

## DEFINING RELATIONS OF 3-DIMENSIONAL QUADRATIC AS-REGULAR ALGEBRAS

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ABSTRACT. Classification of AS-regular algebras is one of the main interests in non-commutative algebraic geometry. Recently, a complete list of superpotentials (defining relations) of all 3-dimensional AS-regular algebras which are Calabi-Yau was given by Mori-Smith (the quadratic case) and Mori-Ueyama (the cubic case), however, no complete list of defining relations of all 3-dimensional AS-regular algebras has not appeared in the literature. In this paper, we give all possible defining relations of 3-dimensional quadratic AS-regular algebras. Moreover, we classify them up to isomorphism and up to graded Morita equivalence in terms of their defining relations in the case that their point schemes are not elliptic curves. In the case that their point schemes are elliptic curves, we give conditions for isomorphism and graded Morita equivalence in terms of geometric data.

### 1. INTRODUCTION

Classification of Artin-Schelter regular (AS-regular) algebras is one of the main interests in noncommutative algebraic geometry. It was originally defined by Artin-Schelter [1], and in that paper, it was attempted to classify 3-dimensional AS-regular algebras generated in degree 1, partially using computer programs. It was shown in [1] that every 3-dimensional AS-regular algebra generated in degree 1 has either 3 generators and 3 quadratic defining relations (the quadratic case), or 2 generators and 2 cubic defining relations (the cubic case). In each case, a list of defining relations (in fact potentials in the modern terminology) of “generic” 3-dimensional AS-regular algebras was given in [1, Table (3.11)] (the quadratic case) and in [1, Table (3.9)] (the cubic case). Soon after, Artin-Tate-Van den Bergh [2] found a nice one-to-one correspondence between the set of 3-dimensional AS-regular algebras  $A$  and the set of regular geometric pairs  $(E, \sigma)$  where  $E$  is a scheme and  $\sigma \in \text{Aut}_k E$ , so the classification of 3-dimensional AS-regular algebras reduces to the classification of regular geometric pairs. A list of regular geometric pairs corresponding to “generic” 3-dimensional AS-regular algebras was given in [2, 4.13]. (A complete list of regular geometric pairs “up to graded Morita equivalence” in the quadratic case was given in [4, Table 1]. See Remark 3.3.) This work convinced us that algebraic geometry is very useful to study

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even noncommutative algebras, and is considered as a starting point of the research field noncommutative algebraic geometry.

Although the next natural project is to classify 4-dimensional AS-regular algebras, which has been in fact very active until now, some “non-generic” 3-dimensional AS-regular algebras were also studied ([14], [16], etc.). Recently, a complete list of superpotentials (defining relations) of all 3-dimensional AS-regular algebras which are “Calabi-Yau” was given in [13] (the quadratic case) and in [15] (the cubic case), however, no complete list of defining relations of “all” 3-dimensional AS-regular algebras has not appeared in the literature. So the goal of our project is

- (I) to give a complete list of defining relations of “all” 3-dimensional quadratic AS-regular algebras,
- (II) to classify them up to isomorphism in terms of their defining relations, and
- (III) to classify them up to graded Morita equivalence in terms of their defining relations.

In this paper, we completed our project in the case that the point scheme is not an elliptic curve.

This paper is organized as follows: In Section 2, we recall the definitions of a twisted algebra from [19], a geometric algebra from [12], and an AS-regular algebra from [1]. In Section 3, we give a complete list of defining relations of 3-dimensional quadratic AS-regular algebras whose point schemes are not elliptic curves, and classify them up to isomorphism and up to graded Morita equivalence in terms of their defining relations (see Theorems 3.1, 3.2). In particular, in the case that the point scheme is a nodal cubic curve, we found a new algebra which is not isomorphic to any algebra classified in [16] (see Remark 3.4). Finally, in Section 4, we give a complete list of defining relations of geometric algebras whose point schemes are elliptic curves (which include 3-dimensional quadratic AS-regular algebras whose point schemes are elliptic curves), and conditions for isomorphism and graded Morita equivalence in terms of geometric data (see Theorems 4.9, 4.16, 4.20).

## 2. PRELIMINARY

Throughout this paper, we fix an algebraically closed field  $k$  of characteristic zero, and assume that a graded  $k$ -algebra is an  $\mathbb{N}$ -graded algebra  $A = \bigoplus_{i \in \mathbb{N}} A_i$ . A *connected graded* algebra is a graded algebra  $A = \bigoplus_{i \in \mathbb{N}} A_i$  such that  $A_0 = k$ . We denote by  $\text{GrMod } A$  the category of graded right  $A$ -modules. Morphisms in  $\text{GrMod } A$  are right  $A$ -module homomorphisms preserving degrees. We say that two graded algebras  $A$  and  $A'$  are *graded Morita equivalent* if the categories  $\text{GrMod } A$  and  $\text{GrMod } A'$  are equivalent.

**2.1. Twisted Algebras.** For a graded algebra  $A$ , Zhang [19] introduced a notion of twisted algebra  $A^\varphi$  of  $A$  by a graded algebra automorphism  $\varphi \in \text{GrAut}_k A$ . In this paper, we only define a twisted algebra for a quadratic algebra. A quadratic algebra  $A$  is of the form  $T(V)/(R)$  where  $V$  is a finite-dimensional  $k$ -vector space,  $T(V)$  is the tensor algebra of  $V$ ,  $R \subset V \otimes_k V$  is a subspace and  $(R)$  is the two-sided ideal of  $T(V)$  generated by  $R$ . We denote the general linear group of  $V$  by  $\text{GL}(V)$ . It is easy to check the following lemma.

**Lemma 2.1.** *Let  $A = T(V)/(R)$  and  $A' = T(V)/(R')$  be quadratic algebras with  $R, R' \subset V \otimes_k V$ . Then  $A \cong A'$  if and only if there is  $\phi \in \text{GL}(V)$  such that  $R' = (\phi \otimes \phi)(R)$ .*

**Definition 2.2.** Let  $V$  be a finite-dimensional  $k$ -vector space and  $A = T(V)/(R)$  a quadratic algebra with  $R \subset V \otimes_k V$ .

- (1) For  $\phi \in \text{GL}(V)$ , we define the twisted algebra  $A^\phi := T(V)/(R^\phi)$  of  $A$  by  $\phi$  where  $R^\phi := (\phi \otimes \text{id})(R) \subset V \otimes_k V$ .
- (2) For  $\varphi \in \text{GrAut}_k A$ , we define the twisted algebra  $A^\varphi := A^{\varphi|_V}$  of  $A$  by  $\varphi$  where  $\varphi|_V \in \text{GL}(V)$ .

For a quadratic algebra  $A$  and  $\phi \in \text{GL}(V)$ , it follows from the definition that  $(A^\phi)^{\phi^{-1}} = A$ . If  $\varphi \in \text{GrAut}_k A$ , then  $\varphi \in \text{GrAut}_k A^\varphi$  and  $(A^\varphi)^{\varphi^{-1}} = A$ . Since  $A^\varphi$  is isomorphic to the twisted algebra defined in [19], the following theorem is shown.

**Theorem 2.3** ([19, Theorem 3.1]). *Let  $V$  be a finite-dimensional  $k$ -vector space and  $A = T(V)/(R)$  a quadratic algebra with  $R \subset V \otimes V$ . If  $\varphi \in \text{GrAut}_k A$ , then  $\text{GrMod } A \cong \text{GrMod } A^\varphi$ .*

*Remark 2.4.* Let  $A = T(V)/(R)$  be a quadratic algebra and  $\phi \in \text{GL}(V)$ . If  $(\phi \otimes \phi)(R) = R$ , then  $\phi$  extends to  $\bar{\phi} \in \text{GrAut}_k A$ , so  $\text{GrMod } A \cong \text{GrMod } A^{\bar{\phi}} = \text{GrMod } A^\phi$ . However, when  $(\phi \otimes \phi)(R) \neq R$ ,  $A$  may not be graded Morita equivalent to  $A^\phi$  (See Example 4.21).

**2.2. Geometric Algebras.** Let  $V$  be a finite dimensional  $k$ -vector space. The equivalence relation on  $V \setminus \{0\}$  is defined by

$$u \sim v \iff \text{there exists } \lambda \in k^\times \text{ with } u = \lambda v.$$

The projective space associated to  $V$  is defined by

$$\mathbb{P}(V) := V \setminus \{0\} / \sim .$$

For  $\phi \in \text{GL}(V)$ , the map  $\bar{\phi}^* : \mathbb{P}(V^*) \rightarrow \mathbb{P}(V^*)$  defined by  $\bar{\phi}^*(\bar{\xi}) = \overline{\phi^*(\xi)}$  is an automorphism where  $\phi^* : V^* \rightarrow V^*$  is the dual map of  $\phi$ . For  $\phi, \psi \in \text{GL}(V)$ , the map  $\phi \times \psi : V \times V \rightarrow V \otimes_k V$  defined by  $(\phi \times \psi)(v, w) = \phi(v) \otimes \psi(w)$

is a bilinear map and induces a linear map  $\phi \otimes \psi : V \otimes_k V \rightarrow V \otimes_k V$  by  $(\phi \otimes \psi)(v \otimes w) = \phi(v) \otimes \psi(w)$  where  $v, w \in V$ . For  $g = \sum v_i \otimes w_i \in V \otimes_k V$ , we write

$$g(p, q) = \sum \xi(v_i)\eta(w_i)$$

where  $p = \bar{\xi}, q = \bar{\eta} \in \mathbb{P}(V^*)$ . Note that the zero set of  $R \subset V \otimes_k V$ ,

$$\mathcal{V}(R) := \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid g(p, q) = 0 \text{ for any } g \in R\}$$

is well-defined.

In [12], the notion of *geometric algebra* was introduced.

**Definition 2.5** ([12]). A geometric pair  $(E, \sigma)$  consists of a projective variety  $E \subset \mathbb{P}(V^*)$  and  $\sigma \in \text{Aut}_k E$ . Let  $A = T(V)/(R)$  be a quadratic algebra with  $R \subset V \otimes_k V$ .

- (1) We say that  $A$  satisfies (G1) if there exists a geometric pair  $(E, \sigma)$  such that

$$\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}.$$

In this case, we write  $\mathcal{P}(A) = (E, \sigma)$ , and call  $E$  the *point scheme* of  $A$ .

- (2) We say that  $A$  satisfies (G2) if there exists a geometric pair  $(E, \sigma)$  such that

$$R = \{f \in V \otimes_k V \mid f(p, \sigma(p)) = 0 \text{ for any } p \in E\}.$$

In this case, we write  $A = \mathcal{A}(E, \sigma)$ .

- (3) A quadratic algebra  $A$  is called *geometric* if  $A$  satisfies both (G1) and (G2) with  $A = \mathcal{A}(\mathcal{P}(A))$ .

If  $A$  satisfies (G1), then  $A$  determines the pair  $(E, \sigma)$ . Conversely, if  $A$  satisfies (G2), then  $A$  is determined by the pair  $(E, \sigma)$ . When we say that  $\mathcal{A}(E, \sigma)$  is geometric, we tacitly assume that  $\mathcal{P}(\mathcal{A}(E, \sigma)) = (E, \sigma)$  so that the point scheme of  $\mathcal{A}(E, \sigma)$  is  $E$ .

Note that, for  $g = \sum v_i \otimes w_i \in V \otimes_k V$ ,  $\phi, \psi \in \text{GL}(V)$  and  $p, q \in \mathbb{P}(V^*)$ ,  $((\phi \otimes \psi)(g))(p, q) = 0$  if and only if  $g(\overline{\phi^*}(p), \overline{\psi^*}(q)) = 0$ .

**Proposition 2.6.** *Let  $E \subset \mathbb{P}(V^*)$  be a projective variety,  $\sigma \in \text{Aut}_k E$  and  $\phi \in \text{GL}(V)$ . Suppose that  $\overline{\phi^*} \in \text{Aut}_k \mathbb{P}(V^*)$  restricts to  $\overline{\phi^*} \in \text{Aut}_k E$ . Let  $A = T(V)/(R)$  be a quadratic algebra with  $R \subset V \otimes_k V$ .*

- (1)  $\mathcal{A}(E, \sigma \overline{\phi^*}) = \mathcal{A}(E, \sigma)^\phi$ .
- (2) If  $\mathcal{P}(A) = (E, \sigma)$ , then  $\mathcal{P}(A^\phi) = (E, \sigma \overline{\phi^*})$ .
- (3) If  $A$  is *geometric* with  $\mathcal{P}(A) = (E, \sigma)$ , then  $A^\phi$  is *geometric* with  $\mathcal{P}(A^\phi) = (E, \sigma \overline{\phi^*})$ .

*Proof.* (1) By (G2), we can write  $\mathcal{A}(E, \sigma) = T(V)/(R_1)$  where

$$R_1 = \{f \in V \otimes_k V \mid f(p, \sigma(p)) = 0 \text{ for any } p \in E\},$$

and  $\mathcal{A}(E, \sigma\overline{\phi^*}) = T(V)/(R_2)$  where

$$R_2 = \{f \in V \otimes_k V \mid f(p, \sigma\overline{\phi^*}(p)) = 0 \text{ for any } p \in E\}.$$

Since  $\overline{\phi^*} \in \text{Aut}_k E$ ,

$$\begin{aligned} f \in R_2 &\iff f(p, \sigma\overline{\phi^*}(p)) = 0 \text{ for any } p \in E \\ &\iff f\left(\overline{(\phi^*)}^{-1}(p), \sigma(p)\right) = 0 \text{ for any } p \in E \\ &\iff ((\phi^{-1} \otimes \text{id})(f))(p, \sigma(p)) = 0 \text{ for any } p \in E \\ &\iff (\phi^{-1} \otimes \text{id})(f) \in R_1 \\ &\iff f \in (\phi \otimes \text{id})(R_1) =: R_1^\phi, \end{aligned}$$

so  $R_2 = R_1^\phi$ .

- (2) Suppose that  $\mathcal{P}(A) = (E, \sigma)$ , that is,  $\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}$ . Since  $\overline{\phi^*} \in \text{Aut}_k E$ ,

$$\begin{aligned} (p, q) \in \mathcal{V}(R^\phi) &\iff g(p, q) = 0 \text{ for any } g \in R^\phi \\ &\iff ((\phi \otimes \text{id})(f))(p, q) = 0 \text{ for any } f \in R \\ &\iff f(\overline{\phi^*}(p), q) = 0 \text{ for any } f \in R \\ &\iff (\overline{\phi^*}(p), q) \in \mathcal{V}(R) \\ &\iff q = \sigma\overline{\phi^*}(p), p \in E \\ &\iff (p, q) \in \{(p, \sigma\overline{\phi^*}(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}, \end{aligned}$$

so  $\mathcal{P}(A^\phi) = (E, \sigma\overline{\phi^*})$ .

- (3) Suppose that  $A$  is geometric with  $\mathcal{P}(A) = (E, \sigma)$ . Since  $\mathcal{P}(A) = (E, \sigma)$ ,  $\mathcal{P}(A^\phi) = (E, \sigma\overline{\phi^*})$  by (2). Since  $A = \mathcal{A}(\mathcal{P}(A)) = \mathcal{A}(E, \sigma)$ ,  $\mathcal{A}(\mathcal{P}(A^\phi)) = \mathcal{A}(E, \sigma\overline{\phi^*}) = \mathcal{A}(E, \sigma)^\phi = A^\phi$  by (1). □

**Definition 2.7.** Let  $X, Y \subset \mathbb{P}(V)$  be two projective varieties. We say that  $X$  and  $Y$  are *projectively equivalent* if there exists an isomorphism  $\phi : X \rightarrow Y$  which extends to an automorphism of  $\mathbb{P}(V)$ . We call  $\phi$  a *projective equivalence* from  $X$  to  $Y$ .

The following theorem tells us that classifying geometric algebras is equivalent to classifying geometric pairs.

**Theorem 2.8** ([12, Remark 4.9], cf. [2]). *Let  $A = \mathcal{A}(E, \sigma)$  and  $A' = \mathcal{A}(E', \sigma')$  be geometric algebras. Then  $A$  is isomorphic to  $A'$  as graded  $k$ -algebras if and only if there is a projective equivalence  $\phi$  from  $E$  to  $E'$ , such*

that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ \sigma \downarrow & & \downarrow \sigma' \\ E & \xrightarrow{\phi} & E' \end{array}$$

**Theorem 2.9** ([12, Theorem 4.7]). *Let  $A = \mathcal{A}(E, \sigma)$  and  $A' = \mathcal{A}(E', \sigma')$  be geometric algebras. Then  $\text{GrMod } A \cong \text{GrMod } A'$  if and only if there exists a sequence  $\{\phi_i\}_{i \in \mathbb{Z}}$  of projective equivalences from  $E$  to  $E'$  such that the following diagram commute for all  $i \in \mathbb{Z}$ :*

$$\begin{array}{ccc} E & \xrightarrow{\phi_i} & E' \\ \sigma \downarrow & & \downarrow \sigma' \\ E & \xrightarrow{\phi_{i+1}} & E' \end{array}$$

**2.3. AS-regular algebras.** Artin and Schelter [1] defined a class of regular algebras which are main objects of study in noncommutative algebraic geometry.

**Definition 2.10** ([1]). A connected graded algebra  $A$  is called a *d-dimensional Artin-Schelter regular* (simply *AS-regular*) algebra if  $A$  satisfies the following conditions:

- (i)  $\text{gldim } A = d < \infty$ ,
- (ii)  $\text{GKdim } A := \inf\{\alpha \in \mathbb{R} \mid \dim_k(\sum_{i=0}^n A_i) \leq n^\alpha \text{ for all } n \gg 0\} < \infty$ ,  
and,
- (iii) (*Gorenstein condition*)  $\text{Ext}_A^i(k, A) = \begin{cases} k & (i = d), \\ 0 & (i \neq d). \end{cases}$

A 3-dimensional AS-regular algebra  $A$  finitely generated in degree 1 is one of the following forms:

$$A = k\langle x, y, z \rangle / (f_1, f_2, f_3)$$

where  $f_i$  are homogeneous polynomials of degree 2 (the quadratic case), or

$$A = k\langle x, y \rangle / (g_1, g_2)$$

where  $g_i$  are homogeneous polynomials of degree 3 (the cubic case) (see [1, Theorem 1.5]). Our main focus of this paper is to study 3-dimensional quadratic AS-regular algebras.

**Theorem 2.11** ([2]). *Every 3-dimensional quadratic AS-regular algebra  $A$  is geometric. Moreover, the point scheme  $E$  of  $A$  is either  $\mathbb{P}^2$  or a cubic divisor in  $\mathbb{P}^2$ .*

*Remark 2.12.* In the above theorem,  $E \subset \mathbb{P}^2$  could be a non-reduced cubic divisor in  $\mathbb{P}^2$ . See [12, Definition 4.3] for the definition of a geometric algebra in the case that  $E$  is non-reduced.

We call a geometric pair  $(E, \sigma)$  *regular* if  $(E, \sigma) = \mathcal{P}(A)$  for some 3-dimensional quadratic AS-regular algebra  $A$ . The above theorem shows that the classification of 3-dimensional quadratic AS-regular algebras reduces to the classification of regular geometric pairs.

The types of regular geometric pairs are defined in [14] which are slightly modified from the original types defined in [1] and [2]. We extend the types defined in [14] as follows (since  $\text{Aut}_k \mathbb{P}^{n-1} \cong \text{PGL}_n(k)$ , we often identify  $\sigma \in \text{Aut}_k \mathbb{P}^{n-1}$  with the representing matrix  $\sigma \in \text{PGL}_n(k)$ ):

- (1) Type P:  $E$  is  $\mathbb{P}^2$ , and  $\sigma \in \text{Aut}_k \mathbb{P}^2 \cong \text{PGL}_3(k)$  (Type P is divided into Type  $P_i$  ( $i = 1, 2, 3$ ) in terms of the Jordan canonical form of  $\sigma$ ).
- (2-1) Type  $S_1$ :  $E$  is a triangle, and  $\sigma$  stabilizes each component.
- (2-2) Type  $S_2$ :  $E$  is a triangle, and  $\sigma$  interchanges two of its components.
- (2-3) Type  $S_3$ :  $E$  is a triangle, and  $\sigma$  circulates three components.
- (3-1) Type  $S'_1$ :  $E$  is a union of a line and a conic meeting at two points, and  $\sigma$  stabilizes each component and two intersection points.
- (3-2) Type  $S'_2$ :  $E$  is a union of a line and a conic meeting at two points, and  $\sigma$  stabilizes each component and interchanges two intersection points.
- (4-1) Type  $T_1$ :  $E$  is a union of three lines meeting at one point, and  $\sigma$  stabilizes each component.
- (4-2) Type  $T_2$ :  $E$  is a union of three lines meeting at one point, and  $\sigma$  interchanges two of its components.
- (4-3) Type  $T_3$ :  $E$  is a union of three lines meeting at one point, and  $\sigma$  circulates three components.
- (5) Type  $T'$ :  $E$  is a union of a line and a conic meeting at one point, and  $\sigma$  stabilizes each component.
- (6) Type CC:  $E$  is a cuspidal cubic curve.
- (7) Type NC:  $E$  is a nodal cubic curve (Type NC is divided into Type  $NC_i$  ( $i = 1, 2$ )).
- (8) Type WL:  $E$  is a union of a double line and a line (Type WL is divided into Type  $WL_i$  ( $i = 1, 2, 3$ )).
- (9) Type TL:  $E$  is a triple line (Type TL is divided into Type  $TL_i$  ( $i = 1, 2, 3, 4$ )).
- (10) Type EC:  $E$  is an elliptic curve.

**Example 2.13** ([12, Example 4.10]). 3-dimensional quadratic AS-regular algebras  $A = \mathcal{A}(E, \sigma)$  of Type  $S_1$  are classified by the following steps.

Step 0: Since  $E$  is a union of three lines making a triangle,  $E$  is projectively equivalent to  $\mathcal{V}(xyz) = \mathcal{V}(x) \cup \mathcal{V}(y) \cup \mathcal{V}(z)$ , so we may assume that  $E = \mathcal{V}(xyz) = \mathcal{V}(x) \cup \mathcal{V}(y) \cup \mathcal{V}(z)$  by Theorem 2.8.

Step 1: Since  $\sigma \in \text{Aut}_k E$  stabilizes each component,  $\sigma \in \text{Aut}_k E$  is given by

$$\begin{aligned}\sigma|_{\mathcal{V}(x)}(0 : b : c) &= (0 : b : \alpha c), \\ \sigma|_{\mathcal{V}(y)}(a : 0 : c) &= (\beta a : 0 : c), \\ \sigma|_{\mathcal{V}(z)}(a : b : 0) &= (a : \gamma b : 0),\end{aligned}$$

where  $\alpha, \beta, \gamma \in k$  and  $\alpha\beta\gamma \neq 0, 1$ .

Step 2: By using (G2) condition in Definition 2.5, we can compute the defining relation of  $A = \mathcal{A}(E, \sigma)$  as

$$yz - \alpha zy, \quad zx - \beta xz, \quad xy - \gamma yx.$$

Let  $A'$  be another algebra of Type  $S_1$  with the defining relations

$$yz - \alpha' zy, \quad zx - \beta' xz, \quad xy - \gamma' yx,$$

where  $\alpha', \beta', \gamma' \in k$  and  $\alpha'\beta'\gamma' \neq 0, 1$ .

Step 3: By Theorem 2.8, we can show that  $A \cong A'$  as graded  $k$ -algebras if and only if

$$(\alpha', \beta', \gamma') = \begin{cases} (\alpha, \beta, \gamma), (\beta, \gamma, \alpha), (\gamma, \alpha, \beta), \\ (\alpha^{-1}, \gamma^{-1}, \beta^{-1}), (\beta^{-1}, \alpha^{-1}, \gamma^{-1}), (\gamma^{-1}, \beta^{-1}, \alpha^{-1}). \end{cases}$$

Step 4: By Theorem 2.9, we can show that  $\text{GrMod} A \cong \text{GrMod} A'$  if and only if  $\alpha'\beta'\gamma' = (\alpha\beta\gamma)^{\pm 1}$ .

The purpose of this paper is to expand the above example to the remaining types.

### 3. DEFINING RELATIONS FOR NON TYPE EC ALGEBRAS

The following theorem lists all possible defining relations of algebras in each type up to isomorphism except for Type EC.

**Theorem 3.1** ([5, 9, 10, 11, 17]). *Let  $A = \mathcal{A}(E, \sigma)$  be a 3-dimensional quadratic AS-regular algebra. For each type except for Type EC, the following table describes*

- (I): *the defining relations of  $A$ , and*
- (II): *the conditions to be isomorphic as graded algebras in terms of their defining relations. (see Example 2.13.)*



In the following table, if  $X \neq Y$  or  $i \neq j$ , then Type  $X_i$  algebra is not isomorphic to any Type  $Y_j$  algebra.

Type	(I) defining relations ( $\alpha, \beta, \gamma \in k$ )	(II) condition to be graded algebra isomorphic
$P_1$	$\begin{cases} \alpha xy - \beta yx, \\ \beta yz - \gamma zy, & (\alpha\beta\gamma \neq 0) \\ \gamma zx - \alpha xz \end{cases}$	$(\alpha' : \beta' : \gamma') = \begin{cases} (\alpha : \beta : \gamma), (\alpha : \gamma : \beta), \\ (\beta : \alpha : \gamma), (\beta : \gamma : \alpha), \\ (\gamma : \alpha : \beta), (\gamma : \beta : \alpha) \end{cases} \text{ in } \mathbb{P}^2$
$P_2$	$\begin{cases} xy - yx + y^2, \\ xz - \alpha zx + \alpha zy, \\ yz - \alpha zy & (\alpha \neq 0) \end{cases}$	$\alpha' = \alpha$
$P_3$	$\begin{cases} xy - yx + y^2 - zx, \\ xz + yz - zx, \\ zy - yz - z^2 \end{cases}$	—————
$S_1$	$\begin{cases} yz - \alpha zy, \\ zx - \beta xz, & (\alpha\beta\gamma \neq 0, 1) \\ xy - \gamma yx \end{cases}$	$(\alpha', \beta', \gamma') = \begin{cases} (\alpha, \beta, \gamma), (\beta, \gamma, \alpha), \\ (\gamma, \alpha, \beta), (\alpha^{-1}, \gamma^{-1}, \beta^{-1}), \\ (\beta^{-1}, \alpha^{-1}, \gamma^{-1}), (\gamma^{-1}, \beta^{-1}, \alpha^{-1}) \end{cases}$
$S_2$	$\begin{cases} zx - \alpha yz, \\ xz - \beta zy, & (\alpha\beta \neq 0) \\ x^2 + \alpha\beta y^2 \end{cases}$	$(\alpha' : \beta') = (\alpha : \beta) \text{ in } \mathbb{P}^1$
$S_3$	$\begin{cases} yx - \alpha z^2, \\ zy - \beta x^2, & (\alpha\beta\gamma \neq 0, 1) \\ xz - \gamma y^2 \end{cases}$	$\alpha'\beta'\gamma' = \alpha\beta\gamma$
$S'_1$	$\begin{cases} xy - \beta yx, \\ x^2 + yz - \alpha zy, \\ zx - \beta xz & (\alpha\beta^2 \neq 0, 1) \end{cases}$	$(\alpha', \beta') = (\alpha, \beta), (\alpha^{-1}, \beta^{-1})$
$S'_2$	$\begin{cases} xy - zx, \\ yx - xz, \\ x^2 + y^2 + z^2 \end{cases}$	—————

$T_1$	$\begin{cases} xy - yx, \\ xz - zx - \beta x^2 \\ \quad + (\beta + \gamma)yx, \\ yz - zy - \alpha y^2 \\ \quad + (\alpha + \gamma)xy \\ (\alpha + \beta + \gamma \neq 0) \end{cases}$	$\begin{aligned} & (\alpha' : \beta' : \gamma') \\ & = \begin{cases} (\alpha : \beta : \gamma), (\alpha : \gamma : \beta), \\ (\beta : \alpha : \gamma), (\beta : \gamma : \alpha), \\ (\gamma : \alpha : \beta), (\gamma : \beta : \alpha) \end{cases} \quad \text{in } \mathbb{P}^2 \end{aligned}$
$T_2$	$\begin{cases} x^2 - y^2, \\ xz - zy - \beta xy \\ \quad + (\beta + \gamma)y^2, \\ yz - zx - \alpha yx \\ \quad + (\alpha + \gamma)x^2 \\ (\alpha + \beta + \gamma \neq 0) \end{cases}$	$(\alpha' + \beta' : \gamma') = (\alpha + \beta : \gamma) \text{ in } \mathbb{P}^1$
$T_3$	$\begin{cases} x^2 - xy + y^2, \\ xz + zy, \\ yx - yz + zx - zy \end{cases}$	<hr style="width: 100%; border: 0.5px solid black;"/>
$T'$	$\begin{cases} \alpha x^2 + \beta(\alpha + \beta)xy - xz \\ \quad + zx - (\alpha + \beta)zy, \\ xy - yx - \beta y^2, \\ 2\beta xy - \beta^2 y^2 + yz - zy \\ (\alpha + 2\beta \neq 0) \end{cases}$	$(\alpha' : \beta') = (\alpha : \beta) \text{ in } \mathbb{P}^1$
CC	$\begin{cases} -3x^2 - 2xy + xz - zx \\ \quad + 2zy, \\ -xy + yx + y^2, \\ 3x^2 + y^2 + yz - zy \end{cases}$	<hr style="width: 100%; border: 0.5px solid black;"/>
$NC_1$	$\begin{cases} xy - \alpha yx, \\ \frac{\alpha^3 - 1}{\alpha} x^2 + \alpha zy - yz, \\ \frac{\alpha^3 - 1}{\alpha} y^2 + \alpha xz - zx \\ (\alpha(\alpha^3 - 1) \neq 0) \end{cases}$	$\alpha' = \alpha^{\pm 1}$
$NC_2$	$\begin{cases} xz - 2yx + zy, \\ zx - 2xy + yz, \\ y^2 + x^2 \end{cases}$	<hr style="width: 100%; border: 0.5px solid black;"/>

WL <sub>1</sub>	$\begin{cases} \alpha xy - yx, \\ \alpha xz - \gamma yx - zx, \\ zy - yz + (1 + \gamma)y^2 \end{cases}$ $(\alpha \neq 0, 1)$	$(\alpha', \gamma') = (\alpha, \gamma)$
WL <sub>2</sub>	$\begin{cases} xy - yx, \\ xz - \gamma yx - zx, \\ zy - yz + (1 + \gamma)y^2 \end{cases}$	$\gamma' = \gamma$
WL <sub>3</sub>	$\begin{cases} xy - yx, \\ xz - x^2 - \gamma yx - zx, \\ xy + zy - yz \\ \quad + (1 + \gamma)y^2 \end{cases}$	$\gamma' = \gamma$
TL <sub>1</sub>	$\begin{cases} xy - \alpha yx, \\ xz - \alpha^{-1}zx, \\ \alpha^{-1}zy - \alpha yz + x^2 \end{cases}$ $(\alpha \neq 0)$	$\alpha' = \alpha^{\pm 1}$
TL <sub>2</sub>	$\begin{cases} xy - yx - \beta x^2, \\ xz - zx - yx, \\ zy - yz - \beta xz + x^2 + y^2 \end{cases}$	$\beta' = \pm \beta$
TL <sub>3</sub>	$\begin{cases} xy + yx, \\ xz + zx - yx, \\ zy - yz - x^2 - y^2 \end{cases}$	_____
TL <sub>4</sub>	$\begin{cases} xy + yx, \\ xz - zx - x^2, \\ zy - yz + xy + x^2 \end{cases}$	_____

The following theorem lists all possible defining relations of algebras in each type up to graded Morita equivalence except for Type EC.

**Theorem 3.2** ([5, 9, 10, 11, 17]). *Let  $A = \mathcal{A}(E, \sigma)$  be a 3-dimensional quadratic AS-regular algebra. For each type except for Type EC, the following table describes*

- (I): *the defining relations of  $A$ , and*
- (III): *the conditions to be graded Morita equivalent in terms of their defining relations. (see Example 2.13. )*

*In the following table, if  $X \neq Y$ , then Type  $X$  algebra is not graded Morita equivalent to any Type  $Y$  algebra.*

Type	(I) defining relations ( $\alpha, \beta, \gamma \in k$ )	(III) condition to be graded Morita equivalence
P	$\begin{cases} xy - yx, \\ yz - zy, \\ zx - xz \end{cases}$	_____
S	$\begin{cases} yz - \alpha zy, \\ zx - \beta xz, \\ xy - \gamma yx \end{cases} \quad (\alpha\beta\gamma \neq 0, 1)$	$\alpha'\beta'\gamma' = (\alpha\beta\gamma)^{\pm 1}$
S'	$\begin{cases} xy - \beta yx, \\ x^2 + yz - \alpha zy, \\ zx - \beta xz \end{cases} \quad (\alpha\beta^2 \neq 0, 1)$	$\alpha'\beta'^2 = (\alpha\beta^2)^{\pm 1}$
T	$\begin{cases} xy - yx, \\ xz - zx - x^2, \\ yz - zy - y^2 \end{cases}$	_____
T'	$\begin{cases} x^2 - xz + zx - zy, \\ xy - yx, \\ yz - zy \end{cases}$	_____
CC	$\begin{cases} -3x^2 - 2xy + xz - zx \\ \quad \quad \quad + 2zy, \\ -xy + yx + y^2, \\ 3x^2 + y^2 + yz - zy \end{cases}$	_____
NC	$\begin{cases} xy - \alpha yx, \\ \frac{\alpha^3 - 1}{\alpha} x^2 + \alpha zy - yz, \\ \frac{\alpha^3 - 1}{\alpha} y^2 + \alpha xz - zx \end{cases} \quad (\alpha(\alpha^3 - 1) \neq 0)$	$\alpha'^3 = \alpha^{\pm 3}$
WL	$\begin{cases} xy + yx, \\ xz + zx, \\ zy - yz + y^2 \end{cases}$	_____
TL	$\begin{cases} xy - yx, \\ xz - zx, \\ zy - yz + x^2 \end{cases}$	_____

*Remark 3.3.* Since  $\text{GrMod } A \cong \text{GrMod } A'$  if and only if  $\bar{A} \cong \bar{A}'$  as  $\mathbb{Z}$ -algebras where  $\bar{A} := \bigoplus_{i,j \in \mathbb{Z}} A_{j-i}$  by [18], the above table agrees with [4, Table 1].

If  $E$  is reduced, then Theorem 3.1 and Theorem 3.2 are proved by the following five steps (see Example 2.13):

Step 0: Fix a defining relation of  $E$ .

Step 1: Find all automorphisms  $\sigma$  of  $E$ .

Step 2: Find the defining relations of  $\mathcal{A}(E, \sigma)$  for each  $\sigma \in \text{Aut}_k E$  by using (G2) condition in Definition 2.5.

Step 3: Classify them up to isomorphism of graded algebras in terms of their defining relations by using Theorem 2.8.

Step 4: Classify them up to graded Morita equivalence in terms of their defining relations by using Theorem 2.9.

For Type  $P_i$  ( $i = 1, 2, 3$ ), Type  $S_i$  ( $i = 1, 2, 3$ ), Type  $S'_i$  ( $i = 1, 2$ ), Type  $T_i$  ( $i = 1, 2, 3$ ) and Type  $T'$ , the above five steps were completed in [11] and [10]. For Type CC and Type  $NC_i$  ( $i = 1, 2$ ), Step 1 was completed in [17], and Step 2, Step 3 and Step 4 were completed in [5].

We briefly explain the method in [17]. Let  $E$  be an irreducible variety and  $\pi: \tilde{E} \rightarrow E$  a normalization of  $E$ . Then, for any  $\sigma \in \text{Aut}_k E$ , there exists a unique  $\tilde{\sigma} \in \text{Aut}_k \tilde{E}$  such that  $\sigma \circ \pi = \pi \circ \tilde{\sigma}$ , i.e., the following diagram commutes:

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\pi} & E \\ \tilde{\sigma} \downarrow & & \downarrow \sigma \\ \tilde{E} & \xrightarrow{\pi} & E \end{array}$$

In fact, the assignment  $\sigma \mapsto \tilde{\sigma}$  is an injective group homomorphism from  $\text{Aut}_k E$  to  $\text{Aut}_k \tilde{E}$ .

For example, let  $A = \mathcal{A}(E, \sigma)$  be a Type NC algebra.

Step 0: Since  $E$  is a nodal cubic curve, we may assume that  $E = \mathcal{V}(x^3 + y^3 + xyz)$ .

Step 1: A normalization  $\pi: \mathbb{P}^1 = \tilde{E} \rightarrow E$  is given by

$$\pi(a: b) = (a^2b: ab^2: -a^3 - b^3).$$

Since  $\sigma$  fixes the singular point  $(0: 0: 1) \in E$  and  $\pi^{-1}((0: 0: 1)) = \{(1: 0), (0: 1)\} \subset \mathbb{P}^1$ , either  $\tilde{\sigma}$  fixes both  $(1: 0)$  and  $(0: 1)$  so that  $\tilde{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$

for  $0 \neq \alpha \in k$ , or  $\tilde{\sigma}$  switches  $(1: 0)$  and  $(0: 1)$  so that  $\tilde{\sigma} = \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}$  for  $0 \neq \beta \in k$ . In each case, the corresponding  $\sigma$  is given as

$$\sigma_1(x: y: z) = (\alpha xy: \alpha^2 y^2: (\alpha^3 - 1)x^2 + \alpha^3 yz) \quad (\alpha^3 \neq 0, 1)$$

or

$$\sigma_2(x: y: z) = (\beta y^2: \beta^2 xy: (1 - \beta^3)x^2 + yz) \quad (\beta^3 \neq 0, 1).$$

*Remark 3.4.* We call the above  $\mathcal{A}(E, \sigma_i)$  *Type  $NC_i$  algebras* ( $i = 1, 2$ ). Type  $NC_1$  algebras are isomorphic to algebras given in [16, Theorem 2.2], however, Type  $NC_2$  algebras are not isomorphic to any algebra in [16, Theorem 2.2]. In fact, the above  $\sigma_1$  was in [16], but  $\sigma_2$  was overlooked in [16].

To prove Theorem 3.1 and Theorem 3.2 when  $E$  is a non-reduced cubic in  $\mathbb{P}^2$ , we use the following key lemma.

**Lemma 3.5** ([3, Theorem 8.16 (iii)]). (1) *If  $A$  is a 3-dimensional quadratic AS-regular algebra of Type WL, then there exists  $\varphi \in \text{GrAut}_k A$  such that*

$$A^\varphi \cong B_1 := k\langle x, y, z \rangle / (xy - yx, xz - zx, zy - yz + xz), \text{ or}$$

$$A^\varphi \cong B_2 := k\langle x, y, z \rangle / (xy - yx, xz - zx, zy - yz + y^2).$$

(2) *If  $A$  is a 3-dimensional quadratic AS-regular algebra of Type TL, then there exists  $\varphi \in \text{GrAut}_k A$  such that*

$$A^\varphi \cong B_3 := k\langle x, y, z \rangle / (xy - yx, xz - zx, zy - yz + x^2).$$

Since  $B = A^\varphi$  if and only if  $A = B^{\varphi^{-1}}$  by [19, Proposition 2.5 (2)], for Type WL algebras and Type TL algebras, Theorem 3.1 and Theorem 3.2 are proved by the following four steps:

Step 1: Find all graded algebra automorphisms  $\varphi^{-1}$  of  $B_i$  ( $i = 1, 2, 3$ ) in Lemma 3.5.

Step 2: Find the defining relations of  $B_i^{\varphi^{-1}}$  by using Definition 2.2.

Step 3: Classify them up to isomorphism of graded algebras in terms of their defining relations by using Lemma 2.1.

Step 4: Classify them up to graded Morita equivalence in terms of their defining relations by using Theorem 2.3.

Step 1 and Step 2 were completed in [9] and, Step 3 and Step 4 were completed in [5].

#### 4. DEFINING RELATIONS FOR TYPE EC ALGEBRAS

Throughout this section, let  $E$  be an elliptic curve in  $\mathbb{P}^2$ . Our aim in this section is to find  $\text{Aut}_k E$  and to compute the defining relations of  $\mathcal{A}(E, \sigma)$  where  $\sigma \in \text{Aut}_k E$ .

It is well-known that the  $j$ -invariant  $j(E)$  classifies elliptic curves up to projective equivalence.

**Theorem 4.1** ([7, Theorem IV 4.1 (b)]). *Let  $E$  and  $E'$  be two elliptic curves in  $\mathbb{P}^2$ . Then  $E$  and  $E'$  are projectively equivalent if and only if  $j(E) = j(E')$ .*

Let  $X$  be a scheme and  $Y$  a subscheme of  $X$ . We define

$$\text{Aut}_k(X, Y) := \{\phi \in \text{Aut}_k X \mid \phi|_Y \in \text{Aut}_k Y\}.$$

We view an element of  $\text{Aut}_k(X, Y)$  in two ways, that is, as an automorphism of  $X$  which restricts to an automorphism of  $Y$  and as an automorphism of  $Y$  which extends to an automorphism of  $X$ . In particular, if  $Y = \{p\}$ , then we write  $\text{Aut}_k(X, Y) = \text{Aut}_k(X, p)$ .

**Theorem 4.2** ([7, Corollary IV 4.7]). *Let  $E$  be an elliptic curve in  $\mathbb{P}^2$ . For every  $p \in E$ ,*

$$|\text{Aut}_k(E, p)| = \begin{cases} 2 & \text{if } j(E) \neq 0, 12^3, \\ 6 & \text{if } j(E) = 0, \\ 4 & \text{if } j(E) = 12^3. \end{cases}$$

For each point  $o \in E$ , we can define an addition on  $E$  so that  $E$  is an abelian group with the identity element  $o$  and, for  $p \in E$ , the map  $\sigma_p$  defined by  $\sigma_p(q) := p + q$  is a scheme automorphism of  $E$ , called the *translation* by a point  $p$ . We write  $(E, o)$  when we view  $E$  as an abelian group with the identity element  $o \in E$ .

In this paper, we use the *Hesse form*  $E_\lambda := \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$  where  $\lambda \in k$ . It is known that  $E_\lambda$  is an elliptic curve in  $\mathbb{P}^2$  if and only if  $\lambda^3 \neq 1$ . The *j-invariant* of  $E_\lambda$  is given by the formula

$$j(E_\lambda) = \frac{27\lambda^3(\lambda^3 + 8)^3}{(\lambda^3 - 1)^3}$$

([6, Proposition 2.16]).

Every elliptic curve in  $\mathbb{P}^2$  is projectively equivalent to  $E_\lambda$  for some  $\lambda$  with  $\lambda^3 \neq 1$  ([6, Corollary 2.18]).

**Theorem 4.3** ([6, Theorem 2.11]). *Let  $E_\lambda$  be an elliptic curve of the Hesse form in  $\mathbb{P}^2$  and  $o_\lambda := (1 : -1 : 0) \in E_\lambda$ . The group structure on  $(E_\lambda, o_\lambda)$  is given as follows : for  $p = (a : b : c)$  and  $q = (\alpha : \beta : \gamma) \in E_\lambda$ ,*

$$p + q := \begin{cases} (ac\beta^2 - b^2\alpha\gamma : bca^2 - a^2\beta\gamma : ab\gamma^2 - c^2\alpha\beta) & \text{if } p \neq q, \\ (a^3b - bc^3 : ac^3 - ab^3 : b^3c - a^3c) & \text{if } p = q. \end{cases}$$

Throughout this paper, we fix the above group structure on  $E_\lambda$  with the identity  $o_\lambda := (1 : -1 : 0) \in E_\lambda$ .

#### 4.1. Automorphism groups.

**Lemma 4.4** ([7, Lemma IV 4.9]). *Let  $(E, o)$  and  $(E', o')$  be two elliptic curves in  $\mathbb{P}^2$ . If  $\varphi : E \rightarrow E'$  is a morphism of schemes sending  $o$  to  $o'$ , then  $\varphi$  is also a group homomorphism.*

We set the following notations:

- (i)  $T := \{\sigma_p \in \text{Aut}_k E \mid p \in E\}$  and  $T_\lambda := \{\sigma_p \in \text{Aut}_k E_\lambda \mid p \in E_\lambda\}$ .  
(ii)  $G := \text{Aut}_k(E, o)$  and  $G_\lambda := \text{Aut}_k(E_\lambda, o_\lambda)$ .

For  $\sigma_p \in T$  and  $\tau \in G$ , it is easy to see that  $\tau\sigma_p\tau^{-1} = \sigma_{\tau(p)} \in T$ .

**Proposition 4.5** (cf. [4, Section 6]). *Suppose that  $(E, o)$  is an elliptic curve in  $\mathbb{P}^2$ . If  $\Phi : G \rightarrow \text{Aut } T$  is the group homomorphism defined by  $\Phi_\tau(\sigma_p) = \sigma_{\tau(p)}$  for  $\tau \in G$  and  $\sigma_p \in T$ , then  $\text{Aut}_k E \cong T \rtimes_\Phi G$ .*

**Theorem 4.6.** *Let  $E_\lambda$  be an elliptic curve in  $\mathbb{P}^2$ . A generator  $\tau_\lambda$  of  $G_\lambda$  is given by*

$$\begin{cases} \tau_\lambda(a : b : c) := (b : a : c) & \text{if } j(E_\lambda) \neq 0, 12^3, \\ \tau_\lambda(a : b : c) := (b : a : c\varepsilon) & \text{if } \lambda = 0 \text{ (so that } j(E_\lambda) = 0), \\ \tau_\lambda(a : b : c) := (a\varepsilon^2 + b\varepsilon + c : a\varepsilon + b\varepsilon^2 + c : a + b + c) & \text{if } \lambda = 1 + \sqrt{3} \text{ (so that } j(E_\lambda) = 12^3), \end{cases}$$

where  $\varepsilon$  is a primitive 3rd root of unity. In particular,  $G_\lambda$  is the subgroup of  $\text{Aut}_k(\mathbb{P}^2, E_\lambda)$ .

*Proof.* (i) If  $j(E_\lambda) \neq 0, 12^3$ , then  $|G_\lambda| = 2$  by Theorem 4.2. Let  $\tau_\lambda =$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{PGL}_3(k) \cong \text{Aut}_k \mathbb{P}^2. \text{ If } p = (a : b : c) \in E_\lambda, \text{ then}$$

$\tau_\lambda(p) = (b : a : c) \in E_\lambda$ , so  $\tau_\lambda \in \text{Aut}_k(\mathbb{P}^2, E_\lambda)$ . Since  $\tau_\lambda(o_\lambda) = o_\lambda$ , we have  $\tau_\lambda \in G_\lambda$ . By calculations,  $|\tau_\lambda| = 2$ , so  $G_\lambda = \langle \tau_\lambda \rangle$ .

(ii) If  $\lambda = 0$  so that  $E_\lambda = \mathcal{V}(x^3 + y^3 + z^3)$ , then  $j(E_\lambda) = 0$ , so  $|G_\lambda| = 6$

by Theorem 4.2. Let  $\tau_\lambda = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \in \text{PGL}_3(k) \cong \text{Aut}_k \mathbb{P}^2$ , where

$\varepsilon$  is a primitive 3rd root of unity. If  $p = (a : b : c) \in E_\lambda$ , then  $\tau_\lambda(p) = (b : a : c\varepsilon) \in E_\lambda$ , so  $\tau_\lambda \in \text{Aut}_k(\mathbb{P}^2, E_\lambda)$ . Since  $\tau_\lambda(o_\lambda) = o_\lambda$ , we have  $\tau_\lambda \in G_\lambda$ . By calculations,  $|\tau_\lambda| = 6$ , so  $G_\lambda = \langle \tau_\lambda \rangle$ .

(iii) If  $\lambda = 1 + \sqrt{3}$  so that  $E_\lambda = \mathcal{V}(x^3 + y^3 + z^3 - 3(1 + \sqrt{3})xyz)$ , then

$j(E_\lambda) = 12^3$ , so  $|G_\lambda| = 4$  by Theorem 4.2. Let  $\tau_\lambda = \begin{pmatrix} \varepsilon^2 & \varepsilon & 1 \\ \varepsilon & \varepsilon^2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in$

$\text{PGL}_3(k) \cong \text{Aut}_k \mathbb{P}^2$ . If  $p = (a : b : c) \in E_\lambda$ , then  $\tau_\lambda(p) = (a\varepsilon^2 + b\varepsilon + c : a\varepsilon + b\varepsilon^2 + c : a + b + c)$ . Since

$$\begin{aligned} & (a\varepsilon^2 + b\varepsilon + c)^3 + (a\varepsilon + b\varepsilon^2 + c)^3 + (a + b + c)^3 \\ & \quad - 3(1 + \sqrt{3})(a\varepsilon^2 + b\varepsilon + c)(a\varepsilon + b\varepsilon^2 + c)(a + b + c) \end{aligned}$$



$$\begin{aligned}
 &= 3(a^3 + b^3 + c^3) + 18abc - 3(1 + \sqrt{3})(a^3 + b^3 + c^3 - 3abc) \\
 &= -3\sqrt{3}(a^3 + b^3 + c^3) + 9\sqrt{3}(1 + \sqrt{3})abc \\
 &= -3\sqrt{3}(a^3 + b^3 + c^3 - 3(1 + \sqrt{3})abc) \\
 &= 0,
 \end{aligned}$$

we have  $\tau_\lambda(p) \in E_\lambda$ , so  $\tau_\lambda \in \text{Aut}_k(\mathbb{P}^2, E_\lambda)$ . Since  $\tau_\lambda(o_\lambda) = o_\lambda$ , we have  $\tau_\lambda \in G_\lambda$ . By calculations,  $|\tau_\lambda| = 4$ , so  $G_\lambda = \langle \tau_\lambda \rangle$ .  $\square$

We fix the above generator  $\tau_\lambda$  of  $G_\lambda$  for the rest of the paper.

## 4.2. Defining Relations.

**Lemma 4.7.** *Every 3-dimensional quadratic AS-regular algebra  $A = \mathcal{A}(E, \sigma)$  of Type EC is isomorphic to  $\mathcal{A}(E_\lambda, \sigma_p \tau_\lambda^i)$  where  $\lambda \in k$  with  $\lambda^3 \neq 1$ ,  $p \in E_\lambda$  and  $i \in \mathbb{Z}$ .*

*Proof.* By Theorem 4.1, there exists  $\lambda \in k$  such that  $E$  and  $E_\lambda$  are projectively equivalent. If we set  $\sigma' := \phi \sigma \phi^{-1} \in \text{Aut}_k E_\lambda$  where  $\phi : E \rightarrow E_\lambda$  is a projective equivalence, then the diagram

$$\begin{array}{ccc}
 E & \xrightarrow{\phi} & E_\lambda \\
 \sigma \downarrow & & \downarrow \sigma' \\
 E & \xrightarrow{\phi} & E_\lambda
 \end{array}$$

commutes, so  $\mathcal{A}(E, \sigma) \cong \mathcal{A}(E_\lambda, \sigma')$  by [14, Lemma 2.6 (1)]. By Proposition 4.5 and Theorem 4.6, there exist  $p \in E_\lambda$  and  $i \in \mathbb{Z}$  such that  $\sigma' = \sigma_p \tau_\lambda^i$  where  $\langle \tau_\lambda \rangle = G_\lambda = \text{Aut}_k(E_\lambda, o_\lambda)$ , so  $A \cong \mathcal{A}(E_\lambda, \sigma_p \tau_\lambda^i)$ .  $\square$

We can compute the defining relations of 3-dimensional quadratic AS-regular algebras of Type EC by using the defining relations of a 3-dimensional Sklyanin algebra

$$\mathcal{A}(E, \sigma_p) = k\langle x, y, z \rangle / (ayz + bzy + cx^2, axz + bxz + cy^2, axy + byx + cz^2)$$

where  $p = (a : b : c) \in \mathbb{P}^2$ . We say that a geometric algebra  $A$  is of Type EC if the point scheme of  $A$  is an elliptic curve.

**Lemma 4.8.** *Let  $E_\lambda$  be an elliptic curve in  $\mathbb{P}^2$  where  $\lambda^3 \neq 1$ ,  $p = (a : b : c) \in E_\lambda$  and  $i \in \mathbb{Z}$ . Then  $\mathcal{A}(E_\lambda, \sigma_p \tau_\lambda^i)$  is a geometric algebra of Type EC if and only if  $abc \neq 0$ .*

*Proof.* If  $abc \neq 0$ , then  $((a^3 + b^3 + c^3)/3abc)^3 = \lambda^3 \neq 1$ , that is,  $(a^3 + b^3 + c^3)^3 \neq (3abc)^3$ , so  $\mathcal{A}(E_\lambda, \sigma_p)$  is a 3-dimensional quadratic AS-regular algebra of Type EC by [2, Section 1]. Since  $\mathcal{A}(E_\lambda, \sigma_p)$  is a geometric algebra of Type





**4.3. Classification up to graded algebra isomorphism.** By [6, Corollary 2.18], for every  $E$ , there exists  $\lambda \in k$  with  $\lambda^3 \neq 1$  such that  $E_\lambda$  and  $E$  are projectively equivalent. If  $\psi : E_\lambda \rightarrow E$  is a projective equivalence, then  $\Psi : \text{Aut}_k E_\lambda \rightarrow \text{Aut}_k E$  defined by  $\Psi(\sigma) := \psi\sigma\psi^{-1}$  is a group isomorphism. If  $o := \psi(o_\lambda)$ , then  $\psi : (E_\lambda, o_\lambda) \rightarrow (E, o)$  is a group isomorphism by Lemma 4.4, and  $\Psi(\sigma_p) = \psi\sigma_p\psi^{-1} = \sigma_{\psi(p)} \in T$  for  $\sigma_p \in T_\lambda$ . For the rest paper, we fix

- (a) a projective equivalence  $\psi : E_\lambda \rightarrow E$ ,
- (b) the group isomorphism  $\Psi : \text{Aut}_k E_\lambda \rightarrow \text{Aut}_k E$  defined by

$$\Psi(\sigma) := \psi\sigma\psi^{-1},$$

- (c) the identity element  $o := \psi(o_\lambda)$  of  $E$ , and
- (d) the generator  $\tau := \Psi(\tau_\lambda)$  of  $G = \text{Aut}_k(E, o)$ .

We set the following notations:

- (i)  $E[3] := \{p \in E \mid 3p = o\}$  and  $E_\lambda[3] := \{p \in E_\lambda \mid 3p = o_\lambda\}$ .
- (ii)  $T[3] := \{\sigma \in T \mid \sigma^3 = \text{id}_E\} = \{\sigma_p \in T \mid p \in E[3]\}$  and  
 $T_\lambda[3] := \{\sigma \in T_\lambda \mid \sigma^3 = \text{id}_{E_\lambda}\} = \{\sigma_p \in T_\lambda \mid p \in E_\lambda[3]\}$ .
- (iii)  $d := |G|$  and  $d_\lambda := |G_\lambda|$ .
- (iv)  $F_i := \{p - \tau^i(p) \in E \mid p \in E[3]\}$  for  $i \in \mathbb{Z}_d$  and  
 $F_{\lambda,i} := \{p - \tau_\lambda^i(p) \in E_\lambda \mid p \in E_\lambda[3]\}$  for  $i \in \mathbb{Z}_{d_\lambda}$ .

It is easy to check the following lemma.

**Lemma 4.11.** *The following hold.*

- (1)  $E[3] = \psi(E_\lambda[3])$ .
- (2)  $F_i = \psi(F_{\lambda,i})$ .
- (3)  $\text{Aut}_k(\mathbb{P}^2, E) = \Psi(\text{Aut}_k(\mathbb{P}^2, E_\lambda))$ .
- (4)  $G = \Psi(G_\lambda)$ .
- (5)  $T = \Psi(T_\lambda)$ .
- (6)  $T[3] = \Psi(T_\lambda[3])$ .

**Theorem 4.12.** *The following hold.*

- (1)  $\text{Aut}_k(\mathbb{P}^2, E) \cap T = T[3]$ .
- (2)  $G \leq \text{Aut}_k(\mathbb{P}^2, E)$ .
- (3)  $\text{Aut}_k(\mathbb{P}^2, E) \cong T[3] \rtimes G$ .

*Proof.* (1) See [12, Lemma 5.3].

(2) Since  $G_\lambda \leq \text{Aut}_k(\mathbb{P}^2, E_\lambda)$  by Theorem 4.6,  $G = \Psi(G_\lambda) \leq \Psi(\text{Aut}_k(\mathbb{P}^2, E_\lambda)) = \text{Aut}_k(\mathbb{P}^2, E)$  by Lemma 4.11 (3) and (4).

(3) Since  $\text{Aut}_k E \cong T \rtimes G$  by Proposition 4.5 and  $G \leq \text{Aut}_k(\mathbb{P}^2, E)$  by (2), for  $\sigma_p\tau^i \in \text{Aut}_k E$ ,  $\sigma_p\tau^i \in \text{Aut}_k(\mathbb{P}^2, E)$  if and only if  $\sigma_p \in \text{Aut}_k(\mathbb{P}^2, E)$  if and only if  $\sigma_p \in T[3]$  by (1), so  $\text{Aut}_k(\mathbb{P}^2, E) \cong T[3] \rtimes G$ .

□

*Remark 4.13.* Theorem 4.12 (2) depends of the special choice of the identity element  $o \in E$ . In fact, if we choose an arbitrary point  $p \in E$ , then it is hardly the case that  $\text{Aut}_k(E, p) \leq \text{Aut}_k(\mathbb{P}^2, E)$ .

**Lemma 4.14.** *Let  $E$  be an elliptic curve in  $\mathbb{P}^2$ ,  $p \in E$  and  $i \in \mathbb{Z}$ . Then  $\mathcal{A}(E, \sigma_p \tau^i)$  is a geometric algebra of Type EC if and only if  $p \in E \setminus E[3]$ .*

*Proof.* For  $q = (a : b : c) \in E_\lambda$ ,  $q \in E_\lambda \setminus E_\lambda[3]$  if and only if  $abc \neq 0$  if and only if  $\mathcal{A}(E_\lambda, \sigma_q \tau_\lambda^i)$  is a geometric algebra of Type EC by Lemma 4.8, so

$$\begin{aligned} \mathcal{A}(E, \sigma_p \tau^i) &\cong \mathcal{A}(E_\lambda, \Psi^{-1}(\sigma_p \tau^i)) \\ &= \mathcal{A}(E_\lambda, \Psi^{-1}(\sigma_p) \Psi^{-1}(\tau)^i) \\ &= \mathcal{A}(E_\lambda, \sigma_{\psi^{-1}(p)} \tau_\lambda^i) \end{aligned}$$

is a geometric algebra of Type EC if and only if  $\psi^{-1}(p) \in E_\lambda \setminus E_\lambda[3]$  if and only if  $p \in E \setminus E[3]$ .  $\square$

We use the following two formulas.

**Lemma 4.15.** *For  $\sigma_p \tau^i, \sigma_q \tau^j$  and  $\sigma_r \tau^l \in \text{Aut}_k E$ ,*

$$(4.1) \quad (\sigma_q \tau^j)(\sigma_r \tau^l)(\sigma_p \tau^i)^{-1} = \sigma_{q+\tau^j(r)-\tau^{l+j-i}(p)} \tau^{l+j-i},$$

and

$$(4.2) \quad (\sigma_q \tau^j)^{-1}(\sigma_r \tau^l)(\sigma_p \tau^i) = \sigma_{\tau^{-j}(-q+r+\tau^l(p))} \tau^{l+i-j}.$$

*Proof.* By calculations.  $\square$

By Proposition 4.5, for  $\sigma_p \tau^i, \sigma_q \tau^j \in \text{Aut}_k E \cong T \rtimes G$ ,  $\sigma_p \tau^i = \sigma_q \tau^j$  if and only if  $p = q$  in  $E$  and  $i = j$  in  $\mathbb{Z}_d$ ,

**Theorem 4.16.** *Let  $E$  be an elliptic curve in  $\mathbb{P}^2$ ,  $p, q \in E \setminus E[3]$  and  $i, j \in \mathbb{Z}_d$ . Then  $\mathcal{A}(E, \sigma_p \tau^i) \cong \mathcal{A}(E, \sigma_q \tau^j)$  if and only if  $i = j$  and  $q = \tau^l(p) + r$  where  $r \in F_i$  and  $l \in \mathbb{Z}_d$ .*

*Proof.* Since  $\mathcal{A}(E, \sigma_p \tau^i)$  and  $\mathcal{A}(E, \sigma_q \tau^j)$  are geometric algebras of Type EC by Lemma 4.14,  $\mathcal{A}(E, \sigma_p \tau^i) \cong \mathcal{A}(E, \sigma_q \tau^j)$  if and only if there is  $\varphi = \sigma_s \tau^l \in \text{Aut}_k(\mathbb{P}^2, E)$  where  $s \in E[3]$  and  $l \in \mathbb{Z}_d$  such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E \\ \sigma_p \tau^i \downarrow & & \downarrow \sigma_q \tau^j \\ E & \xrightarrow{\varphi} & E \end{array}$$

commutes by Theorem 2.8, that is,

$$(\sigma_q \tau^j)(\sigma_s \tau^l)(\sigma_p \tau^i)^{-1} = \sigma_s \tau^l.$$

By Lemma 4.15 (4.1),  $(\sigma_q \tau^j)(\sigma_s \tau^l)(\sigma_p \tau^i)^{-1} = \sigma_{q+\tau^j(s)-\tau^{l+j-i}(p)} \tau^{l+j-i}$ , so we have  $q + \tau^j(s) - \tau^{l+j-i}(p) = s$  and  $l + j - i = l$ , that is,  $q = \tau^l(p) + s - \tau^i(s)$  and  $i = j$ . By the definition of  $F_i$ ,  $s - \tau^i(s) \in F_i$ , so  $\mathcal{A}(E, \sigma_p \tau^i) \cong \mathcal{A}(E, \sigma_q \tau^j)$  if and only if  $i = j$  and  $q = \tau^l(p) + r$  where  $r \in F_i$  and  $l \in \mathbb{Z}_d$ .  $\square$

By [6], we label the elements of  $E_\lambda[3]$  by

$$\begin{aligned} p_0 &:= o_\lambda := (1 : -1 : 0), & p_1 &:= (1 : -\varepsilon : 0), & p_2 &:= (1 : -\varepsilon^2 : 0), \\ p_3 &:= (1 : 0 : -1), & p_4 &:= (1 : 0 : -\varepsilon), & p_5 &:= (1 : 0 : -\varepsilon^2), \\ p_6 &:= (0 : 1 : -1), & p_7 &:= (0 : 1 : -\varepsilon), & p_8 &:= (0 : 1 : -\varepsilon^2). \end{aligned}$$

We calculate  $F_{\lambda,i} = \{p_l - \tau_\lambda^i(p_l) \in E_\lambda \mid 0 \leq l \leq 8\}$  for each  $i \in \mathbb{Z}_{d_\lambda}$ .

**Lemma 4.17.** (1) *If  $j(E_\lambda) \neq 0, 12^3$ , then*

$$F_{\lambda,i} = \begin{cases} \{p_0\} & \text{if } i = 0, \\ E_\lambda[3] & \text{otherwise.} \end{cases}$$

(2) *If  $\lambda = 0$ , then*

$$F_{\lambda,i} = \begin{cases} \{p_0\} & \text{if } i = 0, \\ \langle p_1 \rangle = \{p_0, p_1, p_2\} & \text{if } i = 2, 4, \\ E_\lambda[3] & \text{otherwise.} \end{cases}$$

(3) *If  $\lambda = 1 + \sqrt{3}$ , then*

$$F_{\lambda,i} = \begin{cases} \{p_0\} & \text{if } i = 0, \\ E_\lambda[3] & \text{otherwise.} \end{cases}$$

*Proof.* By calculations.  $\square$

**Example 4.18.** Fix  $\lambda \in k$  such that  $\lambda^3 \neq 1$  and  $j(E_\lambda) \neq 0, 12^3$  and let  $p = (a : b : c) \in E_\lambda = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$  such that  $abc \neq 0$ . If  $A = \mathcal{A}(E_\lambda, \sigma_p)$ , then

$$A = k\langle x, y, z \rangle / (ayz + bzy + cx^2, azx + bxz + cy^2, axy + byx + cz^2),$$

and  $A$  is a 3-dimensional Sklyanin algebra. If  $A' = \mathcal{A}(E_\lambda, \sigma_{-p})$  where  $-p = (b : a : c)$ , then

$$A' = k\langle x, y, z \rangle / (byz + azy + cx^2, bzx + axz + cy^2, bxy + ayx + cz^2),$$

and  $A'$  is also a 3-dimensional Sklyanin algebra. If  $A'' = \mathcal{A}(E_\lambda, \sigma_p \tau_\lambda)$ , then

$$A'' = k\langle x, y, z \rangle / (axz + bzy + cyx, azx + byz + cxy, ay^2 + bx^2 + cz^2).$$

If  $A''' = \mathcal{A}(E_\lambda, \sigma_{p+p_3} \tau_\lambda)$  where  $p_3 := (1 : 0 : -1) \in E_\lambda[3]$ , then

$$A''' = k\langle x, y, z \rangle / (bxz + czy + ayx, bzx + cyz + axy, by^2 + cx^2 + az^2).$$

By Theorem 4.16, since  $-p = \tau_\lambda(p)$  and  $p_3 \in E_\lambda[3] = F_{\lambda,1}$ ,  $A \cong A'$  and  $A'' \cong A'''$  but no other pairs are isomorphic.

**4.4. Classification up to graded Morita equivalence.** We recall that  $o := \psi(o_\lambda)$  and  $\tau := \Psi(\tau_\lambda) \in G = \text{Aut}_k(E, o)$ . Since  $\tau$  is also a group automorphism of  $(E, o)$ , it follows that  $\tau(E[3]) = E[3]$ .

**Lemma 4.19.** *For  $p \in E$  and  $l \in \mathbb{Z}$ , if  $p - \tau^l(p) \in E[3]$ , then  $p - \tau^{nl}(p) \in E[3]$  for any  $n \in \mathbb{Z}$ .*

*Proof.* If  $n = 0$ , then  $p - \tau^{nl}(p) = p - p = o \in E[3]$ .

For any  $n \geq 1$ , we can write

$$p - \tau^{nl}(p) = \sum_{i=0}^{n-1} \tau^{il}(p - \tau^l(p)).$$

Since  $p - \tau^l(p) \in E[3]$ ,  $\tau^{il}(p - \tau^l(p)) \in E[3]$  for  $1 \leq i \leq n-1$ , so  $p - \tau^{nl}(p) \in E[3]$ .

If  $n \leq -1$ , then  $p - \tau^{nl}(p) = -\tau^{nl}(p - \tau^{-nl}(p))$ . Since  $-n \geq 1$  and  $p - \tau^{-nl}(p) \in E[3]$ , it follows that  $p - \tau^{nl}(p) \in E[3]$  for any  $n \leq -1$ .  $\square$

**Theorem 4.20.** *Let  $p, q \in E \setminus E[3]$  and  $i, j \in \mathbb{Z}_d$ . Then  $\text{GrMod } \mathcal{A}(E, \sigma_p \tau^i) \cong \text{GrMod } \mathcal{A}(E, \sigma_q \tau^j)$  if and only if  $p - \tau^{j-i}(p) \in E[3]$  and there exist  $r \in E[3]$  and  $l \in \mathbb{Z}_d$  such that  $q = \tau^l(p) + r$ .*

*Proof.* Suppose that  $\text{GrMod } \mathcal{A}(E, \sigma_p \tau^i) \cong \text{GrMod } \mathcal{A}(E, \sigma_q \tau^j)$ . Since  $\mathcal{A}(E, \sigma_p \tau^i)$  and  $\mathcal{A}(E, \sigma_q \tau^j)$  are geometric algebras of Type EC by Lemma 4.14, there exists a sequence  $\{\phi_n\}_{n \in \mathbb{Z}}$  of  $\text{Aut}_k(\mathbb{P}^2, E)$  such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi_n} & E \\ \sigma_p \tau^i \downarrow & & \downarrow \sigma_q \tau^j \\ E & \xrightarrow{\phi_{n+1}} & E \end{array}$$

commutes for  $n \in \mathbb{Z}$  by Theorem 2.9. By Theorem 4.12 (3), there exist  $r \in E[3]$  and  $l \in \mathbb{Z}_d$  such that  $\phi_0 = \sigma_r \tau^l$ . Since the diagrams

$$\begin{array}{ccc} E & \xrightarrow{\phi_{-1}} & E & & E & \xrightarrow{\sigma_r \tau^l} & E \\ \sigma_p \tau^i \downarrow & & \downarrow \sigma_q \tau^j & & \sigma_p \tau^i \downarrow & & \downarrow \sigma_q \tau^j \\ E & \xrightarrow{\sigma_r \tau^l} & E & & E & \xrightarrow{\phi_1} & E \end{array}$$

commute,

$$\begin{aligned} \phi_{-1} &= (\sigma_q \tau^j)^{-1} (\sigma_r \tau^l) (\sigma_p \tau^i) \\ &= \sigma_{\tau^{-j}(-q+r+\tau^l(p))} \tau^{l+i-j} \end{aligned}$$

and

$$\begin{aligned}\phi_1 &= (\sigma_q \tau^j)(\sigma_r \tau^l)(\sigma_p \tau^i)^{-1} \\ &= \sigma_{q+\tau^j(r)-\tau^{l+j-i}(p)} \tau^{l+j-i}\end{aligned}$$

by Lemma 4.15. Since  $\phi_{-1}, \phi_1 \in \text{Aut}_k(\mathbb{P}^2, E)$ , we have

$$\begin{aligned}\tau^{-j}(-q + r + \tau^l(p)) &\in E[3], \\ q + \tau^j(r) - \tau^{l+j-i}(p) &\in E[3],\end{aligned}$$

that is,

$$\begin{aligned}s &:= -q + r + \tau^l(p) = \tau^j(\tau^{-j}(-q + r + \tau^l(p))) \in E[3], \\ t &:= q + \tau^j(r) - \tau^{l+j-i}(p) \in E[3].\end{aligned}$$

By the first condition, we have  $q = \tau^l(p) + r - s$  where  $r - s \in E[3]$ . Since  $s + t = r + \tau^j(r) + \tau^l(p) - \tau^{l+j-i}(p) \in E[3]$ , we have

$$\begin{aligned}p - \tau^{j-i}(p) &= \tau^{-l}(\tau^l(p) - \tau^{l+j-i}(p)) \\ &= \tau^{-l}(s + t - r - \tau^j(r)) \in E[3].\end{aligned}$$

Conversely, suppose that  $p - \tau^{j-i}(p) \in E[3]$  and  $q = \tau^l(p) + r$  where  $r \in E[3]$  and  $l \in \mathbb{Z}_d$ . By Theorem 4.16, we have

$$\mathcal{A}(E, \sigma_q \tau^j) = \mathcal{A}(E, \sigma_{\tau^l(p)+r} \tau^j) = \mathcal{A}(E, \sigma_{\tau^l(p+\tau^{-l}(r))} \tau^j) \cong \mathcal{A}(E, \sigma_{p+\tau^{-l}(r)} \tau^j).$$

To show

$$\text{GrMod } \mathcal{A}(E, \sigma_p \tau^i) \cong \text{GrMod } \mathcal{A}(E, \sigma_q \tau^j),$$

it is enough to show

$$\text{GrMod } \mathcal{A}(E, \sigma_p \tau^i) \cong \text{GrMod } \mathcal{A}(E, \sigma_{p+s} \tau^j)$$

where  $s = \tau^{-l}(r) \in E[3]$ . Since  $p \in E \setminus E[3]$ ,  $p + s \in E \setminus E[3]$ , so  $\mathcal{A}(E, \sigma_p \tau^i)$  and  $\mathcal{A}(E, \sigma_{p+s} \tau^j)$  are geometric algebras of Type EC by Lemma 4.14. We construct a sequence of automorphisms  $\{\phi_n\}_{n \in \mathbb{Z}}$  of  $\text{Aut}_k(\mathbb{P}^2, E)$ . We set  $\phi_0 := \text{id}_E$ . For each  $n \geq 1$ , we define  $\phi_n$  inductively as

$$\phi_n := \sigma_{r_n} \tau^{n(j-i)},$$

where  $r_n := p - \tau^{n(j-i)}(p) + s + \tau^j(r_{n-1})$  and  $r_0 := o$ . For any  $n \geq 0$ , if  $r_n \in E[3]$ , then  $r_{n+1} := p - \tau^{(n+1)(j-i)}(p) + s + \tau^j(r_n) \in E[3]$  by Lemma 4.19.

Next, for  $n \leq -1$ , we construct automorphisms  $\phi_n \in \text{Aut}_k(\mathbb{P}^2, E)$ . For each  $n \geq 1$ , we define  $\phi_{-n}$  inductively as

$$\phi_{-n} := \sigma_{r_{-n}} \tau^{-n(j-i)},$$



where  $r_{-n} := \tau^{(n-1)i-nj}(p - \tau^{(n-1)(j-i)}(p) + \tau^{(n-1)(j-i)}(-s + r_{-(n-1)}))$  and  $r_0 := o$ . For any  $n \geq 0$ , if  $r_{-n} \in E[3]$ , then  $r_{-(n+1)} := \tau^{ni-(n+1)(j-i)}(p - \tau^{n(j-i)}(p) + \tau^{n(j-i)}(-s + r_{-n})) \in E[3]$  by Lemma 4.19. By this construction, we have the sequence of automorphisms  $\{\phi_n\}_{n \in \mathbb{Z}}$  of  $\text{Aut}_k(\mathbb{P}^2, E)$  such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi_n} & E \\ \sigma_p \tau^i \downarrow & & \downarrow \sigma_{p+s} \tau^j \\ E & \xrightarrow{\phi_{n+1}} & E \end{array}$$

commutes for each  $n \in \mathbb{Z}$ , so  $\text{GrMod } \mathcal{A}(E, \sigma_p \tau^i) \cong \text{GrMod } \mathcal{A}(E, \sigma_{p+s} \tau^j)$  by Theorem 2.9.  $\square$

**Example 4.21.** We use the same graded algebras  $A$  and  $A''$  as in Example 4.18 so that  $A \not\cong A''$ . It follows from Theorem 4.20 that  $\text{GrMod } A \cong \text{GrMod } A''$  if and only if  $2p = p - \tau_\lambda(p) \in E_\lambda[3]$  if and only if  $p \in E_\lambda[6]$ . From [6], we see that  $|E_\lambda[6]| = 36$ , so  $A$  and  $A'' = A^{\phi_\lambda}$  are rarely graded Morita equivalent (see Remark 2.4).

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