2).
 (z) Assume that $j \in \{1, 2, \dots, n(i) - 1\}$. Then $e_{i,j}J_R$ is projective by the definition of a well-indexed set $\{e_{i,j}\}_{i=1, j=1}^{m \ n(i)}$ of left Harada ring. So $e_{k,l} = e_{i,j+1}$ and ξ is an R -isomorphism because ξ is an R -epimorphism. But, since $l = 1$, $\text{Ker } \xi \neq 0$ by (1), a contradiction. Hence $j = n(i)$.

(ii) Let $l \neq 1$. Then $\xi : e_{k,l}R_R \rightarrow e_{i,j}J_R$ is an R -isomorphism by (1). So $(k, l) = (i, j + 1)$ and $j < n(i)$ by the definition of a well-indexed set $\{e_{i,j}\}_{i=1, j=1}^{m \ n(i)}$ of left Harada ring.

- (3) By (1), (2)(i)(x). (We only note that, if $S(RRe_{s,t}) \cong T(RRe_{k,l})$, then $l = 1$ by Lemma A (II)(5).)

(II) We see that ζ is a left H -epimorphism and (1),(2),(3) hold by the same way as in (I).

- (4) For each $i = 1, 2, \dots, m$ and $j = 2, 3, \dots, n(i)$, we put $\xi \stackrel{\text{put}}{:=} \theta_{i,j} : e_{i,j}R_R \rightarrow e_{i,j-1}J_R$. And there exists a left R -epimorphism

$$\zeta \stackrel{\text{put}}{:=} (\xi(e_{i,j}))_R : RRe_{i,j-1} \rightarrow RJe_{i,j}.$$

So $RRe_{i,n(i)}/J^{n(i)}e_{i,n(i)}$ is uniserial. □

Using Lemma 2.1, we characterize left (right) H -epimorphisms.

Theorem 2.2.

(I) Suppose that $\zeta : RRe_{i,j} \rightarrow RJe_{k,l}$ is a left H -epimorphism. And, if $RRe_{i,j}$ is injective, we let $(e_{p,1}R, Re_{i,j})$ be an i -pair. Then the following hold.

- (1) (i) Suppose that $e_{k,l}R_R$ is injective, i.e., $l = 1$. Then $j = n(i)$, i.e., $\zeta : RRe_{i,n(i)} \rightarrow RJe_{k,1}$.
 (ii) Suppose that $e_{k,l}R_R$ is not injective, i.e., $l \neq 1$. Then $(k, l) = (i, j + 1)$ ($j < n(i)$), i.e., $\zeta : RRe_{i,j} \rightarrow RJe_{i,j+1}$.

$$(2) \quad (i) \quad \text{Ker } \zeta = S(R_R) e_{i,j} = \begin{cases} \bigoplus_{q=1}^{n(p)} S(e_{p,q} R_R) = S_{n(p)}(R R e_{i,j}) \neq 0 \\ \text{and it is uniserial as a left } R\text{-module} & (\text{if } R R e_{i,j} \text{ is injective}) \\ 0 & (\text{if } R R e_{i,j} \text{ is not injective}) \end{cases}$$

(ii) If $R R e_{i,j}$ is injective, then, for each $q = 1, 2, \dots, n(p)$,
 $S(e_{p,q} R_R) = S(e_{p,q} R_R) e_{i,j} = S(e_{p,q} R e_{i,j})$.

(II) Suppose that $\xi : e_{i,j} R_R \rightarrow e_{k,l} J_R$ is a right H -epimorphism. And, if $e_{i,j} R_R$ is injective, we put $I_i \stackrel{\text{put}}{:=} \{ (p, q) \mid S(R R e_{p,q}) \cong T(R R e_{i,1}) \}$ and let n' be the number of elements in I_i . Then the following hold.

(1) (i) Suppose that $e_{i,j} R_R$ is injective, i.e., $j = 1$. Then $l = n(k)$, i.e., $\xi : e_{i,1} R_R \rightarrow e_{k,n(k)} J_R$.

(ii) Suppose that $e_{i,j} R_R$ is not injective, i.e., $j \geq 2$. Then $(k, l) = (i, j - 1)$ ($l < n(k)$), i.e., $\xi : e_{i,j} R_R \rightarrow e_{i,j-1} J_R$.

$$(2) \quad (i) \quad \text{Ker } \xi = e_{i,j} S(R R) = \begin{cases} \bigoplus_{(p,q) \in I_i} e_{i,1} S(R R e_{p,q}) = S_{n'}(e_{i,1} R_R) \neq 0 \\ \text{and it is uniserial as a right } R\text{-module} & (\text{if } j = 1) \\ 0 & (\text{if } j \neq 1) \end{cases}$$

(ii) If $e_{i,j} R_R$ is injective, i.e., $j = 1$, then, for each $(p, q) \in I_i$,
 $S(R R e_{p,q}) = e_{i,1} S(R R e_{p,q}) = S(e_{i,1} R e_{p,q})$.

Proof.

(I) (1) By Lemma 2.1 (I) (1), (2)(i)(z), (ii), we can define a right H -epimorphism

$$\xi \stackrel{\text{put}}{:=} ((e_{i,j}) \zeta)_L : e_{k,l} R_R \rightarrow e_{i,j} J_R$$

which satisfies the following.

(i') Suppose that $e_{k,l} R_R$ is injective, i.e., $l = 1$. Then $j = n(i)$, i.e., $\xi : e_{k,1} R_R \rightarrow e_{i,n(i)} J_R$.

(ii') Suppose that $e_{k,l} R_R$ is not injective, i.e., $l \neq 1$. Then $(k, l) = (i, j + 1)$ ($j < n(i)$), i.e., $\xi : e_{i,j+1} R_R \rightarrow e_{i,j} J_R$ and it is a right R -isomorphism.

Using Lemma 2.1 (II) for the ξ , either the following (i) or (ii) holds.

(i) $e_{k,l} R_R$ is injective, i.e. $l = 1$, $j = n(i)$ and

$$\zeta = (((e_{i,n(i)}) \zeta)_L (e_{k,1}))_R = (\xi(e_{k,1}))_R : R R e_{i,n(i)} \rightarrow R J e_{k,1}.$$

(ii) $e_{k,l}R_R$ is not injective, i.e., $l \neq 1$, $(k, l) = (i, j + 1)$ and
 $\zeta = ((e_{i,j})\zeta)_L(e_{i,j+1})_R = (\xi(e_{i,j+1}))_R : {}_RRe_{i,j} \rightarrow {}_RJe_{i,j+1}$.

(2) Since $\zeta = ((e_{i,j})\zeta)_L(e_{k,l})_R$, the statements follow from Lemma 2.1 (II) (1), (2).

(II) (1) By Lemma 2.1 (II), we can define a left H -epimorphism

$$\zeta \stackrel{\text{put}}{:=} (\xi(e_{i,j}))_R : {}_RRe_{k,l} \rightarrow {}_RJe_{i,j}.$$

So, using Lemma 2.1 (I) (1), (2)(i)(z), (ii) for the ζ , the statements hold.

(2) Since $\xi = (e_{k,l})(\xi(e_{i,j}))_R)_L$, the statements follow from Lemma 2.1 (I) (1), (2)(i)(x), (y).

□

By the definition of a well-indexed set $\{e_{i,j}\}_{i=1,j=1}^{m,n(i)}$ of left Harada ring,

$$e_{i,n(i)}R, e_{i,n(i)-1}R, \dots, e_{i,1}R \quad (i = 1, 2, \dots, m)$$

are right co- H -sequences of R . And, from Theorem 2.2, we obtain the following characterization left co- H -sequences of R using the same set $\{e_{i,j}\}_{i=1,j=1}^{m,n(i)}$.

Corollary 2.3. Every left co- H -sequence of R is of the form

$$Re_{i_1,s}, Re_{i_1,s+1}, \dots, Re_{i_1,n(i_1)}, Re_{i_2,1}, Re_{i_2,2}, \dots, Re_{i_2,n(i_2)}, Re_{i_3,1}, \dots, Re_{i_u,t},$$

where $1 \leq i_1, i_2, \dots, i_u \leq m$, $1 \leq s \leq n(i_1)$ and $1 \leq t \leq n(i_u)$.

Proof. By Theorem 2.2 (I) (1). □

Example 2.4. Let R be a basic indecomposable Nakayama ring with a complete set $\{g_i\}_{i=1}^7$ of orthogonal primitive idempotents which satisfies

(i) $T(g_i J_R) \cong T(g_{i+1} R_R)$ for any $i = 1, 2, \dots, 6$, and

(ii) $c(g_1 R_R) = 10$, $c(g_2 R_R) = 9$,

$$c(g_3 R_R) = 10, \quad c(g_4 R_R) = 9,$$

$$c(g_5 R_R) = 11, \quad c(g_6 R_R) = 10, \quad c(g_7 R_R) = 9,$$

where $c(M)$ means the composition length of an R -module M .

We put

$$e_{1,1} \stackrel{\text{put}}{:=} g_1, \quad e_{1,2} \stackrel{\text{put}}{:=} g_2, \quad e_{2,1} \stackrel{\text{put}}{:=} g_3, \quad e_{2,2} \stackrel{\text{put}}{:=} g_4, \quad e_{3,1} \stackrel{\text{put}}{:=} g_5, \quad e_{3,2} \stackrel{\text{put}}{:=} g_6, \quad e_{3,3} \stackrel{\text{put}}{:=} g_7.$$

And $\{e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}, e_{3,1}, e_{3,2}, e_{3,3}\}$ is a left well-indexed set of R and

$$(e_{1,1}R, Re_{2,1}), \quad (e_{2,1}R, Re_{3,1}), \quad (e_{3,1}R, Re_{1,1})$$

are i -pairs and

$$\begin{aligned} &Re_{1,2}, Re_{2,1} \\ &Re_{2,2}, Re_{3,1} \\ &Re_{3,2}, Re_{3,3}, Re_{1,1} \end{aligned}$$

are left co- H -sequences.

3. w - H -EPIMORPHISMS

In section 2, we characterize H -epimorphisms in Theorem 2.2 and left co- H -sequences by a well-indexed set $\{e_{i,j}\}_{i=1, j=1}^{m, n(i)}$ of left Harada ring as a corollary (Corollary 2.3). In this section, we characterize w - H -epimorphisms.

Lemma 3.1. *Let $Rf_1, Rf_2, \dots, Rf_{n'}$ be a left co- H -sequence and let $f_1 = e_{k,l}$.*

- (1) *Suppose that $l \geq 2$. Then we have a left H -epimorphism $\zeta' : {}_RRe_{k,l-1} \rightarrow {}_RJf_1$.*
- (2) *Suppose that $l = 1$. If there exists a left H -epimorphism $\zeta' : {}_RRe_{i,j} \rightarrow {}_RJe_{k,1} = {}_RJf_1$, then $j = n(i)$, i.e., $\zeta' : {}_RRe_{i,n(i)} \rightarrow {}_RJf_1$.*

Proof.

- (1) By Lemma 2.1 (II), $(\theta_{k,l}(e_{k,l}))_R : {}_RRe_{k,l-1} \rightarrow {}_RJe_{k,l}$ is a left H -epimorphism.
- (2) By Theorem 2.2 (I)(1).

□

Now we further consider a left H -epimorphism the codomain of which is the Jacobson radical of the first term of some left co- H -sequence.

Lemma 3.2.

- (I) Let $Rf_1, Rf_2, \dots, Rf_{n'}$ be a left co- H -sequence. And suppose that there exists a left H -epimorphism $\zeta' : {}_RRf_0 \rightarrow {}_RJf_1$. Then the following hold.
 - (0) Then ${}_RRf_0$ is injective.
So we let $(e_{k,1}R, Rf_0)$ be an i -pair. Then the following hold.
 - (1) $\text{Ker } \zeta' = S_{n(k)}({}_RRf_0)$.
 - (2) ${}_RRf_0/S_{j-1}({}_RRf_0) \cong E(T({}_RRe_{k,j}))$ for any $j = 1, 2, \dots, n(k)$.
 - (3) $S_{n(k)+1}({}_RRf_0)$ is uniserial as a left R -module.

Further we let $(e_{l,1}R, Rf_{n'})$ be an i -pair. Then the following hold.

- (4) $S\left({}_R Rf_0/S_{n(k)+j-1}({}_R Rf_0)\right) \cong T({}_R Re_{l,j})$ for any $j = 1, 2, \dots, n(l)$.
- (II) Suppose that $\xi' : e_{l,1}R_R \rightarrow e_{k,n(k)}J_R$ is a right H -epimorphism and we let both $Rg_1, Rg_2, \dots, Rg_{n_l} = Re_{\sigma(l),\rho(l)}$ and $Rh_1, Rh_2, \dots, Rh_{n_k} = Re_{\sigma(k),\rho(k)}$ be left co- H -sequences. Then the following hold.
- (1) $\text{Ker } \xi' = S_{n_l}(e_{l,1}R_R)$.
- (2) $e_{l,1}R/S_{j-1}(e_{l,1}R_R) \cong E(T(g_{n_l-j+1}R_R))$ for any $j = 1, 2, \dots, n_l$.
- (3) $S_{n_l+1}(e_{l,1}R_R)$ is uniserial as a right R -module.
- (4) $S(e_{l,1}R_R/S_{n_l+j-1}(e_{l,1}R_R)) \cong T(h_{n_k-j+1}R_R)$ for any $j = 1, 2, \dots, n_k$.

Proof.

- (I) (0) By (CHS3) in the definition of a left co- H -sequence and Theorem 2.2 (I)(2)(i).
- (1) By Theorem 2.2 (I)(2)(i).
- (2) By Lemma A (I)(1).
- (3) $S_{n(k)}({}_R Rf_0)$ is uniserial by Lemma A (I)(1). And $S_{n(k)+1}({}_R Rf_0)/S_{n(k)}({}_R Rf_0)$ is simple by (i) since ${}_R Rf_1$ is colocal.
- (4) By (1), ${}_R Rf_0/S_{n(k)}({}_R Rf_0) \cong {}_R Jf_1 \cong {}_R J^{n'}f_{n'}$. And, for any $j = 1, 2, \dots, n(l)$, $S\left({}_R Rf_{n'}/S_{j-1}({}_R Rf_{n'})\right) \cong T({}_R Re_{l,j})$ by Lemma A (I)(1) since $(e_{l,1}R, Rf_{n'})$ is an i -pair. So the statement holds.
- (II) We see by the same way as in (I).

□

Next we consider colocal pairs in two-sided Harada rings.

Lemma 3.3. *Let $Rf_1, Rf_2, \dots, Rf_{n'}$ be a left co- H -sequence and let $(e_{l,1}R, Rf_{n'})$ be an i -pair. Then the following hold.*

- (I) (0) $S(e_{l,j}Rf_s)$ is defined for any $j = 1, 2, \dots, n(l)$ and any $s = 1, 2, \dots, n'$.
- (1) Suppose that there exists a left H -epimorphism $\zeta' : {}_R Rg_{n''} \rightarrow {}_R Jf_1$ and we let $Rg_1, Rg_2, \dots, Rg_{n''}$ be a left co- H -sequence. Then $S(e_{l,j}Rg_t)$ is defined for any $j = 1, 2, \dots, n(l)$ and any $t = 1, 2, \dots, n''$.

- (2) We further suppose that there exists a left H -epimorphism $\zeta'' : {}_R R h_{n'''} \rightarrow {}_R J g_1$ and we let $R h_1, R h_2, \dots, R h_{n'''}$ be a left co- H -sequence. Then $S(e_{l,j} R h_u)$ is defined for any $j = 1, 2, \dots, n(l)$ and any $u = 1, 2, \dots, n'''$.
- (II) (1) Suppose that there exists a right H -epimorphism $\xi' : e_{l',1} R_R \rightarrow e_{l',n(l')} J_R$. Then $S(e_{l',t} R f_j)$ is defined for any $t = 1, 2, \dots, n(l')$ and any $j = 1, 2, \dots, n'$.
- (2) We further suppose that there exists a right H -epimorphism $\xi'' : e_{l'',1} R_R \rightarrow e_{l'',n(l'')} J_R$. Then $S(e_{l'',u} R f_j)$ is defined for any $u = 1, 2, \dots, n(l'')$ and any $j = 1, 2, \dots, n'$.

Proof.

- (I) (0) $S(e_{l,1} R f_s)$ is defined by Lemma A (II)(3). So $S(e_{l,j} R f_s {}_R (f_s))$ is simple since $e_{l,j} R_R \cong e_{l,1} J_R^{j-1}$. On the other hand, $S({}_R (e_{l,j}) e_{l,j} R f_{n'})$ is defined by Lemma A (I)(3). So $S({}_R (e_{l,j}) e_{l,j} R f_s)$ is simple since ${}_R R f_s \cong {}_R J^{n'-s} f_{n'}$. Hence $S(e_{l,j} R f_s)$ is defined.

- (1) First we consider the case ${}_R R g_{n''} \cong {}_R R f_{n'}$. Then $g_s = f_s$. So the statement holds from (0).

Next we consider the case ${}_R R g_{n''} \not\cong {}_R R f_{n'}$. Let $(e_{k,1} R, R g_{n''})$ be an i -pair. Then

$$r_{R g_{n''}}(e_{l,j} R) = S_{n(k)+j-1}(R g_{n''})$$

and

$$S(R g_{n''}/r_{R g_{n''}}(e_{l,j} R)) \cong T({}_R R e_{l,j})$$

by Lemma 3.2 (I)(2),(4). So

$${}_R R g_{n''}/r_{R g_{n''}}(e_{l,j} R) \cong {}_R J^{n'} f_{n'}/S_{j-1}({}_R R f_{n'})$$

and

$$E({}_R R g_{n''}/r_{R g_{n''}}(e_{l,j} R)) \cong {}_R R f_{n'}/S_{j-1}({}_R R f_{n'})$$

by Lemma 3.2 (I)(1) and Lemma A (I)(1) since $R f_1, R f_2, \dots, R f_{n'}$ is a left co- H -sequence. Therefore

$${}_R R g_t/r_{R g_t}(e_{l,j} R) \cong {}_R J^{n'+(n''-t)} f_{n'}/S_{j-1}({}_R R f_{n'})$$

and

$$E({}_R R g_t/r_{R g_t}(e_{l,j} R)) \cong {}_R R f_{n'}/S_{j-1}({}_R R f_{n'})$$

since $R g_1, R g_2, \dots, R g_{n''}$ is a left co- H -sequence. So ${}_R R g_t/r_{R g_t}(e_{l,j} R)$ is quasi-injective. Hence $S(e_{l,j} R g_t)$ is defined by [5, Corollary 1.6].

- (2) If ${}_R R h_{n'''} \cong {}_R R f_{n'}$, the statement holds from (0). So we assume that ${}_R R h_{n'''} \not\cong {}_R R f_{n'}$. Let $(e_{k',1} R, R h_{n'''})$ be an i -pair. Then

the follong (i), (ii), (iii), (iv), (v), (vi) hold by Lemma 3.2 (I)(1),(2),(4) and Lemma A (I)(1).

- (i) $r_{Rh_{n''''}}(e_{l,j}R) = S_{n(k')+n(k)+j-1}(RRh_{n''''})$
- (ii) $S(RRh_{n''''}/r_{Rh_{n''''}}(e_{l,j}R)) \cong T(RRel_{l,j})$
- (iii) $RRh_{n''''}/r_{Rh_{n''''}}(e_{l,j}R) \cong RJ^{n'+n''}f_{n'}/S_{j-1}(RRf_{n'})$
- (iv) $E(RRh_{n''''}/r_{Rh_{n''''}}(e_{l,j}R)) \cong RRf_{n'}/S_{j-1}(RRf_{n'})$
- (v) $RRh_u/r_{Rh_u}(e_{l,j}R) \cong RJ^{n'+n''+(n'''-u)}f_{n'}/S_{j-1}(RRf_{n'})$
- (vi) $E(RRh_u/r_{Rh_u}(e_{l,j}R)) \cong RRf_{n'}/S_{j-1}(RRf_{n'})$

So $RRh_u/r_{Rh_u}(e_{l,j}R)$ is quasi-injective and $S(e_{l,j}Rh_u)$ is defined by [5, Corollary 1.6].

(II) We see by the left-right symmetrical argument of (I). □

Using Lemma 3.3, last we generalize Lemma 2.1 to w - H -epimorphisms.

Proposition 3.4. Let f_1, f_2, \dots, f_{u+1} be distinct elements in $\{e_{i,j}\}_{i=1,j=1}^{m,n(i)}$. Suppose that

$$\zeta \stackrel{put}{:=} \zeta_1 \zeta_2 \cdots \zeta_u : RRf_1 \rightarrow RJ^u f_{u+1}$$

is a left w - H -epimorphism, where $\zeta_i : RRf_i \rightarrow RJf_{i+1}$ is a left H -epimorphism for $i = 1, 2, \dots, u$. For each $i = 1, 2, \dots, u$, we consider a right H -epimorphism

$$\xi_i \stackrel{put}{:=} ((f_i)\zeta_i)_L : f_{i+1}R \rightarrow f_iJ$$

given in Lemma 2.1 (I). And we put

$$\xi \stackrel{put}{:=} ((f_1)\zeta)_L : f_{u+1}R \rightarrow f_1J^u.$$

Further we put

$$X \stackrel{put}{:=} \{i \in \{2, 3, \dots, u+1\} \mid f_i R_R \text{ is injective}\}.$$

And, for each $i \in X$, put $I_i \stackrel{put}{:=} \{(p, q) \mid S(RRe_{p,q}) \cong T(RRf_i)\}$, let $(f_i R, Rg_i)$ be an i -pair, let n'_i be the length of a left co- H -sequence with the last term Rg_i and put $n' \stackrel{put}{:=} \sum_{i \in X} n'_i$.

Then the following hold.

- (1) ξ is a right w - H -epimorphism with $\xi = \xi_1 \xi_2 \cdots \xi_u$.
- (2) $\text{Ker } \xi = \bigoplus_{i \in X} \bigoplus_{(p,q) \in I_i} S(f_{u+1}Re_{p,q}) = S_{n'}(f_{u+1}R_R)$ and it is uniserial as a right R -module.

We note that the left-right symmetric statement of (I) also holds for a right w - H -epimorphism $\xi \stackrel{\text{put}}{:=} \xi_1 \xi_2 \cdots \xi_u : f_{u+1}R_R \rightarrow f_1J_R^u$, where $\xi_i : f_{i+1}R_R \rightarrow f_iJ_R$ is a right H -epimorphism for $i = 1, 2, \dots, u$.

Proof.

$$\begin{aligned}
 (1) \quad \xi(f_{u+1}) &= ((f_1)\zeta)_L(f_{u+1}) \\
 &= ((f_1)\zeta_1\zeta_2 \cdots \zeta_u)_L(f_{u+1}) \\
 &= ((f_1)((f_1)\zeta_1)_R((f_2)\zeta_2)_R \cdots ((f_u)\zeta_u)_R)_L(f_{u+1}) \\
 &= f_1 \cdot (f_1)\zeta_1 \cdot (f_2)\zeta_2 \cdots \cdots \cdot (f_u)\zeta_u \cdot f_{u+1} \\
 &= ((f_1)\zeta_1)_L((f_2)\zeta_2)_L \cdots ((f_u)\zeta_u)_L(f_{u+1}) \\
 &= \xi_1 \xi_2 \cdots \xi_u(f_{u+1}).
 \end{aligned}$$

So $\xi = \xi_1 \xi_2 \cdots \xi_u$. Therefore ξ is a right w - H -epimorphism.

- (2) If f_2R_R is not injective, i.e., $2 \notin X$, then $\text{Ker } \xi_1 = 0$ by Lemma 2.1 (I)(1). If f_2R_R is injective, i.e., $2 \in X$, then

$$\text{Ker } \xi_1 = \bigoplus_{(p,q) \in I_2} S(f_2Re_{p,q}) = S_{n'_2}(f_2R_R)$$

and it is uniserial as a right R -module by Lemma 2.1 (I)(1), (2)(i)(x), (y).

Next, with respect f_2 and f_3 , we consider the following four cases.

- (i) If both f_2R_R and f_3R_R are not injective, i.e., $2, 3 \notin X$, then $\xi_2^{-1}(\text{Ker } \xi_1) = 0$ by Lemma 2.1 (I)(1).
- (ii) If f_2R_R is not injective and f_3R_R is injective, i.e., $2 \notin X$ and $3 \in X$, then

$$\begin{aligned}
 \xi_2^{-1}(\text{Ker } \xi_1) &= \xi_2^{-1}(0) \\
 &= \bigoplus_{(p,q) \in I_3} S(f_3Re_{p,q}) \\
 &= S_{n'_3}(f_3R_R)
 \end{aligned}$$

and it is uniserial as a right R -module by Lemma 2.1 (I)(1), (2)(i)(x), (y).

- (iii) If f_2R_R is injective but f_3R_R is not injective, i.e., $2 \in X$ and $3 \notin X$, then

$$\begin{aligned}
 \xi_2^{-1}(\text{Ker } \xi_1) &= \xi_2^{-1}(\bigoplus_{(p,q) \in I_2} S(f_2Re_{p,q})) \\
 &= \bigoplus_{(p,q) \in I_2} S(f_3Re_{p,q} R_{(e_{p,q})}) \\
 &= S_{n'_2}(f_3R_R)
 \end{aligned}$$

and it is uniserial as a right R -module by Lemma 2.1 (I)(1), (2)(i)(x), (y).

Further $S(f_3Re_{p,q} R_{(e_{p,q})}) = S(f_3Re_{p,q})$ for any $(p, q) \in I_2$ by Lemma 3.3 (II)(1).

(iv) If both f_2R_R and f_3R_R are injective, i.e., $2, 3 \in X$, then

$$\begin{aligned} & \xi_2^{-1}(\text{Ker } \xi_1) \\ &= \xi_2^{-1}(\oplus_{(p,q) \in I_2} S(f_2Re_{p,q})) \\ &= (\oplus_{(p,q) \in I_2} S(f_3Re_{p,q} R_{(e_{p,q})})) \oplus (\oplus_{(p',q') \in I_3} S(f_3Re_{p',q'})) \\ &= S_{n'_2+n'_3}(f_3R_R) \end{aligned}$$

and it is uniserial as a right R -module by Lemma 2.1 (I)(1), (2)(i)(x), (y).

Further $S(f_3Re_{p,q} R_{(e_{p,q})}) = S(f_3Re_{p,q})$ for any $(p, q) \in I_2$ by Lemma 3.3 (II)(1).

Inductively we obtain

$$\text{Ker } \xi = \oplus_{i \in X} \oplus_{(p,q) \in I_i} S(f_{u+1}Re_{p,q}) = S_{n'}(f_{u+1}R_R)$$

and it is uniserial as a right R -module. □

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