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# A WEAK EULER FORMULA FOR $l$-ADIC GALOIS DOUBLE ZETA VALUES

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Abstract. The fact that the double zeta values $\zeta(n,m)$ can be written in terms of zeta values, whenever $n+m$ is odd is attributed to Euler. We shall show the weak version of this result for the $l$-adic Galois realization.

Contents



## 0. Introduction.

The fact that the double zeta values

$$\zeta(a+1,b+1)=\int_0^1 \frac{dz}{1-z},(\frac{dz}{z})^a,\frac{dz}{1-z},(\frac{dz}{z})^b=\sum_{i_2>i_1\geq 1}\frac{1}{i_2^{b+1}}\frac{1}{i_1^{a+1}}$$

can be written as a linear combination with rational coefficients of products of two zeta values $\zeta(\alpha)\zeta(\beta)$ with $\alpha$ and $\beta$ positive integers and of $\zeta(a+1+b+1)$, whenever $a+1+b+1$ is a positive odd integer seems to be attributed to Euler. In fact, in the papers quoted below, it is said that Euler found a formula for $\zeta(a+1,b+1)$ in terms of the Riemann zeta function, whenever $a+1+b+1$ is a positive odd integer. (See [3, page 71] and [8, page 275], where the Euler paper [2] is cited. In [7] the author claims to give rigorous proof of the Euler results. However we have not found the explicit formulation of the result mentioned at the beginning of the section, neither in [2], nor in [3], [8] or in any other papers on multiple zeta values we have consulted. Still there are so many papers on the subject that one can easily miss some of them.)

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The examples of the above mentioned equality are

$$\zeta(1,2) = \sum_{n_1>n_2\geq 1} \frac{1}{n_1^2 n_2} = \zeta(3)$$

(see [8, page 275]) and

$$\zeta(2,3) = 3\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5)$$

(see [5, page 215], notice however that our notation is different from the notation in [5]). On the other side, one should be able to show, perhaps somewhere in future, that $\zeta(3,5)$ cannot be written as a linear combination with rational coefficients of products of two zeta values at positive integers and of zeta values at positive integers. Notice that $3+5$ is even.

The purpose of this paper is to prove a weak analogue of this result for $l$-adic Galois multiple zeta values. Let us explain what we mean by this statement.

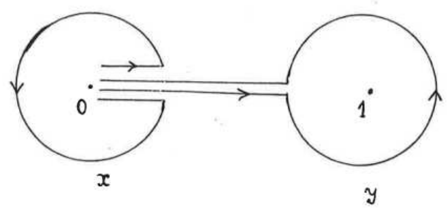
Let us fix a rational prime number $p$. Let us denote by

$$\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}, \overrightarrow{01})$$

the maximal pro-$p$ quotient of the étale fundamental group of $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}$ based at the tangential point $\overrightarrow{01}$. The Galois group

$$G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

acts on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}, \overrightarrow{01})$ (see [1] and [9]). Let us fix an embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}$. Let $\pi$ be the canonical path from $\overrightarrow{01}$ to $\overrightarrow{10}$ on $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}$, the interval $[0,1] \subset \mathbb{C}$. Let $x$ and $y$ be generators of $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}, \overrightarrow{01})$ as on the picture.




Picture 1

Let

$$E : \pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}, \overrightarrow{01}) \to \mathbb{Q}_p\{\{X,Y\}\}$$

be a continuous multiplicative map defined by

$$E(x) := \exp(X) \text{ and } E(y) := \exp(Y)\,.$$

For $\sigma \in G_{\mathbb{Q}}$, let us define a power series

$$\Lambda_\pi(\sigma) := E(\pi^{-1} \cdot \sigma(\pi)) \in \mathbb{Q}_p\{\{X, Y\}\}\,.$$

The coefficients of the power series $\Lambda_\pi$ or $\log\Lambda_\pi$, considered as functions on $G_{\mathbb{Q}}$, are analogues of multiple zeta values (see [14, page 119], where we have pointed this analogy). Let us denote by

$$\lambda_{YX^aYX^b}$$

the coefficient at $YX^aYX^b$ of the power series $\Lambda_\pi$. This coefficient we view as an analogue of the multiple zeta value $\zeta(a+1, b+1)$. The analogue of the result attributed to Euler and mentioned at the beginning of the section will be the following conjecture, which we state only for elements of $G_{\mathbb{Q}(\mu_{p^\infty})}$.

**Conjecture A.** *If $a+1+b+1$ is odd then $\lambda_{YX^aYX^b}$ is a linear combination over $\mathbb{Q}$ of products $\lambda_{YX^\alpha} \cdot \lambda_{YX^\beta}$ with $\alpha + \beta \leq a + b$ and of $\lambda_{YX^\alpha}$ with $\alpha \leq a + b + 1$.*

The actual result we shall prove is much weaker, hence "weak Euler formula" in the title of the paper. We denote by $\mathbb{N}$ the set of non-negative integers. Let us define a subfield of $\bar{\mathbb{Q}}$,

$$\mathcal{K}_1 := \mathbb{Q}(\mu_{p^\infty})((1 - \xi_{p^n}^i)^{\frac{1}{p^m}} \mid n, m \in \mathbb{N},\, 0 < i < p^n)\,.$$

We shall prove the following result as well as its generalization.

**Theorem B.** *Let us assume that $a + 1 + b + 1$ is odd. If $\sigma \in G_{\mathcal{K}_1}$ then $\lambda_{YX^aYX^b}(\sigma) = 0$.*

It seems clear that Conjecture A and Theorem B and its generalizations can be proved using the Drinfeld-Ihara-Deligne relations: the $\mathbb{Z}/2\mathbb{Z}$-relation $\Lambda_\pi(X, Y) \cdot \Lambda_\pi(Y, X) = 1$, the $\mathbb{Z}/3\mathbb{Z}$-relation and the $\mathbb{Z}/5\mathbb{Z}$-relation (see [10]), in the same way as in [17] we have calculated the coefficients $\lambda_{YX^{2n-1}}$.

The proof we present in this paper is however different. In [18] we have shown that the coefficient

$$\lambda_{YX^{a_1}YX^{a_2}\ldots YX^{a_r}}$$

of $\Lambda_\pi$ at $YX^{a_1}YX^{a_2}\ldots YX^{a_r}$ can be expressed as an integral on $(\mathbb{Z}_p)^r$ against the measure which we denoted by $G_r(\overrightarrow{10})$. We shall show that the measures $G_r(\overrightarrow{10})$ satisfy also some symmetry relations. These symmetry relations of the measures $G_r(\overrightarrow{10})$ will imply Theorem B and its generalizations.

It is also possible to prove Conjecture A, as well as its generalization for $\sigma \in G_{\mathbb{Q}}$, in this way, but the calculations will be much more complicated.

## 1. The power series associated with Galois action.

Let $V$ be a smooth algebraic variety over a number field $K$ and let $V_{\bar{K}} := V \times_K \bar{K}$. Let $v$ be a point of $V$ with values in an algebraic closed field. We denote by

$$\pi_1(V_{\bar{K}}, v)$$

the maximal pro-$p$ quotient of the étale fundamental group of $V_{\bar{K}}$ based at $v$. Assume that $v$ and $w$ are $K$-points or generic points (tangential points) "defined" over $K$. The Galois group $G_K := \mathrm{Gal}(\bar{K}/K)$ acts on $\pi_1(V_{\bar{K}}, v)$ and on the $\pi_1(V_{\bar{K}}, v)$-torsor of pro-$p$ étale paths on $V_{\bar{K}}$ from $v$ to $w$. Let $\gamma$ be a pro-$p$ étale path on $V_{\bar{K}}$ from $x$ to $y$. For any $\sigma \in G_K$ we define

$$\mathfrak{f}_\gamma(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) \in \pi_1(V_{\bar{K}}, v)\,.$$

The function $\mathfrak{f}_\gamma : G_K \to \pi_1(V_{\bar{K}}, v)$ is a cocycle and have the following properties:

a) naturality – if $g : V \to W$ is a smooth algebraic morphism defined over $K$ then

$$g_*(\mathfrak{f}_\gamma(\sigma)) = \mathfrak{f}_{g_*(\gamma)}(\sigma)\,, \tag{1.1}$$

where $g_*$ is the map induced by $g$ on étale fundamental groups and on torsors of paths;

b) compatibility with composition of paths – if $\alpha$ is a path from $x$ to $y$ and $\beta$ from $y$ to $z$, we denote by $\beta \cdot \alpha$ the composed path from $x$ to $z$. Then we have

$$\mathfrak{f}_{\beta\cdot\alpha}(\sigma) = \alpha^{-1} \cdot \mathfrak{f}_\beta(\sigma) \cdot \alpha \cdot \mathfrak{f}_\alpha(\sigma)\,. \tag{1.2}$$

c) Hence we get that

$$\mathfrak{f}_{\alpha^{-1}}(\sigma) = \alpha \cdot \mathfrak{f}_\alpha(\sigma)^{-1} \cdot \alpha^{-1} \tag{1.3}$$

(see [13, pages 117-118]).

We assume that $K \subset \bar{\mathbb{Q}}$. We recall that we have fixed an embedding of $\bar{\mathbb{Q}}$ into the field of complex numbers $\mathbb{C}$.

We denote by $V(\mathbb{C})$ the set of $\mathbb{C}$-points of $V$. Then $V(\mathbb{C})$ is a complex variety.

A $K$-point $v$ of $V$ or a tangential point $v$ defined over $K$ (called a $K$-rational tangential base point on $V$ in [12, (1.1) Definition]) determines a corresponding point of $V(\mathbb{C})$, which we denote also by $v$. We denote by $\pi_1(V(\mathbb{C}), v)$ the fundamental group of the topological space $V(\mathbb{C})$ based at $v$.

We have the comparison homomorphism

$$\pi_1(V(\mathbb{C}), v) \to \pi_1(V_{\bar{K}}, v)\,,$$

which induces a canonical isomorphism of the pro-$p$ completion of $\pi_1(V(\mathbb{C}), v)$ onto $\pi_1(V_{\bar{K}}, v)$ (see [4, Exposé X, Corollaire 1.8 and Exposé XII, Corollaire 5.2]). In the sequel elements of $\pi_1(V(\mathbb{C}), v)$ we shall identify with the corresponding elements of $\pi_1(V_{\bar{K}}, v)$.

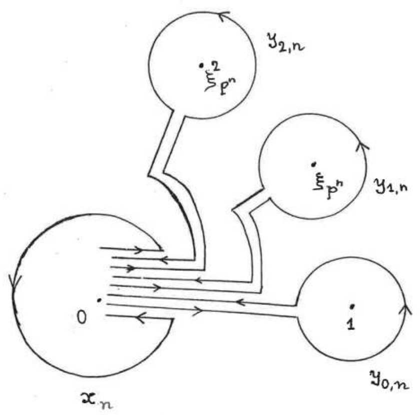
We denote by $\mu_{p^n}$ the subgroup of all $p^n$-th roots of 1 in $\bar{\mathbb{Q}}$. We set $\mu_{p^\infty} := \bigcup_{n=1}^{\infty} \mu_{p^n}$. Let us denote

$$\xi_{p^n} := \exp\left(\frac{2\pi i}{p^n}\right).$$

Let us set

$$V_n := \mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0, \infty\} \cup \mu_{p^n}).$$

Let $x_n$ (loop around 0) and $y_{k,n}$ (loop around $\xi^k_{p^n}$) for $k \in \mathbb{Z}/p^n\mathbb{Z}$ be the standard generators of $\pi_1(V_n, \overrightarrow{01})$ as on Picture 2.




Picture 2

Let

$$\mathcal{Y}_n$$

be a set of non-commuting variables $X_n$ and $Y_{k,n}$ for $k \in \mathbb{Z}/p^n\mathbb{Z}$ and let

$$\mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$$

be a $\mathbb{Q}_p$-algebra of non-commutative formal power series on elements of $\mathcal{Y}_n$. Let

$$E_n : \pi_1(V_n, \overrightarrow{01}) \to \mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$$

be a continuous, multiplicative map defined by

$$E_n(x_n) := \exp X_n \text{ and } E_n(y_{k,n}) := \exp Y_{k,n} \text{ for } k \in \mathbb{Z}/p^n\mathbb{Z}.$$

It follows from the Baker-Campbell-Hausdorff formula (see [11, Theorem 5.19]) that for any $g \in \pi_1(V_n, \overrightarrow{01})$, $E_n(g) = \exp G$ for some $G \in \mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$, which is a (possibly infinite) sum of homogeneous Lie elements of $\mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$. Further such a Lie series we shall call a Lie element.

Let $\mathcal{M}_n$ be the set of all monomials in non-commuting variables belonging to $\mathcal{Y}_n$. If $w \in \mathcal{M}_n$ we denote by $\deg w$ the degree of $w$ as a monomial in variables $X_n$ and $Y_{k,n}$ for $k \in \mathbb{Z}/p^n\mathbb{Z}$ and by $\deg_{\mathcal{Y}} w$ the degree of $w$ in variables $Y_{k,n}$ for $k \in \mathbb{Z}/p^n\mathbb{Z}$.

Let $\pi_n$ be the canonical path from $\overrightarrow{01}$ to $\frac{1}{p^n}\overrightarrow{10}$ on $V_n$ (the interval $[0,1]$). Let $w \in \mathcal{M}_n$. For $\sigma \in G_{\mathbb{Q}}$ we set

$$\Lambda_{\pi_n}(\sigma) := E_n(\mathfrak{f}_{\pi_n}(\sigma))$$

and we define coefficients $\lambda_w^{(n)}$ and $li_w^{(n)}$ by the equalities

$$\Lambda_{\pi_n}(\sigma) = 1 + \sum_{w \in \mathcal{M}_n} \lambda_w^{(n)}(\sigma) w \in \mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$$

and

$$\log\Lambda_{\pi_n}(\sigma) = \sum_{w \in \mathcal{M}_n} li_w^{(n)}(\sigma) w \in \mathbb{Q}_p\{\{\mathcal{Y}_n\}\}\,.$$

If $n = 0$ we shall usually omit the index 0 and we shall write

$$\Lambda_{\pi}(\sigma) = 1 + \sum_{w \in \mathcal{M}} \lambda_w(\sigma) w \in \mathbb{Q}_p\{\{X, Y\}\}\,.$$

We shall also omit $\sigma$ from the formulas to stress that $\lambda_w^{(n)}$ and $li_w^{(n)}$ are functions on $G_{\mathbb{Q}}$.

The study of the coefficients $\lambda_w$ and $li_w$ of the power series $\Lambda_\pi$ and $\log\Lambda_\pi$ is the principal aim of the paper. Let $w = YX^{n_1}YX^{n_2}\dots YX^{n_k}$. The coefficients $\lambda_w$ or $li_w$ we view as analogues of the multiple zeta values $\zeta(n_1 + 1, n_2 + 1, \dots, n_k + 1)$. For example in [17, Proposition 3.1] it is shown that

$$\lambda_{YX^{2n-1}} = li_{YX^{2n-1}} = -\frac{B_{2n}}{2 \cdot (2n)!}(\chi^{2n} - 1)\,,$$

where $B_{2n}$ is the $2n$-th Bernoulli number and $\chi : \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \to \mathbb{Z}_p^\times$ is the $p$-cyclotomic character.

In the next proposition we present one of our two main tools used in the paper. The proposition below is a special case of Propositions 2.3 and 2.5 in [18].

**Proposition 1.1.** *Let $\sigma \in G_{\mathbb{Q}}$ and let $r \geq 1$.*

i) *The family of functions*

$$\left\{G_r^{(n)}(\sigma) : (\mathbb{Z}/p^n\mathbb{Z})^r \ni (i_1, \dots, i_r) \mapsto \lambda^{(n)}_{Y_{i_1,n}Y_{i_2,n}\dots Y_{i_r,n}}(\sigma) \in \mathbb{Q}_p\right\}_{n \in \mathbb{N}}$$

*forms a measure $G_r(\sigma)$ on $(\mathbb{Z}_p)^r$ with values in $\mathbb{Q}_p$.*

ii) *Let* $w = X^{n_0}YX^{n_1}YX^{n_2}\dots YX^{n_r}$. *Then we have*

$$\lambda_w = \Big(\prod_{k=0}^{r} n_k!\Big)^{-1}\int_{(\mathbb{Z}_p)^r}(-x_1)^{n_0}(x_1-x_2)^{n_1}\dots(x_{r-1}-x_r)^{n_{r-1}}x_r^{n_r}\,dG_r(x_1,\dots,x_r)\,.$$

iii) *Let* $0\le i_k<p^n$ *for* $k=1,2,\dots,r$ *and let* $\mathbf{i}=(i_1,i_2,\dots,i_r)$. *Let* $w = X_n^{n_0}Y_{i_1,n}X_n^{n_1}Y_{i_2,n}\cdots X_n^{n_{r-1}}Y_{i_r,n}X_n^{n_r}$ *be in* $\mathcal{M}_n$. *Then*

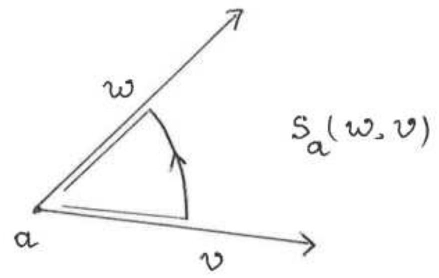
$$\lambda_w^{(n)} =$$

$$\Big(\prod_{k=0}^{r} n_k!\Big)^{-1}\int_{\mathbf{i}+(p^n\mathbb{Z}_p)^r}(-x_1)^{n_0}(x_1-x_2)^{n_1}\dots(x_{r-1}-x_r)^{n_{r-1}}x_r^{n_r}\,dG_r(x_1,\dots,x_r)\,.$$

The point iii) of the proposition is not proved in [18], but it follows immediately from the proof of Proposition 2.5 in [18]. There is an analogous formula for the coefficients $li_w$ (see [18]). In [18] the measure $G_r$ was denoted by $G_r(\overrightarrow{10})$.

## 2. The rhombus relation.

In this section we present our second tool. We start with few definitions. Let $a\in\mathbb{P}^1(\mathbb{C})$ and let $v$, $w$ be two tangent vectors at $a$ such that $||v||=||w||$. We denote by $s_a(w,v)$ a path on $\mathbb{P}^1_{\mathbb{C}}\setminus\{a\}$ from $v$ to $w$ in an infinitesimal neighbourhood of the point $a$. This path is an arc in the opposite clockwise sense (see Picture 3).




Picture 3

Let us define a morphism

$$k_n : V_n \to V_n$$

by $k_n(\mathfrak{z}) = 1/\mathfrak{z}$. Let us set

$$q_n := k_n(\pi_n)^{-1}\,.$$

Let

$$R_{n,1} : V_n \to V_n$$

be given by $R_{n,1}(\mathfrak{z}) = \xi_{p^n}\mathfrak{z}$.

Let us set

$$s = s_0(\overrightarrow{0\xi_{p^n}}, \overrightarrow{01})^{-1},\ t = s_1(\frac{1}{p^n}\overrightarrow{10}, \frac{1}{p^n}\overrightarrow{1\infty})^{-1},\ \eta = R_{n,1}(k_n(s)) \text{ and}$$
$$e = R_{n,1}(k_n(t)).$$

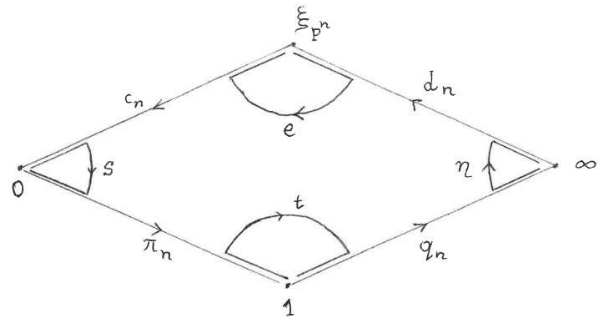
We set also

$$c_n = R_{n,1}(\pi_n)^{-1} \text{ and } d_n = R_{n,1}(q_n)^{-1} = R_{n,1}(k_n(\pi_n))\,.$$

Observe that the composition of paths

$$s \cdot c_n \cdot e \cdot d_n \cdot \eta \cdot q_n \cdot t \cdot \pi_n = 1 \tag{2.1}$$

in $\pi_1(V_n, \overrightarrow{01})$ (see Picture 4 or [16, page 167], where a similar composition of paths appears).




PICTURE 4

The picture 4 has a shape of an octagon or a rhombus, hence a name of a relation we shall deduce. Let us set

$$\alpha_1 := t \cdot \pi_n,\ \alpha_2 := q_n \cdot \alpha_1,\ \alpha_3 := \eta \cdot \alpha_2,$$
$$\alpha_4 := d_n \cdot \alpha_3,\ \alpha_5 := e \cdot \alpha_4 \text{ and } \alpha_6 := c_n \cdot \alpha_5\,.$$

**Proposition 2.1.** (Octagon relation) *On the group $G_{\mathbb{Q}(\mu_{p^n})}$ we have*

$$\alpha_6^{-1}\cdot\mathfrak{f}_s\cdot\alpha_6\cdot\alpha_5^{-1}\cdot\mathfrak{f}_{c_n}\cdot\alpha_5\cdot\alpha_4^{-1}\cdot\mathfrak{f}_e\cdot\alpha_4\cdot\alpha_3^{-1}\cdot\mathfrak{f}_{d_n}\cdot\alpha_3\cdot\alpha_2^{-1}\cdot\mathfrak{f}_\eta\cdot\alpha_2\cdot\alpha_1^{-1}\cdot\mathfrak{f}_{q_n}\cdot\alpha_1\cdot\pi_n^{-1}\cdot\mathfrak{f}_t\cdot\pi_n\cdot\mathfrak{f}_{\pi_n} = 1\,.$$

*Proof.* The proposition follows immediately from the formula (1.2) applied several times to the equality (2.1). □

**Lemma 2.2.** *On the group $G_{\mathbb{Q}}$ we have*

$$\mathfrak{f}_{q_n} = q_n^{-1} \cdot \big((k_n)_*(\mathfrak{f}_{\pi_n}^{-1})\big) \cdot q_n \,.$$

*Proof.* The lemma follows from the naturality property (1.1) and the formula (1.3). □

**Lemma 2.3.**

a) *On the subgroup $G_{\mathbb{Q}(\mu_{p^n})}$ of $G_{\mathbb{Q}}$ we have*
   - i) $\mathfrak{f}_{c_n} = c_n^{-1} \cdot \big((R_{n,1})_*(\mathfrak{f}_{\pi_n}^{-1})\big) \cdot c_n$ ;
   - ii) $\mathfrak{f}_{d_n} = (R_{n,1})_*\big((k_n)_*(\mathfrak{f}_{\pi_n})\big)$ ;
   - iii) $\mathfrak{f}_{\eta} = (R_{n,1})_*\big((k_n)_*(\mathfrak{f}_s)\big)$ .

b) *On the subgroup $G_{\mathbb{Q}(\mu_{p^\infty})}$ of $G_{\mathbb{Q}}$ we have*
   - iv) $\mathfrak{f}_s = \mathfrak{f}_e = \mathfrak{f}_\eta = \mathfrak{f}_t = 1$ .

*Proof.* The morphism $R_{n,1}$ commutes with the action of the Galois group $G_{\mathbb{Q}(\mu_{p^n})}$. Hence the points i), ii) and iii) of the proposition follow. Studying the effect of $\mathfrak{f}_s(\sigma)$ on the test functions $\mathfrak{z}^{\frac{1}{p^n}}$ one shows that $\mathfrak{f}_s(\sigma) = x_n^{\frac{1}{p^n}(1-\chi(\sigma))}$ for $\sigma \in G_{\mathbb{Q}(\mu_{p^n})}$. Therefore it follows that $\mathfrak{f}_s(\sigma) = 1$ for $\sigma \in G_{\mathbb{Q}(\mu_{p^\infty})}$. Hence we get that $\mathfrak{f}_\eta(\sigma) = 1$ for $\sigma \in G_{\mathbb{Q}(\mu_{p^\infty})}$. In the same way one shows that $\mathfrak{f}_e = 1$ and $\mathfrak{f}_t = 1$ on the subgroup $G_{\mathbb{Q}(\mu_{p^\infty})}$ of $G_{\mathbb{Q}}$. □

The elements $x_n, y_{0,n}, \ldots, y_{p^n-1,n}$ are free generators of a free pro-$p$ group $\pi_1(V_n, \overrightarrow{01})$. Hence the element $\mathfrak{f}_{\pi_n}$ is a convergent infinite product of commutators in these generators as the group $\pi_1(V_n, \overrightarrow{01})$ is pro-unipotent. We shall write
$\mathfrak{f}_{\pi_n} = \mathfrak{f}_{\pi_n}(x_n, y_{0,n}, \ldots, y_{p^n-1,n})$ to indicate this dependence on generators. Moreover if $g : \pi_1(V_n, \overrightarrow{01}) \to G$ is a continuous morphism of groups then

$$g_*(\mathfrak{f}_{\pi_n}(x_n, y_{0,n}, \ldots, y_{p^n-1,n})) = \mathfrak{f}_{\pi_n}(g_*(x_n), g_*(y_{0,n}), \ldots, g_*(y_{p^n-1,n})),$$

as $x_n$ and $y_{k,n}$ for $k \in \mathbb{Z}/p^n\mathbb{Z}$ are free generators of the free pro-$p$ group $\pi_1(V_n, \overrightarrow{01})$.

**Proposition 2.4.** (Rhombus relation) *We have the following equality on the subgroup $G_{\mathbb{Q}(\mu_{p^\infty})}$ of $G_{\mathbb{Q}}$ :*

$$\mathfrak{f}_{\pi_n}^{-1}\big(\alpha_6^{-1}\cdot((R_{n,1})_*(x_n))\cdot\alpha_6, \alpha_6^{-1}\cdot((R_{n,1})_*(y_{0,n}))\cdot\alpha_6, \ldots, \alpha_6^{-1}\cdot((R_{n,1})_*(y_{p^n-1,n}))\cdot\alpha_6\big)\cdot$$
$$\mathfrak{f}_{\pi_n}\big(\alpha_3^{-1} \cdot ((R_{n,1} \circ k_n)_*(x_n)) \cdot \alpha_3, \alpha_3^{-1} \cdot ((R_{n,1} \circ k_n)_*(y_{0,n})) \cdot \alpha_3, \ldots\big)\cdot$$
$$\mathfrak{f}_{\pi_n}^{-1}\big(\alpha_2^{-1} \cdot ((k_n)_*(x_n)) \cdot \alpha_2, \alpha_2^{-1} \cdot ((k_n)_*(y_{0,n})) \cdot \alpha_2, \ldots\big) \cdot \mathfrak{f}_{\pi_n}(x_n, y_{0,n}, y_{1,n}, \ldots) = 1\,.$$

*Proof.* The proposition follows from Proposition 2.1 and Lemmas 2.2 and 2.3. □

Let us define

$$z_n := \alpha_2^{-1} \cdot (k_n)_*(x_n) \cdot \alpha_2.$$

Then

$$y_{0,n} \cdot x_n \cdot y_{p^n-1,n} \cdot \ldots \cdot y_{2,n} \cdot y_{1,n} \cdot z_n = 1$$

in $\pi_1(V_n, \overrightarrow{01})$. Now we shall describe the maps induced by $k_n$ and $R_{n,1}$ on fundamental groups. For our purpose the following result will be sufficient.

**Lemma 2.5.** *We have the following equalities and congruences in the group* $\pi_1(V_n, \overrightarrow{01})$ *modulo the commutator subgroup* $\big(\pi_1(V_n, \overrightarrow{01}), \pi_1(V_n, \overrightarrow{01})\big)$.

i) $\alpha_2^{-1} \cdot (k_n)_*(x_n) \cdot \alpha_2 = y_{1,n}^{-1} \cdot y_{2,n}^{-1} \cdot \ldots \cdot y_{p^n-1,n}^{-1} \cdot x_n^{-1} \cdot y_{0,n}^{-1} = z_n$,
$\alpha_2^{-1} \cdot (k_n)_*(y_{i,n}) \cdot \alpha_2 \equiv y_{-i,n}$ for $i \in \mathbb{Z}/p^n\mathbb{Z}$,

ii) $\alpha_3^{-1} \cdot (R_{n,1} \circ k_n)_*(x_n) \cdot \alpha_3 = z_n$,
$\alpha_3^{-1} \cdot (R_{n,1} \circ k_n)_*(y_{i,n}) \cdot \alpha_3 \equiv y_{-i+1,n}$ for $i \in \mathbb{Z}/p^n\mathbb{Z}$,

iii) $\alpha_6^{-1} \cdot (R_{n,1})_*(x_n) \cdot \alpha_6 = x_n$,
$\alpha_6^{-1} \cdot (R_{n,1})_*(y_{i,n}) \cdot \alpha_6 \equiv y_{i+1,n}$ for $i \in \mathbb{Z}/p^n\mathbb{Z}$.

*Proof.* We view the loops $x_n$, $z_n$ and $y_{k.n}$ for $k \in \mathbb{Z}/p^n\mathbb{Z}$ as the elements of the first homology group $H_1(V_n(\mathbb{C}), \mathbb{Z})$. But then the congruences of the lemma are equalities in $H_1(V_n(\mathbb{C}); \mathbb{Z})$ and they are clear. □

We finish the section with the following technical lemma.

**Lemma 2.6.** *Let* $f(x_n, y_{0,n}, \ldots, y_{p^n-1,n}) \in \pi_1(V_n, \overrightarrow{01})$ *and let*

$$E_n\big(f(x_n, y_{0,n}, \ldots, y_{p^n-1,n})\big) = F(X_n, Y_{0,n}, \ldots, Y_{p^n-1,n}).$$

*Let* $\alpha$ *and* $\beta_0, \ldots, \beta_{p^n-1}$ *be elements of* $\pi_1(V_n, \overrightarrow{01})$ *and let* $E_n(\alpha) = \exp A$ *and* $E_n(\beta_k) = \exp B_k$ $(0 \le k \le p^n - 1)$ *for some Lie elements* $A, B_0, \ldots, B_{p^n-1}$ *in* $\mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$. *Then* $E_n\big(f(\alpha, \beta_0, \ldots, \beta_{p^n-1})\big) = F(A, B_0, \ldots, B_{p^n-1})$.

*Proof.* Let $\phi : \pi_1(V_n, \overrightarrow{01}) \to \pi_1(V_n, \overrightarrow{01})$ be a continuous morphism of pro-$p$ groups defined by $\phi(x_n) = \alpha$ and $\phi(y_{k,n}) = \beta_k$ for $k \in \mathbb{Z}/p^n\mathbb{Z}$. Let us define a continuous morphism of $\mathbb{Q}_p$-algebras $\Phi : \mathbb{Q}_p\{\{\mathcal{Y}_n\}\} \to \mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$ by $\Phi(X_n) = A$ and $\Phi(Y_{k,n}) = B_k$ for $k \in \mathbb{Z}/p^n\mathbb{Z}$. Then we have $E_n \circ \phi = \Phi \circ E_n$. Hence it follows that

$$\begin{aligned} &E_n\big(f(\alpha, \beta_0, \ldots, \beta_{p^n-1})\big) = E_n(\phi\big(f(x_n, y_{0,n}, \ldots, y_{p^n-1,n})\big) \\ &= \Phi(E_n\big(f(x_n, y_{0,n}, \ldots, y_{p^n-1,n})\big) = \Phi(F(X_n, Y_{0,n}, \ldots, Y_{p^n-1,n})) \\ &= F(A, B_0, \ldots, B_{p^n-1}). \qquad \square \end{aligned}$$

## 3. Filtration of the group Gal($\bar{\mathbb{Q}}/\mathbb{Q}$).

With the action of the Galois group $G_{\mathbb{Q}}$ on fundamental groups one can associate several filtrations of $G_{\mathbb{Q}}$. We shall define a filtration associated with the action of $G_{\mathbb{Q}}$ on fundamental groups of the tower of coverings $\{V_n \to V_0\}_{n\in\mathbb{N}}$. Let us set

$$\mathcal{L}_0 := G_{\mathbb{Q}(\mu_{p^\infty})}$$

and

$$\mathcal{L}_k := \{\sigma \in \mathcal{L}_0 \mid \forall n \in \mathbb{N},\ \forall w \in \mathcal{M}_n,\ \deg w = \deg_{\mathcal{Y}} w\ \le k \Rightarrow \lambda_w^{(n)}(\sigma) = 0\}$$

for $k > 0$.

We denote by

$$\mathcal{I}_n$$

the augmentation ideal of $\mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$.

**Lemma 3.1.** *We have*

i) $\mathcal{L}_{k+1} \subset \mathcal{L}_k$ *for* $k \ge 0$*;*
ii) *the subsets* $\mathcal{L}_k$ *of* $G_{\mathbb{Q}(\mu_{p^\infty})}$ *are closed subgroups of* $G_{\mathbb{Q}(\mu_{p^\infty})}$.

*Proof.* The first point is clear from the very definition. It rests to show the point ii). Let us take $\tau, \sigma \in \mathcal{L}_k$. It follows from the cocycle formula $\mathfrak{f}_{\pi_n}(\tau\sigma) = \mathfrak{f}_{\pi_n}(\tau)\cdot\tau_*(\mathfrak{f}_{\pi_n}(\sigma))$ (see [13, Proposition 1.0.7]) that

$$\Lambda_{\pi_n}(\tau\sigma) = \Lambda_{\pi_n}(\tau)\cdot\tau_*(\Lambda_{\pi_n}(\sigma))\,. \tag{3.1}$$

It follows from [15, Proposition 15.1.7] that $\tau_*(X_n) = X_n$ and $\tau_*(Y_{i,n}) = \exp(-F_i(\tau))\cdot Y_{i,n}\cdot\exp(F_i(\tau))$ ($i \in \mathbb{Z}/p^n\mathbb{Z}$) for some Lie element $F_i(\tau) \in \mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$. Let $(X_n) + \mathcal{I}_n^{k+1}$ be the ideal of $\mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$ generated by $X_n$ and $\mathcal{I}_n^{k+1}$. Then it follows from Lemma 2.6 and the definition of the filtration $\{\mathcal{L}_k\}_{k\in\mathbb{N}}$ that

$$\Lambda_{\pi_n}(\tau)\cdot\tau_*(\Lambda_{\pi_n}(\sigma)) \equiv 1 \ \text{ modulo }\ (X_n) + \mathcal{I}_n^{k+1}\,.$$

Therefore the equality (3.1) implies that

$$\Lambda_{\pi_n}(\tau\sigma) \equiv 1 \ \text{ modulo }\ (X_n) + \mathcal{I}_n^{k+1}\,.$$

Hence it follows that $\tau\sigma \in \mathcal{L}_k$. In the similar way one shows that $\tau^{-1} \in \mathcal{L}_k$ if $\tau \in \mathcal{L}_k$.

The groups $G_{\mathbb{Q}}$ and $\pi_1(V_n, \overrightarrow{01})$ are equipped with their natural profinite topologies. A finite dimensional vector space over $\mathbb{Q}_p$ is naturally a topological locally compact normed vector space over $\mathbb{Q}_p$. Observe that

$$\mathbb{Q}_p\{\{\mathcal{Y}_n\}\} = \varprojlim_k \mathbb{Q}_p\{\{\mathcal{Y}_n\}\}/\mathcal{I}_n^k\,.$$

We equipped $\mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$ with the topology of the inverse limit of finite dimensional topological vector spaces over $\mathbb{Q}_p$. Then the maps $\mathfrak{f}_{\pi_n} : G_{\mathbb{Q}} \to \pi_1(V_n, \overrightarrow{01})$ and $E_n : \pi_1(V_n, \overrightarrow{01}) \to \mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$ are continuous. Therefore the coefficients $\lambda_w^{(n)}$ for $w \in \mathcal{M}_n$ are continuous functions on $G_{\mathbb{Q}}$ with values in $\mathbb{Q}_p$. The finite subsets of $\mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$ are closed. Hence $(\lambda_w^{(n)})^{-1}(\{0\})$ is a closed subset of $G_{\mathbb{Q}}$. The subgroup $G_{\mathbb{Q}(\mu_{p^\infty})}$ is a closed subgroup of $G_{\mathbb{Q}}$. Therefore $\mathcal{L}_k$ is a closed subgroup of $G_{\mathbb{Q}}$ and of $G_{\mathbb{Q}(\mu_{p^\infty})}$, as an intersection of a family of closed subsets of $G_{\mathbb{Q}}$. □

**Lemma 3.2.** *Let $r$ be a positive integer. Let $0 \le i_\alpha < p^n$ for $\alpha = 1, 2, \ldots, r$. Let*

$$w = X_n^{n_0} Y_{i_1,n} X_n^{n_1} Y_{i_2,n} X_n^{n_2} \ldots X_n^{n_{r-1}} Y_{i_r,n} X_n^{n_r} \in \mathcal{M}_n\,.$$

*Then the coefficient $\lambda_w^{(n)}$ vanishes on $\mathcal{L}_k$ for $k \ge r$.*

*Proof.* Let $\mathbf{i} = (i_1, i_2, \ldots, i_r)$. It follows from the Proposition 1.1, iii) that

$$\lambda_w^{(n)} =$$

$$(\prod_{i=0}^{r} n_i!)^{-1} \int_{\mathbf{i}+(p^n\mathbb{Z}_p)^r} (-x_1)^{n_0}(x_1 - x_2)^{n_1} \ldots (x_{r-1} - x_r)^{n_{r-1}} x_r^{n_r}\, dG_r(x_1, \ldots, x_r)\,.$$

The assumptions that $\sigma \in \mathcal{L}_k$ and $k \ge r$ imply that the measures $G_r(\sigma)$ are zero measures. Hence the coefficient $\lambda_w^{(n)}$ vanishes on $\mathcal{L}_k$. □

## 4. Symmetries of the measures $G_r$.

We recall that $\mathcal{I}_n$ is the augmentation ideal of $\mathbb{Q}_p\{\{\mathcal{Y}_n\}\}$. The subgroups $\Gamma^k\pi$ of the lower central series of a group $\pi$ are defined recursively by

$$\Gamma^1 := \pi,\ \ \Gamma^{k+1}\pi := (\Gamma^k\pi, \pi)\ \text{ for }\ k = 1, 2 \ldots.$$

(see [11, Section 5.3.]).

**Lemma 4.1.** *For $k \ge 1$ we have*

$$E_n\big(\Gamma^k \pi_1(V_n, \overrightarrow{01})\big) \subset 1 + \mathcal{I}_n^k\,.$$

*Proof.* Let $a, b \in \pi_1(V_n, \overrightarrow{01})$. It follows from the Baker-Campbell-Hausdorff formula (see [11, Theorem 5.19]) that $E_n(a) = \exp A$ and $E_n(b) = \exp B$ for some Lie elements $A, B \in \mathbb{Q}\{\{\mathcal{Y}_n\}\}$. Applying three times the Baker-Campbell-Hausdorff formula we get that

$$E_n(aba^{-1}b^{-1}) = \exp D$$

for some $D \in \mathbb{Q}\{\{\mathcal{Y}_n\}\}$ satisfying $D \equiv A \cdot B - B \cdot A$ modulo $\mathcal{I}_n^3$. Observe that $A \cdot B - B \cdot A \in \mathcal{I}_n^2$. Hence the lemma holds for $k = 2$. Repeating the above arguments one shows the statement of the lemma for any $k$. □

The principal result of this section is the following theorem.

**Theorem 4.2.** *Let $r \geq 1$ and let $\sigma \in \mathcal{L}_{r-1}$. Then*

$$G_r(x_1, x_2, \dots, x_r)(\sigma) - G_r(-x_1, -x_2, \dots, -x_r)(\sigma) +$$
$$G_r(-x_1+1, -x_2+1, \dots, -x_r+1)(\sigma) - G_r(x_1-1, x_2-1, \dots, x_r-1)(\sigma) = 0\,.$$

*Proof.* To simplify the notation let us set $\mathbf{i} = (i_1, i_2, \dots, i_r) \in (\mathbb{Z}/p^n\mathbb{Z})^r$ and $\mathbf{1} = (1, 1, \dots, 1) \in (\mathbb{Z}/p^n\mathbb{Z})^r$. Then $-\mathbf{i}$, $\mathbf{i}+\mathbf{1}$ and $-\mathbf{i}+\mathbf{1}$ are in $(\mathbb{Z}/p^n\mathbb{Z})^r$. If $\mathbf{i} = (i_1, i_2, \dots, i_r) \in (\mathbb{Z}/p^n\mathbb{Z})^r$ we denote by $Y_{\mathbf{i}}$ the product $Y_{i_1,n}Y_{i_2,n}\dots Y_{i_r,n}$.

It follows from Lemma 3.2 that on the subgroup $\mathcal{L}_{r-1}$ of $G_{\mathbb{Q}}$ we have

$$\Lambda_{\pi_n} \equiv 1 + \sum_{\mathbf{i} \in (\mathbb{Z}/p^n\mathbb{Z})^r} \lambda_{\mathbf{i}}^{(n)} Y_{\mathbf{i}} \text{ modulo } \mathcal{I}_n^{r+1}\,. \tag{4.1}$$

It follows from Lemma 2.6 that

$$E_n(\mathfrak{f}_{\pi_n}(\alpha_2^{-1} \cdot ((k_n)_*(x_n)) \cdot \alpha_2, \alpha_2^{-1} \cdot ((k_n)_*(y_{0,n})) \cdot \alpha_2, \dots)) =$$
$$\Lambda_{\pi_n}(\log E_n(\alpha_2^{-1} \cdot ((k_n)_*(x_n)) \cdot \alpha_2), \log E_n(\alpha_2^{-1} \cdot ((k_n)_*(y_{0,n})) \cdot \alpha_2), \dots)\,.$$

Lemma 2.5 and Lemma 4.1 imply that

$$\log E_n(\alpha_2^{-1} \cdot ((k_n)_*(y_{i,n})) \cdot \alpha_2) \equiv Y_{-i,n} \text{ modulo } \mathcal{I}_n^2$$

for $i \in \mathbb{Z}/p^n\mathbb{Z}$. Hence it follows from the congruence (4.1) that

$$E_n(\mathfrak{f}_{\pi_n}(\alpha_2^{-1} \cdot ((k_n)_*(x_n)) \cdot \alpha_2, \alpha_2^{-1} \cdot ((k_n)_*(y_{0,n})) \cdot \alpha_2, \dots)) \equiv 1 + \sum_{\mathbf{i} \in (\mathbb{Z}/p^n\mathbb{Z})^r} \lambda_{\mathbf{i}}^{(n)} Y_{-\mathbf{i}} \text{ modulo } \mathcal{I}_n^{r+1}\,, \tag{4.2}$$

on the subgroup $\mathcal{L}_{r-1}$ of $G_{\mathbb{Q}}$. In the similar way we show that

$$E_n(\mathfrak{f}_{\pi_n}(\alpha_3^{-1} \cdot ((R_{n,1} \circ k_n)_*(x_n)) \cdot \alpha_3, \alpha_3^{-1} \cdot ((R_{n,1} \circ k_n)_*(y_{0,n})) \cdot \alpha_3, \dots)) \equiv 1 + \sum_{\mathbf{i} \in (\mathbb{Z}/p^n\mathbb{Z})^r} \lambda_{\mathbf{i}}^{(n)} Y_{-\mathbf{i}+\mathbf{1}} \text{ modulo } \mathcal{I}_n^{r+1} \tag{4.3}$$

and

$$E_n(\mathfrak{f}_{\pi_n}(\alpha_6^{-1} \cdot ((R_{n,1})_*(x_n)) \cdot \alpha_6, \alpha_6^{-1} \cdot ((R_{n,1})_*(y_{0,n})) \cdot \alpha_6, \dots)) \equiv 1 + \sum_{\mathbf{i} \in (\mathbb{Z}/p^n\mathbb{Z})^r} \lambda_{\mathbf{i}}^{(n)} Y_{\mathbf{i}+\mathbf{1}} \text{ modulo } \mathcal{I}_n^{r+1} \tag{4.4}$$

on the subgroup $\mathcal{L}_{r-1}$ of $G_{\mathbb{Q}}$. Hence it follows from Proposition 2.4 and the congruences (4.1) - (4.4) that for $\sigma \in \mathcal{L}_{r-1}$ we have

$$\Big(1 - \sum_{\mathbf{i}\in(\mathbb{Z}/p^n\mathbb{Z})^r} \lambda_{\mathbf{i}}^{(n)}(\sigma)Y_{\mathbf{i}+\mathbf{1}}\Big) \cdot \big(1 + \sum_{\mathbf{i}\in(\mathbb{Z}/p^n\mathbb{Z})^r} \lambda_{\mathbf{i}}^{(n)}(\sigma)Y_{-\mathbf{i}+\mathbf{1}}\big)\cdot$$

$$\big(1 - \sum_{\mathbf{i}\in(\mathbb{Z}/p^n\mathbb{Z})^r} \lambda_{\mathbf{i}}^{(n)}(\sigma)Y_{-\mathbf{i}}\big) \cdot \big(1 + \sum_{\mathbf{i}\in(\mathbb{Z}/p^n\mathbb{Z})^r} \lambda_{\mathbf{i}}^{(n)}(\sigma)Y_{\mathbf{i}}\big) \equiv 1 \text{ modulo } \mathcal{I}_n^{r+1}\,.$$

Comparing coefficients at monomials $Y_{\mathbf{i}} = Y_{i_1,n}\dots Y_{i_r,n}$ we get the identity of measures. □

## 5. Euler relations.

Now we shall formulate and prove our main result. We recall from Introduction and from section 1 that

$$\lambda_{X^{n_0}YX^{n_1}YX^{n_2}\dots X^{n_{r-1}}YX^{n_r}}(\sigma)$$

are the coefficients of the power series $\Lambda_\pi(\sigma) \in \mathbb{Q}_p\{\{X,Y\}\}$, where $\pi$ is the canonical path from $\overrightarrow{01}$ to $\overrightarrow{10}$ on $\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}$.

**Theorem 5.1.** *Let $r$ be a positive integer. Let $n_i$ be non-negative integers for $0 \le i \le r$. Let $\sigma \in \mathcal{L}_{r-1}$ and let $w = X^{n_0}YX^{n_1}YX^{n_2}\dots X^{n_{r-1}}YX^{n_r}$. If $\sum_{i=0}^r n_i$ is odd then*

$$\lambda_w(\sigma) = 0\,.$$

*Proof.* Let us set

$$d\mu(x_1,\dots,x_r)(\sigma) := d\big(G_r(x_1,\dots,x_r)(\sigma) - G_r(-x_1,\dots,-x_r)(\sigma)+$$

$$G_r(-x_1+1,\dots,-x_r+1)(\sigma) - G_r(x_1-1,\dots,x_r-1)(\sigma)\big)\,.$$

Let $\sigma \in \mathcal{L}_{r-1}$. Let $d, n_1,\dots,n_{r-1},q$ be any sequence of length $r+1$ of non-negative integers. It follows from Theorem 4.2 that

$$\text{(5.1)}\quad \int_{(\mathbb{Z}_p)^r} (-x_1)^d(x_1-x_2)^{n_1}\dots(x_{r-1}-x_r)^{n_{r-1}}x_r^q\,d\mu(x_1,\dots,x_r)(\sigma) = 0\,.$$

Let us set $m = \sum_{i=1}^{r-1} n_i$. Let us define a polynomial

$$P_{d,q}(x_1,x_r) :=$$

$$(-1)^d x_1^d x_r^q - (-1)^{m+q}x_1^d x_r^q + (-1)^{m+q}(x_1-1)^d(x_r-1)^q - (-1)^d(x_1+1)^d(x_r+1)^q\,.$$

After changes of variables in the last three integrals in the formula (5.1) we get
(5.2)
$$\int_{(\mathbb{Z}_p)^r}(x_1-x_2)^{n_1}\dots(x_{r-1}-x_r)^{n_{r-1}}P_{d,q}(x_1,x_r)dG_r(x_1,\dots,x_r)(\sigma)=0\,.$$

We define an order $\prec$ on the set $\mathbb{N}^2$ by
$$(d,q)\prec(d_1,q_1)$$
if $d<d_1$ or $d=d_1$ and $q<q_1$. Observe that the set $\mathbb{N}^2$ is well-ordered by $\prec$.

For $(d,q)\in\mathbb{N}^2$, let $\mathcal{P}_{d,q}$ be the following proposition:

*Let $w=X^dYX^{n_1}YX^{n_2}\dots X^{n_{r-1}}YX^q$ and let $m=\sum_{i=1}^{r-1}n_i$. If $d+m+q$ is odd then $\lambda_w(\sigma)=0$ for $\sigma\in\mathcal{L}_{r-1}$.*

In order to prove the theorem it is enough to prove the propositions $\mathcal{P}_{d,q}$ for all $(d,q)\in\mathbb{N}^2$. We shall prove the theorem by the method of transfinite induction applied to the set $\mathbb{N}^2$ well ordered by $\prec$ (see [6, Chapter VII]).

Observe that
$$P_{0,2}(x_1,x_r)=\begin{cases}-4x_r & \text{if } m \text{ is even,}\\ -2 & \text{if } m \text{ is odd .}\end{cases}$$

The identity (5.2) holds for any sequence of length $r+1$ of non-negative integers $d,n_1,\dots,n_{r-1},q$. In particular, in the case $m$ is odd, $d=0$ and $q=2$ we have
$$\int_{(\mathbb{Z}_p)^r}(x_1-x_2)^{n_1}\dots(x_{r-1}-x_r)^{n_{r-1}}\cdot(-2)\,dG_r(x_1,..,x_r)=0$$
on $\mathcal{L}_{r-1}$. In the case $m$ is even, $d=0$ and $q=2$ we have
$$\int_{(\mathbb{Z}_p)^r}(x_1-x_2)^{n_1}\dots(x_{r-1}-x_r)^{n_{r-1}}\cdot(-4x_r)\,dG_r(x_1,..,x_r)=0$$
on $\mathcal{L}_{r-1}$. Observe that if $m$ is even then $m+1$ is odd. Hence we have shown the propositions $\mathcal{P}_{0,0}$ and $\mathcal{P}_{0,1}$.

In order to do an inductive step we need to consider two cases. The first case is that of an element, which has a direct predecessor and the second case is that of an element, which has no direct predecessor.

Let us assume that the propositions $\mathcal{P}_{x,y}$ are true for all pairs $(x,y)\prec(d,a+1)$. Let us assume that $d+m+a+1$ is odd. We have
$$P_{d,a+2}(x_1,x_r)=$$

$$(-1)^{m+a+2}\Big(\sum_{i=0}^{d}\sum_{j=0}^{a+2}\binom{d}{i}\binom{a+2}{j}\big((-1)^{i+j}-1\big)x_1^{d-i}x_r^{a+2-j}\Big).$$

Observe that $(-1)^{i+j}-1=0$ if $i+j$ is even and $(-1)^{i+j}-1=-2$ if $i+j$ is odd. If $i+j$ is odd and $(i,j)\neq(0,1)$ then

$$\int_{(\mathbb{Z}_p)^r} x_1^{d-i}(x_1-x_2)^{n_1}\dots(x_{r-1}-x_r)^{n_{r-1}}x_r^{a+2-j}\,dG_r(x_1,..,x_r)=0 \tag{5.3}$$

on $\mathcal{L}_{r-1}$ by the inductive assumption because then $d-i+m+a+2-j$ is odd and $(d-i,a+2-j)\prec(d,a+1)$. Hence it follows from the equality (5.2) applied to the polynomial $P_{d,a+2}(x_1,x_r)$ and from (5.3) that

$$\int_{(\mathbb{Z}_p)^r} x_1^{d}(x_1-x_2)^{n_1}\dots(x_{r-1}-x_r)^{n_{r-1}}x_r^{a+1}\,dG_r(x_1,..,x_r)=0$$

on $\mathcal{L}_{r-1}$. Therefore we have shown the proposition $\mathcal{P}_{d,a+1}$.

Let us assume that the propositions $\mathcal{P}_{x,y}$ are true for all pairs $(x,y)\prec(d+1,0)$. Let us assume that $d+1+m$ is odd. We have

$P_{d+1,1}(x_1,x_r)=$

$$(-1)^{m+1}\Big(\sum_{i=1,\ i\,\mathrm{odd}}^{d+1}\binom{d+1}{i}(-2)x_1^{d+1-i}\Big)x_r+(-1)^{m}\Big(\sum_{i=0,\ i\,\mathrm{even}}^{d+1}\binom{d+1}{i}2x_1^{d+1-i}\Big).$$

If $i$ is odd and $1\le i\le d+1$ then $d+1-i+m+1$ is odd. By the inductive assumption

$$\int_{(\mathbb{Z}_p)^r} x_1^{d+1-i}(x_1-x_2)^{n_1}\dots(x_{r-1}-x_r)^{n_{r-1}}x_r\,dG_r(x_1,\dots,x_r)=0 \tag{5.4}$$

on $\mathcal{L}_{r-1}$ for $i$ odd and $1\le i\le d+1$ because then $(d+1-i,1)\prec(d+1,0)$ and $d+1-i+m+1$ is odd.

By the inductive assumption we have also

$$\int_{(\mathbb{Z}_p)^r} x_1^{d+1-i}(x_1-x_2)^{n_1}\dots(x_{r-1}-x_r)^{n_{r-1}}\,dG_r(x_1,\dots,x_r)=0 \tag{5.5}$$

on $\mathcal{L}_{r-1}$ for $i$ even and $2\le i\le d+1$ because then $(d+1-i,0)\prec(d+1,0)$ and $d+1-i+m$ is odd. Hence it follows from the equality (5.2) applied to the polynomial $P_{d+1,1}(x_1,x_r)$ and from (5.4) and (5.5) that

$$\int_{(\mathbb{Z}_p)^r} x_1^{d+1}(x_1-x_2)^{n_1}\dots(x_{r-1}-x_r)^{n_{r-1}}\,dG_r(x_1,\dots,x_r)=0$$

on $\mathcal{L}_{r-1}$. Hence the proposition $\mathcal{P}_{d+1,0}$ is true. Therefore by the principle of transfinite induction the theorem is true. □

*Proof of Theorem B.* Observe that

$$\mathcal{L}_1 = \{\sigma \in G_{\mathbb{Q}(\mu_{p^\infty})} \mid \forall n \in \mathbb{N},\ \forall\, 0 \le i < p^n,\ \lambda^{(n)}_{Y_{i,n}}(\sigma) = 0\}.$$

The functions $\lambda^{(n)}_{Y_{i,n}} : G_{\mathbb{Q}(\mu_{p^\infty})} \to \mathbb{Z}_p$ are Kummer characters $\kappa(1 - \xi^i_{p^n})$ associated with $1 - \xi^i_{p^n}$ for $n \in \mathbb{N}$ and $0 \le i < p^n$. Hence they vanish on the Galois group $G_{\mathcal{K}_1}$ of the field $\mathcal{K}_1$. Therefore we have $G_{\mathcal{K}_1} \subset \mathcal{L}_1$. Hence Theorem B follows from Theorem 5.1. □

**Corollary 5.2.** *Let $r$ be a positive integer. Let $0 \le i_k < p^n$ for $k = 1, \dots, r$ and let $\mathbf{i} = (i_1, \dots, i_r)$. Let $\mathbf{1} = (1, 1, \dots, 1) \in (\mathbb{Z}_p)^r$. Let $n_0, \dots, n_r$ be any sequence of length $r+1$ of non negative integers. Let $m = n_0 + \ldots + n_r$. Let $F(x_1, \dots, x_r) = (x_1 - x_2)^{n_1} \ldots (x_{r-1} - x_r)^{n_{r-1}}$. Let $\sigma \in \mathcal{L}_{r-1}$ and let $n \ge 0$. Then we have*

$$\int_{\mathbf{i}+p^n(\mathbb{Z}_p)^r} (-x_1)^{n_0} F(x_1, \dots, x_r) x_r^{n_r}\, dG_r(x_1, .., x_r)(\sigma) +$$
$$(-1)^{m+1} \int_{-\mathbf{i}+p^n(\mathbb{Z}_p)^r} (-x_1)^{n_0} F(x_1, \dots, x_r) x_r^{n_r}\, dG_r(x_1, .., x_r)(\sigma) +$$
$$(-1)^{m} \int_{-\mathbf{i}+\mathbf{1}+p^n(\mathbb{Z}_p)^r} (-x_1+1)^{n_0} F(x_1, \dots, x_r) (x_r - 1)^{n_r}\, dG_r(x_1, .., x_r)(\sigma) +$$
$$(-1) \int_{\mathbf{i}-\mathbf{1}+p^n(\mathbb{Z}_p)^r} (-x_1-1)^{n_0} F(x_1, \dots, x_r) (x_r + 1)^{n_r}\, dG_r(x_1, .., x_r)(\sigma) = 0\,.$$

*Proof.* We calculate integrals over the set $\mathbf{i} + p^n(\mathbb{Z}_p)^r$ against the measure $\mu$ from the proof of Theorem 5.1. After changes of variables in the last three integrals we get the result. □

Below we shall rewrite the formula from Corollary 5.2 in terms of coefficients $\lambda^{(n)}_w$.

**Corollary 5.3.** *Let $n_0, \dots, n_r$ be any sequence of length $r+1$ of non-negative integers. Let $m = n_0 + \ldots + n_r$. Let $n \ge 0$. Let*

$$w = X_n^{n_0} Y_{i_1,n} X_n^{n_1} Y_{i_2,n} X_n^{n_2} \dots X_n^{n_{r-1}} Y_{i_r,n} X_n^{n_r},$$
$$w_1 = X_n^{n_0} Y_{-i_1,n} X_n^{n_1} Y_{-i_2,n} X_n^{n_2} \dots X_n^{n_{r-1}} Y_{-i_r,n} X_n^{n_r},$$

*and let*

$$v = Y_{-i_1+1,n} X_n^{n_1} Y_{-i_2+1,n} X_n^{n_2} \dots X_n^{n_{r-1}} Y_{-i_r+1,n},$$
$$u = Y_{i_1-1,n} X_n^{n_1} Y_{i_2-1,n} X_n^{n_2} \dots X_n^{n_{r-1}} Y_{i_r-1,n}.$$

*Let $v(a,b) = X_n^a v X_n^b$ and $u(a,b) = X_n^a u X_n^b$.*

i) *Let $\sigma \in \mathcal{L}_{r-1}$. Then*

$$\lambda^{(n)}_{w}(\sigma) + (-1)^{m+1}\lambda^{(n)}_{w_1}(\sigma) + (-1)^m \sum_{k=0}^{n_0}\sum_{j=0}^{n_r}\binom{n_0}{k}\binom{n_r}{j}\lambda^{(n)}_{v(n_0-k,n_r-j)}(\sigma)$$
$$+ (-1)\sum_{k=0}^{n_0}\sum_{j=0}^{n_r}(-1)^j\binom{n_0}{k}\binom{n_r}{j}\lambda^{(n)}_{u(n_0-k,n_r-j)}(\sigma) = 0$$

*and*

$$\lambda^{(n)}_{w}(\sigma) + (-1)^{m+1}\lambda^{(n)}_{w_1}(\sigma) + (-1)^m\lambda^{(n)}_{v(n_0,n_1)}(\sigma) - \lambda^{(n)}_{u(n_0,n_r)}(\sigma) \equiv 0$$

*modulo* $\mathcal{I}_n^{m+r-1}$.

ii) *Let $\sigma \in \mathcal{L}_{r-1}$ and let $n_0 = n_r = 0$. Then we have*

$$\lambda^{(n)}_{w}(\sigma) + (-1)^{m+1}\lambda^{(n)}_{w_1}(\sigma) + (-1)^m\lambda^{(n)}_{v(n_0,n_1)}(\sigma) - \lambda^{(n)}_{u(n_0,n_r)}(\sigma) = 0.$$

*Proof.* We apply the binomial formula to the polynomials $(-x_1+1)^{n_0}$, $(x_r-1)^{n_r}$, $(-x_0-1)^{n_0}$ and $(x_r+1)^{n_r}$ in the formula of Corollary 5.2. Then the formula in the point i) of the corollary follows immediately from Proposition 1.1, iii).

Let $k+j>0$. Then $\deg v(n_0-k,n_r-j) < m+r$ and $\deg u(n_0-k,n_r-j) < m+r$. Hence it follows the congruence in the point i). The point ii) is a special case of the point i). □

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