Central limit theorems for non-symmetric random walks on covering graphs

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Contents

Chapter 1

Introduction

Random walks are one of the most fundamental classes of stochastic processes and well-studied topics in harmonic analysis, geometry, graph theory and group theory, to say nothing of probability theory. These are defined to be time-homogeneous Markov chains whose transition probability is adapted to the structures of the underlying state space. From the probabilistic and geometric perspectives, many authors have been tried to study *long time asymptotics* of random walks in various settings. In particular, a *central limit theorem* (CLT), that is, a generalization of the Laplace–de Moivre theorem, must be a central problem and is studied intensively and extensively. Roughly speaking, the CLT asserts that the limiting distribution of random walks under an appropriate scaling of space and time is nothing but the normal distribution. Furthermore, a *functional CLT* (*Donsker's invariance principle*) is well-known as a stronger assertion and it means that the distribution of a corresponding rescaled path-valued process converges to that of Brownian motion. These mathematical backgrounds basically motivate author's study. For the classical results on random walks, see Spitzer [66]. We refer to Woess [79] for rich results on random walks on infinite state spaces with extensive references therein. See also Lawler–Limic [48] for relation between random walks and potential theory and Barlow [5] for properties of heat kernels of random walks.

Our main concerns of this thesis are long time asymptotics of random walks on infinite graphs. In particular, we pay much attention to geometric features of the graph such as the *periodicity* and the *volume growth*, which play important role to obtain the asymptotics (see e.g., Spitzer [66] and Woess [79]). A *covering graph* of a finite graph, which is a discrete analogue of covering spaces, is a basic and typical example equipped with the above two geometric features. In this study, we usually employ ideas from the method of homogenization. Generally speaking, homogenization theory is a method which relates a periodic system to the corresponding homogenized system through a scaling relation between the time and the underlying state space (cf. Bensoussan–Lions–Papanicolaou [8]). However, since the notion of the scale change on graphs is not defined, it is not possible to apply this method directly to the case where the underlying space is an infinite graph. Therefore, it is necessary to find a realization of the graph, preserving the geometric features, in a space on which a scaling is defined.

We now focus on an infinite graph which is equipped with the periodicity. A typical example of such infinite graphs is a *crystal lattice*, that is, a covering graph X of a finite graph X_0 whose covering transformation group Γ is finitely generated and abelian. It is regarded as a generalization of the square lattice, the triangular lattice, the hexagonal lattice, the dice lattice and so on (see Figure 1.1). We remark that the crystal lattice has

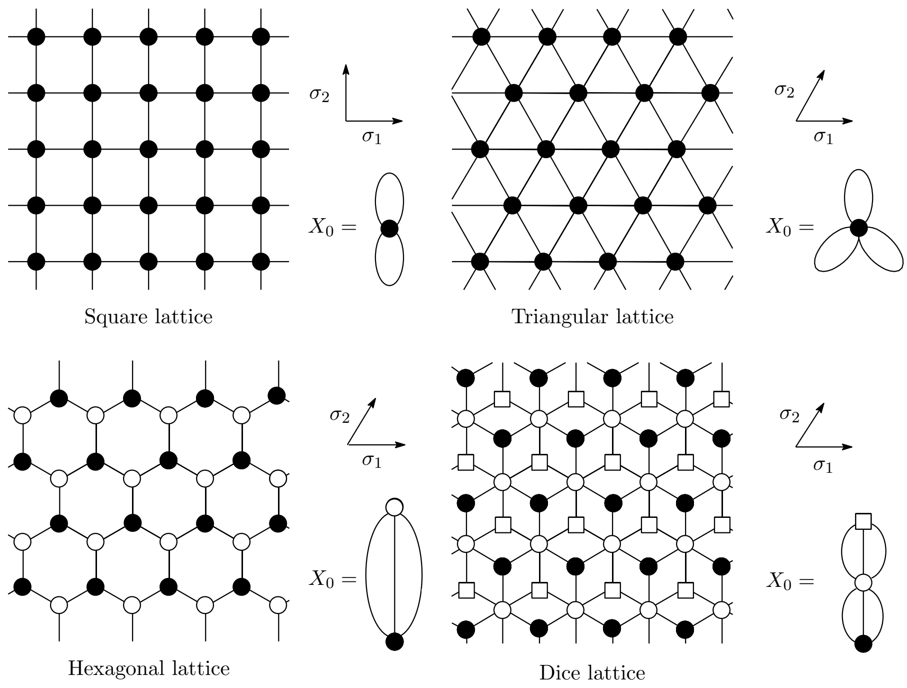

Figure 1.1: Crystal lattices with the covering transformation group $\Gamma = \langle \sigma_1, \sigma_2 \rangle \cong \mathbb{Z}^2$

inhomogeneous local structures though it has a periodic global structure. Let us briefly review the history of the study of random walks on crystal lattices. In Kotani–Shirai–Sunada [43], an asymptotic behavior of the n-step transition probability of symmetric random walks on crystal lattices was obtained. As mentioned above, there is an essential difficulty to establish CLTs for random walks on crystal lattices, because such a graph does not have any appropriate spatial scaling. In order to overcome this difficulty, Kotani and Sunada [41] introduced the notion of *standard realization* of a crystal lattice X, which is a discrete harmonic map Φ_0 from X into the Euclidean space $\Gamma \otimes \mathbb{R}$ equipped with the *Albanese metric* associated with the given transition probability. It characterizes an equilibrium configuration of X in a geometric point of view. In Kotani–Sunada [40], they discussed the relation between the standard realization of X and the CLT for symmetric random walks on X. As the scaling limit, they captured a homogenized Laplacian on $\Gamma \otimes \mathbb{R}$. In terms of probability theory, it means that, for fixed $0 \leq t \leq 1$, a sequence of $\Gamma \otimes \mathbb{R}$-valued random variables $\{n^{-1/2}\Phi_0(w_{[nt]})\}_{n=1}^{\infty}$ converges to B_t as $n \to \infty$ in law. Here $\{w_n\}_{n=0}^{\infty}$ is the given symmetric random walk on X and $(B_t)_{0 \leq t \leq 1}$ is a standard

Brownian motion on $\Gamma \otimes \mathbb{R}$ equipped with the Albanese metric. In their proof, both the symmetry of the given random walk $\{w_n\}_{n=0}^{\infty}$ and the harmonicity of the realization Φ_0 play an important role to show the convergence of the sequence of infinitesimal generators associated with $\{n^{-1/2}\Phi_0(w_{[nt]}) : 0 \le t \le 1\}_{n=1}^{\infty}$. Indeed, these properties are effectively used to delete a diverging drift term as $n \to \infty$ from the homogenized Laplacian. See also Kotani [38] for the proof of CLT for magnetic transition operator on X via this technique. Moreover, Kotani and Sunada [42] obtained the large deviation principle (LDP) for random walks on X (see also Section 2.6). Among these papers, they developed a hybrid field of several traditional disciplines including graph theory, geometry, discrete group theory and probability theory. Since this new field, called *discrete geometric analysis*, was introduced by Sunada, it has been making new interactions with many other fields. For example, Le Jan employed discrete geometric analysis effectively in a series of recent studies of Markov loops (see e.g., [49, 50]). We refer to Sunada [70, 71] for recent progress of discrete geometric analysis.

On the other hand, it turns out that the notion of volume growth affects the long time asymptotics of random walks on finitely generated groups or Cayley graphs of them. Suppose a finitely generated group Γ with the generating set $\{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \dots, \gamma_\ell^{\pm 1}\}$ satisfies

$$\#\{\gamma_{k_1}^{\varepsilon_1}\gamma_{k_2}^{\varepsilon_2}\cdots\gamma_{k_n}^{\varepsilon_n} \mid k_i = 1, 2, \dots, \ell,\ \varepsilon_i = 1, -1,\ i = 1, 2, \dots, n\} \le C \cdot V(n) \qquad (n \in \mathbb{N})$$

for some constant $C > 0$ and some function $V(n)$. If $V(n) \le n^d\,(n \in \mathbb{N})$ for some $d \in \mathbb{N}$, then we call Γ a group of *polynomial volume growth*. Otherwise, we call it a group of *superpolynomial volume growth*. Generally, it is difficult to characterize a finitely generated group itself in terms of its volume growth. For example, all non-amenable groups have *exponential* volume growth, however there are also many amenable groups of exponential volume growth. In fact, this kind of difficulty comes from the diversity and complexity of the algebraic structures of finitely generated groups. We refer to Saloff-Coste [65] for basic problems and results for random walks on such groups including the case of superpolynomial volume growth. On the contrary, there is a remarkable theorem on a group of polynomial volume growth due to Gromov, which asserts that it is essentially characterized as a nilpotent group (cf. Gromov [25] and Ozawa [59]). Hence, we find a large number of papers on long time asymptotics of symmetric random walks on state spaces with a nilpotent structure. We refer to Wehn [78], Tutubalin [75] and Stroock–Varadhan [68] for related early works, Raugi [63], Pap [61], Watkins [77] and Alexopoulos [3] for CLTs for centered random walks on nilpotent Lie groups. See also Breuillard [10] for an overview of random walks on Lie groups with extensive references. For local CLTs on nilpotent Lie groups, Alexopoulos [1, 2], Breuillard [11], Diaconis–Hough [17] and Hough [28] may be consulted.

In view of these developments, it is natural to ask whether the long time asymptotic of random walks on a covering graph X whose covering transformation group Γ is a finitely generated group of polynomial volume growth is obtained or not. The graph X is regarded as a generalization of a crystal lattice or the Cayley graph of a finitely generated group of polynomial volume growth. A typical example of such Γ is the 3-dimensional (3D) discrete Heisenberg group $\Gamma = \mathbb{H}^3(\mathbb{Z})$ (see Figure 1.2). Thanks to Gromov's theorem mentioned above, Γ has a finite extension of a torsion free nilpotent subgroup $\widetilde{\Gamma} \lhd \Gamma$. Therefore, X is regarded as a covering graph of the finite quotient graph $\widetilde{\Gamma}\backslash X$ whose covering transformation group is $\widetilde{\Gamma}$. Hence, we may assume that X is a covering graph of a finite graph X_0 whose covering transformation group Γ is a finitely generated, torsion free nilpotent group of step r ($r \in \mathbb{N}$) without loss of generality. We now mention a few related works on long time asymptotics of random walks on a Γ-*nilpotent covering graph* X. Ishiwata [29] discussed symmetric random walks on X and extended the notion of standard realization of crystal lattices to the nilpotent case, so that the similar problems to the case of crystal lattices could be considered. As a result, a semigroup CLT was obtained through the standard realization Φ_0 of X into a nilpotent Lie group $G = G_\Gamma$ such that Γ is isomorphic to a cocompact lattice in G equipped with a scalar multiplication called a one-parameter family of canonical *dilations* $(\tau_\varepsilon)_{\varepsilon>0}$ (cf. Malcév [56]). More precisely, he captured the homogenized sub-Laplacian on G associated with the Albanese metric on $\mathfrak{g}^{(1)}$ as the CLT-scaling limit. Here $\mathfrak{g}^{(1)}$ stands for the generating part of the Lie algebra $\mathfrak{g}$ of G. We note that the diverging drift term appears only in $\mathfrak{g}^{(1)}$-direction due to the basic property of the dilation operator. Hence, it is sufficient to introduce the notion of harmonicity of the realization Φ_0 only on $\mathfrak{g}^{(1)}$ for proving the CLT in the nilpotent case. In spite of such developments, long time asymptotics of *non-symmetric* random walks on nilpotent covering graphs have not been studied sufficiently though an LDP on X was obtained in Tanaka [72] (see also Section 2.6).

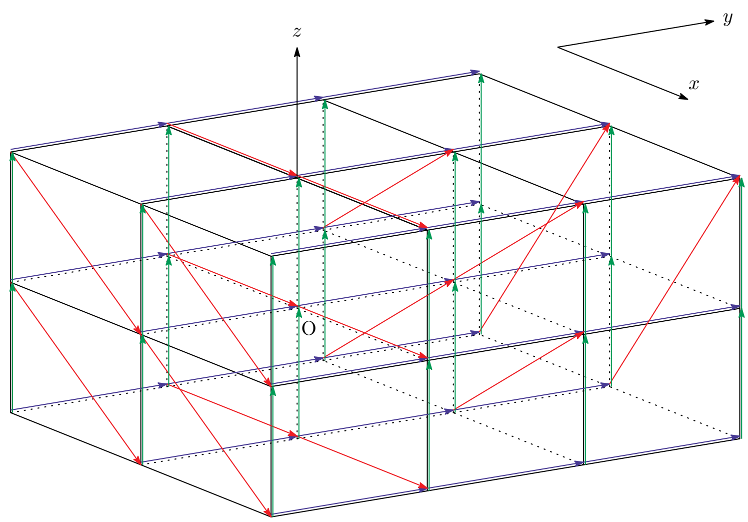

Figure 1.2: A part of the Cayley graph of $\Gamma = \mathbb{H}^3(\mathbb{Z})$

If we consider the non-symmetric case, the same method as the symmetric case does not work well for proving CLTs because the diverging drift term arising from the non-symmetry of the given random walk does not vanish. To overcome this difficulty, Ishiwata, Kawabi and Kotani [31] introduced two kinds of schemes for proving functional CLTs (FCLTs) for a non-symmetric random walk $\{w_n\}_{n=0}^{\infty}$ on a crystal lattice X. One is to replace the usual transition operator by the *transition-shift operator*, which "deletes" the diverging drift term. Combining this scheme with a modification of the harmonicity of the realization Φ_0, they proved that a sequence $\big\{n^{-1/2}\big(\Phi_0(w_{[nt]}) - [nt]\rho_{\mathbb{R}}(\gamma_p)\big); 0 \leq t \leq 1\big\}_{n=1}^{\infty}$ converges in law to a $\Gamma \otimes \mathbb{R}$-valued standard Brownian motion $(B_t)_{t\geq 0}$ as $n \to \infty$. Here $\rho_{\mathbb{R}}(\gamma_p) \in \Gamma \otimes \mathbb{R}$ is the so-called *asymptotic direction* which appears in the law of large numbers for the random walk $\{\Phi_0(w_n)\}_{n=0}^{\infty}$ on $\Gamma \otimes \mathbb{R}$ (see Proposition 2.5.1). The other is to introduce a one-parameter family of $\Gamma \otimes \mathbb{R}$-valued random walks $(\xi^{(\varepsilon)})_{0\leq\varepsilon\leq 1}$ which "weakens" the diverging drift term, where this family interpolates the original non-symmetric random walk $\xi_n^{(1)} := \Phi_0(w_n)$ $(n = 0, 1, 2, \ldots)$ and the symmetrized one $\xi^{(0)}$. Putting $\varepsilon = n^{-1/2}$ and letting $n \to \infty$, we capture a drifted Brownian motion $(B_t + \rho_{\mathbb{R}}(\gamma_p)t)_{0\leq t\leq 1}$ as the limit of a sequence $\big\{n^{-1/2}\xi_{[nt]}^{(n^{-1/2})}; 0 \leq t \leq 1\big\}_{n=1}^{\infty}$. See Trotter [74] for related early works. It is worth mentioning that this scheme is well-known in the study of the hydrodynamic limit of weakly asymmetric exclusion processes. See e.g., Kipnis–Landim [36], Tanaka [72] and references therein. In Alexopoulos [2], a non-centered random walk on a finitely generated group of polynomial volume growth Γ is discussed. For the same reason as above, in the non-centered case, the diverging drift term prevents us from obtaining CLTs. He introduced another kind of scheme to avoid this problem. It is to establish a measure-change formula for the given non-centered transition probability on Γ, to "change" the situation into the drftless one. We note that it corresponds to a kind of Girsanov's formula on Γ. As an application of this scheme, he proved a CLT and a generalization of the Berry–Esseen type estimate for non-centered random walks on Γ.

The main purpose of this thesis is to investigate long time asymptotics of non-symmetric random walks on covering graphs in view of the three schemes explained above. We now state frameworks and results with the organization of this thesis.

Chapter 2: We lay the foundations that will be needed in all subsequent chapters. We give several definitions, notations and properties of graphs and random walks, as well as those of function spaces on a metric space in Section 2.1. We review basic materials on nilpotent Lie groups and corresponding Lie algebras in Section 2.2. In particular, the notion of limit group of a nilpotent Lie group is introduced, which is defined by a certain deformation of the original Lie-group product through the dilation operator. Note that it plays a very important role to establish main results in Chapters 4 and 5. Section 2.3 concerns with two notions on nilpotent Lie groups. One is the Carnot–Carathéodory metric, which is an intrinsic metric appeared in the context of sub-Riemannian geometry. The other is homogeneous norms, which is compatible with dilations and behaves like a "norm" on G. In Section 2.4, we summarize the theory of discrete geometric analysis on finite graphs which was developed by Kotani and Sunada. After that, we apply the theory to introduce the notion of modified harmonic realization of both a crystal lattice and a nilpotent covering graph (Definitions 2.4.5 and 2.4.6). As is well-known, there is an important relation between the notion of martingale and that of harmonicity. In Section

2.5, such relations for Markov chains with values in both a crystal lattice and a nilpotent covering graph are clearly stated (Lemmas 2.5.1 and 2.5.3). Finally, in Section 2.6, we summarize the known results on LDP on covering graphs due to Kotani–Sunada [42, 39] and Tanaka [72], with a relation between the LDPs and geometric aspects such as the Gromov–Hausdorff limit of scaled covering graphs.

Chapter 3: The content of this chapter is based on author's paper [58], discussing a measure-change formula for non-symmetric random walks on a Γ-crystal lattice X. In Section 3.1, we establish the measure-change formula by using a variational method due to Alexopoulos [2]. We introduce a function $F : X_0 \times \mathrm{Hom}(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R}$ by (3.1.1), where $X_0 = \Gamma \backslash X$ is the quotient graph. We show that, for a fixed vertex $x \in X_0$, there exists a unique minimizer $\lambda_* = \lambda_*(x) \in \mathrm{Hom}(\Gamma, \mathbb{R})$ of the function F (Lemma 3.1.1). By using this minimizer $\lambda_*(x)$, we then construct a new transition probability $\mathfrak{p}$ on the crystal lattice such that it is still non-symmetric but the asymptotic direction $\rho_{\mathbb{R}}(\gamma_{\mathfrak{p}})$ vanishes (see (3.1.4) for the definition). This means that, under the new transition probability $\mathfrak{p}$, the modified harmonic realization Φ_0 is regarded as the harmonic realization. We apply the measure-change formula to give yet another approach to the proof of CLTs (Lemma 3.2.3 and Theorem 3.2.1) for non-symmetric random walks on a crystal lattice in Section 3.2. More precisely, we show that, in a Hölder space over $\Gamma \otimes \mathbb{R}$, a sequence $\{n^{-1/2}\Phi_0(w^{(\mathfrak{p})}_{[nt]}) \,:\, 0 \leq t \leq 1\}_{n=1}^{\infty}$ converges in law to a $\Gamma \otimes \mathbb{R}$-valued standard Brownian motion $(B_t)_{0 \leq t \leq 1}$ as $n \to \infty$. Here $\{w^{(\mathfrak{p})}_n\}_{n=0}^{\infty}$ is the random walk on X governed by the changed transition probability $\mathfrak{p}$. In the proof, the diverging drift term vanishes thanks to the ($\mathfrak{p}$-)harmonicity of the realization Φ_0. Moreover, an asymptotic relation between the given n-step transition probability and the changed one is also discussed (see Theorem 3.2.5). The measure-change formula is regarded as a discrete analogue of Girsanov's formula, which is well-studied in stochastic analysis. Indeed, in Fujita [23], a discrete Girsanov's formula for non-symmetric random walks on $\mathbb{Z}^1$ was established. We discuss a relation between our formula and the above Girsanov's formula in the case where the quotient graph is a bouquet graph in Section 3.3.

Chapter 4: This chapter is based on author's paper [32], which is jointwork with Satoshi Ishiwata and Hiroshi Kawabi. We establish CLTs for non-symmetric random walks on a Γ-nilpotent covering graph X by using the transition-shift scheme mentioned above. We give settings and statements of main results in Section 4.1. Let $\Phi_0 : X \longrightarrow G = G_{\Gamma}$ be the modified standard realization of X, where the Lie algebra $\mathfrak{g}$ of G is equipped with the Albanese metric. Since the modified harmonicity of Φ_0 is defined only on $\mathfrak{g}^{(1)}$, we remark that the modified harmonic realization Φ_0 has the ambiguity except for the component corresponding to $\mathfrak{g}^{(1)}$. Through the map Φ_0, in Section 4.2, we obtain a semigroup CLT (Theorem 4.1.2), which means that the n-th iteration of the "transition shift operator" converges to a diffusion semigroup on G as $n \to \infty$ with a suitable scale change on G. The infinitesimal generator $-\mathcal{A}$ of the diffusion semigroup is the homogenized sub-Laplacian with a non-trivial $\mathfrak{g}^{(2)}$-valued drift $\beta(\Phi_0)$ arising from the non-symmetry of the given random walk, where $\mathfrak{g}^{(2)} := [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]$. The drift $\beta(\Phi_0)$ seems to depend on the choice of a modified harmonic realization Φ_0 due to the $\mathfrak{g}^{(2)}$-ambiguity mentioned above. On the contrary, we show that it is independent of the choice of Φ_0 (Proposition 4.2.3). Furthermore, by imposing an additional natural condition **(C)**, we prove an FCLT in a Hölder space over G (Theorem 4.1.3) in Section 4.3. Note that the FCLT is much stronger than Theorem 4.1.2. Roughly speaking, we capture a G-valued diffusion process associated with $-\mathcal{A}$ through the CLT-scaling limit of the non-symmetric random walk on X. We call the condition **(C)** the *centered condition*. As is emphasized in Breuillard [10, Section 6], the situation of the non-centered case is quite different from the centered case and thus some technical difficulties arise to obtain CLTs. That is why there are few papers which discuss CLTs for non-centered random walks on nilpotent Lie groups. We obtain, in Theorem 4.1.2, a semigroup CLT for the non-centered random walk $\{\xi_n := \Phi_0(w_n)\}_{n=0}^{\infty}$ on G with a canonical dilation $\tau_{n^{-1/2}}$, while Crépel–Raugi [15] and Raugi [63] proved similar CLTs for the random walk to (4.1.6) with spatial scalings whose orders are higher than $\tau_{n^{-1/2}}$. On the other hand, we need to assume the centered condition **(C)** to obtain an FCLT (Theorem 4.1.3) for $\{\xi_n\}_{n=0}^{\infty}$ in the Hölder topology, which is stronger than the uniform topology. In Section 4.4, we extend the measure-change method established in Section 3 to the nilpotent case and establish a CLT and an FCLT (Theorems 4.4.2 and 4.4.3) as generalizations of Theorems 4.1.2 and Theorem 4.1.3.

Let us give another motivation of this study from rough path theory, which will be discussed in Section 4.5. It is known that rough path theory was initiated by Lyons in [54] to discuss line integrals and ordinary differential equations (ODEs) driven by an irregular path such as a sample path of Brownian motion $B = (B_t)_{0 \le t \le 1}$ on $\mathbb{R}^d$. Rough path theory makes us possible to handle a Stratonovich type stochastic differential equation (SDE) driven by Brownian motion B as a deterministic ODE driven by standard *Brownian rough path* (i.e., Stratonovich enhanced Brownian motion) $\mathbf{B} = (B, \mathbb{B})$, where $\mathbf{B}$ is a couple of Brownian motion B itself and its Stratonovich iterated integral $\mathbb{B}$. Thus, rough path theory provides a new insight to the usual SDE-theory and it has developed rapidly in stochastic analysis. For more details on an overview of rough path theory and its applications to stochastic analysis, see Lyons–Qian [55], Friz–Victoir [22] and Friz–Hairer [19]. In the rough path framework, several authors have studied Donsker-type invariance principles. Among them, Breuillard–Friz–Huesmann [12] first studied this problem for Brownian rough path. Namely, they captured Stratonovich enhanced Brownian motion $\mathbf{B} = (B, \mathbb{B})$ on $\mathbb{R}^d$ as the usual CLT-scaling limit of the natural rough path lift of an $\mathbb{R}^d$-valued random walk with the centered condition. We also refer to Bayer–Friz [6] for applications to cubature and Chevyrev [14] for a recent study on an extension to the case of Lévy processes. Here we should note that there are good approximations to Brownian motion which do not converge to $\mathbf{B}$ but instead to a *distorted Brownian rough path* $\overline{\mathbf{B}} = (B, \mathbb{B} + \boldsymbol{\beta})$, where $\boldsymbol{\beta}$ is an anti-symmetric perturbation of $\mathbb{B}$. For example, Friz–Gassiat–Lyons [18] constructed such a rough path called *magnetic Brownian rough path* as the small mass limit of the natural rough path lift of a physical Brownian motion on $\mathbb{R}^d$ in a magnetic field. Through this approximation, they showed an effect of the magnetic field appears explicitly in the anti-symmetric perturbation term $\boldsymbol{\beta}$. See also e.g., Lejay–Lyons [51] and Friz–Oberhauser [21] for related results on this topic. In view of the background described above, we discuss a random walk approximation of the distorted Brownian rough path $\overline{\mathbf{B}}$ from a perspective of discrete geometric analysis. Since the unique *Lyons extension* of $\overline{\mathbf{B}}$ of order r $(r \geq 2)$ can be regarded as a diffusion process on a free step-r nilpotent Lie group $\mathbb{G}^{(r)}(\mathbb{R}^d)$, we obtain such a diffusion process in Corollary 4.5.4 through the CLT-scaling limit of a non-symmetric random walk on a nilpotent covering graph X as a direct application of Theorem 4.1.3. Besides, we observe that the non-symmetry of the random walk on X affects the anti-symmetric perturbation term of $\overline{\mathbb{B}}$ explicitly. Recently, Lopusanschi–Simon [53] and Lopusanschi–Orenshtein [52] proved a similar invariance principle for $\overline{\mathbf{B}}$ to ours. However, they did not discuss an explicit relation between the perturbation term, called the *area anomaly*, and the non-symmetry of the given random walk. In view of that, Corollary 4.5.4 gives a new approach to such an invariance principle in that we pay much attention to the non-symmetry of random walks on X.

Finally, in Section 4.6, we concern with an FCLT for a non-symmetric random walk $\{w_n\}_{n=0}^{\infty}$ on X through a non-harmonic realization $\Phi : X \longrightarrow G$, though the modified harmonicity of realizations play an important role in the proof of the FCLT (Theorem 4.1.3). We employ the so-called *corrector method*, which is often used in the study of invariance principles on random media (see e.g., Kumagai [45]). By noting the definition of the ($\mathfrak{g}^{(1)}$-)modified harmonic realization Φ_0, we introduce the $\mathfrak{g}^{(1)}$-corrector of a non-harmonic realization Φ by the difference of $\mathfrak{g}^{(1)}$-components of Φ and Φ_0. In fact, we notice that this corrector is easy to estimate thanks to the periodicity of these realizations. By using the estimation, we show that, under the centered condition, the sequence of stochastic processes given by the geodesic interpolation of the G-valued scaled random walk $\{\tau_{n^{-1/2}}\Phi(w_k)\}_{k=0}^{n}$ also converges to the same diffusion as captured in Theorem 4.1.3. See Theorem 4.6.2 for details.

Chapter 5: This chapter is based on author's paper [33], which is jointwork with Satoshi Ishiwata and Hiroshi Kawabi. As a continuation of Section 4, we study another kind of CLTs for a non-symmetric random walk $\{w_n\}_{n=0}^{\infty}$ on a Γ-nilpotent covering graph X by applying the scheme for weakening the diverging drift term. Settings and statements of main results are given in Section 5.1. We first introduce a one-parameter family of transition probabilities $(p_\varepsilon)_{0\leq\varepsilon\leq1}$ on X as the linear interpolation between the given transition probability $p_1 := p$ and the symmetrized one p_0, that is, $p_\varepsilon := p_0 + \varepsilon(p - p_0)$ $(0 \leq \varepsilon \leq 1)$. For each ε, we take a modified harmonic realization $\Phi_0^{(\varepsilon)} : X \longrightarrow G$ associated with the transition probability p_ε, and define a one-parameter family of G-valued random walks $(\xi^{(\varepsilon)})_{0\leq\varepsilon\leq1}$ by $\xi_n^{(\varepsilon)} := \Phi_0^{(\varepsilon)}(w_n)$ $(n = 0, 1, 2, \ldots)$. In Section 5.2, several properties of the family of modified harmonic realizations $(\Phi_0^{(\varepsilon)})_{0\leq\varepsilon\leq1}$ are discussed. In the proof of a main result (Theorem 5.1.1), a $\mathfrak{g}^{(2)}$-valued drift $\beta_{(\varepsilon)}(\Phi_0^{(\varepsilon)})$, which is like $\beta(\Phi_0)$ in Section 4, will appear in the limiting infinitesimal generator and we need to know the behavior of it as $\varepsilon \searrow 0$. We show that the sequence of $\mathfrak{g}^{(2)}$-valued drift vanishes as $\varepsilon \searrow 0$ under a natural condition **(A1)**. See Proposition 5.2.1. As a result, by putting $\varepsilon = n^{-1/2}$ and letting $n \to \infty$, we prove a CLT (Theorem 5.1.1) for the family of G-valued random walks $\{\xi^{(n^{-1/2})}\}_{n=1}^{\infty}$ in Section 5.3. Furthermore, in Section 5.4, we show that a sequence $\big\{\tau_{n^{-1/2}}\big(\xi_{[nt]}^{(n^{-1/2})}\big) \;:\; 0 \le t \le 1\big\}_{n=1}^{\infty}$ converges in law to a G-valued diffusion process as $n \to \infty$ under suitable assumptions **(A1)** and **(A2)**. See Theorem 5.1.2. Here the diffusion process is generated by the homogenized sub-Laplacian with the $\mathfrak{g}^{(1)}$-valued drift $\rho_{\mathbb{R}}(\gamma_p)$ defined on G equipped with the Albanese metric $g_0^{(0)}$ associated with the symmetrized transition probability p_0. To our best knowledge, there seems to be few results on CLTs in the nilpotent setting in which a $\mathfrak{g}^{(1)}$-valued drift appears in the infinitesimal generator of the limiting diffusion. On the other hand, as we have already mentioned, there are many papers on CLTs in which $\mathfrak{g}^{(2)}$-valued drift like $\beta(\Phi_0)$ appears in the infinitesimal generator of the limiting diffusion. In view of these circumstances, the study of the long time asymptotics of random walks on more general graphs by applying our "weakening" scheme would be an interesting problem. In closing this section, we summarize the limiting infinitesimal generators and limiting diffusions captured in Chapters 4 and 5, as well as them on crystal lattices captured in Ishiwata–Kawabi–Kotani [31].

Chapter 6: This chapter is based on the author's paper [32], which is jointwork with Satoshi Ishiwata and Hiroshi Kawabi. We give several concrete examples of non-symmetric random walks on Γ-nilpotent covering graphs in the case where Γ is the 3D discrete Heisenberg group $\mathbb{H}^3(\mathbb{Z})$. We review some basics on $\mathbb{H}^3(\mathbb{Z})$ in Section 6.1. We consider a non-symmetric random walk on the 3D Heisenberg triangular lattice (resp. the 3D Heisenberg dice lattice) in Section 6.2 (resp. in Section 6.3), as a generalization of the triangular lattice (resp. the dice lattice) to the nilpotent case. In both sections, explicit calculations on several quantities of random walks and several figures are given.

Chapter 2

Preliminaries

2.1 Notations

Let $X = (V, E)$ be a locally finite, connected and oriented graph, where V is the set of all vertices and E is the set of all oriented edges. The graph X possibly have multiple edges or loops and is equipped with the discrete topology induced by the graph distance. For an edge $e \in E$, we denote by $o(e)$ and $t(e)$ the origin and the terminus of e, respectively. The inverse edge of $e \in E$ is defined by an edge, say $\overline{e}$, satisfying $o(\overline{e}) = t(e)$ and $t(\overline{e}) = o(e)$. Let E_x be the set of all edges whose origin is $x \in V$, that is, $E_x = \{e \in E \mid o(e) = x\}$. A path c in X of length n is a sequence $c = (e_1, e_2, \dots, e_n)$ of n edges $e_1, e_2, \dots, e_n \in E$ with $o(e_{i+1}) = t(e_i)$ for $i = 1, 2, \dots, n-1$. We denote by $\Omega_{x,n}(X)$ the set of all paths in X of length $n \in \mathbb{N} \cup \{\infty\}$ starting from $x \in V$. Put $\Omega_x(X) = \Omega_{x,\infty}(X)$ for simplicity.

We introduce a *transition probability*, that is, a function $p : E \longrightarrow [0, 1]$ satisfying

$$\sum_{e \in E_x} p(e) = 1 \qquad (x \in V) \quad \text{and} \quad p(e) + p(\overline{e}) > 0 \qquad (e \in E).$$

The value $p(e)$ represents the probability that a particle at the origin $o(e)$ moves to the terminus $t(e)$ along the edge $e \in E$ in a unit time. The random walk associated with p is the X-valued time-homogeneous Markov chain $(\Omega_x(X), \mathbb{P}_x, \{w_n\}_{n=0}^{\infty})$, where $\mathbb{P}_x$ is the probability measure on $\Omega_x(X)$ satisfying

$$\mathbb{P}_x\big(\{c = (e_1, e_2, \dots, e_n, *, *, \dots)\}\big) = p(e_1)p(e_2)\cdots p(e_n) \qquad \big(c \in \Omega_x(X)\big)$$

and $w_n(c) := o(e_{n+1})$ for $n \in \mathbb{N} \cup \{0\}$ and $c = (e_1, e_2, \dots, e_n, \dots) \in \Omega_x(X)$.

We define the *transition operator* L associated with the transition probability p by

$$Lf(x) := \sum_{e \in E_x} p(e) f\big(t(e)\big) \qquad (x \in V,\ f : V \longrightarrow \mathbb{R})$$

and the n-step transition probability $p(n, x, y)$ by

$$p(n, x, y) := L^n \delta_y(x) \qquad (n \in \mathbb{N},\ x, y \in V),$$

where δ_y stands for the Dirac delta function at y. We put $p(c) = p(e_1)p(e_2)\cdots p(e_n)$ for $c = (e_1, e_2, \dots, e_n) \in \Omega_{x,n}(X)$. If there is a function $m : V \longrightarrow (0, \infty)$ such that $p(e)m\big(o(e)\big) = p(\overline{e})m\big(t(e)\big)$ for $e \in E$, then the random walk is called (m-)*symmetric* or *reversible*, and the function m is called a *reversible measure*. Note that m is determined up to constant multiplication.

For a metric space $\mathcal{T}$, we denote by $C_\infty(\mathcal{T})$ the space of all continuous functions $f : \mathcal{T} \longrightarrow \mathbb{R}$ vanishing at infinity with the usual sup-norm $\|f\|_\infty^{\mathcal{T}} = \sup_{x\in\mathcal{T}} |f(x)|$. We also denoted by $C_0(\mathcal{T})$ the space of all continuous functions which are supported compactly.

Throughout this thesis, C denotes a positive constant that may change from line to line and $O(\cdot)$ stands for the Landau symbol. If the dependence of C and $O(\cdot)$ are significant, we denote them like $C(N)$ and $O_N(\cdot)$, respectively.

2.2 Nilpotent Lie groups and its limit groups

Let us review some properties of nilpotent Lie groups and the corresponding limit group. For more details, see e.g., Alexopoulos [1] and Ishiwata [29]. We also refer to Alexopoulos [2, 3] Crépel–Raugi [15] and Goodman [24] for related topics.

Let $(G, \cdot)$ be a connected and simply connected nilpotent Lie group of step r and $(\mathfrak{g}, [\cdot, \cdot])$ the corresponding Lie algebra. Note that the exponential map $\exp : \mathfrak{g} \longrightarrow G$ is globally defined and thus $\log = \exp^{-1} : G \longrightarrow \mathfrak{g}$ is also globally defined.

We now construct a new product $*$ on G in the following manner. Set $\mathfrak{n}_1 := \mathfrak{g}$ and $\mathfrak{n}_{k+1} := [\mathfrak{g}, \mathfrak{n}_k]$ for $k \in \mathbb{N}$. Since $\mathfrak{g}$ is nilpotent, we have

$$\mathfrak{g} = \mathfrak{n}_1 \supset \mathfrak{n}_2 \supset \cdots \supset \mathfrak{n}_r \supsetneq \mathfrak{n}_{r+1} = \{\mathbf{0}_{\mathfrak{g}}\}.$$

The integer r is called the *step number* of $\mathfrak{g}$ or G. We define the subspace $\mathfrak{g}^{(k)}$ of $\mathfrak{g}$ by $\mathfrak{n}_k = \mathfrak{g}^{(k)} \oplus \mathfrak{n}_{k+1}$ for $k = 1, 2, \dots, r$. Then the Lie algebra $\mathfrak{g}$ is decomposed as $\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$ and each $Z \in \mathfrak{g}$ is uniquely written as $Z = Z^{(1)} + Z^{(2)} + \cdots + Z^{(r)}$, where $Z^{(k)} \in \mathfrak{g}^{(k)}$ for $k = 1, 2, \dots, r$. We define a map $\tau_\varepsilon^{(\mathfrak{g})} : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\tau_\varepsilon^{(\mathfrak{g})}(Z) := \varepsilon Z^{(1)} + \varepsilon^2 Z^{(2)} + \cdots + \varepsilon^r Z^{(r)} \qquad (\varepsilon \geq 0,\, Z \in \mathfrak{g})$$

and also define a Lie bracket product $[\![\cdot, \cdot]\!]$ on $\mathfrak{g}$ by

$$[\![Z_1, Z_2]\!] := \lim_{\varepsilon \searrow 0} \tau_\varepsilon^{(\mathfrak{g})}\big[\tau_{1/\varepsilon}^{(\mathfrak{g})}(Z_1), \tau_{1/\varepsilon}^{(\mathfrak{g})}(Z_2)\big] \qquad (Z_1, Z_2 \in \mathfrak{g}).$$

We introduce a map $\tau_\varepsilon : G \longrightarrow G$, called the *dilation operator* on G, by

$$\tau_\varepsilon(g) := \exp\big(\tau_\varepsilon^{(\mathfrak{g})}\big(\log(g)\big)\big) \qquad (\varepsilon \geq 0,\, g \in G),$$

which, roughly speaking, gives the scalar multiplication on G. We note that τ_ε may not be a group homomorphism, though it is a diffeomorphism on G. The inverse map of τ_ε is given by $\tau_{1/\varepsilon}$ for $\varepsilon > 0$. By making use of the dilation map τ_ε, a Lie-group product $*$ on G is defined as follows:

$$g * h := \lim_{\varepsilon \searrow 0} \tau_\varepsilon\big(\tau_{1/\varepsilon}(g) \cdot \tau_{1/\varepsilon}(h)\big) \qquad (g, h \in G). \tag{2.2.1}$$

The Lie group $G_\infty = (G, *)$ is called the *limit group* of $(G, \cdot)$. Note that the Lie group G is *stratified* of step r in the sense that $(\mathfrak{g}, [\![\cdot, \cdot]\!])$ is decomposed as $\mathfrak{g} = \bigoplus_{k=1}^r \mathfrak{g}^{(k)}$ satisfying

$$[\![\mathfrak{g}^{(k)}, \mathfrak{g}^{(\ell)}]\!] \begin{cases} \subset \mathfrak{g}^{(k+\ell)} & (k + \ell \leq r), \\ = \mathbf{0}_{\mathfrak{g}} & (k + \ell > r), \end{cases}$$

and the subspace $\mathfrak{g}^{(1)}$ generates $\mathfrak{g}$. The Lie algebra $\mathfrak{g}_\infty$ of $G_\infty = (G, *)$ coincides with $(\mathfrak{g}, [\![\cdot, \cdot]\!])$ (cf. [29, Lemma 2.1]). It should be noted that the dilation map $\tau_\varepsilon : G \longrightarrow G$ is a group automorphism on $(G, *)$ (see [29, Lemma 2.1]). The exponential map $\exp : \mathfrak{g}_\infty \longrightarrow G_\infty$ coincides with the original exponential map $\exp : \mathfrak{g} \longrightarrow G$. Furthermore, for any $g \in G$, the inverse element of g in $(G, \cdot)$ coincides with the inverse element in $(G, *)$.

We set $d_k = \dim_{\mathbb{R}} \mathfrak{g}^{(k)}$ for $k = 1, 2, \ldots, r$ and $d = d_1 + d_2 + \cdots + d_r$. For $k = 1, 2, \ldots, r$, we denote by $\{X_1^{(k)}, X_2^{(k)}, \ldots, X_{d_k}^{(k)}\}$ a basis of the subspace $\mathfrak{g}^{(k)}$. We introduce several kinds of global coordinate systems in G through $\exp : \mathfrak{g} \longrightarrow G$. We identify the nilpotent Lie group G with $\mathbb{R}^d$ as a differentiable manifold by

- *canonical* $(\cdot)$*-coordinates of the first kind* :

$$\mathbb{R}^d \ni (g^{(1)}, g^{(2)}, \ldots, g^{(r)}) \longmapsto g = \exp\Big(\sum_{k=1}^r \sum_{i=1}^{d_k} g_i^{(k)} X_i^{(k)}\Big) \in G,$$

- *canonical* $(\cdot)$*-coordinates of the second kind* :

$$\begin{aligned} \mathbb{R}^d \ni & (g^{(1)}, g^{(2)}, \ldots, g^{(r)}) \\ \longmapsto & \; g = \exp\big(g_{d_r}^{(r)} X_{d_r}^{(r)}\big) \cdot \exp\big(g_{d_r-1}^{(r)} X_{d_r-1}^{(r)}\big) \cdot \cdots \cdot \exp\big(g_1^{(r)} X_1^{(r)}\big) \\ & \quad \cdot \exp\big(g_{d_{r-1}}^{(r-1)} X_{d_{r-1}}^{(r-1)}\big) \cdot \exp\big(g_{d_{r-1}-1}^{(r-1)} X_{d_{r-1}-1}^{(r-1)}\big) \cdot \cdots \cdot \exp\big(g_1^{(r-1)} X_1^{(r-1)}\big) \\ & \quad \cdots \cdot \exp\big(g_{d_1}^{(1)} X_{d_1}^{(1)}\big) \cdot \exp\big(g_{d_1-1}^{(1)} X_{d_1-1}^{(1)}\big) \cdot \cdots \cdot \exp\big(g_1^{(1)} X_1^{(1)}\big) \in G, \end{aligned}$$

- *canonical* $(*)$*-coordinates of the second kind* :

$$\begin{aligned} \mathbb{R}^d \ni & (g_*^{(1)}, g_*^{(2)}, \ldots, g_*^{(r)}) \\ \longmapsto & \; g = \exp\big(g_{d_r *}^{(r)} X_{d_r}^{(r)}\big) * \exp\big(g_{d_r-1 *}^{(r)} X_{d_r-1}^{(r)}\big) * \cdots * \exp\big(g_{1*}^{(r)} X_1^{(r)}\big) \\ & \quad * \exp\big(g_{d_{r-1}*}^{(r-1)} X_{d_{r-1}}^{(r-1)}\big) * \exp\big(g_{d_{r-1}-1*}^{(r-1)} X_{d_{r-1}-1}^{(r-1)}\big) * \cdots * \exp\big(g_{1*}^{(r-1)} X_1^{(r-1)}\big) \\ & \quad * \cdots * \exp\big(g_{d_1 *}^{(1)} X_{d_1}^{(1)}\big) * \exp\big(g_{d_1-1*}^{(1)} X_{d_1-1}^{(1)}\big) * \cdots * \exp\big(g_{1*}^{(1)} X_1^{(1)}\big) \in G_\infty, \end{aligned}$$

where we write $g^{(k)} = (g_1^{(k)}, g_2^{(k)}, \ldots, g_{d_k}^{(k)}) \in \mathbb{R}^{d_k}$ for $k = 1, 2, \ldots, r$.

We give the relations between the deformed product and the given product on G as an easy application of the Campbell–Baker–Hausdorff (CBH) formula

$$\log\big(\exp(Z_1)\cdot\exp(Z_2)\big) = Z_1 + Z_2 + \frac{1}{2}[Z_1, Z_2] + \cdots \qquad (Z_1, Z_2 \in \mathfrak{g}). \tag{2.2.2}$$

The following is straightforward from the definition of the deformed product.

$$\log(g * h)\big|_{\mathfrak{g}^{(k)}} = \log(g \cdot h)\big|_{\mathfrak{g}^{(k)}} \qquad (g, h \in G,\ k = 1, 2). \tag{2.2.3}$$

We notice that the relation above does not hold in general for $k = 3, 4, \dots, r$. The following identities give us a comparison between $(\cdot)$-coordinates and $(*)$-coordinates. For $g \in G$, we have the following.

$$g_{i*}^{(k)} = g_i^{(k)} \quad (i = 1, 2, \dots, d_k,\ k = 1, 2), \tag{2.2.4}$$

$$g_{i*}^{(k)} = g_i^{(k)} + \sum_{0<|K|\le k-1} C_K \mathcal{P}^K(g) \qquad (i = 1, 2, \dots, d_k,\ k = 3, 4, \dots, r) \tag{2.2.5}$$

for some constant C_K, where K stands for a multi-index $\big((i_1, k_1), (i_2, k_2), \dots, (i_\ell, k_\ell)\big)$ with length $|K| := k_1 + k_2 + \cdots + k_\ell$ and $\mathcal{P}^K(g) := g_{i_1}^{(k_1)} \cdot g_{i_2}^{(k_2)} \cdots g_{i_\ell}^{(k_\ell)}$. The invariances (2.2.3) and (2.2.4) play an important role to obtain main results. For $g, h \in G$, we also have

$$(g * h)_{i*}^{(k)} = (g \cdot h)_i^{(k)} \qquad (i = 1, 2, \dots, d_k,\ k = 1, 2), \tag{2.2.6}$$

$$(g * h)_{i*}^{(k)} = (g \cdot h)_i^{(k)} + \sum_{\substack{|K_1|+|K_2|\le k-1 \\ |K_2|>0}} C_{K_1,K_2} \mathcal{P}_*^{K_1}(g) \mathcal{P}^{K_2}(g \cdot h)$$
$$(i = 1, 2, \dots, d_k,\ k = 3, 4, \dots, r) \tag{2.2.7}$$

by using (2.2.4) and (2.2.5), where $\mathcal{P}_*^K(g) := g_{i_1}^{(k_1)} * g_{i_2}^{(k_2)} * \cdots * g_{i_\ell}^{(k_\ell)}$. See [29, Section 2] for more details.

2.3 Carnot–Carathéodory metric and homogeneous norms

As is well-known, a nilpotent Lie group G is a candidate of the typical sub-Riemannian manifolds, which is a certain generalization of a Riemannian manifold. The notion of the Carnot–Carathéodory metric naturally appears when we investigate distances between two points in G. It is an important intrinsic metric in this context and is degenerate in the sense that we only go along curves which are tangent to a "horizontal subspace" of the tangent space of G. We discuss several properties of the Carnot–Carathéodory metric on a nilpotent Lie group G in this section. Note that the definition of such an intrinsic metric in more general setting is found in some references. See e.g., Varopoulos–Saloff-Coste–Coulhon [76] for details.

We start with the definition of the *Carnot–Carathéodory metric* on G.

Definition 2.3.1 *We endow G with the Carnot–Carathéodory metric d_{CC}, which is an intrinsic metric defined by*

$$d_{\mathrm{CC}}(g,h) := \inf\left\{ \int_0^1 \|\dot{w}_t\|_{\mathfrak{g}^{(1)}}\,dt \,\middle|\, \begin{array}{c} w \in \mathrm{Lip}([0,1];G),\ w_0 = g,\ w_1 = h, \\ w \textit{ is tangent to } \mathfrak{g}^{(1)} \end{array} \right\} \tag{2.3.1}$$

for $g, h \in G$, where we write $\mathrm{Lip}([0,1];G)$ *for the set of all Lipschitz continuous paths and $\|\cdot\|_{\mathfrak{g}^{(1)}}$ stands for a norm on $\mathfrak{g}^{(1)}$.*

We see that the subspace $\mathfrak{g}^{(1)}$ satisfies the so-called *Hörmander condition* in $\mathfrak{g}$, that is, $L_{\mathfrak{g}^{(1)}}(g) = T_gG$ for any $g \in G$, where $L_{\mathfrak{g}^{(1)}}(g)$ denotes the evaluation of $\mathfrak{g}^{(1)}$ at $g \in G$. The Carnot-Carathéodory metric is then well-defined in the sense that $d_{\mathrm{CC}}(g,h) < \infty$ for every $g, h \in G$, thanks to the Hörmander condition on $\mathfrak{g}^{(1)}$ (cf. Mitchell [57]). Furthermore, the topology induced by the Carnot-Carathéodory metric d_{CC} coincides with the original one of G. We emphasize that d_{CC} is behaved well under dilations. More precisely, we have

$$d_{\mathrm{CC}}\big(\tau_\varepsilon(g), \tau_\varepsilon(h)\big) = \varepsilon d_{\mathrm{CC}}(g,h) \qquad (\varepsilon \geq 0,\ g, h \in G). \tag{2.3.2}$$

We now present the notion of *homogeneous norm* on G. The one-parameter group of dilations $(\tau_\varepsilon)_{\varepsilon \geq 0}$ allows us to consider scalar multiplications on nilpotent Lie groups. We replace the usual Euclidean norms by the following functions.

Definition 2.3.2 *A continuous function $\|\cdot\| : G \longrightarrow [0,\infty)$ is called a homogeneous norm on G if*
(i) $\|g\| = 0$ *if and only if* $g = \mathbf{1}_G$*, and*
(ii) $\|\tau_\varepsilon g\| = \varepsilon\|g\|$ *for* $\varepsilon \geq 0$ *and* $g \in G$*.*

One of the typical examples of homogeneous norms is given by the Carnot–Carathéodory metric d_{CC}. We define a continuous function $\|\cdot\|_{\mathrm{CC}} : G \longrightarrow [0,\infty)$ by

$$\|g\|_{\mathrm{CC}} := d_{\mathrm{CC}}(\mathbf{1}_G, g) \qquad (g \in G).$$

Then $\|\cdot\|_{\mathrm{CC}}$ is a homogeneous norm on G thanks to (2.3.2). Another basic homogeneous norm is given in the following way. We denote by $\{X_1^{(k)}, X_2^{(k)}, \dots, X_{d_k}^{(k)}\}$ a basis of $\mathfrak{g}^{(k)}$ for $k = 1, 2, \dots, r$. We introduce a norm $\|\cdot\|_{\mathfrak{g}^{(k)}}$ on $\mathfrak{g}^{(k)}$ by the usual Euclidean one. If $Z \in \mathfrak{g}$ is decomposed as $Z = Z^{(1)} + Z^{(2)} + \cdots + Z^{(r)}$ ($Z^{(k)} \in \mathfrak{g}^{(k)}$), we define a function $\|\cdot\|_{\mathfrak{g}} : \mathfrak{g} \longrightarrow [0,\infty)$ by

$$\|Z\|_{\mathfrak{g}} := \sum_{k=1}^r \|Z^{(k)}\|_{\mathfrak{g}^{(k)}}^{1/k}.$$

We set $\|g\|_{\mathrm{Hom}} := \|\log(g)\|_{\mathfrak{g}}$ for $g \in G$. We then observe that $\|\cdot\|_{\mathrm{Hom}}$ is a homogeneous norm on G. The homogenuity (ii) leads to the most important fact that all homogeneous norms on G are equivalent, which is similar to the case of norms on Euclidean space. More precisely, we have the following.

Proposition 2.3.3 (cf. Goodman [24]) *If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two homogeneous norms on G, then there exists a constant $C > 0$ such that*

$$\frac{1}{C}\|g\|_1 \leq \|g\|_2 \leq C\|g\|_1 \qquad (g \in G).$$

For more details, we also refer to Bonfiglioli–Lanconelli–Uguzzoni [9].

2.4 Discrete geometric analysis

2.4.1 Discrete geometric analysis on graphs

We present some basics of discrete geometric analysis on graphs due to Kotani–Sunada [42] or Sunada [69, 70, 71]. We consider a finite graph $X_0 = (V_0, E_0)$ and an irreducible random walk on X_0 associated with a non-negative transition probability $p : E_0 \longrightarrow [0, 1]$. We find a unique positive function $m : V_0 \longrightarrow (0, 1]$, which is called the *invariant probability measure* on V_0, by applying the Perron–Frobenius theorem. We put $\widetilde{m}(e) := p(e)m\big(o(e)\big)$ for $e \in E_0$. We easily see that $\widetilde{m}$ has the following properties:

$$\sum_{e \in E_0} \widetilde{m}(e) = 1, \qquad m(x) = \sum_{e \in (E_0)_x} \widetilde{m}(e) \qquad (x \in V).$$

We define the symmetry and the non-symmetry of the random walk on X_0.

Definition 2.4.1 *The random walk on X_0 is said to be (m-)symmetric if $\widetilde{m}(e) = \widetilde{m}(\overline{e})$ holds for $e \in E_0$. Otherwise, it is called (m-)non-symmetric.*

We define the 0-chain group and the 1-chain group of X_0 by

$$C_0(X_0, \mathbb{R}) := \Big\{ \sum_{x \in V_0} a_x x \,\Big|\, a_x \in \mathbb{R} \Big\}, \quad C_1(X_0, \mathbb{R}) := \Big\{ \sum_{e \in E_0} a_e e \,\Big|\, a_e \in \mathbb{R},\, \overline{e} = -e \Big\},$$

respectively. The boundary operator $\partial : C_1(X_0, \mathbb{R}) \longrightarrow C_0(X_0, \mathbb{R})$ is defined by the linear map satisfying $\partial(e) = t(e) - o(e)$ for $e \in E_0$. Note that $\partial(\overline{e}) = -\partial(e)$ for $e \in E_0$ due to $\overline{e} = -e$. The first homology group $\mathrm{H}_1(X_0, \mathbb{R})$ is defined by $\mathrm{Ker}\,(\partial) \subset C_1(X_0, \mathbb{R})$, which is a vector space over $\mathbb{R}$ whose dimension is $|E_0|/2 - |V_0| + 1$. We also define the 0-cochain group and the 1-cochain group by

$$C^0(X_0, \mathbb{R}) := \{f : V_0 \longrightarrow \mathbb{R}\}, \quad C^1(X_0, \mathbb{R}) := \{\omega : E_0 \longrightarrow \mathbb{R} \,|\, \omega(\overline{e}) = -\omega(e)\},$$

respectively. An element of $C^1(X_0, \mathbb{R})$ is also called a 1-form on X_0. We equip $C^0(X_0, \mathbb{R})$ and $C^1(X_0, \mathbb{R})$ with inner products given by

$$\langle f_1, f_2 \rangle_0 = \sum_{x \in V_0} f_1(x) f_2(x) \qquad \big(f_1, f_2 \in C^0(X_0, \mathbb{R})\big),$$
$$\langle \omega_1, \omega_2 \rangle_1 = \frac{1}{2} \sum_{e \in E_0} \omega_1(e) \omega_2(e) \qquad \big(\omega_1, \omega_2 \in C^1(X_0, \mathbb{R})\big),$$

respectively. We introduce the difference operator $d : C^0(X_0, \mathbb{R}) \longrightarrow C^1(X_0, \mathbb{R})$ by the linear map satisfying $df(e) = f\big(t(e)\big) - f\big(o(e)\big)$ for $f \in C^0(X_0, \mathbb{R})$ and $e \in E_0$. Note that $df(\overline{e}) = -df(e)$ for $f \in C^0(X_0, \mathbb{R})$ and $e \in E_0$. The first cohomology group $\mathrm{H}^1(X_0, \mathbb{R})$ is defined by $C^1(X_0, \mathbb{R})/\mathrm{Im}\,(d)$. By the discrete analogue of the Poincaré duality theorem, we have $\mathrm{H}^1(X_0, \mathbb{R}) = \big(\mathrm{H}_1(X_0, \mathbb{R})\big)^*$. We define an operator $\delta_p : C^1(X_0, \mathbb{R}) \longrightarrow C^0(X_0, \mathbb{R})$ associated with the trasition probability p by

$$\delta_p \omega(x) := -\sum_{e \in (E_0)_x} p(e)\omega(e) \qquad (x \in V_0).$$

Then the *transition operator* $L : C^0(X_0, \mathbb{R}) \longrightarrow C^0(X_0, \mathbb{R})$ associated with the transition probability p is defined by

$$Lf(x) := (I - \delta_p d)f(x) = \sum_{e \in (E_0)_x} p(e) f\big(t(e)\big) \qquad \big(f \in C^0(X_0, \mathbb{R}),\ x \in V_0\big).$$

Since the operator $I - L$ is regarded as a discrete analogue of the Laplacian, the operators d and δ_p play roles of the exterior differentiation and its formal adjoint. However, $\delta_p : C^1(X_0, \mathbb{R}) \longrightarrow C^0(X_0, \mathbb{R})$ is the adjoint operator of $d : C^0(X_0, \mathbb{R}) \longrightarrow C^1(X_0, \mathbb{R})$ if and only if the random walk on X_0 is (m-)symmetric (cf. [42, page 852]). We now introduce a 1-chain

$$\gamma_p := \sum_{e \in E_0} \widetilde{m}(e)e \in C_1(X_0, \mathbb{R}).$$

We present several properties of γ_p.

Lemma 2.4.2 (cf. [42, Proposition 2.1]) (1) $\partial(\gamma_p) = 0$, *that is,* $\gamma_p \in \mathrm{H}_1(X_0, \mathbb{R})$.
(2) *The random walk on* X_0 *is* (m-)*symmetric if and only if* $\gamma_p = 0$.

Proof. By definition, we see

$$\partial(\gamma_p) = \sum_{e \in E_0} \widetilde{m}(e)t(e) - \sum_{e \in E_0} \widetilde{m}(e)o(e).$$

Since

$$\begin{aligned}
\sum_{e \in E_0} \widetilde{m}(e)t(e) &= \sum_{e \in E_0} \widetilde{m}(\overline{e})o(e) \\
&= \sum_{x \in V_0} x \sum_{e \in (E_0)_x} p(\overline{e})m\big(t(e)\big) = \sum_{x \in V_0} m(x)x, \\
\sum_{e \in E_0} \widetilde{m}(e)o(e) &= \sum_{x \in V_0} x \sum_{e \in (E_0)_x} p(e)m\big(o(e)\big) \\
&= \sum_{x \in V_0} m(x)x \sum_{e \in (E_0)_x} p(e) = \sum_{x \in V_0} m(x)x,
\end{aligned}$$

we obtain the first item. The second one readily follows from

$$\gamma_p = \frac{1}{2}\sum_{e \in E_0} \big(\widetilde{m}(e) - \widetilde{m}(\overline{e})\big)e = 0 \quad \Longleftrightarrow \quad \widetilde{m}(e) = \widetilde{m}(\overline{e}) \qquad (e \in E_0).$$

This completes the proof. ∎

This 1-cycle γ_p is called the *homological direction*, which is regarded as a quantity to measure the homological drift of the given random walk on X_0. To see this, we review the following:

Proposition 2.4.3 (cf. Sunada [69]) *Let $\mathcal{V}$ be a vector space over $\mathbb{R}$ and $f : E_0 \longrightarrow \mathcal{V}$ a map. We define a sequence of $\mathcal{V}$-valued random variables $\{\eta_i\}_{i=1}^{\infty}$ by*

$$\eta_i(c) := f(e_i) \qquad \big(c = (e_1, e_2, \dots,) \in \Omega_x(X_0)\big).$$

Then we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \eta_i(c) = \sum_{e\in E_0} \widetilde{m}(e) f(e) \quad \mathbb{P}_x\text{-a.s.}$$

Proof. Let $\big(\Omega_x(X_0), \mathbb{P}_x, w = \{w_n\}_{n=0}^{\infty}\big)$ be an irreducible Markov chain with values in X_0. We write $\mathbb{Q}_x$ for the probability measure on $\Omega_x(X)$ induced by $\widetilde{m}$. Thanks to the positivity of m, we see that $\mathbb{P}_x$-almost sure events are $\mathbb{Q}_x$-almost sure ones and vice versa. Since X_0 is finite, this Markov chain is recurrent and therefore it is ergodic on $\big(\Omega_x(X_0), \mathbb{P}_x\big)$. Namely, the probability space $\big(\Omega_x(X_0), \mathbb{Q}_x\big)$ with the shift operator $T : \Omega_x(X_0) \longrightarrow \Omega_x(X_0)$ given by $Tc = T(e_n)_{n=1}^{\infty} = (e_{n+1})_{n=1}^{\infty}$ is a measure-preserving dynamical system (cf. Klenke [37, Theorem 20.29]). Note that $\eta_n = \eta_1 \circ T^{n-1}$ for $n = 2, 3, \dots$. By applying the Birkhoff individual ergodic theorem, we have

$$\frac{1}{n}\sum_{i=1}^{n} \eta_i = \frac{1}{n}\sum_{i=1}^{n} \eta_1 \circ T^{i-1} \longrightarrow \mathbb{E}^{\mathbb{Q}_x}[\eta_0] = \sum_{e\in E_0} \widetilde{m}(e) f(e) \quad \mathbb{P}_x\text{-a.s.}$$

as $n \to \infty$, which completes the proof. ∎

Indeed, taking $\mathcal{V} = C_1(X_0, \mathbb{R})$ and $f(e) = e$ for $e \in E_0$ in Proposition 2.4.3 immediately leads to the law of large numbers (LLN) on $C_1(X_0, \mathbb{R})$.

$$\lim_{n\to\infty} \frac{1}{n}(e_1 + e_2 + \cdots + e_n) = \gamma_p, \quad \mathbb{P}_x\text{-a.e. } c = (e_1, e_2, \dots, e_n, \dots) \in \Omega_x(X_0).$$

We introduce the notion of modified harmonic 1-form on X_0, which is the discrete analogue of that of harmonic forms on Riemannian manifolds. A 1-form $\omega \in C^1(X_0, \mathbb{R})$ is said to be *modified harmonic* if

$$\delta_p \omega(x) + \langle \gamma_p, \omega \rangle = 0 \qquad (x \in V_0), \tag{2.4.1}$$

where $\langle \gamma_p, \omega \rangle := {}_{C_1(X_0,\mathbb{R})}\langle \gamma_p, \omega \rangle_{C^1(X_0,\mathbb{R})}$ is constant as a function on V_0. We denote by $\mathcal{H}^1(X_0)$ the space of modified harmonic 1-forms and equip it with the inner product and the norm given by

$$\langle\!\langle \omega_1, \omega_2 \rangle\!\rangle_p := \sum_{e\in E_0} \widetilde{m}(e)\omega_1(e)\omega_2(e) - \langle \gamma_p, \omega_1 \rangle \langle \gamma_p, \omega_2 \rangle \qquad \big(\omega_1, \omega_2 \in \mathcal{H}^1(X_0)\big),$$

$$\|\omega\|_{\mathcal{H}^1(X_0)} := \Big(\sum_{e\in E_0} \widetilde{m}(e)\omega(e)^2 - \langle \gamma_p, \omega \rangle^2 \Big)^{1/2} \qquad \big(\omega \in \mathcal{H}^1(X_0)\big)$$

associated with the transition probability p. The following proposition is noting but the discrete analogue of the Hodge–Kodaira theorem.

Proposition 2.4.4 (cf. [42, Lemma 5.2]) *The linear map* $\varphi : \mathcal{H}^1(X_0) \longrightarrow \mathrm{H}^1(X_0, \mathbb{R})$ *defined by*

$$\varphi(\omega) := [\omega] \qquad \big(\omega \in \mathcal{H}^1(X_0)\big)$$

gives an isomorphism of $\mathcal{H}^1(X_0)$ *onto* $\mathrm{H}^1(X_0, \mathbb{R})$.

For the sake of completeness, we give the proof of Proposition 2.4.4.

Proof. Suppose that $[\omega] = 0$, that is, $\omega = df$ for some $f \in C^0(X_0, \mathbb{R})$. Thanks to the fact that $\omega = df$ is modified harmonic and $\partial(\gamma_p) = 0$, we have

$$\delta_p \omega = \delta_p df = -{}_{C_1(X_0,\mathbb{R})}\langle \gamma_p, df \rangle_{C^1(X_0,\mathbb{R})} = -{}_{C_0(X_0,\mathbb{R})}\langle \partial(\gamma_p), f \rangle_{C^0(X_0,\mathbb{R})} = 0.$$

Therefore, it follows that $Lf = (I - \delta_p d)f = f$. Since X_0 is connected, we see that the function f is constant and thus $\omega = df = 0$, which leads to the injectivity of φ.

For the surjectivy of φ, we show that, for any $\omega \in C^1(X_0, \mathbb{R})$, there is $f \in C^0(X_0, \mathbb{R})$ such that $\omega + df \in \mathcal{H}^1(X_0)$. It is sufficient to find f satisfying $(I - L)f = -\big(\langle \gamma_p, \omega \rangle + \delta_p \omega\big)$. For this sake, we only to show

$$\big\langle \langle \gamma_p, \omega \rangle + \delta_p \omega, m \big\rangle_0 = 0,$$

by noting $\mathrm{Im}\,(I - L) = \big(\mathrm{Ker}\,(I - {}^tL)\big)^{\perp} = (\mathbb{R}m)^{\perp}$. The left-hand side is written as

$$\langle \gamma_p, \omega \rangle \langle 1, m \rangle_0 + \langle \delta_p \omega, m \rangle_0 = \langle \gamma_p, \omega \rangle + \langle \delta_p \omega, m \rangle_0$$

Therefore, we have

$$\langle \delta_p \omega, m \rangle_0 = -\sum_{x \in V_0} m(x) \sum_{e \in (E_0)_x} p(e)\omega(e) = -\langle \gamma_p, \omega \rangle,$$

which completes the proof of Proposition 2.4.4. ∎

2.4.2 Modified harmonic realization of a crystal lattice

Let Γ be a finitely generated abelian group. Suppose that Γ is torsion free. Then we may assume $\Gamma \cong \mathbb{Z}^d$ without loss of generality, where $d = \mathrm{rank}\,\Gamma$. Now let $X = (V, E)$ be a Γ-crystal lattice. Namely, X is a covering graph of a finite graph X_0 whose covering transformation group is Γ. The graph X_0 is also represented as $X_0 = \Gamma \backslash X$, the quotient graph of X. Let $p : E_0 \longrightarrow [0, 1]$ and $m : V_0 \longrightarrow (0, 1]$ be a transition probability on X_0 and the normalized invariant measure on X_0, respectively. We write $p : E \longrightarrow [0, 1]$ and $m : V \longrightarrow (0, 1]$ for the Γ-invariant lifts of $p : E_0 \longrightarrow [0, 1]$ and $m : V_0 \longrightarrow (0, 1]$, respectively. Namely,

$$p(\gamma e) = p(e), \quad m(\gamma x) = m(x) \qquad (\gamma \in \Gamma,\, e \in E,\, x \in V).$$

Let $\pi_1(X_0)$ be the fundamental group of X_0. Then we find a canonical surjective homomorphism $\rho : \pi_1(X_0) \longrightarrow \Gamma$ by the general theory of covering spaces. This map gives rise to a surjective homomorphism $\rho : \mathrm{H}_1(X_0, \mathbb{Z}) \longrightarrow \Gamma$, where $\mathrm{H}_1(X_0, \mathbb{Z})$ stands for the first homology group of X_0 with $\mathbb{Z}$-coefficients. Then we have a surjective linear map $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{Z}) \otimes \mathbb{R} \cong \mathrm{H}_1(X_0, \mathbb{R}) \longrightarrow \Gamma \otimes \mathbb{R}$. We consider the transpose ${}^t\rho_{\mathbb{R}} : \mathrm{Hom}(\Gamma, \mathbb{R}) \longrightarrow \mathrm{H}^1(X_0, \mathbb{R})$, which is a injective linear map. Here $\mathrm{Hom}(\Gamma, \mathbb{R})$ denotes the space of homomorphisms from Γ into $\mathbb{R}$. By noting Proposition 2.4.4, we induce a flat metric g_0 associated with the transition probability p on the Euclidean space $\Gamma \otimes \mathbb{R}$ through the following diagram:

$$\begin{tikzcd}
(\Gamma \otimes \mathbb{R}, g_0) \arrow[d, leftrightarrow, "\text{dual}"] & \mathrm{H}_1(X_0, \mathbb{R}) \arrow[l, two heads, "\rho_{\mathbb{R}}"'] \arrow[d, leftrightarrow, "\text{dual}"] \\
\mathrm{Hom}(\Gamma, \mathbb{R}) \arrow[r, hook, "{}^t\rho_{\mathbb{R}}"'] & \mathrm{H}^1(X_0, \mathbb{R}) \cong \big(\mathcal{H}^1(X_0), \langle\langle \cdot, \cdot \rangle\rangle_p\big).
\end{tikzcd}$$

This metric g_0 is called the *Albanese metric* on $\Gamma \otimes \mathbb{R}$.

From now on, we realize the crystal lattice X into the continuous model $(\Gamma \otimes \mathbb{R}, g_0)$ in the following manner. A *periodic realization* of X into $\Gamma \otimes \mathbb{R}$ is defined by a piecewise linear map $\Phi : X \longrightarrow \Gamma \otimes \mathbb{R}$ satisfying

$$\Phi(\sigma x) = \Phi(x) + \sigma \otimes 1 \qquad (\sigma \in \Gamma,\, x \in V).$$

. We review the definition of the modified harmonicity of the periodic realization of a crystal lattice X.

Definition 2.4.5 (cf. [42, page 854]) *The periodic realization Φ_0 is said to be modified harmonic if*

$$L\Phi_0(x) - \Phi_0(x) = \rho_{\mathbb{R}}(\gamma_p) \qquad (x \in V).$$

We note that this equation is also written as

$$\sum_{e \in E_x} p(e)\Big\{\Phi_0\big(t(e)\big) - \Phi_0\big(o(e)\big)\Big\} = \rho_{\mathbb{R}}(\gamma_p) \qquad (x \in V). \tag{2.4.2}$$

Furthermore, such a realization is uniquely determined up to translation. Indeed, if Φ_0 and Φ_0' are two modified harmonic realizations, then we see

$$L\big(\Phi_0(x) - \Phi_0'(x)\big) = \Phi_0(x) - \Phi_0'(x) \qquad (x \in V)$$

and it follows from the connectedness of X that $\Phi_0 - \Phi_0'$ is constant. We call the quantity $\rho_{\mathbb{R}}(\gamma_p)$ the *asymptotic direction* of the given random walk on X_0. We should emphasize that $\gamma_p = 0$ implies $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$. However, the converse does not always hold. If we equip $\Gamma \otimes \mathbb{R}$ with the Albanese metric, then the modified harmonic realization $\Phi_0 : X \longrightarrow (\Gamma \otimes \mathbb{R}, g_0)$ is especially called the *modified standard realization.*

We define a special periodic realization $\Phi_0 : X \longrightarrow \Gamma \otimes \mathbb{R}$ by

$$ {}_{\mathrm{Hom}(\Gamma,\mathbb{R})}\big\langle \omega, \Phi_0(x) \big\rangle_{\Gamma\otimes\mathbb{R}} = \int_{x_*}^{x} \widetilde{\omega} \qquad \big(x \in V,\, \omega \in \mathrm{Hom}(\Gamma,\mathbb{R})\big), \tag{2.4.3} $$

where x_* is a fixed reference point satisfying $\Phi_0(x_*) = \mathbf{0}$ and $\widetilde{\omega}$ is the lift of ω to X. Here

$$ \int_{x_*}^{x} \widetilde{\omega} = \int_c \widetilde{\omega} := \sum_{i=1}^{n} \widetilde{\omega}(e) $$

for a path $c = (e_1, \dots, e_n)$ with $o(e_1) = x_*$ and $t(e_n) = x$. It should be noted that this line integral does not depend on the choice of a path c. Then we immediately see that the periodic realization defined by (2.4.3) enjoys (2.4.2). See [31, Section 3.1].

2.4.3 Modified harmonic realization of a nilpotent covering graph

We introduce a notion of the modified harmonic realization of a nilpotent covering graph as an extension of Kotani–Sunada [42] and Ishiwata [29].

Let Γ be a torsion free, finitely generated nilpotent group of step r and $X = (V, E)$ a Γ-*nilpotent covering graph*, that is, a covering graph of a finite graph X_0 with the covering transformation group Γ. We denote by $\pi : X \longrightarrow X_0$ the covering map. Let $p : E_0 \longrightarrow [0, 1]$ and $m : V_0 \longrightarrow (0, 1]$ be a transition probability on X_0 and the normalized invariant measure on X_0, respectively. We write $p : E \longrightarrow [0, 1]$ and $m : V \longrightarrow (0, 1]$ for the Γ-invariant lifts of $p : E_0 \longrightarrow [0, 1]$ and $m : V_0 \longrightarrow (0, 1]$, respectively.

As in the case of crystal lattices, we would like to realize the nilpotent covering graph X into some continuous state space equipped with a scalar multiplication. Malcév's theorem [56] asserts that there exists a connected and simply connected nilpotent Lie group $G = G_\Gamma$ of step r such that Γ is isomorphic to a cocompact lattice in G. Namely, Γ is a discrete subgroup of G such that $\Gamma\backslash G$ is compact and $\mu(\Gamma\backslash G) < \infty$ for a Haar measure on G. Let $\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$ be the corresponding Lie algebra. We denote by $\widehat{\pi} : G \longrightarrow G/[G, G]$ the canonical projection. Since Γ is a cocompact lattice in G, the subset $\widehat{\pi}(\Gamma) \subset G/[G, G]$ is also a lattice in $G/[G, G]$ (cf. Malcév [56] and Raghunathan [62]). By $\mathfrak{g}^{(1)} \cong G/[G, G]$, the subgroup $\widehat{\pi}(\Gamma)$ is regarded as a lattice in $\mathfrak{g}^{(1)}$.

We take a canonical surjective homomorphism $\rho : \pi_1(X_0) \longrightarrow \Gamma$ and this map gives rise to a surjective homomorphism $\rho : \mathrm{H}_1(X_0, \mathbb{Z}) \longrightarrow \Gamma/[\Gamma, \Gamma] \cong \widehat{\pi}(\Gamma)$ by abelianization. Then we have a surjective linear map $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{R}) \longrightarrow \widehat{\pi}(\Gamma) \otimes \mathbb{R} \cong \mathfrak{g}^{(1)}$. We identify $\mathrm{Hom}(\widehat{\pi}(\Gamma), \mathbb{R})$ with a subspace of $\mathrm{H}^1(X_0, \mathbb{R})$ by using the transposed map ${}^t\rho_{\mathbb{R}}$. We restrict the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_p$ on $\mathrm{H}^1(X_0, \mathbb{R})$ to the subspace $\mathrm{Hom}(\widehat{\pi}(\Gamma), \mathbb{R})$ and take it up the dual inner product $\langle \cdot, \cdot \rangle_{alb}$ on $\widehat{\pi}(\Gamma) \otimes \mathbb{R}$. Then, as in the case of crystal lattices, a flat metric g_0 is induced on $\mathfrak{g}^{(1)}$ and we call it the *Albanese metric* on $\mathfrak{g}^{(1)}$. This procedure can be summarized as follows:

$$\begin{tikzcd}
(\mathfrak{g}^{(1)}, g_0) \arrow[r, phantom, "\cong"] \arrow[d, leftrightarrow, "\text{dual}"] & \widehat{\pi}(\Gamma) \otimes \mathbb{R} \arrow[d, leftrightarrow, "\text{dual}"] & \mathrm{H}_1(X_0, \mathbb{R}) \arrow[l, two heads, "\rho_{\mathbb{R}}"'] \arrow[d, leftrightarrow, "\text{dual}"] \\
\mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R}) \arrow[r, phantom, "\cong"] & \mathrm{Hom}(\widehat{\pi}(\Gamma), \mathbb{R}) \arrow[r, hook, "{}^t\rho_{\mathbb{R}}"'] & \mathrm{H}^1(X_0, \mathbb{R}) \cong \big(\mathcal{H}^1(X_0), \langle\!\langle \cdot, \cdot \rangle\!\rangle_p\big).
\end{tikzcd}$$

A map $\Phi : X \longrightarrow G$ is said to be a *periodic realization* of X when it satisfies

$$\Phi(\gamma x) = \gamma \cdot \Phi(x) \qquad (\gamma \in \Gamma,\, x \in V).$$

We are now in a position to give the definition of the modified harmonicity of the realization of X, as a generalization of [42, 29].

Definition 2.4.6 (cf. [32, 33]) *The realization Φ_0 is said to be modified harmonic if*

$$\sum_{e \in E_x} p(e) \log \Big(\Phi_0\big(o(e)\big)^{-1} \cdot \Phi_0\big(t(e)\big)\Big)\Big|_{\mathfrak{g}^{(1)}} = \rho_{\mathbb{R}}(\gamma_p) \qquad (x \in V). \tag{2.4.4}$$

Note that such a realization is uniquely determined up to $\mathfrak{g}^{(1)}$-translation. We also call the quantity $\rho_{\mathbb{R}}(\gamma_p)$ the ($\mathfrak{g}^{(1)}$-)*asymptotic direction* of the given random walk on X_0. If we equip $\mathfrak{g}^{(1)}$ with the Albanese metric g_0, then the modified harmonic realization $\Phi_0 : X \longrightarrow G$ is called the *modified standard realization.*

Remark 2.4.7 *The modified harmonic realization $\Phi_0 : X \longrightarrow G$ has the ambiguity of the components corresponding to the subspace $\mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$, though $\mathfrak{g}^{(1)}$-components completely controlled by* (2.4.4) *up to $\mathfrak{g}^{(1)}$-translation. However, it is sufficient to establish CLTs for non-symmetric random walks on X. Indeed, in showing CLTs of semigroup-type in Sections* 3 *and* 4*, the modified harmonicity* (2.4.4) *will be used effectively to handle the diverging drift term which appears in $\mathfrak{g}^{(1)}$. See the proof of Theorems* 4.1.2 *and* 5.1.1.

Fix a reference point $x_* \in V$ and define a realization $\Phi_0 : X \longrightarrow G$ by

$${}_{\mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R})}\Big\langle \omega,\, \log\big(\Phi_0(x)\big)\big|_{\mathfrak{g}^{(1)}} \Big\rangle_{\mathfrak{g}^{(1)}} = \int_{x_*}^{x} \widetilde{\omega} \qquad \big(\omega \in \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R}),\, x \in V\big), \tag{2.4.5}$$

where $\widetilde{\omega}$ is the lift of $\omega = {}^t\rho_{\mathbb{R}}(\omega) \in \mathrm{H}^1(X_0, \mathbb{R})$ to X. The following lemma asserts that such Φ_0 enjoys the modified harmonicity (2.4.4).

Lemma 2.4.8 (cf. [32, Lemma 3.2]) *The periodic realization $\Phi_0 : X \longrightarrow G$ defined by* (2.4.5) *is the modified harmonic realization.*

Proof. For each $\omega = {}^t\rho_{\mathbb{R}}(\omega) \in \mathrm{H}^1(X_0, \mathbb{R}) \cong \mathcal{H}^1(X_0)$ and $x \in V$, Equation (2.4.5) yields

$$\begin{aligned}
&{}_{\mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R})}\Big\langle \omega, \sum_{e \in E_x} p(e) \log\Big(\Phi_0\big(o(e)\big)^{-1} \cdot \Phi_0\big(t(e)\big)\Big)\Big|_{\mathfrak{g}^{(1)}}\Big\rangle_{\mathfrak{g}^{(1)}} \\
&= \sum_{e \in E_x} p(e)\, {}_{\mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R})}\Big\langle \omega, \log\big(\Phi_0\big(t(e)\big)\big)\Big|_{\mathfrak{g}^{(1)}} - \log\big(\Phi_0\big(o(e)\big)\big)\Big|_{\mathfrak{g}^{(1)}}\Big\rangle_{\mathfrak{g}^{(1)}} \\
&= \sum_{e \in E_x} p(e)\widetilde{\omega}(e) \\
&= -(\delta_p \omega)\big(\pi(x)\big) \\
&= \langle \gamma_p, \omega \rangle = {}_{\mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R})}\big\langle \omega, \rho_{\mathbb{R}}(\gamma_p)\big\rangle_{\mathfrak{g}^{(1)}}.
\end{aligned}$$

This gives the desired equation (2.4.4). ∎

2.5 Markov chains

Let us consider a time-homogeneous Markov chain $(\Omega_x(X), \mathbb{P}_x, \{w_n\}_{n=0}^\infty)$ with values in a Γ-covering graph X, where Γ is a torsion free, finitely generated group. Let $\pi_n : \Omega_x(X) \longrightarrow \Omega_{x,n}(X)\,(n \in \mathbb{N} \cup \{0\})$ be a projection defined by $\pi_n(c) := (e_1, e_2, \dots, e_n)$ for $c = (e_1, e_2, \dots, e_n, \dots) \in \Omega_x(X)$. Denote by $\{\mathcal{F}_n\}_{n=0}^\infty$ the filtration such that $\mathcal{F}_0 = \{\emptyset, \Omega_x(X)\}$ and $\mathcal{F}_n := \sigma\big(\pi_n^{-1}(A) \,\big|\, A \subset \Omega_{x,n}(X)\big)$ for $n \in \mathbb{N}$. We mention that $\mathcal{F}_n$ is a sub-σ-algebra of $\mathcal{F}_\infty := \bigvee_{n=0}^\infty \mathcal{F}_n$ for $n \in \mathbb{N}$.

Suppose first that Γ is abelian, that is, X is a crystal lattice. We denote by $\Phi : X \longrightarrow \Gamma \otimes \mathbb{R}$ a periodic realization of X. We then have the $\Gamma \otimes \mathbb{R}$-valued Markov chain $(\Omega_x(X), \mathbb{P}_x, \{\xi_n\}_{n=0}^\infty)$ defined by $\xi_n(c) := \Phi\big(w_n(c)\big)$ for $n \in \mathbb{N} \cup \{0\}$ and $c \in \Omega_x(X)$, through the map Φ. By applying the ergodic theorem, we easily verify that the law of large numbers on $\Gamma \otimes \mathbb{R}$

$$\lim_{n \to \infty} \frac{1}{n}\xi_n(\cdot) = \rho_{\mathbb{R}}(\gamma_p), \quad \mathbb{P}_x\text{-a.s.} \tag{2.5.1}$$

holds.

The notion of martingales plays a crucial role in the theory of stochastic processes. We give a certain characterization of modified harmonic realizations of crystal lattices in view of martingale theory. Indeed, we have the following:

Lemma 2.5.1 (cf. [42, Proposition 5.3]) *A periodic realization $\Phi_0 : X \longrightarrow \Gamma \otimes \mathbb{R}$ is the modified harmonic realization if and only if the $\Gamma \otimes \mathbb{R}$-valued stochastic process $\big\{\xi_n - n\rho_{\mathbb{R}}(\gamma_p)\big\}_{n=0}^\infty$ is an $\{\mathcal{F}_n\}$-martingale.*

The similar assertion to Lemma 2.5.1 holds in the case where Γ is nilpotent, that is, X is a Γ-nilpotent covering graph. Let $G = G_\Gamma$ be the nilpotent Lie group in which Γ is embedded as a cocompact lattice. We denote by $\Phi : X \longrightarrow G$ a Γ-equivariant realization. Then this map yields a G-valued Markov chain $(\Omega_x(X), \mathbb{P}_x, \{\xi_n\}_{n=0}^\infty)$ defined by $\xi_n(c) := \Phi\big(w_n(c)\big)$ for $n \in \mathbb{N} \cup \{0\}$ and $c \in \Omega_x(X)$. This gives rise to the $\mathfrak{g}$-valued random walk

$$\Xi_n(c) := \log\big(\xi_n(c)\big) = \log\big(\Phi\big(w_n(c)\big)\big) \qquad \big(n \in \mathbb{N} \cup \{0\},\, c \in \Omega_x(X)\big).$$

Note that an LLN on $\mathfrak{g}^{(1)}$ holds as in the case of crystal lattices.

Lemma 2.5.2 *As $n \to \infty$, we have*

$$\frac{1}{n}\Xi_n(\cdot)\big|_{\mathfrak{g}^{(1)}} \longrightarrow \rho_{\mathbb{R}}(\gamma_p), \quad \mathbb{P}_x\text{-}a.s. \tag{2.5.2}$$

Proof. Without loss of generality, we may put $\Phi(x) = \mathbf{1}_G$. For $c = (e_1, e_2, \dots) \in \Omega_x(X_0)$, we write

$$\Xi_n(c)\big|_{\mathfrak{g}^{(1)}} = \sum_{i=1}^n \Big\{ \log\big(\Phi\big(t(e_i)\big)\big)\big|_{\mathfrak{g}^{(1)}} - \log\big(\Phi\big(o(e_i)\big)\big)\big|_{\mathfrak{g}^{(1)}} \Big\}.$$

We take a basis $\{X_1^{(1)}, X_2^{(1)}, \dots, X_{d_1}^{(1)}\}$ of $\mathfrak{g}^{(1)}$ and put

$$F_k(e) := \log\big(\Phi\big(t(e)\big)\big)\big|_{X_k^{(1)}} - \log\big(\Phi\big(o(e)\big)\big)\big|_{X_k^{(1)}} \qquad (e \in E).$$

We fix $k = 1, 2, \dots, d_1$. Then we easily see that $F_k : E \longrightarrow \mathbb{R}$ satisfies $F_k(\overline{e}) = -F_k(e)$ for $e \in E$ and the Γ-invariance. Therefore, we apply Proposition 2.4.3 to obtain

$$\lim_{n\to\infty} \frac{1}{n}\Xi_n(c)\big|_{X_k^{(1)}} = \sum_{e \in E_0} \widetilde{m}(e) F_k(\widetilde{e}) = {}_{C_1(X_0,\mathbb{R})}\langle \gamma_p, F_k \rangle_{C^1(X_0,\mathbb{R})}, \quad \mathbb{P}_x\text{-a.s.},$$

where $\widetilde{e}$ stands for a lift of e to X. Let us take any $\mathbf{z} \in \mathrm{H}_1(X_0, \mathbb{R})$ and represent it as a closed path $c_0 = (e_1, e_2, \dots, e_\ell)$. Then we see

$${}_{C_1(X_0,\mathbb{R})}\langle \mathbf{z}, F_k \rangle_{C^1(X_0,\mathbb{R})} = \sum_{i=1}^n F_k(e_i) = \log\big(\Phi\big(t(c)\big)\big)\big|_{X_k^{(1)}} - \log\big(\Phi\big(o(c)\big)\big)\big|_{X_k^{(1)}} = \rho_{\mathbb{R}}(z)\big|_{X_k^{(1)}},$$

where c is a lift of c_0 to X. By taking $\mathbf{z} = \gamma_p \in \mathrm{H}_1(X_0, \mathbb{R})$, we conclude (2.5.2). ∎

In closing this subsection, we state a relation between the modified harmonicity and martingales in the nilpotent setting. We will use the following in the proof of Lemma 4.3.2 and Lemma 5.4.3.

Lemma 2.5.3 (cf. [32, Lemma 3.3]) *Let $\{X_1^{(1)}, X_2^{(1)}, \dots, X_{d_1}^{(1)}\}$ be a basis of $\mathfrak{g}^{(1)}$. Then a Γ-equivariant realization $\Phi_0 : X \longrightarrow G$ is the modified harmonic realization if and only if the stochastic process*

$$\big\{\Xi_n\big|_{X_i^{(1)}} - n\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}\big\}_{n=0}^\infty \qquad (i = 1, 2, \dots, d_1),$$

with values in $\mathbb{R}$, is an $\{\mathcal{F}_n\}$-martingale.

Proof. Suppose that Φ_0 is modified harmonic. For $n \in \mathbb{N}$ and $A \in \mathcal{F}_n$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_x}\Big[\Xi_{n+1}\big|_{X_i^{(1)}} - (n+1)\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}\,;\,A\Big]\\
&= \sum_{c\in\Omega_x(X)} p(c)\Big\{\log\Big(\Phi_0\big(t(e_{n+1})\big)\Big)\big|_{X_i^{(1)}} - (n+1)\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}\Big\}\mathbf{1}_A(c)\\
&= \sum_{c'\in\Omega_{x,n}(X)} p(c')\sum_{e\in E_{t(c')}} p(e)\bigg[\Big\{\log\Big(\Phi_0\big(t(e)\big)\Big)\big|_{X_i^{(1)}} - \rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}\Big\} - n\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}\bigg]\mathbf{1}_A(c'),
\end{aligned}$$

where $\mathbb{E}^{\mathbb{P}_x}$ stands for the expectation with respect to the probability measure $\mathbb{P}_x$. In terms of the modified harmonicity of Φ_0, this is equal to

$$\begin{aligned}
&\sum_{c'\in\Omega_{x,n}(X)} p(c')\Big\{\log\Big(\Phi_0\big(o(e_{n+1})\big)\Big)\big|_{X_i^{(1)}} - n\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}\Big\}\mathbf{1}_A(c')\\
&= \mathbb{E}^{\mathbb{P}_x}\Big[\Xi_n\big|_{X_i^{(1)}} - n\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}\,;\,A\Big]
\end{aligned}$$

Thus it follows that the process $\big\{\Xi_n\big|_{X_i^{(1)}} - n\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}\big\}_{n=0}^{\infty}$ is an $\{\mathcal{F}_n\}$-martingale. The converse is obvious from the argument above. ∎

2.6 Large deviation principles

Large deviation principles (LDP) are one of the most fundamental and important limit theorems and well-studied topics in probability theory as well as the LLNs and the CLTs. Before mentioning the results on LDPs on covering graphs, we start with a quick review of LDPs by using a simple exqample. Let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of $\mathbb{R}$-valued i.i.d. random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with mean μ and variance σ^2. We set $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \in \mathbb{N}$. We now assume that an LLN holds for $\{\xi_n\}_{n=1}^{\infty}$, that is,

$$\mathbb{P}\Big(\lim_{n\to\infty}\frac{1}{n}S_n = \mu\Big) = 1.$$

However, LDPs concern with how *exponentially* fast the probability that “rare” events such as

$$\mathbb{P}\Big(\frac{1}{n}S_n > x\Big) \qquad (x \geq \mu)$$

occur decays as $n \to \infty$, though such probability tends to zero as $n \to \infty$ by the LLN. More precisely, the LDP finds a lower semi-continuous function $I : \mathbb{R} \to [0, \infty]$, called the *rate function*, satisfying

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\frac{1}{n}S_n > x\Big) = -I(x) \qquad (x \geq \mu).$$

Note that such LDP is known as *Cramér's theorem*, which is one of the most fundamental formulations in the theory of large deviations.

Let us go back to related results on LDPs on covering graphs. Kotani and Sunada [42] established an LDP on a Γ-crystal lattice $X = (V, E)$ and discussed a relation with the pointed Gromov–Hausdorff limit of crystal lattices from a geometric perspective. See also Kotani [39] for related topic on the LDP, and Gromov [26] and Pansu [60] for the existence of the Gromov–Hausdorff limit in this setting. We fix a periodic realization $\Phi : X \longrightarrow \Gamma \otimes \mathbb{R}$ (not necessarily harmonic) and consider a $\Gamma \otimes \mathbb{R}$-valued Markov chain $(\Omega_x(X), \mathbb{P}_x, \{\xi_n = \Phi(w_n)\}_{n=0}^{\infty})$ for $x \in V$. For $\lambda \in \mathrm{Hom}(\Gamma, \mathbb{R}) \cong \mathbb{R}^d$, we set

$$\beta(\lambda) := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}^{\mathbb{P}_x}\big[\exp\big(\lambda(\xi_n)\big)\big].$$

Note that the existence of the limit in the right-hand side is always guaranteed. Moreover, $\beta : \mathrm{Hom}(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R}$ is analytic and its Hessian is positive definite. We now define a function $I : \Gamma \otimes \mathbb{R} \longrightarrow \mathbb{R} \cup \{\infty\}$ by the Fenchel–Legendre transform of β, that is,

$$I(\xi) := \sup_{\lambda \in \mathrm{Hom}(\Gamma, \mathbb{R})} \big\{\lambda(\xi) - \beta(\lambda)\big\} \qquad (\xi \in \mathbb{R}^d).$$

It is not difficult to see that I is lower semi-continuous. Then we have the following LDP for the random walk $\{\xi_n\}_{n=0}^{\infty}$.

Proposition 2.6.1 (cf. Kotani–Sunada [42, Proposition 1.5]) *An LDP holds for the $\Gamma \otimes \mathbb{R}$-valued random walk $\{\xi_n\}_{n=0}^{\infty}$ with the rate function $I : \Gamma \otimes \mathbb{R} \longrightarrow \mathbb{R} \cup \{\infty\}$. Namely, for any Borel measurable subset $A \subset \Gamma \otimes \mathbb{R}$, we have*

$$\begin{aligned} -\inf_{\xi \in A^\circ} I(\xi) &\leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_x\Big(\frac{1}{n}\xi_n \in A\Big) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_x\Big(\frac{1}{n}\xi_n \in A\Big) \leq -\inf_{\xi \in \overline{A}} I(\xi), \end{aligned}$$

where A° and $\overline{A}$ stands for the interior and the closure of A, respectively.

As a generalization of the above result to the nilpotent case, Tanaka [72] also established an LDP and discussed a similar geometric relation to the case of crystal lattices. For related results on an LDP on nilpotent groups, we refer to Baldi–Caremelino [4]. Let $X = (V, E)$ be a Γ-nilpotent covering graph and consider a G-valued Markov chain $(\Omega_x(X), \mathbb{P}_x, \{\xi_n = \Phi(w_n)\}_{n=0}^{\infty})$ for $x \in V$, where G is a nilpotent Lie group such that Γ is isomorphic to a cocompact lattice in G and $\Phi : X \longrightarrow G$ a Γ-equivariant realization. Let $h : G \longrightarrow G_\infty$ be a canonical diffeomorphism. Then an LDP for the G_∞-valued random walk $\{\tau_{1/n} h(\xi_n)\}_{n=0}^{\infty}$ is now stated as follows:

Proposition 2.6.2 (cf. Tanaka [72, Theorem 1.1]) *An LDP holds for the G_∞-valued random walk $\{\tau_{1/n} h(\xi_n)\}_{n=0}^{\infty}$ with a rate function $I : G_\infty \longrightarrow \mathbb{R} \cup \{\infty\}$. Namely, for any Borel measurable subset $A \subset G_\infty$, we have*

$$\begin{aligned} -\inf_{\xi \in A^\circ} I(\xi) &\leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_x\Big(\tau_{1/n} h(\xi_n) \in A\Big) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_x\Big(\tau_{1/n} h(\xi_n) \in A\Big) \leq -\inf_{\xi \in \overline{A}} I(\xi). \end{aligned}$$

We emphasize that, in this case, the rate function $I : G_\infty \longrightarrow \mathbb{R}$ is hard to write down explicitly. Because the proof of Proposition 2.6.2 is done by using an LDP on a $\mathfrak{g}^{(1)}$-valued absolutely continuous path space and several well-known lemmas in LDP theory (the contraction principle and transfer lemma, see e.g., Dembo–Zeitouni [16]).

Let $\mathcal{D}_I := \{g \in G_\infty \,|\, I(g) < \infty\}$ be the effective domain of the rate function I. Tanaka [72] also gave a geometric characterization of $\mathcal{D}_I$ in terms of the Carnot–Carathéodory metric d_{CC}.

Proposition 2.6.3 (cf. Tanaka [72, Theorem 1.2])

$$\mathcal{D}_I = \overline{B}_{d_{\mathrm{CC}}}(\mathbf{1}_G) := \{g \in G_\infty \,|\, d_{\mathrm{CC}}(g, \mathbf{1}_G) \leq 1\}.$$

On the other hand, the pointed Γ-nilpotent covering graph (X, x) endowed with the scaled graph distance εd converges to $(G_\infty, d_{\mathrm{CC}}, \mathbf{1}_G)$ as $\varepsilon \searrow 0$ in the sense of pointed Gromov–Hausdorff topology (cf. Pansu [60]).

Before closing this subsection, we briefly mention a relation between these two propositions putting an attention to the convergence above. The effective domain $\mathcal{D}_I$ is regarded as the set of points to which $\tau_{1/n}h(\xi_n)$ is "close" for sufficiently large n with some positive probability. We can check that

$$\lim_{n\to\infty} \frac{d_{\mathrm{CC}}(\mathbf{1}_G, \tau_{1/n}h(\xi_n))}{d(x, w_n)/n} = 1.$$

On the other hand, if the trajectory of the random walk on X is geodesic, then we see $d(x, w_n) = n$ and $d_{\mathrm{CC}}(\mathbf{1}_G, \tau_{1/n}h(\xi_n)) \to 1$ as $n \to \infty$. Thus, we see that $\tau_{1/n}h(\xi_n)$ converges to a point in $\partial\overline{B}_{d_{\mathrm{CC}}}(\mathbf{1}_G)$. This means that the G_∞-valued random walk $\{h(\xi_n)\}_{n=0}^\infty$ tends to infinity as $n \to \infty$ and $\tau_{1/n}h(\xi_n)$ converges to a point in $\partial\mathcal{D}_I$. The LDP detects such a rare event, though the probability that the event occurs may be zero.

Chapter 3

A measure-change formula for non-symmetric random walks on crystal lattices and its application

3.1 A measure-change technique

Throughout this chapter, Let Γ be a finitely generated abelian group of rank d with no torsions and X a Γ-crystal lattice with $X_0 := \Gamma\backslash X$. Suppose that a time-homogeneous Markov chain $(\Omega_x(X_0), \mathbb{P}_x, \{w_n\}_{n=0}^{\infty})$ governed by a *positive* transition probability $p : E_0 \longrightarrow (0, 1]$ is given, to avoid several technical difficulty.

Let $\Phi_0 : X \longrightarrow \Gamma \otimes \mathbb{R} \cong \mathbb{R}^d$ be the modified harmonic realization. For brevity, write

$$\lambda[\mathbf{x}]_{\Gamma\otimes\mathbb{R}} := {}_{\mathrm{Hom}(\Gamma,\mathbb{R})}\langle \lambda, \mathbf{x}\rangle_{\Gamma\otimes\mathbb{R}} \qquad \big(\lambda \in \mathrm{Hom}(\Gamma, \mathbb{R}),\, \mathbf{x} \in \Gamma \otimes \mathbb{R}\big),$$
$$d\Phi_0(e) := \Phi_0\big(t(e)\big) - \Phi_0\big(o(e)\big) \qquad (e \in E).$$

We take an orthonormal basis $\{\omega_1, \omega_2, \dots, \omega_d\}$ in $\mathrm{Hom}(\Gamma, \mathbb{R})\big(\subset (\mathcal{H}^1(X_0), \langle\!\langle \cdot, \cdot \rangle\!\rangle_p)\big)$ and denote by $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d\}$ its dual basis in $\Gamma \otimes \mathbb{R}$. Namely, $\omega_i[\mathbf{v}_j]_{\Gamma\otimes\mathbb{R}} = \delta_{ij}$ for $i, j = 1, 2, \dots, d$. We note that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d\}$ is an orthonormal basis of $\Gamma \otimes \mathbb{R}$ with respect to the Albanese metric g_0 associated with p. We may identify $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_d\omega_d \in \mathrm{Hom}(\Gamma, \mathbb{R})$ with $(\lambda_1, \lambda_2, \dots, \lambda_d) \in \mathbb{R}^d$. Furthermore, we write $x_i := \omega_i[\mathbf{x}]_{\Gamma\otimes\mathbb{R}}$, $\Phi_0(x)_i := \omega_i[\Phi_0(x)]_{\Gamma\otimes\mathbb{R}}$ and $\partial_i := \partial/\partial\lambda_i$ for $i = 1, 2, \dots, d$ and $x \in V$.

The purpose of this section is to establish a measure-change formula of the non-symmetric transition probability by applying a variational method given by Alexopoulos [2]. Let us consider a function $F = F_x(\lambda) : V_0 \times \mathrm{Hom}(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R}$ defined by

$$F_x(\lambda) := \sum_{e\in(E_0)_x} p(e) \exp\Big({}_{\mathrm{Hom}(\Gamma,\mathbb{R})}\big\langle \lambda, d\Phi_0(\widetilde{e})\big\rangle_{\Gamma\otimes\mathbb{R}}\Big), \tag{3.1.1}$$

for $x \in V_0$ and $\lambda \in \mathrm{Hom}(\Gamma, \mathbb{R})$. We easily see that $F = F_x(\lambda)$ is positive on $V_0 \times \mathrm{Hom}(\Gamma, \mathbb{R})$ with $F_x(\mathbf{0}) = 1$ for $x \in V_0$. The following lemma plays a significant role to construct the changed transition probability in our setting.

Lemma 3.1.1 *For every $x \in V_0$, the function $F_x(\cdot) : \mathrm{Hom}(\Gamma, \mathbb{R}) \longrightarrow (0, \infty)$ has a unique minimizer $\lambda_* = \lambda_*(x)$.*

Proof. For a fixed $x \in V_0$, we have

$$\begin{aligned}
\partial_i F_x(\lambda) &= \partial_i \Bigg(\sum_{e \in (E_0)_x} p(e) \exp\Big(\lambda\big[d\Phi_0(\widetilde{e})\big]_{\Gamma\otimes\mathbb{R}}\Big)\Bigg) \\
&= \partial_i \Bigg(\sum_{e \in (E_0)_x} p(e) \exp\Big(\sum_{i=1}^{d} \lambda_i \cdot \omega_i \big[d\Phi_0(\widetilde{e})\big]_{\Gamma\otimes\mathbb{R}}\Big)\Bigg) \\
&= \sum_{e \in (E_0)_x} p(e) \exp\Big(\lambda\big[d\Phi_0(\widetilde{e})\big]_{\Gamma\otimes\mathbb{R}}\Big) d\Phi_0(\widetilde{e})_i \qquad \big(i = 1, 2, \dots, d,\ \lambda \in \mathrm{Hom}(\Gamma, \mathbb{R})\big).
\end{aligned}$$

In other words,

$$\begin{aligned}
&\Big(\partial_1 F_x(\lambda), \dots, \partial_d F_x(\lambda)\Big) \\
&= \sum_{e \in (E_0)_x} p(e) \exp\Big(\lambda\big[d\Phi_0(\widetilde{e})\big]_{\Gamma\otimes\mathbb{R}}\Big) d\Phi_0(\widetilde{e}) \ (\in \Gamma \otimes \mathbb{R}) \qquad \big(\lambda \in \mathrm{Hom}(\Gamma, \mathbb{R})\big). \qquad (3.1.2)
\end{aligned}$$

Then we have

$$\partial_i \partial_j F_x(\lambda) = \sum_{e \in (E_0)_x} p(e) \exp\Big(\lambda\big[d\Phi_0(\widetilde{e})\big]_{\Gamma\otimes\mathbb{R}}\Big) d\Phi_0(\widetilde{e})_i d\Phi_0(\widetilde{e})_j$$

for $\lambda \in \mathrm{Hom}(\Gamma, \mathbb{R})$ and $i, j = 1, 2, \dots, d$, by repeating the calculation above. Therefore, it follows that $\big(\partial_i \partial_j F_x(\cdot)\big)_{i,j=1}^{d}$, the *Hessian matrix* of the function $F_x(\cdot)$, is positive definite. Indeed, consider the quadratic form corresponding to the Hessian matrix. Since

$$\begin{aligned}
&\sum_{i,j=1}^{d} \sum_{e \in (E_0)_x} p(e) \exp\Big(\lambda\big[d\Phi_0(\widetilde{e})\big]_{\Gamma\otimes\mathbb{R}}\Big) d\Phi_0(\widetilde{e})_i d\Phi_0(\widetilde{e})_j \xi_i \xi_j \\
&= \sum_{e \in (E_0)_x} p(e) \exp\Big(\lambda\big[d\Phi_0(\widetilde{e})\big]_{\Gamma\otimes\mathbb{R}}\Big) \Big\{ \sum_{i=1}^{d} d\Phi_0(\widetilde{e})_i \xi_i \Big\}^2 \geq 0 \qquad (3.1.3)
\end{aligned}$$

for $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbb{R}^d$ and the transition probability p is positive, we easily see that the Hessian matrix is non-negative definite. By multiplying both sides of (3.1.3) by $m(x)$ and taking the sum which runs over all vertices of X_0, we have

$$\sum_{e \in E_0} \widetilde{m}(e) \exp\Big(\lambda\big[d\Phi_0(\widetilde{e})\big]_{\Gamma\otimes\mathbb{R}}\Big) \Big\{ \sum_{i=1}^{d} d\Phi_0(\widetilde{e})_i \xi_i \Big\}^2 \geq 0, \qquad \big(\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbb{R}^d\big).$$

Suppose now that the left-hand side of (3.1.3) is zero. Then we have

$$\sum_{i=1}^{d} d\Phi_0(\widetilde{e})_i \xi_i = 0 \qquad (e \in E_0).$$

This equation implies $\langle \Phi_0(x), \boldsymbol{\xi}\rangle_{\mathbb{R}^d} = \langle \Phi_0(y), \boldsymbol{\xi}\rangle_{\mathbb{R}^d}$ for all $x, y \in V$, where $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ stands for the standard inner product on $\mathbb{R}^d$. Let $\{\sigma_1, \sigma_2, \dots, \sigma_d\}$ be a set of generators of $\Gamma \cong \mathbb{Z}^d$. It follows from the periodicity of Φ_0 that $\langle \sigma_i, \boldsymbol{\xi}\rangle_{\mathbb{R}^d} = 0$ for $i = 1, 2, \dots, d$. Hence, we conclude $\boldsymbol{\xi} = \mathbf{0}$. Namely, we have proved the positive definiteness of the Hessian matrix.

This implies that the function $F_x(\cdot) : \mathrm{Hom}(\Gamma, \mathbb{R}) \longrightarrow (0, \infty)$ is strictly convex for every $x \in V_0$. Moreover, it is easily observed that

$$\lim_{|\lambda|_{\mathbb{R}^d} \to \infty} F_x(\lambda) = \infty \qquad (x \in X_0),$$

by definition. Consequently, we know that there exists a unique minimizer $\lambda_* = \lambda_*(x) \in \mathrm{Hom}(\Gamma, \mathbb{R})$ of $F_x(\lambda)$ for each $x \in V_0$, thereby completing the proof. ∎

We are in a position to define a new transition probability on X_0. We define a positive function $\mathfrak{p} : E_0 \longrightarrow (0, 1]$ by

$$\mathfrak{p}(e) := \frac{p(e) \exp\Big({}_{\mathrm{Hom}(\Gamma,\mathbb{R})}\big\langle \lambda_*\big(o(e)\big), \Phi_0\big(t(\widetilde{e})\big) - \Phi_0\big(o(\widetilde{e})\big)\big\rangle_{\Gamma \otimes \mathbb{R}}\Big)}{F_{o(e)}\big(\lambda_*(o(e))\big)} \qquad (e \in E_0). \qquad (3.1.4)$$

We easily see that, by definition, the function $\mathfrak{p}$ also gives a positive transition probability on X_0. Thus, the transition probability $\mathfrak{p} : E_0 \longrightarrow (0, 1]$ yields an irreducible random walk $(\Omega_x(X_0), \widehat{\mathbb{P}}_x, \{w_n^{(\mathfrak{p})}\}_{n=0}^{\infty})$ with values in X_0 and so does the random walk $(\Omega_x(X), \widehat{\mathbb{P}}_x, \{w_n^{(\mathfrak{p})}\}_{n=0}^{\infty})$ on X. We then find the normalized invariant measure $\mathfrak{m} : V_0 \longrightarrow (0, 1]$ by applying the Perron-Frobenius theorem again. Put $\widetilde{\mathfrak{m}}(e) := \mathfrak{p}(e)\mathfrak{m}(o(e))$ for $e \in E_0$. We also denote by $\mathfrak{p} : E \longrightarrow (0, 1]$ and $\mathfrak{m} : V \longrightarrow (0, 1]$ the Γ-invariant lifts of $\mathfrak{p} : E_0 \longrightarrow (0, 1]$ and $\mathfrak{m} : V_0 \longrightarrow (0, 1]$, respectively. Let $g_0^{(\mathfrak{p})}$ be the ($\mathfrak{p}$-)Albanese metric on $\Gamma \otimes \mathbb{R}$ associated with the transition probability $\mathfrak{p}$. We take an orthonormal basis $\{\omega_1^{(\mathfrak{p})}, \omega_2^{(\mathfrak{p})}, \dots, \omega_d^{(\mathfrak{p})}\}$ of $\mathrm{Hom}(\Gamma, \mathbb{R})\big(\subset (\mathcal{H}^1(X_0), \langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{p}})\big)$.

We define the transition operator $L_{(\mathfrak{p})} : C_\infty(X) \longrightarrow C_\infty(X)$ associated with the transition probability $\mathfrak{p}$ by

$$L_{(\mathfrak{p})} f(x) := \sum_{e \in E_x} \mathfrak{p}(e) f\big(t(e)\big) \qquad (x \in V).$$

Recalling (3.1.2) and the definition of $\lambda_* = \lambda_*(x)$ yields

$$\Big(\partial_1 F_x\big(\lambda_*(x)\big), \dots, \partial_d F_x\big(\lambda_*(x)\big)\Big) = \sum_{e \in (E_0)_x} p(e) \exp\Big(\lambda_*(x)\big[d\Phi_0(\widetilde{e})\big]_{\Gamma \otimes \mathbb{R}}\Big) d\Phi_0(\widetilde{e}) = \mathbf{0}$$

for every $x \in V_0$. This immediately leads to

$$L_{(\mathfrak{p})}\Phi_0(x) - \Phi_0(x) = \sum_{e \in E_x} \mathfrak{p}(e) d\Phi_0(e) = \mathbf{0} \qquad (x \in V). \qquad (3.1.5)$$

By (3.1.5), one concludes that the given p-modified standard realization $\Phi_0 : X \longrightarrow \Gamma \otimes \mathbb{R}$ in the sense of (2.4.2) is the harmonic realization under the new transition probability $\mathfrak{p}$.

Remark 3.1.2 *Equation* (3.1.5) *readily implies* $\rho_{\mathbb{R}}(\gamma_{\mathfrak{p}}) = \mathbf{0}$*. Furthermore, we emphasize that the transition probability* $\mathfrak{p} : E_0 \longrightarrow (0, 1]$ *coincides with the original one* $p : E_0 \longrightarrow (0, 1]$ *provided that* $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$.

Remark 3.1.3 *In our setting, it is essential to assume that the given transition probability* p *is positive. Because, if it were not for the positivity of* p*, the assertion of Lemma* 4.2.1 *would not hold in general.* (*There is a case where the function* $F_x(\cdot)$ *has no minimizers.*)

3.2 Application to the proof of CLTs

In Ishiwata–Kawabi–Kotani [31], two kinds of CLTs for non-symmetric random walks on a crystal lattice X were established. We give yet another approach to prove an FCLT for them, by using the changed transition probability (3.1.4). We emphasize in advance that the ($\mathfrak{p}$-)harmonicity (3.1.5) plays an important role in the proof of the CLTs.

We fix a reference point $x_* \in V$ such that $\Phi_0(x_*) = \mathbf{0}$ and put

$$\xi_n^{(\mathfrak{p})}(c) := \Phi_0\big(w_n^{(\mathfrak{p})}(c)\big) \qquad \big(n = 0, 1, 2, \dots,\ c \in \Omega_{x_*}(X)\big).$$

We define a measurable map $\mathfrak{X}^{(n)} : \Omega_{x_*}(X) \longrightarrow \big(C_{\mathbf{0}}\big([0,\infty), (\Gamma \otimes \mathbb{R}, g_0^{(\mathfrak{p})})\big), \mu\big)$ by

$$\mathfrak{X}_t^{(n)}(c) := \frac{1}{\sqrt{n}}\Big\{\xi_{[nt]}^{(\mathfrak{p})}(c) + (nt - [nt])\big(\xi_{[nt]+1}^{(\mathfrak{p})}(c) - \xi_{[nt]}^{(\mathfrak{p})}(c)\big)\Big\} \qquad (t \geq 0), \tag{3.2.1}$$

where $C_{\mathbf{0}}\big([0,1], \Gamma\otimes\mathbb{R}\big)$ denotes the set of all continuous paths from $[0,\infty)$ to $\Gamma\otimes\mathbb{R}$ with the compact uniform topology and $\mu = \mu^{(\mathfrak{p})}$ is the Wiener measure on $C_{\mathbf{0}}\big([0,1], (\Gamma\otimes\mathbb{R}, g_0^{(\mathfrak{p})})\big)$. We also denote by $\mathrm{Lip}_{\mathbf{0}}\big([0,1]; (\Gamma\otimes\mathbb{R}, g_0^{(\mathfrak{p})})\big)$ the set of all Lipschitz continuous paths $w : [0,1] \longrightarrow \Gamma\otimes\mathbb{R}$ with $w_0 = \mathbf{0}$. We set

$$\|w\|_{\alpha\text{-Höl}} := \sup_{0\leq s\leq t\leq 1} \frac{\|w_t - w_s\|_{g_0^{(\mathfrak{p})}}}{(t-s)^{\alpha}} \qquad (\alpha < 1/2)$$

and define

$$C_{\mathbf{0}}^{0,\alpha\text{-Höl}}\big([0,\infty), (\Gamma\otimes\mathbb{R}, g_0^{(\mathfrak{p})})\big) := \overline{\mathrm{Lip}_{\mathbf{0}}\big([0,1]; (\Gamma\otimes\mathbb{R}, g_0^{(\mathfrak{p})})\big)}^{\|\cdot\|_{\alpha\text{-Höl}}} \qquad (\alpha < 1/2),$$

which is a Polish space. We write $\mathfrak{P}^{(n)}$ $(n = 1, 2, \dots)$ for the image probability measure on $C_{\mathbf{0}}^{0,\alpha\text{-Höl}}\big([0,\infty), (\Gamma\otimes\mathbb{R}, g_0^{(\mathfrak{p})})\big)$ induced by $\mathfrak{X}^{(n)}$. Then the functional CLT is stated as follows:

Theorem 3.2.1 *The sequence* $\{\mathfrak{X}^{(n)}\}_{n=1}^{\infty}$ *converges in law to a* $(\Gamma\otimes\mathbb{R}, g_0^{(\mathfrak{p})})$*-valued standard Brownian motion* $(B_t^{(\mathfrak{p})})_{t\geq 0}$ *starting from the origin in* $C_{\mathbf{0}}^{0,\alpha\text{-Höl}}\big([0,\infty), (\Gamma\otimes\mathbb{R}, g_0^{(\mathfrak{p})})\big)$.

As the first step, we prove the following, which asserts the convergence of the discrete laplacian on X under the suitable scaling.

Lemma 3.2.2 *For any $f \in C_0^\infty(\Gamma \otimes \mathbb{R})$, as $N \to \infty$ and $\varepsilon \searrow 0$ with $N^2\varepsilon \searrow 0$, we have*

$$\left\| \frac{1}{N\varepsilon^2}(I - L_{(\mathfrak{p})}^N)P_\varepsilon f - P_\varepsilon\Big(\frac{\Delta_{(\mathfrak{p})}}{2}f\Big) \right\|_\infty^X \longrightarrow 0.$$

Here $P_\varepsilon : C_\infty(\Gamma \otimes \mathbb{R}) \longrightarrow C_\infty(X)\,(0 \le \varepsilon \le 1)$ is a scaling operator defined by

$$P_\varepsilon f(x) := f\big(\varepsilon\Phi_0(x)\big) \qquad (x \in X)$$

and $\Delta_{(\mathfrak{p})}$ stands for the positive Laplacian $-\sum_{i=1}^d(\partial^2/\partial x_i^2)$ on $\Gamma \otimes \mathbb{R}$ associated with the $\mathfrak{p}$-Albanese metric $g_0^{(\mathfrak{p})}$.

Proof. For $i, j = 1, 2, \dots, d$ and $N \in \mathbb{N}$, we define a function $A^N(\Phi_0)_{ij} : V \longrightarrow \mathbb{R}$ by

$$A^N(\Phi_0)_{ij}(x) := \sum_{c \in \Omega_{x,N}(X)} \mathfrak{p}(c)\Big(\Phi_0\big(t(c)\big) - \Phi_0(x)\Big)_i\Big(\Phi_0\big(t(c)\big) - \Phi_0(x)\Big)_j \qquad (x \in V),$$

where $\mathfrak{p}(c) := \mathfrak{p}(e_1)\mathfrak{p}(e_2)\cdots\mathfrak{p}(e_N)$ for $c = (e_1, e_2, \dots, e_N) \in \Omega_{x,N}(X)$. Then we have

$$\begin{aligned}(I - L_{(\mathfrak{p})}^N)P_\varepsilon f(x) = &-\varepsilon\sum_{i=1}^d \frac{\partial f}{\partial x_i}\big(\varepsilon\Phi_0(x)\big)\sum_{c \in \Omega_{x,N}(X)} \mathfrak{p}(c)\Big(\Phi_0\big(t(c)\big) - \Phi_0(x)\Big)_i \\ &- \frac{\varepsilon^2}{2}\sum_{1 \le i,j \le d} \frac{\partial^2 f}{\partial x_i \partial x_j}\big(\varepsilon\Phi_0(x)\big)A^N(\Phi_0)_{ij}(x) + O\big((N\varepsilon)^3\big), \qquad (3.2.2)\end{aligned}$$

by applying Taylor's expansion formula. We see that the first term of the right-hand side of (3.2.2) vanishes due to (3.1.5). For $i, j = 1, 2, \dots, d$, we define a function $\mathcal{A}(\Phi_0)_{ij} : V_0 \longrightarrow \mathbb{R}$ by

$$\mathcal{A}(\Phi_0)_{ij}(x) := \sum_{e \in (E_0)_x} \mathfrak{p}(e)d\Phi_0(\widetilde{e})_i d\Phi_0(\widetilde{e})_j \qquad (x \in V).$$

We note that $\mathcal{A}(\Phi_0)_{ij}\big(\pi(x)\big) = A^1(\Phi_0)_{ij}(x)$ for $x \in V$ and $i, j = 1, 2, \dots, d$ thanks to the Γ-invariance of $A^N(\Phi_0)_{ij}$. Then, by using (3.1.5) again, we have

$$A^N(\Phi_0)_{ij}(x) = \sum_{k=0}^{N-1} L_{(\mathfrak{p})}^k\big(\mathcal{A}(\Phi_0)_{ij}\big)\big(\pi(x)\big) \qquad (x \in V).$$

The ergodic theorem for $L_{(\mathfrak{p})}$ (cf. [31, Theorem 3.2]) implies

$$\frac{1}{N}\sum_{k=0}^{N-1} L_{(\mathfrak{p})}^k\big(\mathcal{A}(\Phi_0)_{ij}\big)\big(\pi(x)\big) = \sum_{x \in V_0} \mathfrak{m}(x)\mathcal{A}(\Phi_0)_{ij}(x) + O\Big(\frac{1}{N}\Big).$$

Moreover, (2.4.3) and (3.1.5) lead to

$$\sum_{x \in V_0} \mathfrak{m}(x)\mathcal{A}(\Phi_0)_{ij}(x) = \sum_{e \in E_0} \widetilde{\mathfrak{m}}(e)\omega_i^{(\mathfrak{p})}(e)\omega_j^{(\mathfrak{p})}(e) = \langle\!\langle \omega_i^{(\mathfrak{p})}, \omega_j^{(\mathfrak{p})} \rangle\!\rangle_{\mathfrak{p}} = \delta_{ij}$$

for $i, j = 1, 2, \dots, d$. By putting it all together, we obtain

$$\frac{1}{N\varepsilon^2}(I - L_{(\mathfrak{p})}^N)P_\varepsilon f = P_\varepsilon\Big(\frac{\Delta_{(\mathfrak{p})}}{2}f\Big) + O(N^2\varepsilon) + O\Big(\frac{1}{N}\Big).$$

Finally, by letting $N \to \infty$ and $\varepsilon \searrow 0$ with $N^2\varepsilon \searrow 0$, we complete the proof. ∎

Lemma 3.2.2 immediately leads to the following lemma. (See [31, Theorem 2.1 and Lemma 4.2] for details.)

Lemma 3.2.3 (1) *For any* $f \in C_\infty(\Gamma \otimes \mathbb{R})$, *and* $0 \le s \le t$, *we have*

$$\lim_{n\to\infty} \Big\| L_{(\mathfrak{p})}^{[nt]-[ns]} P_{n^{-1/2}} f - P_{n^{-1/2}} \mathrm{e}^{-(t-s)\Delta_{(\mathfrak{p})}/2} f \Big\|_\infty^X = 0.$$

(2) *We fix* $0 \le t_1 < t_2 < \cdots < t_\ell < \infty \, (\ell \in \mathbb{N})$. *Then, we have*

$$(\mathfrak{X}_{t_1}^{(n)}, \mathfrak{X}_{t_2}^{(n)}, \dots, \mathfrak{X}_{t_\ell}^{(n)}) \xrightarrow{(d)} (B_{t_1}^{(\mathfrak{p})}, B_{t_2}^{(\mathfrak{p})}, \dots, B_{t_\ell}^{(\mathfrak{p})}) \qquad (n \to \infty),$$

where $(B_t^{(\mathfrak{p})})_{t\ge 0}$ *is a* $(\Gamma \otimes \mathbb{R}, g_0^{(\mathfrak{p})})$*-valued standard Brownian motion with* $B_0^{(\mathfrak{p})} = \mathbf{0}$.

Having obtained Lemma 3.2.3, it is sufficient to show the tightness of $\{\mathfrak{P}^{(n)}\}_{n=1}^\infty$ for completing the proof of Theorem 3.2.1.

Lemma 3.2.4 *The sequence* $\{\mathfrak{P}^{(n)}\}_{n=1}^\infty$ *is tight in* $C_{\mathbf{0}}^{0,\alpha\text{-Höl}}\big([0,\infty), (\Gamma \otimes \mathbb{R}, g_0^{(\mathfrak{p})})\big)$.

Proof. By virtue of the celebrated Kolmogorov's criterion, the assertion follows from the existence of a positive constant C independent of n such that

$$\mathbb{E}^{\widehat{\mathbb{P}}_{x_*}}\Big[\big\|\mathfrak{X}_t^{(n)} - \mathfrak{X}_s^{(n)}\big\|_{g_0^{(\mathfrak{p})}}^{4m}\Big] \le C(t-s)^{2m} \qquad (0 \le s \le t,\ m, n \in \mathbb{N}). \tag{3.2.3}$$

For this sake, it is sufficient to show that, for $k, \ell \in \mathbb{N} \cup \{0\}$ with $k \le \ell$, there is a constant $C > 0$ independent of n such that

$$\mathbb{E}^{\widehat{\mathbb{P}}_{x_*}}\Big[\big\|\mathfrak{X}_{\frac{\ell}{n}}^{(n)} - \mathfrak{X}_{\frac{k}{n}}^{(n)}\big\|_{g_0^{(\mathfrak{p})}}^{4m}\Big] \le C\Big(\frac{\ell-k}{n}\Big)^{2m} \qquad (0 \le s \le t,\ m, n \in \mathbb{N}). \tag{3.2.4}$$

Indeed, suppose that (3.2.4) holds. For $0 \le s \le t$, we take $k, \ell \in \mathbb{N} \cup \{0\}$ satisfying $k/n \le s < (k+1)/n$ and $\ell/n \le t < (\ell+1)/n$. Since the stochastic process $\mathfrak{X}^{(n)}$ is given by the linear interpolation, we have

$$\begin{aligned} \big\|\mathfrak{X}_{\frac{k+1}{n}}^{(n)} - \mathfrak{X}_s^{(n)}\big\|_{g_0^{(\mathfrak{p})}} &= (k - ns)\big\|\mathfrak{X}_{\frac{k+1}{n}}^{(n)} - \mathfrak{X}_{\frac{k}{n}}^{(n)}\big\|_{g_0^{(\mathfrak{p})}}, \\ \big\|\mathfrak{X}_t^{(n)} - \mathfrak{X}_{\frac{\ell}{n}}^{(n)}\big\|_{g_0^{(\mathfrak{p})}} &= (nt - \ell)\big\|\mathfrak{X}_{\frac{\ell+1}{n}}^{(n)} - \mathfrak{X}_{\frac{\ell}{n}}^{(n)}\big\|_{g_0^{(\mathfrak{p})}}. \end{aligned}$$

By using (3.2.4) and the triangle inequality, we have

$$
\begin{aligned}
&\mathbb{E}^{\widehat{\mathbb{P}}_{x_*}}\Big[\big\|\mathfrak{X}^{(n)}_t - \mathfrak{X}^{(n)}_s\big\|^{4m}_{g_0^{(\mathfrak{p})}}\Big]\\
&\le 3^{4m-1}\Big\{(k+1-ns)^{4m}\cdot C\Big(\frac{1}{n}\Big)^{2m} + C\Big(\frac{\ell-k-1}{n}\Big)^{2m} + (nt-\ell)^{4m}\cdot C\Big(\frac{1}{n}\Big)^{2m}\Big\}\\
&\le C\Big\{\Big(\frac{k+1}{n}-s\Big)^{2m} + \Big(\frac{\ell}{n}-\frac{k+1}{n}\Big)^{2m} + \Big(t-\frac{\ell}{n}\Big)^{2m}\Big\} \le C(t-s)^{2m},
\end{aligned}
$$

which is the desired estimate (3.2.3).

We now show (3.2.4). We put

$$
\|d\Phi_0\|_\infty := \max_{e\in E_0}\|d\Phi_0(\widetilde{e})\|_{g_0^{(\mathfrak{p})}}.
$$

Then we have

$$
\begin{aligned}
&\mathbb{E}^{\widehat{\mathbb{P}}_{x_*}}\Big[\big\|\mathfrak{X}^{(n)}_{\frac{\ell}{n}} - \mathfrak{X}^{(n)}_{\frac{k}{n}}\big\|^{4m}_{g_0^{(\mathfrak{p})}}\Big]\\
&= \Big(\frac{1}{\sqrt{n}}\Big)^{4m}\mathbb{E}^{\widehat{\mathbb{P}}_{x_*}}\Big[\big\|\xi^{(\mathfrak{p})}_\ell - \xi^{(\mathfrak{p})}_k\big\|^{4m}_{g_0^{(\mathfrak{p})}}\Big]\\
&\le Cn^{-2m}\max_{i=1,2,\dots,d}\max_{x\in\mathcal{F}}\Big\{\sum_{c\in\Omega_{x,\ell-k}(X)}\mathfrak{p}(c)\Big(\Phi_0\big(t(c)\big)-\Phi_0(x)\Big)^{4m}_i\Big\},
\end{aligned}
\tag{3.2.5}
$$

where $\mathcal{F}$ stands for the fundamental domain of X containing $x_* \in V$. In terms of $c = (e_1, e_2, \dots, e_{\ell-k}) \in \Omega_{x_*,\ell-k}(X)$, we write

$$
\Big(\Phi_0\big(t(c)\big)-\Phi_0(x)\Big)^{4m}_i = \Big\{\sum_{j=1}^{\ell-k}\Big(d\Phi_0(e_j)\Big)_i\Big\}^{4m}.
$$

We use Lemma 2.5.1 and the Burkholder–Davis–Gundy inequality to obtain

$$
\begin{aligned}
&\sum_{c\in\Omega_{x,\ell-k}(X)}\mathfrak{p}(c)\Big\{\sum_{j=1}^{\ell-k}\Big(d\Phi_0(e_j)\Big)_i\Big\}^{4m}\\
&\le \mathcal{C}^{4m}_{(4m)}\sum_{c\in\Omega_{x,\ell-k}(X)}\mathfrak{p}(c)\Big\{\sum_{j=1}^{\ell-k}\Big(d\Phi_0(e_j)\Big)^2_i\Big\}^{2m}\\
&\le \mathcal{C}^{4m}_{(4m)}\|d\Phi_0\|^{4m}_\infty(\ell-k)^{2m}
\end{aligned}
\tag{3.2.6}
$$

for $i = 1, 2, \dots, d$ and $x \in \mathcal{F}$, where $\mathcal{C}_{(4m)}$ stands for the positive constant which appears in the Burkholder–Davis–Gundy inequality with the exponent $4m$. Combining (3.2.5) with (3.2.6) immediately implies (3.2.4) and this completes the proof. ∎

Let $\mathfrak{p}(n,x,y)$ be the n-step transition probability defined by $\mathfrak{p}(n,x,y) := L^n_{(\mathfrak{p})}\delta_y(x)$ for $n \in \mathbb{N}$ and $x, y \in V$ We are interested in a relation between the n-step transition probabilities $p(n,x,y)$ and $\mathfrak{p}(n,x,y)$. We here give a certain asymptotic formula for $p(n,x,y)$ and $\mathfrak{p}(n,x,y)$ as $n \to \infty$.

Theorem 3.2.5 *There exist some positive constants C_1 and C_2 such that*

$$C_1 p(n,x,y)\exp\big(nM_p\big) \le \mathfrak{p}(n,x,y) \le C_2 p(n,x,y)\exp\big(nM_p\big)$$

for all $n \in \mathbb{N}$ and $x, y \in V$, where

$$M_p := \sum_{\boldsymbol{x}\in V_0} m(\boldsymbol{x})\Big({}_{\mathrm{Hom}(\Gamma,\mathbb{R})}\big\langle \lambda_*(\boldsymbol{x}), \rho_{\mathbb{R}}(\gamma_p)\big\rangle_{\Gamma\otimes\mathbb{R}} - \log F_{\boldsymbol{x}}\big(\lambda_*(\boldsymbol{x})\big)\Big).$$

Proof. For $n \in \mathbb{N}$ and $x, y \in V$, we have

$$\begin{aligned}
\mathfrak{p}(n,x,y) &= \sum_{\substack{(e_1,e_2,\dots,e_n)\in\Omega_{x,n}(X)\\ o(e_1)=x,\, t(e_n)=y}} \mathfrak{p}(e_1)\mathfrak{p}(e_2)\cdots\mathfrak{p}(e_n)\\
&= \sum_{\substack{(e_1,e_2,\dots,e_n)\in\Omega_{x,n}(X)\\ o(e_1)=x,\, t(e_n)=y}} p(e_1)p(e_2)\cdots p(e_n)\cdot\exp\Big(\sum_{i=1}^{n}\lambda_*\big(o(e_i)\big)[d\Phi_0(\widetilde{e}_i)]_{\Gamma\otimes\mathbb{R}}\Big)\\
&\qquad\qquad \times F_{o(e_1)}\big(\lambda_*(o(e_1))\big)^{-1}F_{o(e_2)}\big(\lambda_*(o(e_2))\big)^{-1}\cdots F_{o(e_n)}\big(\lambda_*(o(e_n))\big)^{-1}\\
&= \sum_{\substack{(e_1,e_2,\dots,e_n)\in\Omega_{x,n}(X)\\ o(e_1)=x,\, t(e_n)=y}} p(e_1)p(e_n)\cdots p(e_n)\\
&\qquad\qquad \times\exp\Big(\sum_{i=1}^{n}\lambda_*\big(o(e_i)\big)[d\Phi_0(\widetilde{e}_i)]_{\Gamma\otimes\mathbb{R}} - \sum_{i=1}^{n}\log F_{o(e_i)}\big(\lambda_*(o(e_i))\big)\Big).
\end{aligned}$$

By applying Proposition 2.4.3:

$$\frac{1}{n}\sum_{i=1}^{n} f(e_i) = \sum_{e\in E_0}\widetilde{m}(e)f(e) + O\Big(\frac{1}{n}\Big) \qquad (f : E_0 \longrightarrow \mathbb{R}), \tag{3.2.7}$$

we obtain

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\Big(\lambda_*\big(o(e_i)\big)[d\Phi_0(\widetilde{e}_i)]_{\Gamma\otimes\mathbb{R}} - \log F_{o(e_i)}\big(\lambda_*(o(e_i))\big)\Big)\\
&= \sum_{e\in E_0}\widetilde{m}(e)\Big(\lambda_*\big(o(e)\big)[d\Phi_0(\widetilde{e})]_{\Gamma\otimes\mathbb{R}} - \log F_{o(e)}\big(\lambda_*(o(e))\big)\Big) + O\Big(\frac{1}{n}\Big)\\
&= \sum_{\boldsymbol{x}\in V_0} m(\boldsymbol{x})\Big(\lambda_*(\boldsymbol{x})\Big[\sum_{e\in(E_0)_{\boldsymbol{x}}} d\Phi_0(\widetilde{e})\Big]_{\Gamma\otimes\mathbb{R}} - \log F_{\boldsymbol{x}}\big(\lambda_*(\boldsymbol{x})\big)\Big) + O\Big(\frac{1}{n}\Big)\\
&= \sum_{\boldsymbol{x}\in V_0} m(\boldsymbol{x})\Big(\lambda_*(\boldsymbol{x})\big[\rho_{\mathbb{R}}(\gamma_p)\big]_{\Gamma\otimes\mathbb{R}} - \log F_{\boldsymbol{x}}\big(\lambda_*(\boldsymbol{x})\big)\Big) + O\Big(\frac{1}{n}\Big)
\end{aligned}$$

for $x, y \in V$. Here we used the (2.4.2) for the final line. Finally, we obtain

$$\mathfrak{p}(n,x,y) = p(n,x,y)\exp\Big(n\sum_{\boldsymbol{x}\in V_0} m(\boldsymbol{x})\Big(\lambda_*(\boldsymbol{x})\big[\rho_{\mathbb{R}}(\gamma_p)\big]_{\Gamma\otimes\mathbb{R}} - \log F_{\boldsymbol{x}}\big(\lambda_*(\boldsymbol{x})\big)\Big) + O(1)\Big)$$

for $x, y \in V$. This completes the proof. ∎

Remark 3.2.6 *Let us consider the case where the* Γ*-crystal lattice* X *is given by a covering graph of an* ℓ*-bouquet graph* ($\ell \in \mathbb{N}$) *consisting of one vertex* $\boldsymbol{x} \in V_0$ *and* ℓ*-loops. Without using the ergodic theorem* (3.2.7) *in the proof of* Theorem 3.2.5*, we also obtain*

$$\begin{aligned}\mathfrak{p}(n,x,y) &= \sum_{\substack{(e_1,e_2,\dots,e_n)\in\Omega_{x,n}(X)\\ o(e_1)=x,\, t(e_n)=y}} p(e_1)p(e_2)\cdots p(e_n)\\ &\qquad\times \exp\Big(\sum_{i=1}^n \lambda_*(\boldsymbol{x})[d\Phi_0(\widetilde{e}_i)]_{\Gamma\otimes\mathbb{R}}\Big)\cdot F_{\boldsymbol{x}}\big(\lambda_*(\boldsymbol{x})\big)^{-n}\\ &= p(n,x,y)\exp\Big(\lambda_*(\boldsymbol{x})\big[\Phi_0(y)-\Phi_0(x)\big]_{\Gamma\otimes\mathbb{R}}\Big)\cdot F_{\boldsymbol{x}}\big(\lambda_*(\boldsymbol{x})\big)^{-n} \qquad (3.2.8)\end{aligned}$$

for every $n \in \mathbb{N}$ *and* $x, y \in V$.

3.3 A relation with a discrete analogue of Girsanov's formula

In closing this chapter, we discuss a relation between our formula (3.2.8) and a discrete analogue of Girsanov's theorem due to Fujita [23].

Let $X = (V, E)$ be a crystal lattice covered with a one-bouquet graph $X_0 = (V_0, E_0)$; $V_0 = \{\boldsymbol{x}\}$ and $E_0 = \{e, \overline{e}\}$, by the group action $\Gamma = \langle\sigma\rangle \cong \mathbb{Z}^1$. We consider a random walk on X_0 with the transition probability

$$p(e) = p \quad \text{and} \quad p(\overline{e}) = 1-p \qquad (0 < p < 1).$$

We take a bijective linear map $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{R}) \longrightarrow \Gamma\otimes\mathbb{R}(\cong \mathbb{R}^1)$ by $\rho_{\mathbb{R}}([e]) = \sigma$. Then we have $\gamma_p = (2p-1)[e]$ and $\rho_{\mathbb{R}}(\gamma_p) = (2p-1)\sigma$. Let $\{u\} \subset \mathrm{Hom}(\Gamma,\mathbb{R}) = \big(\mathrm{H}^1(X_0,\mathbb{R}), \langle\!\langle \cdot,\cdot\rangle\!\rangle_p\big)$ be a dual basis of $\{\sigma\otimes 1 = \sigma\} \subset \Gamma\otimes\mathbb{R}$. We easily see that $\langle\!\langle u,u\rangle\!\rangle_p = 4p(1-p)$. Hence the orthogonalization $\{v\} \subset \mathrm{Hom}(\Gamma,\mathbb{R})$ of $\{u\}$ is given by

$$v = \frac{1}{\sqrt{4p(1-p)}}u.$$

To the end, we identify $\lambda v \in \mathrm{Hom}(\Gamma,\mathbb{R})$ with $\lambda \in \mathbb{R}$. We denote by $\{\mathbf{v}\} \subset \Gamma\otimes\mathbb{R}$ the dual basis of $\{v\}$. Then we observe that the realization $\Phi_0 : X \longrightarrow (\Gamma\otimes\mathbb{R}; \{\mathbf{v}\})$ defined by

$$d\Phi_0(\widetilde{e}) := \sigma = \frac{1}{\sqrt{4p(1-p)}}\mathbf{v}$$

is the modified standard realization of X.

We now consider a function $F = F_{\boldsymbol{x}}(\lambda)$ defined by (3.1.1), that is,

$$F_{\boldsymbol{x}}(\lambda) = p\exp\Big(\frac{\lambda}{\sqrt{4p(1-p)}}\Big) + (1-p)\exp\Big(-\frac{\lambda}{\sqrt{4p(1-p)}}\Big) \qquad (\lambda \in \mathbb{R}).$$

Then the minimizer $\lambda_* = \lambda_*(\boldsymbol{x})$ and $F_{\boldsymbol{x}}(\lambda_*)$ are given by

$$\lambda_* = \sqrt{p(1-p)} \log \frac{p-1}{p}, \qquad F_{\boldsymbol{x}}(\lambda_*) = \sqrt{4p(1-p)},$$

respectively. We fix $x \in V$ satisfying $\Phi_0(x) = \mathbf{0}$. For $y \in V$, we write $\Phi_0(y) = k(y)\mathbf{v}$. Then the formula (3.2.8) implies

$$\mathfrak{p}(n,x,y) = p(n,x,y) \cdot \Big(\frac{p-1}{p}\Big)^{-k(y)/2} \cdot \big(\sqrt{4p(1-p)}\big)^{-n} \qquad (n \in \mathbb{N},\, y \in V).$$

In Fujita [23, page 115], the formula above is called a *discrete analogue of Girsanov's theorem* for a non-symmetric random walk $\{Z_n\}_{n=0}^{\infty}$ on $\mathbb{Z}^1$ given by the sum of independent random variables $\{\xi_i\}_{i=1}^{\infty}$ with $\mathbb{P}(\xi_i = 1) = p$ and $\mathbb{P}(\xi_i = -1) = 1 - p$ for $i = 1, 2, \ldots$. Hence we may regard (3.2.8) as a generalization of the discrete Girsanov's theorem to the case of non-symmetric random walks on the ℓ-bouquet graph.

Chapter 4

CLTs of the first kind for non-symmetric random walks on nilpotent covering graphs

4.1 Settings and Statements

Throughout this chapter, suppose that X is a Γ-nilpotent covering graph of a finite graph X_0, that is, Γ is a torsion free, finitely generated nilpotent group of step r. Let G be the connected and simply connected nilpotent Lie group of step r such that Γ is isomorphic to a cocompact lattice in G and $\mathfrak{g} = \bigoplus_{k=1}^{r} \mathfrak{g}^{(k)}$ the corresponding Lie algebra. For notations or properties of random walks on X, nilpotent Lie group G and its Lie algebra, see Section 2.

We now give the settings and statements of CLTs of the first kind in the present section. At the beginning, we need to introduce a special function space in order to discuss CLTs. For $q > 1$, we define

$$C_{\infty,q}(X \times \mathbb{Z}) := \big\{ f = f(x,z) : X \times \mathbb{Z} \longrightarrow \mathbb{R} \,\big|\, f(\cdot, z) \in C_\infty(X),\, \|f\|_{\infty,q} < \infty \big\},$$

where $\|f\|_{\infty,q}$ is a norm on $C_{\infty,q}(X \times \mathbb{Z})$ given by

$$\|f\|_{\infty,q} := \frac{1}{C_q} \sum_{z \in \mathbb{Z}} \frac{\|f(\cdot,z)\|_\infty^X}{1+|z|^q}, \qquad C_q := \sum_{z \in \mathbb{Z}} \frac{1}{1+|z|^q} < \infty.$$

Then we see that $(C_{\infty,q}(X \times \mathbb{Z}), \|\cdot\|_{\infty,q})$ is a Banach space. We introduce the *transition-shift operator* $\mathcal{L}_p : C_{\infty,q}(X \times \mathbb{Z}) \longrightarrow C_{\infty,q}(X \times \mathbb{Z})$ by

$$\mathcal{L}_p f(x,z) := \sum_{e \in E_x} p(e) f\big(t(e), z+1\big) \qquad (x \in V,\ z \in \mathbb{Z}) \tag{4.1.1}$$

and the *approximation operator* $\mathcal{P}_\varepsilon : C_\infty(G) \longrightarrow C_{\infty,q}(X \times \mathbb{Z})$ by

$$\mathcal{P}_\varepsilon f(x,z) := f\Big(\tau_\varepsilon\big(\Phi_0(x) * \exp(-z\rho_{\mathbb{R}}(\gamma_p))\big)\Big) \qquad (0 \le \varepsilon \le 1,\ x \in V,\ z \in \mathbb{Z}). \tag{4.1.2}$$

We give an important property of the family of approximation operators $(\mathcal{P}_\varepsilon)_{0 \le \varepsilon \le 1}$.

Lemma 4.1.1 *Let $q > 1$. Then $\big((C_{\infty,q}(X \times \mathbb{Z}), \|\cdot\|_{\infty,q}; \mathcal{P}_\varepsilon)\big)_{0 \le \varepsilon \le 1}$ is a family of Banach spaces approximating to the Banach space $\big(C_\infty(G), \|\cdot\|_\infty^G\big)$ in the sense of Trotter* [74]:

$$\|\mathcal{P}_\varepsilon f\|_{\infty,q} \le \|f\|_\infty^G \quad \textit{and} \quad \lim_{\varepsilon \searrow 0} \|\mathcal{P}_\varepsilon f\|_{\infty,q} = \|f\|_\infty^G \qquad \big(f \in C_\infty(G)\big).$$

Proof. The former assertion follows from

$$\|\mathcal{P}_\varepsilon f\|_{\infty,q} = \frac{1}{C_q} \sum_{z \in \mathbb{Z}} \frac{\|f(\cdot, z)\|_\infty}{1 + |z|^q} \le \frac{1}{C_q} \sum_{z \in \mathbb{Z}} \frac{\|f\|_\infty}{1 + |z|^q} = \|f\|_\infty.$$

We prove the latter one. Let $g_0 \in G$ be an element which attains $\|f\|_\infty = \sup_{g \in G} |f(g)|$. We fix $z \in \mathbb{Z}$. Then we have

$$\|\mathcal{P}_\varepsilon f(\cdot, z)\|_\infty \ge |f(g_0)| - \inf_{x \in X} \Big| f(g_0) - f\Big(\tau_\varepsilon\big(\Phi_0(x) * \exp(-z\rho_{\mathbb{R}}(\gamma_p))\big)\Big)\Big|.$$

On the other hand, we have

$$\begin{aligned}
&\inf_{x \in X} d_{\mathrm{CC}}\Big(g_0, \tau_\varepsilon\big(\Phi_0(x) * \exp(-z\rho_{\mathbb{R}}(\gamma_p))\big)\Big) \\
&= \varepsilon \inf_{x \in X} d_{\mathrm{CC}}\Big(\tau_{1/\varepsilon}(g_0), \Phi_0(x) * \exp(-z\rho_{\mathbb{R}}(\gamma_p))\Big) < \varepsilon M
\end{aligned}$$

for some $M = M(z) > 0$. From the continuity of f, for any $\delta > 0$, there exists $\delta' > 0$ such that $d_{\mathrm{CC}}(g_0, h) < \delta'$ implies $|f(g_0) - f(h)| < \delta$. By choosing a sufficiently small $\varepsilon > 0$, we have

$$d_{\mathrm{CC}}\Big(g_0, \tau_\varepsilon\big(\Phi_0(x_*) * \exp(-z\rho_{\mathbb{R}}(\gamma_p))\big)\Big) < \delta'$$

for some $x_* \in X$. Then we have

$$\begin{aligned}
&\inf_{x \in X} \Big| f(g_0) - f\Big(\tau_\varepsilon\big(\Phi_0(x) * \exp(-z\rho_{\mathbb{R}}(\gamma_p))\big)\Big)\Big| \\
&\le \Big| f(g_0) - f\Big(\tau_\varepsilon\big(\Phi_0(x_*) * \exp(-z\rho_{\mathbb{R}}(\gamma_p))\big)\Big)\Big| < \delta
\end{aligned}$$

and this implies $\lim_{\varepsilon \searrow 0} \|\mathcal{P}_\varepsilon f(\cdot, z)\|_\infty = \|f\|_\infty$ for $z \in \mathbb{Z}$. By using the dominated convergence theorem, we obtain $\lim_{\varepsilon \searrow 0} \|\mathcal{P}_\varepsilon f\|_{\infty,r} = \|f\|_\infty$. This completes the proof. ∎

We extend each $Z \in \mathfrak{g}$ as a left invariant vector field Z_* on G as follows:

$$Z_* f(g) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f\big(g * \exp(\varepsilon Z)\big) \qquad \big(f \in C^\infty(G),\, g \in G\big).$$

We put

$$\beta(\Phi_0) := \sum_{e \in E_0} \widetilde{m}(e) \log\Big(\Phi_0\big(o(\widetilde{e})\big)^{-1} \cdot \Phi_0\big(t(\widetilde{e})\big) \cdot \exp(-\rho_{\mathbb{R}}(\gamma_p))\Big)\Big|_{\mathfrak{g}^{(2)}},$$

where $\widetilde{e}$ stands for a lift of $e \in E_0$ to X. It should be noted that $\gamma_p = 0$ implies $\beta(\Phi_0) = \mathbf{0}_{\mathfrak{g}}$. However, even if $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}_{\mathfrak{g}}$, the quantity $\beta(\Phi_0)$ does not vanish in general. Furthermore,

$\beta(\Phi_0)$ does not depend on the choice of the $\mathfrak{g}^{(2)}$-component of the modified harmonic realization $\Phi_0 : X \longrightarrow G$, though it has the ambiguity in the component corresponding to $\mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$. See Proposition 4.2.3 for details and Chapter 6 for a concrete example.

Then the following is a semigroup-type CLT of the first kind.

Theorem 4.1.2 *For $q > 4r + 1$, the following hold:*
(1) *For $0 \le s \le t$ and $f \in C_\infty(G)$, we have*

$$\lim_{n\to\infty} \Big\| \mathcal{L}_p^{[nt]-[ns]} \mathcal{P}_{n^{-1/2}} f - \mathcal{P}_{n^{-1/2}} \mathrm{e}^{-(t-s)\mathcal{A}} f \Big\|_{\infty,q} = 0, \tag{4.1.3}$$

where $(\mathrm{e}^{-t\mathcal{A}})_{t\ge 0}$ is the C_0-semigroup with the infinitesimal generator $\mathcal{A}$ on $C_0^\infty(G)$ defined by

$$\mathcal{A} := -\frac{1}{2}\sum_{i=1}^{d_1} V_{i*}^2 - \beta(\Phi_0)_*, \tag{4.1.4}$$

where $\{V_1, V_2, \dots, V_{d_1}\}$ denotes an orthonormal basis of $(\mathfrak{g}^{(1)}, g_0)$.
(2) *Let μ be a Haar measure on G. Fix $z \in \mathbb{Z}$. Then, for any sequence $\{x_n\}_{n=1}^\infty \subset V$ with*

$$\lim_{n\to\infty} \tau_{n^{-1/2}} \Big(\Phi_0(x_n) * \exp\big(- z\rho_{\mathbb{R}}(\gamma_p)\big)\Big) = g \in G$$

and for any $f \in C_\infty(G)$, we have

$$\lim_{n\to\infty} \mathcal{L}_p^{[nt]} \mathcal{P}_{n^{-1/2}} f(x_n, z) = \mathrm{e}^{-t\mathcal{A}} f(g) = \int_G \mathcal{H}_t(h^{-1} * g) f(h)\, \mu(dh) \qquad (t > 0), \tag{4.1.5}$$

where $\mathcal{H}_t(g)$ is a fundamental solution to the heat equation with drift

$$\Big(\frac{\partial}{\partial t} + \mathcal{A}\Big) u(t,g) = 0 \qquad (t > 0,\ g \in G).$$

We now fix a reference point $x_* \in V$ such that $\Phi_0(x_*) = \mathbf{1}_G$ and put

$$\xi_n(c) := \Phi_0\big(w_n(c)\big) \qquad \big(n \in \mathbb{N} \cup \{0\},\ c \in \Omega_{x_*}(X)\big).$$

We then have a G-valued random walk $(\Omega_{x_*}(X), \mathbb{P}_{x_*}, \{\xi_n\}_{n=0}^\infty)$ starting from $\mathbf{1}_G$. For $t \ge 0$, we define a map $\mathcal{X}_t^{(n)} : \Omega_{x_*}(X) \longrightarrow G$ by

$$\mathcal{X}_t^{(n)}(c) := \tau_{n^{-1/2}} \Big(\xi_{[nt]}(c) * \exp\big(- [nt]\rho_{\mathbb{R}}(\gamma_p)\big)\Big) \qquad \big(n \in \mathbb{N},\ c \in \Omega_{x_*}(X)\big).$$

Denote by $\mathcal{D}_n$ the partition $\{t_k = k/n \,|\, k = 0, 1, \dots, n\}$ of $[0,1]$ for $n \in \mathbb{N}$. We define a G-valued continuous stochastic process $(\mathcal{Y}_t^{(n)})_{0\le t\le 1}$ by the geodesic interpolation of $\{\mathcal{X}_{t_k}^{(n)}\}_{k=0}^n$with respect to the Carnot–Carathéodory metric d_{CC}. It is worth noting that (4.1.5) implies

$$\lim_{n\to\infty} \sum_{c \in \Omega_{x_*}(X)} p(c) f\big(\mathcal{X}_t^{(n)}(c)\big) = \int_G \mathcal{H}_t(h^{-1}) f(h)\, \mu(dh) \qquad \big(f \in C_\infty(G)\big). \tag{4.1.6}$$

Let $d_1 = \dim_{\mathbb{R}} \mathfrak{g}^{(1)}$. We consider an SDE

$$dY_t = \sum_{i=1}^{d_1} V_{i*}(Y_t) \circ dB_t^i + \beta(\Phi_0)_*(Y_t)\, dt, \qquad Y_0 = \mathbf{1}_G, \tag{4.1.7}$$

where $(B_t)_{0\le t\le 1} = (B_t^1, B_t^2, \dots, B_t^{d_1})_{0\le t\le 1}$ is an $\mathbb{R}^{d_1}$-valued standard Brownian motion with $B_0 = \mathbf{0}$. Let $(Y_t)_{0\le t\le 1}$ be the G-valued diffusion process which solves (4.1.7). In Proposition 4.5.3 below, we prove that the infinitesimal generator of $(Y_t)_{0\le t\le 1}$ coincides with $-\mathcal{A}$ defined by (4.1.4). Let $\mathrm{Lip}_{\mathbf{1}_G}([0,1];G)$ be the set of all Lipschitz continuous path $w : [0,1] \longrightarrow G$ such that $w_0 = \mathbf{1}_G$. We also set, for $\alpha < 1/2$,

$$C_{\mathbf{1}_G}^{\alpha\text{-Höl}}([0,1];G) = \Big\{ w \in C_{\mathbf{1}_G}([0,1];G) \;:\; \|w\|_{\alpha\text{-Höl}} := \sup_{s,t\in[0,1]} \frac{d_{\mathrm{CC}}(w_s, w_t)}{|t-s|^\alpha} < \infty \Big\}.$$

We define

$$C_{\mathbf{1}_G}^{0,\alpha\text{-Höl}}([0,1];G) := \overline{\mathrm{Lip}_{\mathbf{1}_G}([0,1];G)}^{\|\cdot\|_{\alpha\text{-Höl}}}, \tag{4.1.8}$$

which is separable in the α-Hölder topology (cf. Friz–Victoir [22, Section 8]). Let $\mathbf{P}^{(n)}$ be the image measure on $C_{\mathbf{1}_G}^{0,\alpha\text{-Höl}}([0,1];G)$ induced by $\mathcal{Y}^{(n)}_{\cdot}$ for $n \in \mathbb{N}$.

We now in a position to present an FCLT of the first kind for the non-symmetric random walk $\{w_n\}_{n=0}^\infty$ on X.

Theorem 4.1.3 *We assume the centered condition* **(C):** $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}_{\mathfrak{g}}$. *Then the sequence* $(\mathcal{Y}_t^{(n)})_{0\le t\le 1}$ $(n = 1, 2, \dots)$ *converges in law to the G-valued diffusion process* $(Y_t)_{0\le t\le 1}$ *in* $C_{\mathbf{1}_G}^{0,\alpha\text{-Höl}}([0,1];G)$ *as* $n \to \infty$ *for all* $\alpha < 1/2$.

Let us make comments on Theorems 4.1.2 and 4.1.3. As is emphasized in Breuillard [10, Section 6], the situation of the non-centered case $\rho_{\mathbb{R}}(\gamma_p) \neq \mathbf{0}_{\mathfrak{g}}$ is quite different from the centered case $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}_{\mathfrak{g}}$ and thus some technical difficulties arise to obtain CLTs. That is why there are few papers which discuss CLTs for non-centered random walks on nilpotent Lie groups. We obtain, in Theorem 4.1.2, a semigroup CLT for the non-centered random walk $\{\xi_n\}_{n=0}^\infty$ on G with a canonical dilation $\tau_{n^{-1/2}}$, while Crépel–Raugi [15] and Raugi [63] proved similar CLTs for the random walk to (4.1.6) with spatial scalings whose orders are higher than $\tau_{n^{-1/2}}$. On the other hand, in the present paper, we need to assume the centered condition **(C)** to obtain an FCLT (Theorem 4.1.3) for $\{\xi_n\}_{n=0}^\infty$ in the Hölder topology, stronger than the uniform topology in $C_{\mathbf{1}_G}([0,1];G)$. In Section 4.5, we mention a method to reduce the non-centered case $\rho_{\mathbb{R}}(\gamma_p) \neq \mathbf{0}_{\mathfrak{g}}$ to the centered case by employing a measure-change technique based on the one discussed in Section 3.

4.2 Proof of Theorem 4.1.2

In what follows, we set

$$d\Phi_0(e) = \Phi_0\big(o(e)\big)^{-1} \cdot \Phi_0\big(t(e)\big) \qquad (e \in E),$$
$$\|d\Phi_0\|_\infty = \max_{e\in E_0} \Big\{ \big\| \log\big(d\Phi_0(\widetilde{e})\big)\big|_{\mathfrak{g}^{(1)}} \big\|_{\mathfrak{g}^{(1)}} + \big\| \log\big(d\Phi_0(\widetilde{e})\big)\big|_{\mathfrak{g}^{(2)}} \big\|_{\mathfrak{g}^{(2)}}^{1/2} \Big\}.$$

Here, we need to care the difference between $d\Phi_0$ above and the one introduced in Section 3.1, though we use the same symbol for simplicity. The difference comes from whether the underlying space is commutative or not. We should mention that

$$\Big(\Phi_0(x)^{-1}\cdot\Phi_0\big(t(c)\big)\Big)_i^{(k)} = O(N^k) \tag{4.2.1}$$

for $x \in V, c \in \Omega_{x,N}(X)$, $i = 1, 2, \dots, d_k$ and $k = 1, 2, \dots, r$. We also write

$$\rho = \rho_{\mathbb{R}}(\gamma_p), \qquad e^{z\rho} = \exp\big(z\rho_{\mathbb{R}}(\gamma_p)\big) \qquad (z \in \mathbb{R})$$

for brevity.

The following lemma is significant to prove Theorem 4.1.2.

Lemma 4.2.1 *Let $f \in C_0^\infty(G)$ and $q > 4r + 1$. Then we have*

$$\Big\|\frac{1}{N\varepsilon^2}\big(I - \mathcal{L}_p^N\big)\mathcal{P}_\varepsilon f - \mathcal{P}_\varepsilon \mathcal{A} f\Big\|_{\infty,q} \longrightarrow 0$$

as $N \to \infty$ and $\varepsilon \searrow 0$ with $N^2\varepsilon \searrow 0$, where $\mathcal{L}_p$ is the transition-shift operator defined by (4.1.1) *and $\mathcal{A}$ is the sub-elliptic operator defined by* (4.1.4).

Proof. We divide the proof into several steps.

Step 1. We first apply Taylor's formula (cf. Alexopoulos [2, Lemma 5.3]) for the $(*)$-coordinates of the second kind to $f \in C_0^\infty(G)$ at $\tau_\varepsilon\big(\Phi_0(x) * e^{-z\rho}\big) \in G$. By recalling that $(G, *)$ is a stratified Lie group, we have

$$\begin{aligned}
&\frac{1}{N\varepsilon^2}(I - \mathcal{L}_p^N)\mathcal{P}_\varepsilon f(x,z)\\
&= -\sum_{(i,k)} \frac{\varepsilon^{k-2}}{N} X_{i*}^{(k)} f\big(\tau_\varepsilon\big(\Phi_0(x) * e^{-z\rho}\big)\big) \sum_{c \in \Omega_{x,N}(X)} p(c)\big(\mathcal{B}_N(x,z,c)\big)_{i*}^{(k)}\\
&\quad - \Big(\sum_{(i_1,k_1)\geq(i_2,k_2)} \frac{\varepsilon^{k_1+k_2-2}}{2N} X_{i_1*}^{(k_1)} X_{i_2*}^{(k_2)} + \sum_{(i_2,k_2)>(i_1,k_1)} \frac{\varepsilon^{k_1+k_2-2}}{2N} X_{i_2*}^{(k_2)} X_{i_1*}^{(k_1)}\Big)\\
&\qquad \times f\Big(\tau_\varepsilon\big(\Phi_0(x) * e^{-z\rho}\big)\Big) \sum_{c \in \Omega_{x,N}(X)} p(c)\big(\mathcal{B}_N(x,z,c)\big)_{i_1*}^{(k_1)}\big(\mathcal{B}_N(x,z,c)\big)_{i_2*}^{(k_2)}\\
&\quad - \sum_{(i_1,k_1),(i_2,k_2),(i_3,k_3)} \frac{\varepsilon^{k_1+k_2+k_3-2}}{6N} \frac{\partial^3 f}{\partial g_{i_1*}^{(k_1)} \partial g_{i_2*}^{(k_2)} \partial g_{i_3*}^{(k_3)}}(\theta) \sum_{c \in \Omega_{x,N}(X)} p(c)\big(\mathcal{B}_N(x,z,c)\big)_{i_1*}^{(k_1)}\\
&\qquad \times \big(\mathcal{B}_N(x,z,c)\big)_{i_2*}^{(k_2)}\big(\mathcal{B}_N(x,z,c)\big)_{i_3*}^{(k_3)} \qquad (x \in V,\ z \in \mathbb{Z}),
\end{aligned} \tag{4.2.2}$$

for some $\theta \in G$ with $|\theta_{i_*}^{(k)}| \leq \varepsilon^k\big|\big(\mathcal{B}_N(x,z,c)\big)_{i*}^{(k)}\big|$ for $i = 1, 2, \dots, d_k$ and $k = 1, 2, \dots, r$, where the summation $\sum_{(i_1,k_1)\geq(i_2,k_2)}$ runs over all (i_1, k_1) and (i_2, k_2) with $k_1 > k_2$ or $k_1 = k_2$, $i_1 \geq i_2$. Here we set

$$\mathcal{B}_N(x,z,c) := e^{z\rho} * \Phi_0(x)^{-1} * \Phi_0\big(t(c)\big) * e^{-(z+N)\rho} \quad \big(N \in \mathbb{N},\ x \in V,\ z \in \mathbb{Z},\ c \in \Omega_{x,N}(X)\big).$$

We denote by $\mathrm{Ord}_\varepsilon(k)$ the terms of the right-hand side of (4.2.2) whose order of ε equals just k. Then (4.2.2) is rewritten as

$$\frac{1}{N\varepsilon^2}(I-\mathcal{L}_p^N)\mathcal{P}_\varepsilon f(x,z)=\mathrm{Ord}_\varepsilon(-1)+\mathrm{Ord}_\varepsilon(0)+\sum_{k\geq 1}\mathrm{Ord}_\varepsilon(k) \qquad (x\in V,\ z\in\mathbb{Z}),$$

where

$$\begin{aligned}
\mathrm{Ord}_\varepsilon(-1)&=-\frac{1}{N\varepsilon}\sum_{i=1}^{d_1}X_{i*}^{(1)}f\big(\tau_\varepsilon\big(\Phi_0(x)*e^{-z\rho}\big)\big)\sum_{c\in\Omega_{x,N}(X)}p(c)\big(\mathcal{B}_N(x,z,c)\big)_{i*}^{(1)},\\
\mathrm{Ord}_\varepsilon(0)&=-\frac{1}{N}\sum_{i=1}^{d_2}X_{i*}^{(2)}f\big(\tau_\varepsilon\big(\Phi_0(x)*e^{-z\rho}\big)\big)\sum_{c\in\Omega_{x,N}(X)}p(c)\Big\{\big(\mathcal{B}_N(x,z,c)\big)_{i*}^{(2)}\\
&\qquad-\frac{1}{2}\sum_{1\leq\lambda<\nu\leq d_1}\big(\mathcal{B}_N(x,z,c)\big)_{\lambda*}^{(1)}\big(\mathcal{B}_N(x,z,c)\big)_{\nu*}^{(1)}[\![X_\lambda^{(1)},X_\nu^{(1)}]\!]\big|_{X_i^{(2)}}\Big\}\\
&\quad-\frac{1}{2N}\sum_{1\leq i,j\leq d_1}X_{i*}^{(1)}X_{j*}^{(1)}f\big(\tau_\varepsilon\big(\Phi_0(x)*e^{-z\rho}\big)\big)\\
&\qquad\times\sum_{c\in\Omega_{x,N}(X)}p(c)\big(\mathcal{B}_N(x,z,c)\big)_{i*}^{(1)}\big(\mathcal{B}_N(x,z,c)\big)_{j*}^{(1)}
\end{aligned}$$

and $\sum_{k\geq1}\mathrm{Ord}_\varepsilon(k)$ is given by the sum of the following three parts:

$$\begin{aligned}
\mathcal{I}_1(\varepsilon,N)&=-\sum_{k\geq3}\sum_{i=1}^{d_k}\frac{\varepsilon^{k-2}}{N}X_{i*}^{(k)}f\big(\tau_\varepsilon\big(\Phi_0(x)*e^{-z\rho}\big)\big)\sum_{c\in\Omega_{x,N}(X)}p(c)\big(\mathcal{B}_N(x,z,c)\big)_{i*}^{(k)},\\
\mathcal{I}_2(\varepsilon,N)&=-\Big(\sum_{\substack{(i_1,k_1)\geq(i_2,k_2)\\k_1+k_2\geq3}}\frac{\varepsilon^{k_1+k_2-2}}{2N}X_{i_1*}^{(k_1)}X_{i_2*}^{(k_2)}+\sum_{\substack{(i_2,k_2)>(i_1,k_1)\\k_1+k_2\geq3}}\frac{\varepsilon^{k_1+k_2-2}}{2N}X_{i_2*}^{(k_2)}X_{i_1*}^{(k_1)}\Big)\\
&\qquad\times f\big(\tau_\varepsilon\big(\Phi_0(x)*e^{-z\rho}\big)\big)\sum_{c\in\Omega_{x,N}(X)}p(c)\big(\mathcal{B}_N(x,z,c)\big)_{i_1*}^{(k_1)}\big(\mathcal{B}_N(x,z,c)\big)_{i_2*}^{(k_2)},\\
\mathcal{I}_3(\varepsilon,N)&=-\sum_{(i_1,k_1),(i_2,k_2),(i_3,k_3)}\frac{\varepsilon^{k_1+k_2+k_3-2}}{6N}\frac{\partial^3 f}{\partial g_{i_1*}^{(k_1)}\partial g_{i_2*}^{(k_2)}\partial g_{i_3*}^{(k_3)}}(\theta)\\
&\qquad\times\sum_{c\in\Omega_{x,N}(X)}p(c)\big(\mathcal{B}_N(x,z,c)\big)_{i_1*}^{(k_1)}\big(\mathcal{B}_N(x,z,c)\big)_{i_2*}^{(k_2)}\big(\mathcal{B}_N(x,z,c)\big)_{i_3*}^{(k_3)}.
\end{aligned}$$

To complete the proof of Lemma 4.2.1, it is sufficient to show the followings:
(1) $\mathrm{Ord}_\varepsilon(-1)=0$.
(2) We have

$$\mathrm{Ord}_\varepsilon(0)=-\mathcal{A}f\big(\tau_\varepsilon\big(\Phi_0(x)*e^{-z\rho}\big)\big)+O\Big(\frac{1}{N}\Big). \tag{4.2.3}$$

(3) As $N\to\infty$ and $\varepsilon\searrow0$ with $N^2\varepsilon\searrow0$, we have

$$\|\mathcal{I}_i(\varepsilon,N)\|_{\infty,q}\longrightarrow0 \qquad (i=1,2,3). \tag{4.2.4}$$

Step 2. We here show (1). We fix $i = 1, 2, \ldots, d_1$. By recalling (2.4.4) and (2.2.3), we have inductively

$$\begin{aligned}
&\sum_{c \in \Omega_{x,N}(X)} p(c)\big(\mathcal{B}_N(x,z,c)\big)^{(1)}_{i*} \\
&= \sum_{c' \in \Omega_{x,N-1}(X)} p(c') \sum_{e \in E_{t(c')}} p(e)\Big\{ \log\Big(\Phi_0(x)^{-1} \cdot \Phi_0\big(t(c')\big) \cdot e^{-(N-1)\rho}\Big)\Big|_{X_i^{(1)}} \\
&\qquad + \log\Big(\Phi_0\big(o(e)\big)^{-1} \cdot \Phi_0\big(t(e)\big) \cdot e^{-\rho}\Big)\Big|_{X_i^{(1)}}\Big\} \\
&= \sum_{c' \in \Omega_{x,N-1}(X)} p(c') \log\Big(\Phi_0(x)^{-1} \cdot \Phi_0\big(t(c')\big) \cdot e^{-(N-1)\rho}\Big)\Big|_{X_i^{(1)}} = 0 \qquad (x \in V,\, z \in \mathbb{Z}).
\end{aligned}$$

Step 3. We prove the item (2). First consider the coefficient of $X^{(2)}_{i_*} f\big(\tau_\varepsilon\big(\Phi_0(x) * e^{-z\rho}\big)\big)$ which is given by

$$\begin{aligned}
&-\frac{1}{N} \sum_{c \in \Omega_{x,N}(X)} p(c)\Big\{\big(\mathcal{B}_N(x,z,c)\big)^{(2)}_{i*} \\
&\qquad - \frac{1}{2} \sum_{1 \le \lambda < \nu \le d_1} \big(\mathcal{B}_N(x,z,c)\big)^{(1)}_{\lambda*}\big(\mathcal{B}_N(x,z,c)\big)^{(1)}_{\nu*} [\![X^{(1)}_\lambda, X^{(1)}_\nu]\!]\big|_{X_i^{(2)}}\Big\} \\
&= -\frac{1}{N} \sum_{c \in \Omega_{x,N}(X)} p(c) \log\big(\mathcal{B}_N(x,z,c)\big)\big|_{X_i^{(2)}} \qquad (x \in V,\, i = 1, 2, \ldots, d_2).
\end{aligned}$$

Let us fix $i = 1, 2, \ldots, d_2$. We then deduce from (2.4.4) and (2.2.3) that, for $x \in V$ and $z \in \mathbb{Z}$,

$$\begin{aligned}
&-\frac{1}{N} \sum_{c \in \Omega_{x,N}(X)} p(c) \log\big(\mathcal{B}_N(x,z,c)\big)\big|_{X_i^{(2)}} \\
&= -\frac{1}{N} \sum_{c' \in \Omega_{x,N-1}(X)} p(c') \sum_{e \in E_{t(c')}} p(e) \log\Big(\big(e^{z\rho} * \Phi_0(x)^{-1} * \Phi_0\big(t(c')\big) * e^{-(z+N-1)\rho}\big) \\
&\qquad * \big(e^{(z+N-1)\rho} * \Phi_0\big(o(e)\big)^{-1} * \Phi_0\big(t(e)\big) * e^{-(z+N)\rho}\big)\Big)\Big|_{X_i^{(2)}} \\
&= -\frac{1}{N} \sum_{c' \in \Omega_{x,N-1}(X)} p(c') \log\Big(e^{z\rho} \cdot \Phi_0(x)^{-1} \cdot \Phi_0(t(c')) \cdot e^{-(z+N-1)\rho}\Big)\Big|_{X_i^{(2)}} \\
&\qquad + \sum_{c' \in \Omega_{x,N-1}(X)} p(c') \sum_{e \in E_{t(c')}} p(e) \log\Big(e^{(z+N-1)\rho} \cdot d\Phi_0(e) \cdot e^{-(z+N)\rho}\Big)\Big|_{X_i^{(2)}} \\
&= -\frac{1}{N} \sum_{k=0}^{N-1} \sum_{c \in \Omega_{x,k}(X)} p(c) \sum_{e \in E_{t(c)}} p(e) \log\Big(e^{(z+k)\rho} \cdot d\Phi_0(e) \cdot e^{-(z+k+1)\rho}\Big)\Big|_{X_i^{(2)}}.
\end{aligned}$$

For $g, h \in G$, we denote by $[g, h] := g \cdot h \cdot g^{-1} \cdot h^{-1}$ the usual commutator of g and h.

Then we have

$$\begin{aligned}
&\sum_{e\in E_{t(c)}} p(e)\log\Big(e^{(z+k)\rho}\cdot d\Phi_0(e)\cdot e^{-(z+k+1)\rho}\Big)\Big|_{X_i^{(2)}}\\
&=\sum_{e\in E_{t(c)}} p(e)\log\Big(\big[e^{(z+k)\rho}, d\Phi_0(e)\big]\cdot d\Phi_0(e)\cdot e^{-\rho}\Big)\Big|_{X_i^{(2)}}\\
&=\sum_{e\in E_{t(c)}} p(e)\log\Big(\big[e^{(z+k)\rho}, d\Phi_0(e)\big]\Big)\Big|_{X_i^{(2)}}+\sum_{e\in E_{t(c)}} p(e)\log\big(d\Phi_0(e)\cdot e^{-\rho}\big)\big|_{X_i^{(2)}}\\
&=\sum_{e\in E_{t(c)}} p(e)\log\big(d\Phi_0(e)\cdot e^{-\rho}\big)\big|_{X_i^{(2)}} \qquad (z\in\mathbb{Z},\ k=0,1,\dots,N-1)
\end{aligned}$$

by again using (2.4.4). Since the function

$$M_i(x):=\sum_{e\in E_x} p(e)\log\big(d\Phi_0(e)\cdot e^{-\rho}\big)\big|_{X_i^{(2)}} \qquad (i=1,2,\dots,d_2,\ x\in V)$$

satisfies $M_i(\gamma x)=M_i(x)$ for $\gamma\in\Gamma$ and $x\in V$ due to the Γ-invariance of p and the Γ-equivariance of Φ_0, there exists a function $\mathcal{M}_i : V_0 \longrightarrow \mathbb{R}$ such that $\mathcal{M}_i\big(\pi(x)\big)=M_i(x)$ for $i=1,2,\dots,d_2$ and $x\in V$. Moreover, we have

$$L^k\mathcal{M}_i\big(\pi(x)\big)=L^kM_i(x) \qquad (k\in\mathbb{N},\ i=1,2,\dots,d_2,\ x\in V)$$

by using the Γ-invariance of p. Then the ergodic theorem (cf. [31, Theorem 3.2]) for the transition operator L gives

$$\begin{aligned}
&-\frac{1}{N}\sum_{c\in\Omega_{x,N}(X)} p(c)\log\big(\mathcal{B}_N(x,z,c)\big)\big|_{X_i^{(2)}}\\
&=-\frac{1}{N}\sum_{k=0}^{N-1}L^kM_i(x)\\
&=-\frac{1}{N}\sum_{k=0}^{N-1}L^k\mathcal{M}_i\big(\pi(x)\big)\\
&=-\sum_{x\in V_0}m(x)\mathcal{M}_i(x)+O\Big(\frac{1}{N}\Big)=-\beta(\Phi_0)\big|_{X_i^{(2)}}+O\Big(\frac{1}{N}\Big) \quad (x\in V,\ z\in\mathbb{Z}).
\end{aligned}\tag{4.2.5}$$

We next consider the coefficient of $X_{i*}^{(1)}X_{j*}^{(1)}f\big(\tau_\varepsilon\big(\Phi_0(x)*e^{-z\rho}\big)\big)$ which is given by

$$-\frac{1}{2N}\sum_{c\in\Omega_{x,N}(X)} p(c)\big(\mathcal{B}_N(x,z,c)\big)_{i*}^{(1)}\big(\mathcal{B}_N(x,z,c)\big)_{j*}^{(1)} \qquad (x\in V,\ z\in\mathbb{Z},\ i,j=1,2,\dots,d_1).$$

Fix $i, j = 1, 2, \dots, d_1$. Then (2.4.4) and (2.2.3) imply

$$
\begin{aligned}
&-\frac{1}{2N}\sum_{c\in\Omega_{x,N}(X)} p(c)\big(\mathcal{B}_N(x,z,c)\big)^{(1)}_{i*}\big(\mathcal{B}_N(x,z,c)\big)^{(1)}_{j*}\\
&=-\frac{1}{2N}\sum_{c'\in\Omega_{x,N-1(X)}} p(c')\sum_{e\in E_{t(c')}} p(e)\\
&\qquad\times\Big\{\log\big(\mathcal{B}_{N-1}(x,z,c')\big)\big|_{X_i^{(1)}}+\log\big(d\Phi_0(e)\cdot e^{-\rho}\big)\big|_{X_i^{(1)}}\Big\}\\
&\qquad\times\Big\{\log\big(\mathcal{B}_{N-1}(x,z,c')\big)\big|_{X_j^{(1)}}+\log\big(d\Phi_0(e)\cdot e^{-\rho}\big)\big|_{X_j^{(1)}}\Big\}\\
&=-\frac{1}{2N}\Big\{\sum_{c'\in\Omega_{x,N-1(X)}} p(c')\log\big(\mathcal{B}_{N-1}(x,z,c')\big)\big|_{X_i^{(1)}}\log\big(\mathcal{B}_{N-1}(x,z,c')\big)\big|_{X_j^{(1)}}\\
&\qquad+\sum_{e\in E_{t(c')}} p(e)\log\big(d\Phi_0(e)\cdot e^{-\rho}\big)\big|_{X_i^{(1)}}\log\big(d\Phi_0(e)\cdot e^{-\rho}\big)\big|_{X_j^{(1)}}\Big\}\\
&=-\frac{1}{2N}\sum_{k=0}^{N-1}\sum_{c\in\Omega_{x,N}(X)} p(c)\sum_{e\in E_{t(c)}} p(e)\log\big(d\Phi_0(e)\cdot e^{-\rho}\big)\big|_{X_i^{(1)}}\log\big(d\Phi_0(e)\cdot e^{-\rho}\big)\big|_{X_j^{(1)}}
\end{aligned}
$$

for $x \in V$ and $z \in \mathbb{Z}$. In the same argument as above, the function $N_{ij} : V \longrightarrow \mathbb{R}$ defined by

$$
N_{ij}(x) := \sum_{e\in E_x} p(e)\log\big(d\Phi_0(e)\big)\big|_{X_i^{(1)}}\log\big(d\Phi_0(e)\big)\big|_{X_j^{(1)}} \qquad (i, j = 1, 2, \dots, d_1,\ x \in V)
$$

is Γ-invariant and then there exists a function $\mathcal{N}_{ij} : V_0 \longrightarrow \mathbb{R}$ such that $\mathcal{N}_{ij}\big(\pi(x)\big) = N_{ij}(x)$ for $x \in V$. We also have

$$
L^k\mathcal{N}_{ij}\big(\pi(x)\big) = L^k N_{ij}(x) \qquad (k \in \mathbb{N},\ i, j = 1, 2, \dots, d_2,\ x \in V)
$$

by using the Γ-invariance of p. Hence, we obtain

$$
\begin{aligned}
&-\frac{1}{2N}\sum_{c\in\Omega_{x,N}(X)} p(c)\big(\mathcal{B}_N(x,z,c)\big)^{(1)}_{i*}\big(\mathcal{B}_N(x,z,c)\big)^{(1)}_{j*}\\
&=-\frac{1}{2N}\sum_{k=0}^{N-1} L^k N_{ij}(x)\\
&=-\frac{1}{2N}\sum_{k=0}^{N-1} L^k \mathcal{N}_{ij}\big(\pi(x)\big)\\
&=-\frac{1}{2}\sum_{x\in V_0} m(x)\mathcal{N}_{ij}(x)+O\Big(\frac{1}{N}\Big)\\
&=-\frac{1}{2}\sum_{e\in E_0}\widetilde{m}(e)\log\big(d\Phi_0(\widetilde{e})\cdot e^{-\rho}\big)\big|_{X_i^{(1)}}\log\big(d\Phi_0(\widetilde{e})\cdot e^{-\rho}\big)\big|_{X_j^{(1)}}+O\Big(\frac{1}{N}\Big). \qquad (4.2.6)
\end{aligned}
$$

by virtue of the ergodic theorem. Recall that $\{V_1, V_2, \dots, V_{d_1}\}$ denotes an orthonormal basis of $(\mathfrak{g}^{(1)}, g_0)$. We especially put $X_i^{(1)} = V_i$ for $i = 1, 2, \dots, d_1$. Let $\{\omega_1, \omega_2, \dots, \omega_{d_1}\} \subset \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R}) \hookrightarrow \mathrm{H}^1(X_0, \mathbb{R})$ be the dual basis of $\{V_1, V_2, \dots, V_{d_1}\}$. Namely, $\omega_i(V_j) = \delta_{ij}$ for $i, j = 1, 2, \dots, d_1$. It follows from (2.4.5) that

$$\begin{aligned}
&\sum_{e \in E_0} \widetilde{m}(e) \log \big(d\Phi_0(\widetilde{e}) \cdot e^{-\rho}\big)\big|_{V_i} \log \big(d\Phi_0(\widetilde{e}) \cdot e^{-\rho}\big)\big|_{V_j} \\
&= \sum_{e \in E_0} \widetilde{m}(e) \log \big(d\Phi_0(\widetilde{e})\big)\big|_{V_i} \log \big(d\Phi_0(\widetilde{e})\big)\big|_{V_j} - \rho_{\mathbb{R}}(\gamma_p)\big|_{V_i} \rho_{\mathbb{R}}(\gamma_p)\big|_{V_j} \\
&= \sum_{e \in E_0} \widetilde{m}(e) {}^t\rho_{\mathbb{R}}(\omega_i)(e) {}^t\rho_{\mathbb{R}}(\omega_j)(e) - \omega_i(\rho_{\mathbb{R}}(\gamma_p))\omega_j(\rho_{\mathbb{R}}(\gamma_p)) \\
&= \sum_{e \in E_0} \widetilde{m}(e)\omega_i(e)\omega_j(e) - \langle \gamma_p, \omega_i \rangle \langle \gamma_p, \omega_j \rangle = \langle\!\langle \omega_i, \omega_j \rangle\!\rangle_p = \delta_{ij}. \qquad (4.2.7)
\end{aligned}$$

Hence, we obtain (4.2.3) by combining (4.2.5) with (4.2.6) and (4.2.7).

Step 4. We show (3) at the last step. We first discuss the estimate of $\mathcal{I}_1(\varepsilon, N)$. By using (2.2.7) and (4.2.1), we have

$$\begin{aligned}
&\Big| \Big(\Phi_0(x)^{-1} * \Phi_0\big(t(c)\big) \Big)^{(k)}_{i*} \Big| \\
&\le C \sum_{\substack{|K_1| + |K_2| \le k \\ |K_2| > 0}} \Big| \mathcal{P}_*^{K_1} \Big(\Phi_0(x)^{-1} \Big) \Big| \Big| \mathcal{P}^{K_2} \Big(\Phi_0(x)^{-1} \cdot \Phi_0\big(t(c)\big) \Big) \Big| \\
&\le C \sum_{\substack{|K_1| + |K_2| \le k \\ |K_2| > 0}} N^{|K_2|} \Big| \mathcal{P}_*^{K_1} \Big(e^{-z\rho} * \big(\Phi_0(x) * e^{-z\rho} \big)^{-1} \Big) \Big|
\end{aligned}$$

for $i = 1, 2, \dots, d_k$ and $k = 1, 2, \dots, r$. Then (2.2.2) implies that there is a continuous function $Q_1 : G \longrightarrow \mathbb{R}$ such that

$$\begin{aligned}
&\Big| \Big(\Phi_0(x)^{-1} * \Phi_0\big(t(c)\big) \Big)^{(k)}_{i*} \Big| \\
&\le |z|^{k-1} Q_1\Big(\tau_\varepsilon \big(\Phi_0(x) * e^{-z\rho} \big) \Big) \sum_{\substack{|K_1| + |K_2| \le k \\ |K_2| > 0}} \varepsilon^{-|K_1|} N^{|K_2|} \qquad (4.2.8)
\end{aligned}$$

for $i = 1, 2, \dots, d_k$ and $k = 1, 2, \dots, r$. Thus, (2.2.2) and (4.2.8) yields

$$\begin{aligned}
&\big| \big(\mathcal{B}_N(x, 0, c) \big)^{(k)}_{i*} \big| \\
&\le C \sum_{\substack{|L_1| + |L_2| = k \\ |L_1|, |L_2| \ge 0}} \Big| \mathcal{P}_*^{L_1} \Big(\Phi_0(x)^{-1} * \Phi_0\big(t(c)\big) \Big) \Big| \big| \mathcal{P}_*^{L_2} \big(e^{-N\rho} \big) \big| \\
&\le C|z|^k Q_2\Big(\tau_\varepsilon \big(\Phi_0(x) * e^{-z\rho} \big) \Big) \sum_{\substack{|L_1| + |L_2| = k \\ |L_1|, |L_2| \ge 0}} N^{|L_2|} \sum_{\substack{|K_1| + |K_2| \le |L_1| \\ |K_2| > 0}} \varepsilon^{-|K_1|} N^{|K_2|} \\
&= C|z|^k Q_2\Big(\tau_\varepsilon \big(\Phi_0(x) * e^{-z\rho} \big) \Big) F(\varepsilon, N) \qquad (4.2.9)
\end{aligned}$$

for some continuous function $Q_2 : G \longrightarrow \mathbb{R}$, where $F(\varepsilon, N)$ denotes the polynomial of ε and N which satisfies $\varepsilon^{k-2}N^{-1}F(\varepsilon, N) \to 0$ as $N \to \infty, \varepsilon \searrow 0$ and $N^2\varepsilon \searrow 0$.

On the other hand, combining (4.2.9) with $\rho_{\mathbb{R}}(\gamma_p) \in \mathfrak{g}^{(1)}$ gives

$$
\begin{aligned}
&\frac{\varepsilon^{k-2}}{N}\big|\big(\mathcal{B}_N(x,z,c)\big)^{(k)}_{i*}\big| \\
&= \frac{\varepsilon^{k-2}}{N}\Big|\Big(\big[e^{z\rho}, \mathcal{B}_N(x,0,c)\big]_* * \mathcal{B}_N(x,0,c)\Big)^{(k)}_{i*}\Big| \\
&\leq C\frac{\varepsilon^{k-2}}{N}\sum_{\substack{|K_1|+|K_2|=k \\ |K_1|,|K_2|\geq 0}}\Big|\mathcal{P}^{K_1}_*\Big(\big[e^{z\rho}, \mathcal{B}_N(x,0,c)\big]_*\Big)\Big|\Big|\mathcal{P}^{K_2}_*\big(\mathcal{B}_N(x,0,c)\big)\Big| \\
&\leq C|z|^{2k}\frac{\varepsilon^{k-2}}{N}Q_3\big(\tau_\varepsilon\big(\Phi_0(x) * e^{-z\rho}\big)\big)F(\varepsilon,N) \\
&\qquad\qquad\qquad \big(i = 1,2,\ldots,d_k,\ k = 3,4,\ldots,r,\ x \in V,\ z \in \mathbb{Z},\ c \in \Omega_{x,N}(X)\big) \qquad (4.2.10)
\end{aligned}
$$

for some continuous function $Q_3 : G \longrightarrow \mathbb{R}$. Hence, we obtain $\|\mathcal{I}_1(\varepsilon, N)\|_{\infty,q} \to 0$ as $N \to \infty, \varepsilon \searrow 0$ and $N^2\varepsilon \searrow 0$ in $C_{\infty,q}(X \times \mathbb{Z})$ by using (4.2.10). This follows from $2k < 2r < q$. In the same argument as above, we also obtain $\|\mathcal{I}_2(\varepsilon, N)\|_{\infty,q} \to 0$ as $N \to \infty$, $\varepsilon \searrow 0$ and $N^2\varepsilon \searrow 0$ in $C_{\infty,q}(X \times \mathbb{Z})$-topology since the order of $|z|$ in $\mathcal{I}_2(\varepsilon, N)$ satisfies $2 \times 2k < 4r < q$.

Finally, we study the estimate of the term $\mathcal{I}_3(\varepsilon, N)$. We recall that $f \in C_0^\infty(G)$ and $\mathrm{supp}\,\partial^3 f/(\partial g^{(k_1)}_{i_1*}\partial g^{(k_2)}_{i_2*}\partial g^{(k_3)}_{i_3*}) \subset \mathrm{supp}\,f$. Therefore, it suffices to show by induction on $k = 1, 2, \ldots, r$ that, if $\varepsilon N < 1$,

$$
\varepsilon^k\big|\big(\mathcal{B}_N(x,z,c)\big)^{(k)}_{i*}\big| \leq |z|^k Q^{(k)}\big(\tau_\varepsilon(\Phi_0(x) * e^{-z\rho}) * \theta\big) \times \varepsilon N \qquad (4.2.11)
$$

for some continuous function $Q^{(k)} : G \longrightarrow \mathbb{R}$, where $\theta \in G$ appears in the remainder term of (4.2.2). The cases $k = 1$ and $k = 2$ are obvious. Suppose that (4.2.11) holds for less than k. Then we have

$$
\varepsilon^k\big|\big(\mathcal{B}_N(x,z,c)\big)^{(k)}_{i*}\big| \leq C\varepsilon^k \sum_{\substack{|K_1|+|K_2|\leq k \\ |K_2|>0}}\Big|\mathcal{P}^{K_1}_*\Big(\Phi_0(x)^{-1}\Big)\Big|\Big|\mathcal{P}^{K_2}\Big(\Phi_0(x)^{-1}\cdot\Phi_0\big(t(c)\big)\Big)\Big|
$$

by using (2.2.7). Since

$$
\big(\Phi_0(x)^{-1}\big)^{(k_1)}_{i_1*} = \Big(e^{-z\rho} * (\tau_{\varepsilon^{-1}}\theta) * \big(\tau_{\varepsilon^{-1}}(\tau_\varepsilon(\Phi_0(x) * e^{-z\rho}) * \theta)^{-1}\big)\Big)^{(k_1)}_{i_1*} \qquad (k_1 \leq k-1),
$$

we have inductively

$$
\big|\big(\Phi_0(x)^{-1}\big)^{(k_1)}_{i_1*}\big| \leq |z|^{k_1} Q\big(\tau_\varepsilon(\Phi_0(x) * e^{-z\rho}) * \theta\big)
$$

for a continuous function $Q : G \longrightarrow \mathbb{R}$ and $k_1 \leq k-1$. We thus obtain

$$
\begin{aligned}
&\varepsilon^k \big| \big(\mathcal{B}_N(x,z,c)\big)^{(k)}_{i*} \big| \\
&\leq C\varepsilon^k \sum_{\substack{|K_1|+|K_2|\leq k \\ |K_2|>0}} N^{|K_2|} \Big| \mathcal{P}_*^{K_1} \Big(e^{-z\rho} * (\tau_{\varepsilon^{-1}}\theta) * \big(\tau_{\varepsilon^{-1}}(\tau_\varepsilon(\Phi_0(x) * e^{-z\rho}) * \theta)^{-1}\big)\Big)\Big| \\
&\leq C|z|^k Q\big(\tau_\varepsilon(\Phi_0(x) * e^{-z\rho}) * \theta\big) \sum_{\substack{|K_1|+|K_2|\leq k \\ |K_2|>0}} \varepsilon^{k-|K_1|+1} N^{|K_2|+1} \\
&\leq |z|^k Q^{(k)}\big(\tau_\varepsilon(\Phi_0(x) * e^{-z\rho}) * \theta\big) \times \varepsilon N
\end{aligned}
$$

for some continuous function $Q^{(k)} : G \longrightarrow \mathbb{R}$. Therefore, (4.2.11) holds for $k = 1, 2, \ldots, r$ and this implies that $\|\mathcal{I}_3(\varepsilon, N)\|_{\infty,q} \to 0$ as $N \to \infty$, $\varepsilon \searrow 0$ and $N^2\varepsilon \searrow 0$ in $C_{\infty,q}(X \times \mathbb{Z})$ since the order of $|z|$ in $\mathcal{I}_3(\varepsilon, N)$ satisfies $3k < 3r < q$. This completes the proof. ∎

We now give the proof of Theorem 4.1.2 by using this lemma. We note that the infinitesimal operator $\mathcal{A}$ in Lemma 4.2.1 enjoys the following property.

Lemma 4.2.2 (cf. Robinson [64, page 304]) *The range of $\lambda - \mathcal{A}$ is dense in $C_\infty(G)$ for some $\lambda > 0$. Namely, $(\lambda - \mathcal{A})\big(C_0^\infty(G)\big)$ is dense in $C_\infty(G)$.*

Proof of Theorem 4.1.2. (1) We follow the argument in Kotani [38, Theorem 4]. Let $N = N(n)$ be the integer satisfying $n^{1/5} \leq N < n^{1/5}+1$ and k_N and r_N be the quotient and the remainder of $([nt]-[ns])/N(n)$, respectively. Note that $r_N < N$. We put $\varepsilon_N := n^{-1/2}$ and $h_N := N\varepsilon_N^2$. Then we have $N = N(n) \to \infty$,

$$
r_N^2\varepsilon_N < N^2\varepsilon_N \leq (1+n^{1/5})^2 \cdot n^{-1/2} \to 0,
$$

and $h_N \leq (1+n^{1/5}) \cdot n^{-1} \to 0$ as $n \to \infty$. We also see that

$$
r_N\varepsilon_N^2 < N\varepsilon_N^2 \leq (1+n^{1/5}) \cdot n^{-1} \to 0 \quad (n \to \infty).
$$

Hence, we have

$$
k_N h_N = \frac{[nt]-[ns]-r_N}{N} \cdot N\varepsilon_N^2 = \big([nt]-[ns]-r_N\big)\varepsilon_N^2 \to t-s \quad (n \to \infty).
$$

Since $C_0^\infty(G) \subset \mathrm{Dom}(\mathcal{A}) \subset C_\infty(G)$ and $C_0^\infty(G)$ is dense in $C_\infty(G)$, the operator $\mathcal{A}$ is densely defined in $C_\infty(G)$. We use this fact and Lemma 4.2.2 to apply Trotter's approximation theorem (cf. Trotter [74] and Kurtz [47]). We obtain, for $f \in C_0^\infty(G)$,

$$
\lim_{n\to\infty} \Big\| \mathcal{L}_p^{Nk_N}\mathcal{P}_{n^{-1/2}}f - \mathcal{P}_{n^{-1/2}}\mathrm{e}^{-(t-s)\mathcal{A}}f \Big\|_{\infty,q} = 0. \tag{4.2.12}
$$

Then Lemma 4.2.1 implies

$$
\lim_{n\to\infty} \Big\| \frac{1}{r_N\varepsilon_N^2}\big(I - \mathcal{L}_p^{r_N}\big)\mathcal{P}_{n^{-1/2}}f - \mathcal{P}_{n^{-1/2}}\mathcal{A}f \Big\|_{\infty,q} = 0 \tag{4.2.13}
$$

for all $f \in C_0^\infty(G)$. We thus have

$$\begin{aligned}
&\Big\|\mathcal{L}_p^{[nt]-[ns]}P_{n^{-1/2}}f - P_{n^{-1/2}}\mathrm{e}^{-(t-s)\mathcal{A}}f\Big\|_{\infty,q}\\
&\leq \Big\|\big(I-\mathcal{L}_p^{r_N}\big)\mathcal{P}_{n^{-1/2}}f\Big\|_{\infty,q} + \Big\|\mathcal{L}_p^{Nk_N}\mathcal{P}_{n^{-1/2}}f - \mathcal{P}_{n^{-1/2}}\mathrm{e}^{-(t-s)\mathcal{A}}f\Big\|_{\infty,q}. \qquad (4.2.14)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\Big\|\big(I-\mathcal{L}_p^{r_N}\big)\mathcal{P}_{n^{-1/2}}f\Big\|_{\infty,q}\\
&\leq r_N\varepsilon_N^2\Big\|\frac{1}{r_N\varepsilon_N^2}\big(I-\mathcal{L}_p^{r_N}\big)\mathcal{P}_{n^{-1/2}}f - \mathcal{P}_{n^{-1/2}}\mathcal{A}f\Big\|_{\infty,q} + r_N\varepsilon_N^2\big\|\mathcal{P}_{n^{-1/2}}\mathcal{A}f\big\|_{\infty,q}\\
&\leq r_N\varepsilon_N^2\Big\|\frac{1}{r_N\varepsilon_N^2}\big(I-\mathcal{L}_p^{r_N}\big)\mathcal{P}_{n^{-1/2}}f - \mathcal{P}_{n^{-1/2}}\mathcal{A}f\Big\|_{\infty,q} + r_N\varepsilon_N^2\big\|\mathcal{A}f\big\|_{\infty}^{G}. \qquad (4.2.15)
\end{aligned}$$

We obtain (4.1.3) for $f \in C_0^\infty(G)$ by combining (4.2.13), (4.2.14) and (4.2.15) with $r_N\varepsilon_N^2 \to 0\,(n\to\infty)$. For $f \in C_\infty(G)$, we also obtain the convergence (4.1.3) by following the same argument as [31, Theorem 2.1].

(2) For $t>0$ and $z \in \mathbb{Z}$, we have

$$\begin{aligned}
&\big|\mathcal{L}_p^{[nt]}\mathcal{P}_{n^{-1/2}}f(x_n,z) - \mathrm{e}^{-t\mathcal{A}}f(g)\big|\\
&\leq \big|\mathcal{L}_p^{[nt]}\mathcal{P}_{n^{-1/2}}f(x_n,z) - \mathcal{P}_{n^{-1/2}}\mathrm{e}^{-t\mathcal{A}}f(x_n,z)\big| + \big|\mathcal{P}_{n^{-1/2}}\mathrm{e}^{-t\mathcal{A}}f(x_n,z) - e^{-t\mathcal{A}}f(g)\big|\\
&\leq (1+|z|^q)\Big\|\mathcal{L}_p^{[nt]}\mathcal{P}_{n^{-1/2}}f - \mathcal{P}_{n^{-1/2}}\mathrm{e}^{-t\mathcal{A}}f\Big\|_{\infty,q}\\
&\qquad + \Big|\mathrm{e}^{-t\mathcal{A}}f\Big(\tau_{n^{-1/2}}\big(\Phi_0(x_n)*\exp(-z\rho_{\mathbb{R}}(\gamma_p))\big)\Big) - \mathrm{e}^{-t\mathcal{A}}f(g)\Big|.
\end{aligned}$$

We thus obtain (4.1.5) by (4.1.3) and the continuity of the function $\mathrm{e}^{-t\mathcal{A}}f : G \longrightarrow \mathbb{R}$. This completes the proof of Theorem 4.1.2. ∎

To the end, we give several properties of $\beta(\Phi_0)$.

Proposition 4.2.3 (1) *If the random walk on X is m-symmetric, then $\beta(\Phi_0) = \mathbf{0}_{\mathfrak{g}}$.*
(2) *Let Φ_0, $\widehat{\Phi}_0 : X \longrightarrow G$ be two modified harmonic realizations. Then*

$$\beta(\Phi_0) = \beta(\widehat{\Phi}_0) - \big[\rho_{\mathbb{R}}(\gamma_p), \log\big(\Phi_0(x)^{-1}\cdot\widehat{\Phi}_0(x)\big)\big]\big|_{\mathfrak{g}^{(2)}} \qquad (x \in V).$$

In particular, if either

- $\log\Phi_0(x_*)\big|_{\mathfrak{g}^{(1)}} = \log\widehat{\Phi}_0(x_*)\big|_{\mathfrak{g}^{(1)}}$ *for some reference point $x_* \in V$, or*
- $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}_{\mathfrak{g}}$

holds, then we have $\beta(\Phi_0) = \beta(\widehat{\Phi}_0)$.

Proof. Assertion (1) is easily obtained as follows:

$$\begin{aligned}
\beta(\Phi_0) &= \frac{1}{2}\sum_{e\in E_0}\Big\{\widetilde{m}(e)\log\big(d\Phi_0(\widetilde{e})\big)\big|_{\mathfrak{g}^{(2)}} + \widetilde{m}(\overline{e})\log\big(d\Phi_0(\widetilde{\overline{e}})\big)\big|_{\mathfrak{g}^{(2)}}\Big\}\\
&= \frac{1}{2}\sum_{e\in E_0}\big(\widetilde{m}(e)-\widetilde{m}(\overline{e})\big)\log\big(d\Phi_0(\widetilde{e})\big)\big|_{\mathfrak{g}^{(2)}} = \mathbf{0}_{\mathfrak{g}}.
\end{aligned}$$

Next we show Assertion (2). We set $\Psi(x) := \Phi_0(x)^{-1} \cdot \widehat{\Phi}_0(x)$ for $x \in V$. We note that the map $\Psi : X \longrightarrow G$ is Γ-invariant. Since the $\mathfrak{g}^{(1)}$-components of Φ_0 and $\widehat{\Phi}_0$ are uniquely determined up to $\mathfrak{g}^{(1)}$-translation, there exists a constant vector $C \in \mathfrak{g}^{(1)}$ such that $\log\big(\Psi(x)\big)\big|_{\mathfrak{g}^{(1)}} = C$ for $x \in V$. Define a function $F_i : V \longrightarrow \mathbb{R}$ by $F_i(x) := \log\big(\Psi(x)\big)\big|_{X_i^{(2)}}$ for $i = 1, 2, \dots, d_2$ and $x \in V$. Then we see that the function F_i is Γ-invariant. Hence, there is a function $\widehat{F}_i : V_0 \longrightarrow \mathbb{R}$ satisfying $\widehat{F}_i\big(\pi(x)\big) = F_i(x)$ for $x \in V$. Then we obtain

$$
\begin{aligned}
\beta(\Phi_0) &= \sum_{e \in E_0} \widetilde{m}(e) \log\Big(\Psi\big(o(\widetilde{e})\big) \cdot \big(d\widehat{\Phi}_0(\widetilde{e}) \cdot e^{-\rho}\big) \cdot e^{\rho} \cdot \Psi\big(t(\widetilde{e})\big)^{-1} \cdot e^{-\rho}\Big)\Big|_{\mathfrak{g}^{(2)}} \\
&= \beta(\widehat{\Phi}_0) - \sum_{e \in E_0} \widetilde{m}(e) \Big\{ \log\Big(\Psi\big(t(\widetilde{e})\big)\Big)\Big|_{\mathfrak{g}^{(2)}} - \log\Big(\Psi\big(o(\widetilde{e})\big)\Big)\Big|_{\mathfrak{g}^{(2)}} \Big\} - [\rho_{\mathbb{R}}(\gamma_p), C]\big|_{\mathfrak{g}^{(2)}} \\
&= \beta(\widehat{\Phi}_0) - \sum_{i=1}^{d_2} \big({}_{C_1(X_0,\mathbb{R})}\langle \gamma_p, d\widehat{F}_i \rangle_{C^1(X_0,\mathbb{R})}\big) X_i^{(2)} - [\rho_{\mathbb{R}}(\gamma_p), C]\big|_{\mathfrak{g}^{(2)}} \\
&= \beta(\widehat{\Phi}_0) - \sum_{i=1}^{d_2} \big({}_{C_0(X_0,\mathbb{R})}\langle \partial(\gamma_p), \widehat{F}_i \rangle_{C^0(X_0,\mathbb{R})}\big) X_i^{(2)} - [\rho_{\mathbb{R}}(\gamma_p), C]\big|_{\mathfrak{g}^{(2)}} \\
&= \beta(\widehat{\Phi}_0) - [\rho_{\mathbb{R}}(\gamma_p), C]\big|_{\mathfrak{g}^{(2)}},
\end{aligned}
$$

where we used (2.2.2) for the second line and $\gamma_p \in \mathrm{H}_1(X_0, \mathbb{R})$ for the fourth line. ∎

4.3 Proof of Theorem 4.1.3

We now assume the centered condition **(C)**: $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}_{\mathfrak{g}}$, throughout this subsection. For $k = 1, 2, \dots, r$, we denote by $(G^{(k)}, \cdot)$ and $(G^{(k)}, *)$ the connected and simply connected nilpotent Lie group of step k and the corresponding limit group whose Lie algebras are $\big(\mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(k)}, [\cdot,\cdot]\big)$ and $\big(\mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(k)}, [\![\cdot,\cdot]\!]\big)$, respectively. For the piecewise smooth stochastic process $(\mathcal{Y}_t^{(n)})_{0 \le t \le 1} = (\mathcal{Y}_t^{(n),1}, \mathcal{Y}_t^{(n),2}, \dots, \mathcal{Y}_t^{(n),r})_{0 \le t \le 1}$ defined in Section 2, we define its truncated process by

$$
\mathcal{Y}_t^{(n;\,k)} = \big(\mathcal{Y}_t^{(n),1}, \mathcal{Y}_t^{(n),2}, \dots, \mathcal{Y}_t^{(n),k}\big) \in G^{(k)} \qquad (0 \le t \le 1,\ k = 1, 2, \dots, r)
$$

in the $(\cdot)$-coordinate system. To complete the proof of Theorem 4.1.3, it is sufficient to show the tightness of $\{\mathbf{P}^{(n)}\}_{n=1}^{\infty}$ (Lemma 4.3.1) and the convergence of the finite dimensional distribution of $\{\mathcal{Y}_{\cdot}^{(n)}\}_{n=1}^{\infty}$ (Lemma 4.3.4).

In the former part of this subsection, we aim to show the following.

Lemma 4.3.1 *Under* **(C)**, *the family* $\{\mathbf{P}^{(n)}\}_{n=1}^{\infty}$ *is tight in* $C_{\mathbf{1}_G}^{0,\alpha\text{-Höl}}([0,1]; G)$, *where* α *is an arbitrary real number less than* $1/2$.

As the first step of the proof of Lemma 4.3.1, we prepare the following lemma.

Lemma 4.3.2 *Let m, n be positive integers. Then there exists a constant $C > 0$ which is independent of n (however, it may depend on m) such that*

$$\mathbb{E}^{\mathbb{P}_{x_*}}\Big[d_{\mathrm{CC}}(\mathcal{Y}_s^{(n;\,2)}, \mathcal{Y}_t^{(n;\,2)})^{4m}\Big] \le C(t-s)^{2m} \qquad (0 \le s \le t \le 1). \tag{4.3.1}$$

Proof. The proof is partially based on Bayer–Friz [6, Proposition 4.3]. We split the proof into several steps.

Step 1. At the beginning, we show

$$\mathbb{E}^{\mathbb{P}_{x_*}}\Big[d_{\mathrm{CC}}(\mathcal{Y}_{t_k}^{(n;\,2)}, \mathcal{Y}_{t_\ell}^{(n;\,2)})^{4m}\Big] \le C\Big(\frac{\ell-k}{n}\Big)^{2m} \qquad \big(n, m \in \mathbb{N},\, t_k, t_\ell \in \mathcal{D}_n\,(k \le \ell)\big) \tag{4.3.2}$$

for some $C > 0$ independent of n (depending on m). By recalling the equivalence of two homogeneous norms $\|\cdot\|_{\mathrm{CC}}$ and $\|\cdot\|_{\mathrm{hom}}$ (cf. Proposition 2.3.3), we readily see that (4.3.2) is equivalent to the existence of positive constants $C^{(1)}$ and $C^{(2)}$ independent of n such that

$$\mathbb{E}^{\mathbb{P}_{x_*}}\Big[\big\|\log\big((\mathcal{Y}_{t_k}^{(n)})^{-1}\cdot\mathcal{Y}_{t_\ell}^{(n)}\big)\big|_{\mathfrak{g}^{(1)}}\big\|_{\mathfrak{g}^{(1)}}^{4m}\Big] \le C^{(1)}\Big(\frac{\ell-k}{n}\Big)^{2m}, \tag{4.3.3}$$

$$\mathbb{E}^{\mathbb{P}_{x_*}}\Big[\big\|\log\big((\mathcal{Y}_{t_k}^{(n)})^{-1}\cdot\mathcal{Y}_{t_\ell}^{(n)}\big)\big|_{\mathfrak{g}^{(2)}}\big\|_{\mathfrak{g}^{(2)}}^{2m}\Big] \le C^{(2)}\Big(\frac{\ell-k}{n}\Big)^{2m}. \tag{4.3.4}$$

Step 2. We now show (4.3.3). We see

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_{x_*}}\Big[\big\|\log\big((\mathcal{Y}_{t_k}^{(n)})^{-1}\cdot\mathcal{Y}_{t_\ell}^{(n)}\big)\big|_{\mathfrak{g}^{(1)}}\big\|_{\mathfrak{g}^{(1)}}^{4m}\Big]\\
&= \Big(\frac{1}{\sqrt{n}}\Big)^{4m}\mathbb{E}^{\mathbb{P}_{x_*}}\Big[\Big(\sum_{i=1}^{d_1}\log\big(\xi_k^{-1}\cdot\xi_\ell\big)\big|_{X_i^{(1)}}^2\Big)^{2m}\Big]\\
&\le \Big(\frac{1}{\sqrt{n}}\Big)^{4m}\cdot d_1^{2m}\max_{i=1,2,\dots,d_1}\max_{x\in\mathcal{F}}\Big\{\sum_{c\in\Omega_{x,\ell-k}(X)} p(c)\log\Big(\Phi_0(x)^{-1}\cdot\Phi_0\big(t(c)\big)\Big)\Big|_{X_i^{(1)}}^{4m}\Big\},
\end{aligned} \tag{4.3.5}$$

where $\mathcal{F}$ stands for the fundamental domain in X containing the reference point $x_* \in V$. For $i = 1, 2, \dots, d_1$, $x \in \mathcal{F}$, $N \in \mathbb{N}$ and $c = (e_1, e_2, \dots, e_N) \in \Omega_{x,N}(X)$, we put

$$\mathcal{M}_N^{(i,x)}(c) = \mathcal{M}_N^{(i,x)}(\Phi_0; c) := \log\Big(\Phi_0(x)^{-1}\cdot\Phi_0\big(t(c)\big)\Big)\Big|_{X_i^{(1)}} = \sum_{j=1}^{N}\log\big(d\Phi_0(e_j)\big)\big|_{X_i^{(1)}}.$$

By Lemma 2.5.3, $\{\mathcal{M}_N^{(i,x)}\}_{N=1}^\infty$ is an $\mathbb{R}$-valued martingale for every $i = 1, 2, \dots, d_1$ and $x \in \mathcal{F}$. Therefore, we apply the Burkholder–Davis–Gundy inequality with the exponent $4m$ to obtain

$$\begin{aligned}
\sum_{c\in\Omega_{x,N}(X)} p(c)\big(\mathcal{M}_N^{(i,x)}(c)\big)^{4m} &= \sum_{c\in\Omega_{x,N}(X)} p(c)\Big(\sum_{j=1}^{N}\log\big(d\Phi_0(e_j)\big)\big|_{X_i^{(1)}}\Big)^{4m}\\
&\le \mathcal{C}_{(4m)}^{4m}\sum_{c\in\Omega_{x,N}(X)} p(c)\Big(\sum_{j=1}^{N}\log\big(d\Phi_0(e_j)\big)\big|_{X_i^{(1)}}^2\Big)^{2m}\\
&\le \mathcal{C}_{(4m)}^{4m}\|d\Phi_0\|_\infty^{4m}N^{2m}
\end{aligned} \tag{4.3.6}$$

for $i=1,2,\dots,d_1$, $x\in\mathcal{F}$ and $N\in\mathbb{N}$, where $\mathcal{C}_{(4m)}$ stands for the positive constant which appears in the Burkholder–Davis–Gundy inequality with the exponent $4m$. In particular, by putting $N=\ell-k$, (4.3.6) leads to

$$\sum_{c\in\Omega_{x,\ell-k}(X)} p(c)\log\Big(\Phi_0(x)^{-1}\cdot\Phi_0\big(t(c)\big)\Big)\Big|_{X_i^{(1)}}^{4m} \le \mathcal{C}_{(4m)}^{4m}\|d\Phi_0\|_\infty^{4m}(\ell-k)^{2m}. \tag{4.3.7}$$

Thus, we obtain

$$\mathbb{E}^{\mathbb{P}_{x*}}\Big[\big\|\log\big((\mathcal{Y}_{t_k}^{(n)})^{-1}\cdot\mathcal{Y}_{t_\ell}^{(n)}\big)\big|_{\mathfrak{g}^{(1)}}\big\|_{\mathfrak{g}^{(1)}}^{4m}\Big] \le d_1^{2m}\mathcal{C}_{(4m)}^{4m}\|d\Phi_0\|_\infty^{4m}\cdot\Big(\frac{\ell-k}{n}\Big)^{2m} = C^{(1)}\Big(\frac{\ell-k}{n}\Big)^{2m}$$

by combining (4.3.5) with (4.3.7), which is the desired estimate (4.3.3).

Step 3. Next we prove (4.3.4). In the similar way to (4.3.5), we also have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_{x*}}\Big[\big\|\log\big((\mathcal{Y}_{t_k}^{(n)})^{-1}\cdot\mathcal{Y}_{t_\ell}^{(n)}\big)\big|_{\mathfrak{g}^{(2)}}\big\|_{\mathfrak{g}^{(2)}}^{2m}\Big]\\
&\le\Big(\frac{1}{n}\Big)^{2m}\cdot d_2^{2m}\max_{i=1,2,\dots,d_2}\max_{x\in\mathcal{F}}\Big\{\sum_{c\in\Omega_{x,\ell-k}(X)} p(c)\log\Big(\Phi_0(x)^{-1}\cdot\Phi_0\big(t(c)\big)\Big)\Big|_{X_i^{(2)}}^{2m}\Big\}.
\end{aligned} \tag{4.3.8}$$

An elementary inequality $(a_1+a_2+\cdots+a_K)^{2m}\le K^{2m-1}(a_1^{2m}+a_2^{2m}\cdots+a_K^{2m})$ yields

$$\begin{aligned}
&\log\Big(\Phi_0(x)^{-1}\cdot\Phi_0\big(t(c)\big)\Big)\Big|_{X_i^{(2)}}^{2m}\\
&=\log\Big(\Phi_0\big(o(e_1)\big)^{-1}\cdot\Phi_0\big(t(e_1)\big)\cdot\cdots\cdot\Phi_0\big(o(e_{\ell-k})\big)^{-1}\cdot\Phi_0\big(t(e_{\ell-k})\big)\Big)\Big|_{X_i^{(2)}}^{2m}\\
&=\Big(\sum_{j=1}^{\ell-k}\log\big(d\Phi_0(e_j)\big)\big|_{X_i^{(2)}}-\frac{1}{2}\sum_{1\le j_1<j_2\le\ell-k}\ \sum_{1\le\lambda<\nu\le d_1}[\![X_\lambda^{(1)},X_\nu^{(1)}]\!]\big|_{X_i^{(2)}}\\
&\qquad\times\Big\{\log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\lambda^{(1)}}\log\big(d\Phi_0(e_{j_2})\big)\big|_{X_\nu^{(1)}}\\
&\qquad\quad-\log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\nu^{(1)}}\log\big(d\Phi_0(e_{j_2})\big)\big|_{X_\lambda^{(1)}}\Big\}\Big)^{2m}\\
&\le 3^{2m-1}\Big\{\Big(\sum_{j=1}^{\ell-k}\log\big(d\Phi_0(e_j)\big)\big|_{X_i^{(2)}}\Big)^{2m}\\
&\qquad+L\max_{1\le\lambda<\nu\le d_1}\Big(\sum_{1\le j_1<j_2\le\ell-k}\log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\lambda^{(1)}}\log\big(d\Phi_0(e_{j_2})\big)\big|_{X_\nu^{(1)}}\Big)^{2m}\\
&\qquad+L\max_{1\le\lambda<\nu\le d_1}\Big(\sum_{1\le j_1<j_2\le\ell-k}\log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\nu^{(1)}}\log\big(d\Phi_0(e_{j_2})\big)\big|_{X_\lambda^{(1)}}\Big)^{2m}\Big\},
\end{aligned} \tag{4.3.9}$$

where we put

$$L:=\frac{1}{2}\max_{i=1,2,\dots,d_2}\max_{1\le\lambda<\nu\le d_1}\big|[\![X_\lambda^{(1)},X_\nu^{(1)}]\!]\big|_{X_i^{(2)}}\big|.$$

We fix $i = 1, 2, \dots, d_2$. Then the Jensen inequality gives

$$\begin{aligned}\Big(\sum_{j=1}^{\ell-k} \log\big(d\Phi_0(e_j)\big)\big|_{X_i^{(2)}}\Big)^{2m} &= (\ell-k)^{2m}\Big(\sum_{j=1}^{\ell-k} \frac{1}{\ell-k}\log\big(d\Phi_0(e_j)\big)\big|_{X_i^{(2)}}\Big)^{2m} \\ &\le (\ell-k)^{2m}\sum_{j=1}^{\ell-k}\frac{1}{\ell-k}\log\big(d\Phi_0(e_j)\big)\big|_{X_i^{(2)}}^{2m} \\ &\le (\ell-k)^{2m}\|d\Phi_0\|_\infty^{4m}. \qquad (4.3.10)\end{aligned}$$

For $1 \le \lambda < \nu \le d_1$, $x \in \mathcal{F}$, $N \in \mathbb{N}$ and $c = (e_1, e_2, \dots, e_N) \in \Omega_{x,N}(X)$, we put

$$\begin{aligned}\widetilde{\mathcal{M}}_N^{(\lambda,\nu,x)}(c) = \widetilde{\mathcal{M}}_N^{(\lambda,\nu,x)}(\Phi_0; c) &:= \sum_{1\le j_1<j_2\le N} \log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\lambda^{(1)}}\log\big(d\Phi_0(e_{j_2})\big)\big|_{X_\nu^{(1)}} \\ &= \sum_{j_2=2}^{N}\log\big(d\Phi_0(e_{j_2})\big)\big|_{X_\nu^{(1)}}\Big(\sum_{j_1=1}^{j_2-1}\log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\lambda^{(1)}}\Big).\end{aligned}$$

We clearly observe that $\{\widetilde{\mathcal{M}}_N^{(\lambda,\nu,x)}\}_{N=1}^\infty$ is an $\mathbb{R}$-valued martingale for every $1 \le \lambda < \nu \le d$ and $x \in \mathcal{F}$ due to Lemma 2.5.3. Hence, we apply the Burkholder–Davis–Gundy inequality with the exponent $2m$ to obtain

$$\begin{aligned}&\sum_{c\in\Omega_{x,N}(X)} p(c)\big(\widetilde{\mathcal{M}}_N^{(\lambda,\nu,x)}(c)\big)^{2m} \\ &\le \mathcal{C}_{(2m)}^{2m}\sum_{c\in\Omega_{x,N}(X)} p(c)\Big\{\sum_{j_2=2}^{N}\log\big(d\Phi_0(e_{j_2})\big)\big|_{X_\nu^{(1)}}^{2}\Big(\sum_{j_1=1}^{j_2-1}\log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\lambda^{(1)}}\Big)^2\Big\}^m \\ &\le \mathcal{C}_{(2m)}^{2m}\sum_{c\in\Omega_{x,N}(X)} p(c)N^m\sum_{j_2=2}^{N}\frac{1}{N-1}\log\big(d\Phi_0(e_{j_2})\big)\big|_{X_\nu^{(1)}}^{2m}\Big(\sum_{j_1=1}^{j_2-1}\log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\lambda^{(1)}}\Big)^{2m} \\ &\le \mathcal{C}_{(2m)}^{2m}N^m\sum_{j_2=2}^{N}\frac{1}{N-1}\Big\{\sum_{c\in\Omega_{x,N}(X)} p(c)\log\big(d\Phi_0(e_{j_2})\big)\big|_{X_\nu^{(1)}}^{4m}\Big\}^{1/2} \\ &\qquad\times\Big\{\sum_{c\in\Omega_{x,N}(X)} p(c)\Big(\sum_{j_1=1}^{j_2-1}\log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\lambda^{(1)}}\Big)^{4m}\Big\}^{1/2} \\ &\le \mathcal{C}_{(2m)}^{2m}\|d\Phi_0\|_\infty^{2m}N^m\sum_{j_2=2}^{N}\frac{1}{N-1}\Big\{\sum_{c\in\Omega_{x,N}(X)} p(c)\Big(\sum_{j_1=1}^{j_2-1}\log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\lambda^{(1)}}\Big)^{4m}\Big\}^{1/2}, \qquad (4.3.11)\end{aligned}$$

where we used Jensen's inequality for the third line and Schwarz' inequality for the fourth line. Then we have

$$\sum_{c\in\Omega_{x,N}(X)} p(c)\Big(\sum_{j_1=1}^{j_2-1}\log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\lambda^{(1)}}\Big)^{4m}$$

$$
\begin{aligned}
&\le \mathcal{C}_{(4m)}^{4m} \sum_{c \in \Omega_{x,N}(X)} p(c) \Big(\sum_{j_1=1}^{j_2-1} \log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\lambda^{(1)}}^2 \Big)^{2m} \\
&= \mathcal{C}_{(4m)}^{4m} (j_2-1)^{2m} \sum_{c \in \Omega_{x,N}(X)} p(c) \Big(\sum_{j_1=1}^{j_2-1} \frac{1}{j_2-1} \log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\lambda^{(1)}}^2 \Big)^{2m} \\
&\le \mathcal{C}_{(4m)}^{4m} j_2^{2m} \sum_{c \in \Omega_{x,N}(X)} p(c) \sum_{j_1=1}^{j_2-1} \frac{1}{j_2-1} \log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\lambda^{(1)}}^{4m} \le \mathcal{C}_{(4m)}^{4m} \|d\Phi_0\|_\infty^{4m} j_2^{2m} \qquad (4.3.12)
\end{aligned}
$$

by applying the Burkholder–Davis–Gundy inequality with the exponent $4m$. It follows from (4.3.11) and (4.3.12) that

$$
\begin{aligned}
&\sum_{c \in \Omega_{x,N}(X)} p(c) \big(\widetilde{\mathcal{M}}_N^{(\lambda,\nu,x)}(c) \big)^{2m} \\
&\le \mathcal{C}_{(2m)}^{2m} \|d\Phi_0\|_\infty^{2m} N^m \sum_{j_2=2}^{N} \frac{1}{N-1} \Big(\mathcal{C}_{(4m)}^{4m} \|d\Phi_0\|_\infty^{4m} j_2^{2m} \Big)^{1/2} \\
&\le \mathcal{C}_{(2m)}^{2m} \mathcal{C}_{(4m)}^{2m} \|d\Phi_0\|_\infty^{4m} N^m \sum_{j_2=2}^{N} \frac{1}{N-1} \cdot N^m = \mathcal{C}_{(2m)}^{2m} \mathcal{C}_{(4m)}^{2m} \|d\Phi_0\|_\infty^{4m} N^{2m}. \qquad (4.3.13)
\end{aligned}
$$

We now put $N = \ell - k$. Then (4.3.13) implies

$$
\begin{aligned}
&\sum_{c \in \Omega_{x,\ell-k}(X)} p(c) \Big\{ \Big(\sum_{1 \le j_1 < j_2 \le \ell-k} \log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\lambda^{(1)}} \log\big(d\Phi_0(e_{j_2})\big)\big|_{X_\nu^{(1)}} \Big)^{2m} \\
&\qquad + \Big(\sum_{1 \le j_1 < j_2 \le \ell-k} \log\big(d\Phi_0(e_{j_1})\big)\big|_{X_\nu^{(1)}} \log\big(d\Phi_0(e_{j_2})\big)\big|_{X_\lambda^{(1)}} \Big)^{2m} \Big\} \\
&\le 2\mathcal{C}_{(2m)}^{2m} \mathcal{C}_{(4m)}^{2m} \|d\Phi_0\|_\infty^{4m} (\ell-k)^{2m} \qquad (1 \le \lambda < \nu \le d_1). \qquad (4.3.14)
\end{aligned}
$$

By combining (4.3.8) with (4.3.9), (4.3.10) and (4.3.14), we obtain

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}_{x*}} \Big[\big\| \log\big((\mathcal{Y}_{t_k}^{(n)})^{-1} \cdot \mathcal{Y}_{t_\ell}^{(n)} \big)\big|_{\mathfrak{g}^{(2)}} \big\|_{\mathfrak{g}^{(2)}}^{2m} \Big] \\
&\le \Big(\frac{1}{n}\Big)^{2m} d_2^{2m} 3^{2m-1} \|d\Phi_0\|_\infty^{4m} \Big\{ 1 + 2L\mathcal{C}_{(2m)}^{2m} \mathcal{C}_{(4m)}^{2m} \Big\} (\ell-k)^{2m} = C^{(2)} \Big(\frac{\ell-k}{n} \Big)^{2m}.
\end{aligned}
$$

This is the desired estimate (4.3.4), and thus we have shown (4.3.2).

Step 4. We finally prove (4.3.1). Suppose that $t_k \le s \le t_{k+1}$ and $t_\ell \le t \le t_{\ell+1}$ for some $1 \le k \le \ell \le n$. Since the stochastic process $\mathcal{Y}_\cdot^{(n)}$ is given by the d_{CC}-geodesic interpolation, we have

$$
\begin{aligned}
d_{\mathrm{CC}}(\mathcal{Y}_s^{(n;2)}, \mathcal{Y}_{t_{k+1}}^{(n;2)}) &= (k - ns) d_{\mathrm{CC}}(\mathcal{Y}_{t_k}^{(n;2)}, \mathcal{Y}_{t_{k+1}}^{(n;2)}), \\
d_{\mathrm{CC}}(\mathcal{Y}_{t_\ell}^{(n;2)}, \mathcal{Y}_t^{(n;2)}) &= (nt - \ell) d_{\mathrm{CC}}(\mathcal{Y}_{t_\ell}^{(n;2)}, \mathcal{Y}_{t_{\ell+1}}^{(n;2)}).
\end{aligned}
$$

By using (4.3.2) and the triangle inequality, we have

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}_{x_*}}\Big[d_{\mathrm{CC}}(\mathcal{Y}_s^{(n;\,2)},\mathcal{Y}_t^{(n;\,2)})^{4m}\Big]\\
&\leq 3^{4m-1}\Big\{(k+1-ns)^{4m}\cdot C\Big(\frac{1}{n}\Big)^{2m}+C\Big(\frac{\ell-k-1}{n}\Big)^{2m}+(nt-\ell)^{4m}\cdot C\Big(\frac{1}{n}\Big)^{2m}\Big\}\\
&\leq C\Big\{(t_{k+1}-s)^{2m}+(t_\ell-t_{k+1})^{2m}+(t-t_\ell)^{2m}\Big\}\leq C(t-s)^{2m},
\end{aligned}
$$

which is the desired estimate (4.3.1) and we have proved Lemma 4.3.2. ∎

In what follows, we write

$$
d\mathcal{Y}_{s,t}^{(n)*}:=(\mathcal{Y}_s^{(n)})^{-1}*\mathcal{Y}_t^{(n)}\qquad(n\in\mathbb{N},\,0\leq s\leq t\leq 1)
$$

for simplicity. By using Lemma 4.3.2, we obtain the following.

Lemma 4.3.3 *For $m,n\in\mathbb{N}$, $k=1,2,\dots,r$ and $\alpha<\frac{2m-1}{4m}$, there exist an $\mathcal{F}_\infty$-measurable set $\Omega_k^{(n)}\subset\Omega_{x_*}(X)$, a non-negative random variable $\mathcal{K}_k^{(n)}\in L^{4m}(\Omega_{x_*}(X)\to\mathbb{R};\mathbb{P}_{x_*})$ such that $\mathbb{P}_{x_*}(\Omega_k^{(n)})=1$ and*

$$
d_{\mathrm{CC}}\big(\mathcal{Y}_s^{(n;\,k)}(c),\mathcal{Y}_t^{(n;\,k)}(c)\big)\leq\mathcal{K}_k^{(n)}(c)(t-s)^\alpha\qquad(c\in\Omega_k^{(n)},\,0\leq s\leq t\leq 1).\tag{4.3.15}
$$

Proof. We partially follow Lyons' original proof (cf. [54, Theorem 2.2.1]) for the extension theorem in rough path theory. We show (4.3.15) by induction on the step number $k=1,2,\dots,r$.

Step 1. In the cases $k=1,2$, we have already obtained (4.3.15) in Lemma 4.3.2. Indeed, (4.3.15) for $k=1,2$ are readily obtained by a simple application of the Kolmogorov–Chentsov criterion with the bound

$$
\|\mathcal{K}_k^{(n)}\|_{L^{4m}(\mathbb{P}_{x_*})}\leq\frac{5C}{(1-2^{-\theta})(1-2^{\alpha-\theta})}\qquad(n,m\in\mathbb{N},\,k=1,2),\tag{4.3.16}
$$

where $\theta=(2m-1)/4m$ and C is a constant independent of n, which appears in the right-hand side of (4.3.1). See e.g., Stroock [67, Theorem 4.3.2] for details.

Step 2. Suppose that (4.3.15) holds up to step k. Then, for $n\in\mathbb{N}$, there are $\mathcal{F}_\infty$-measurable sets $\{\widehat{\Omega}_j^{(n)}\}_{j=1}^k$ and non-negative random variables $\{\widehat{\mathcal{K}}_j^{(n)}\}_{j=1}^k$ such that $\mathbb{P}_{x_*}(\widehat{\Omega}_j^{(n)})=1$ for $j=1,2,\dots,k$ and

$$
\big\|\big(d\mathcal{Y}_{s,t}^{(n)*}(c)\big)^{(j)}\big\|_{\mathbb{R}^{d_j}}\leq\widehat{\mathcal{K}}_j^{(n)}(c)(t-s)^{j\alpha}\qquad(j=1,2,\dots,k,\,c\in\widehat{\Omega}_j^{(n)},\,0\leq s\leq t\leq 1)\tag{4.3.17}
$$

with $\widehat{\mathcal{K}}_j^{(n)}\in L^{4m/j}(\Omega_{x_*}(X)\to\mathbb{R};\mathbb{P}_{x_*})$ for $m\in\mathbb{N}$ and $j=1,2,\dots,k$.

We fix $0 \le s \le t \le 1$ and $n \in \mathbb{N}$. Set $\widehat{\Omega}^{(n)}_{k+1} = \bigcap_{j=1}^{k} \widehat{\Omega}^{(n)}_{j}$. We denote by Δ the partition $\{s = t_0 < t_1 < \cdots < t_N = t\}$ of the time interval $[s,t]$ independent of $n \in \mathbb{N}$. We define two $G^{(k+1)}$-valued random variables $\mathcal{Z}^{(n)}_{s,t}$ and $\mathcal{Z}(\Delta)^{(n)}_{s,t}$ by

$$
\begin{aligned}
\big(\mathcal{Z}^{(n)}_{s,t}\big)^{(j)} &:= \begin{cases} \big(d\mathcal{Y}^{(n)*}_{s,t}\big)^{(j)}, & (j = 1, 2, \dots, k), \\ \mathbf{0} & (j = k+1), \end{cases} \\
\mathcal{Z}(\Delta)^{(n)}_{s,t} &:= \mathcal{Z}^{(n)}_{t_0,t_1} * \mathcal{Z}^{(n)}_{t_1,t_2} * \cdots * \mathcal{Z}^{(n)}_{t_{N-1},t_N},
\end{aligned}
$$

respectively. For $i = 1, 2, \dots, d_{k+1}$, (2.2.2) and (4.3.15) imply

$$
\begin{aligned}
&\Big|\big(\mathcal{Z}(\Delta)^{(n)}_{s,t}(c)\big)^{(k+1)}_{i*} - \big(\mathcal{Z}(\Delta \setminus \{t_{N-1}\})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i*}\Big| \\
&= \Big|\big(\mathcal{Z}^{(n)}_{t_{N-2},t_{N-1}}(c) * \mathcal{Z}^{(n)}_{t_{N-1},t_N}(c)\big)^{(k+1)}_{i*} - \big(\mathcal{Z}^{(n)}_{t_{N-2},t_N}(c)\big)^{(k+1)}_{i*}\Big| \\
&= \Bigg| \sum_{\substack{|K_1|+|K_2|=k+1 \\ |K_1|,|K_2| \ge 0}} C_{K_1,K_2} \mathcal{P}^{K_1}_{*}\big(\mathcal{Z}^{(n)}_{t_{N-2},t_{N-1}}(c)\big) \mathcal{P}^{K_2}_{*}\big(\mathcal{Z}^{(n)}_{t_{N-1},t_N}(c)\big) \Bigg| \\
&\le C \sum_{\substack{|K_1|+|K_2|=k+1 \\ |K_1|,|K_2| \ge 0}} \Big|\mathcal{P}^{K_1}_{*}\big(d\mathcal{Y}^{(n)*}_{t_{N-2},t_{N-1}}(c)\big)\Big| \Big|\mathcal{P}^{K_2}_{*}\big(d\mathcal{Y}^{(n)*}_{t_{N-1},t_N}(c)\big)\Big| \\
&\le \widehat{\mathcal{K}}^{(n)}_{k+1}(c)(t_N - t_{N-2})^{(k+1)\alpha} \le \widehat{\mathcal{K}}^{(n)}_{k+1}(c)\Big(\frac{2}{N-1}(t-s)\Big)^{(k+1)\alpha} \qquad (c \in \widehat{\Omega}^{(n)}_{k+1}),
\end{aligned}
$$

where the random variable $\widehat{\mathcal{K}}^{(n)}_{k+1} : \Omega_{x_*}(X) \longrightarrow \mathbb{R}$ is given by

$$
\begin{aligned}
\widehat{\mathcal{K}}^{(n)}_{k+1}(c) &:= C \sum_{\substack{|K_1|+|K_2|=k+1 \\ |K_1|,|K_2| \ge 0}} \mathcal{Q}^{(n,K_1)}(c)\mathcal{Q}^{(n,K_2)}(c), \\
\mathcal{Q}^{(n,K)}(c) &:= \widehat{\mathcal{K}}^{(n)}_{k_1}(c)\widehat{\mathcal{K}}^{(n)}_{k_2}(c) \cdots \widehat{\mathcal{K}}^{(n)}_{k_\ell}(c) \qquad \big(K = \big((i_1,k_1),(i_2,k_2),\dots,(i_\ell,k_\ell)\big)\big).
\end{aligned}
$$

Note that $\widehat{\mathcal{K}}^{(n)}_{k+1}$ is non-negative and it has the following integrability:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}_{x_*}}\big[(\widehat{\mathcal{K}}^{(n)}_{k+1})^{4m/(k+1)}\big] &\le C \sum_{\substack{k_1,\dots,k_\ell > 0 \\ k_1+k_2+\cdots+k_\ell = k+1}} \mathbb{E}^{\mathbb{P}_{x_*}}\Big[\big(\widehat{\mathcal{K}}^{(n)}_{k_1}\widehat{\mathcal{K}}^{(n)}_{k_2}\cdots\widehat{\mathcal{K}}^{(n)}_{k_\ell}\big)^{4m/(k+1)}\Big] \\
&\le C \sum_{\substack{k_1,\dots,k_\ell > 0 \\ k_1+k_2+\cdots+k_\ell = k+1}} \prod_{\lambda=1}^{\ell} \mathbb{E}^{\mathbb{P}_{x_*}}\Big[\big(\widehat{\mathcal{K}}^{(n)}_{k_\lambda}\big)^{4m/k_\lambda}\Big]^{k_\lambda/(k+1)} < \infty,
\end{aligned}
$$

where we used the generalized Hölder inequality for the second line. By removing points in Δ successively until the partition Δ coincides with $\{s,t\}$, we have

$$
\begin{aligned}
&\Big|\big(\mathcal{Z}(\Delta)^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| \\
&\le \Big|\big(\mathcal{Z}(\Delta\setminus\{t_{N-1}\})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| + \widehat{\mathcal{K}}^{(n)}_{k+1}(c)\Big(\frac{2}{N-1}(t-s)\Big)^{(k+1)\alpha} \\
&\le \Big|\big(\mathcal{Z}(\{s,t\})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| + \sum_{\ell=1}^{N-2}\widehat{\mathcal{K}}^{(n)}_{k+1}(c)\Big(\frac{2}{N-\ell}\Big)^{(k+1)\alpha}(t-s)^{(k+1)\alpha} \\
&\le \Big|\big(\mathcal{Z}^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| + \widehat{\mathcal{K}}^{(n)}_{k+1}(c)2^{(k+1)\alpha}\zeta\big((k+1)\alpha\big)(t-s)^{(k+1)\alpha} \\
&\le \widehat{\mathcal{K}}^{(n)}_{k+1}(c)(t-s)^{(k+1)\alpha} \qquad (i=1,2,\dots,d_{k+1},\ c\in\widehat{\Omega}^{(n)}_{k+1}),
\end{aligned}
\tag{4.3.18}
$$

where $\zeta(z)$ denotes the Riemann zeta function $\zeta(z):=\sum_{n=1}^{\infty}(1/n^z)$ for $z\in\mathbb{R}$.

We will show that the family $\{\mathcal{Z}(\Delta)^{(n)}_{s,t}\}$ satisfies the Cauchy convergence principle. Let $\delta>0$ and take two partitions $\Delta=\{s=t_0<t_1\cdots<t_N=t\}$ and Δ' of $[s,t]$ independent of $n\in\mathbb{N}$ satisfying $|\Delta|,|\Delta'|<\delta$. We set $\widehat{\Delta}:=\Delta\cup\Delta'$ and write

$$
\widehat{\Delta}_\ell=\widehat{\Delta}\cap[t_\ell,t_{\ell+1}]=\{t_\ell=s_{\ell 0}<s_{\ell 1}<\cdots<s_{\ell L_\ell}=t_{\ell+1}\} \qquad (\ell=0,1,\dots,N-1).
$$

By using (4.3.18), we have

$$
\begin{aligned}
&\Big|\big(\mathcal{Z}(\Delta)^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}-\big(\mathcal{Z}(\widehat{\Delta})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| \\
&=\Big|\big(\mathcal{Z}^{(n)}_{t_0,t_1}(c)*\cdots*\mathcal{Z}^{(n)}_{t_{N-1},t_N}(c)\big)^{(k+1)}_{i_*}-\big(\mathcal{Z}(\widehat{\Delta}_0)^{(n)}_{t_0,t_1}(c)*\cdots*\mathcal{Z}(\widehat{\Delta}_{N-1})^{(n)}_{t_{N-1},t_N}(c)\big)^{(k+1)}_{i_*}\Big| \\
&=\Big|\big(\mathcal{Z}^{(n)}_{t_0,t_1}(c)\big)^{(k+1)}_{i_*}+\big(\mathcal{Z}^{(n)}_{t_1,t_2}(c)*\cdots*\mathcal{Z}^{(n)}_{t_{N-1},t_N}(c)\big)^{(k+1)}_{i_*} \\
&\qquad -\big(\mathcal{Z}(\widehat{\Delta}_0)^{(n)}_{t_0,t_1}(c)\big)^{(k+1)}_{i_*}-\big(\mathcal{Z}(\widehat{\Delta}_1)^{(n)}_{t_1,t_2}(c)*\cdots*\mathcal{Z}(\widehat{\Delta}_{N-1})^{(n)}_{t_{N-1},t_N}(c)\big)^{(k+1)}_{i_*}\Big| \\
&\le \widehat{\mathcal{K}}^{(n)}_{k+1}(c)(t_1-t_0)^{(k+1)\alpha}+\Big|\big(\mathcal{Z}^{(n)}_{t_1,t_2}(c)*\cdots*\mathcal{Z}^{(n)}_{t_{N-1},t_N}(c)\big)^{(k+1)}_{i_*} \\
&\qquad -\big(\mathcal{Z}(\widehat{\Delta}_0)^{(n)}_{t_1,t_2}(c)*\cdots*\mathcal{Z}(\widehat{\Delta}_{N-1})^{(n)}_{t_{N-1},t_N}(c)\big)^{(k+1)}_{i_*}\Big| \qquad (i=1,2,\dots,d_{k+1},\ c\in\widehat{\Omega}^{(n)}_{k+1}).
\end{aligned}
$$

Repeating this kind of estimate and recalling $(k+1)\alpha>1$ yield

$$
\begin{aligned}
&\Big|\big(\mathcal{Z}(\Delta)^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}-\big(\mathcal{Z}(\widehat{\Delta})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| \\
&\le \sum_{\ell=0}^{N-1}\widehat{\mathcal{K}}^{(n)}_{k+1}(c)(t_{\ell+1}-t_\ell)^{(k+1)\alpha} \\
&\le \widehat{\mathcal{K}}^{(n)}_{k+1}(c)\Big(\max_{\Delta}(t_{\ell+1}-t_\ell)^{(k+1)\alpha-1}\Big)\sum_{\ell=0}^{N-1}(t_{\ell+1}-t_\ell) \\
&\le \widehat{\mathcal{K}}^{(n)}_{k+1}(c)(t-s)\times\delta^{(k+1)\alpha-1} \qquad (i=1,2,\dots,d_{k+1},\ c\in\widehat{\Omega}^{(n)}_{k+1}).
\end{aligned}
\tag{4.3.19}
$$

We thus obtain

$$\begin{aligned}
&\Big|\big(\mathcal{Z}(\Delta)^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*} - \big(\mathcal{Z}(\Delta')^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| \\
&\le \Big|\big(\mathcal{Z}(\Delta)^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*} - \big(\mathcal{Z}(\widehat{\Delta})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| + \Big|\big(\mathcal{Z}(\widehat{\Delta})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*} - \big(\mathcal{Z}(\Delta')^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| \\
&\le 2\widehat{\mathcal{K}}^{(n)}_{k+1}(c)(t-s) \times \delta^{(k+1)\alpha-1} \longrightarrow 0 \qquad (i=1,2,\dots,d_{k+1},\, c \in \widehat{\Omega}^{(n)}_{k+1})
\end{aligned}$$

as $\delta \searrow 0$ uniformly in $0 \le s \le t \le 1$ by (4.3.19). Therefore, noting the estimate (4.3.18), there exists a random variable

$$\overline{\mathcal{Z}}^{(n)}_{s,t}(c) := \begin{cases} \lim\limits_{|\Delta|\searrow 0} \mathcal{Z}(\Delta)^{(n)}_{s,t}(c) & (c \in \widehat{\Omega}^{(n)}_{k+1}), \\ \mathbf{1}_G & (c \in \Omega_{x_*}(X) \setminus \widehat{\Omega}^{(n)}_{k+1}), \end{cases} \qquad (0 \le s \le t \le 1)$$

satisfying

$$\big\|\big(\overline{\mathcal{Z}}^{(n)}_{s,t}(c)\big)^{(k+1)}\big\|_{\mathbb{R}^{d_{k+1}}} \le \widehat{\mathcal{K}}^{(n)}_{k+1}(c)(t-s)^{(k+1)\alpha} \qquad (c \in \widehat{\Omega}^{(n)}_{k+1}).$$

Our final goal is to show

$$\overline{\mathcal{Z}}^{(n)}_{s,t}(c) = \mathcal{Y}^{(n;\,k+1)}_s(c) * \mathcal{Y}^{(n;\,k+1)}_t(c) \qquad (0 \le s \le t \le 1,\, c \in \widehat{\Omega}^{(n)}_{k+1}).$$

However, it suffices to check that

$$\big(\overline{\mathcal{Z}}^{(n)}_{s,t}(c)\big)^{(k+1)} = \big(d\mathcal{Y}^{(n)*}_{s,t}(c)\big)^{(k+1)} \qquad (0 \le s \le t \le 1,\, c \in \widehat{\Omega}^{(n)}_{k+1}) \tag{4.3.20}$$

by the definition of $\overline{\mathcal{Z}}^{(n)}_{s,t}$. We fix $i = 1,2,\dots,d_{k+1}$ and $c \in \widehat{\Omega}^{(n)}_{k+1}$. Put

$$\Psi^i_{s,t}(c) := \big(d\mathcal{Y}^{(n)*}_{s,t}(c)\big)^{(k+1)}_{i*} - \big(\overline{\mathcal{Z}}^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*} \quad (0 \le s \le t \le 1).$$

Then we easily see that $\Psi^i_{s,t}(c)$ is additive in the sense that

$$\Psi^i_{s,t}(c) = \Psi^i_{s,u}(c) + \Psi^i_{u,t}(c) \qquad (0 \le s \le u \le t \le 1). \tag{4.3.21}$$

Since the piecewise smooth stochastic process $(\mathcal{Y}^{(n)}_t)_{0\le t\le 1}$ is defined by the d_{CC}- geodesic interpolation of $\{\mathcal{X}^{(n)}_{t_k}\}^n_{k=0}$, we know

$$\big\|\big(d\mathcal{Y}^{(n)*}_{s,t}(c)\big)^{(k+1)}\big\|_{\mathbb{R}^{d_{k+1}}} \le \widetilde{\mathcal{K}}^{(n)}_{k+1}(c)(t-s)^{(k+1)\alpha} \qquad (c \in \widetilde{\Omega}^{(n)}_{k+1})$$

for some set $\widetilde{\Omega}^{(n)}_{k+1}$ with $\mathbb{P}_{x_*}(\widetilde{\Omega}^{(n)}_{k+1}) = 1$ and random variable $\widetilde{\mathcal{K}}^{(n)}_{k+1} : \Omega_{x_*}(X) \longrightarrow \mathbb{R}$. Then we have

$$\Big|\Psi^i_{s,t}(c)\Big| \le \big(\widetilde{\mathcal{K}}^{(n)}_{k+1}(c) + \widehat{\mathcal{K}}^{(n)}_{k+1}(c)\big)(t-s)^{(k+1)\alpha} \qquad (0 \le s \le t \le 1,\, c \in \widetilde{\Omega}^{(n)}_{k+1} \cap \widehat{\Omega}^{(n)}_{k+1}).$$

We may write $\widehat{\Omega}^{(n)}_{k+1}$ instead of $\widetilde{\Omega}^{(n)}_{k+1} \cap \widehat{\Omega}^{(n)}_{k+1}$ by abuse of notation, because its probability is equal to one. For any small $\varepsilon > 0$, there is a sufficiently large $N \in \mathbb{N}$ such that $1/N < \varepsilon$.

We obtain, as $\varepsilon \searrow 0$,

$$\begin{aligned}\left|\Psi^i_{0,t}(c)\right| &= \left|\Psi^i_{0,1/N}(c) + \Psi^i_{1/N,2/N}(c) + \cdots + \Psi^i_{[Nt]/N,t}(c)\right| \\ &\le \big(\widetilde{\mathcal{K}}^{(n)}_{k+1}(c) + \widehat{\mathcal{K}}^{(n)}_{k+1}(c)\big)\varepsilon^{(k+1)\alpha-1}\Big\{\underbrace{\frac{1}{N} + \frac{1}{N} + \cdots + \frac{1}{N}}_{[Nt]\text{-times}} + \Big(t - \frac{[Nt]}{N}\Big)\Big\} \\ &= \big(\widetilde{\mathcal{K}}^{(n)}_{k+1}(c) + \widehat{\mathcal{K}}^{(n)}_{k+1}(c)\big)\varepsilon^{(k+1)\alpha-1}t \longrightarrow 0 \qquad (0 \le t \le 1,\, c \in \widehat{\Omega}^{(n)}_{k+1})\end{aligned}$$

by (4.3.21) and $(k+1)\alpha - 1 > 0$. This implies that $\Psi^i_{0,t}(c) = 0$ for $0 \le t \le 1$ and $c \in \widehat{\Omega}^{(n)}_{k+1}$. Therefore, it follows from (4.3.20) that

$$\Psi^i_{s,t}(c) = \Psi^i_{0,t}(c) - \Psi^i_{0,s}(c) = 0 \qquad (0 \le s \le t \le 1,\, c \in \widehat{\Omega}^{(n)}_{k+1}),$$

which means (4.3.19). Consequently, there exist a $\mathcal{F}_\infty$-measurable set $\Omega^{(n)}_{k+1} \subset \Omega_{x_*}(X)$ with probability one and a non-negative random variable $\mathcal{K}^{(n)}_{k+1} \in L^{4m}(\Omega_{x_*}(X) \to \mathbb{R}; \mathbb{P}_{x_*})$ satisfying

$$d_{\mathrm{CC}}\big(\mathcal{Y}^{(n;\,k+1)}_s(c), \mathcal{Y}^{(n;\,k+1)}_t(c)\big) \le \mathcal{K}^{(n)}_{k+1}(c)(t-s)^\alpha \quad (0 \le s \le t \le 1,\, c \in \Omega^{(n)}_{k+1}).$$

This completes the proof of Lemma 4.3.3. ∎

Proof of Lemma 4.3.1. For $m, n \in \mathbb{N}$ and $\widehat{\alpha} < \frac{2m-1}{4m}$, it follows from (4.3.15) that

$$\mathbb{E}^{\mathbb{P}_{x_*}}\Big[d_{\mathrm{CC}}\big(\mathcal{Y}^{(n;\,r)}_s, \mathcal{Y}^{(n;\,r)}_t\big)^{4m}\Big] \le \mathbb{E}^{\mathbb{P}_{x_*}}\big[\big(\mathcal{K}^{(n)}_r\big)^{4m}\big](t-s)^{4m\widehat{\alpha}}$$

for $0 \le s \le t \le 1$. We thus have, by (4.3.16),

$$\mathbb{E}^{\mathbb{P}_{x_*}}\Big[d_{\mathrm{CC}}\big(\mathcal{Y}^{(n;\,r)}_s, \mathcal{Y}^{(n;\,r)}_t\big)^{4m}\Big] \le C(t-s)^{4m\widehat{\alpha}} \qquad (0 \le s \le t \le 1).$$

for a positive constant $C > 0$ independent of $n \in \mathbb{N}$. By applying the Kolmogorov tightness criterion (cf. Friz–Hairer [19, Section 3.1]), we have shown that the family $\{\mathbf{P}^{(n)}\}_{n=1}^\infty$ is tight in $C^{0,\alpha\text{-Höl}}_{\mathbf{1}_G}([0,1]; G)$ for any $\alpha < \frac{4m\widehat{\alpha}-1}{4m} < \frac{1}{2} - \frac{1}{2m}$. Since $m \in \mathbb{N}$ is arbitrary, we conclude that $\{\mathbf{P}^{(n)}\}_{n=1}^\infty$ is tight in $C^{0,\alpha\text{-Höl}}_{\mathbf{1}_G}([0,1]; G)$ for any $\alpha < 1/2$. ∎

We conclude Theorem 4.1.3 by showing the following convergence of the finite dimensional distribution.

Lemma 4.3.4 *Let $\ell \in \mathbb{N}$. For fixed $0 \le s_1 < s_2 < \cdots < s_\ell \le 1$, we have*

$$(\mathcal{Y}^{(n)}_{s_1}, \mathcal{Y}^{(n)}_{s_2}, \ldots, \mathcal{Y}^{(n)}_{s_\ell}) \xrightarrow{(d)} (Y_{s_1}, Y_{s_2}, \ldots, Y_{s_\ell}) \quad (n \to \infty).$$

Proof. We only prove the convergence for $\ell = 2$. General cases ($\ell \ge 3$) can be also proved by repeating the same argument. Put $s = s_1$ and $t = s_2$. Then, by applying Theorem 4.1.2, we obtain $(\mathcal{X}^{(n)}_s, \mathcal{X}^{(n)}_t) \xrightarrow{(d)} (Y_s, Y_t)$ as $n \to \infty$ in the same way as [31, Lemma 4.2].

On the other hand, Lemma 4.3.3 tells us that there exists a non-negative random variable $\mathcal{K}_r^{(n)} \in L^{4m}(\Omega_{x_*}(X) \to \mathbb{R}; \mathbb{P}_{x_*})$ such that

$$d_{\mathrm{CC}}\big(\mathcal{Y}_s^{(n)}(c), \mathcal{Y}_t^{(n)}(c)\big) \le \mathcal{K}_r^{(n)}(c)(t-s)^{\alpha} \quad \mathbb{P}_{x_*}\text{-a.s.} \qquad (0 \le s \le t \le 1).$$

Now suppose that $t_k \le t \le t_{k+1}$ for some $k = 0, 1, \dots, n-1$. For all $\varepsilon > 0$ and sufficiently large $m \in \mathbb{N}$, by using Chebyshev's inequality, we have

$$\begin{aligned}
&\mathbb{P}_{x_*}\Big(d_{\mathrm{CC}}\big(\mathcal{X}_t^{(n)}, \mathcal{Y}_t^{(n)}\big) > \varepsilon\Big)\\
&\le \frac{1}{\varepsilon^{4m}}\mathbb{E}^{\mathbb{P}_{x_*}}\Big[d_{\mathrm{CC}}\big(\mathcal{X}_t^{(n)}, \mathcal{Y}_t^{(n)}\big)^{4m}\Big]\\
&\le \frac{1}{\varepsilon^{4m}}\mathbb{E}^{\mathbb{P}_{x_*}}\Big[d_{\mathrm{CC}}\big(\mathcal{Y}_{t_k}^{(n)}, \mathcal{Y}_{t_{k+1}}^{(n)}\big)^{4m}\Big]\\
&\le \frac{1}{\varepsilon^{4m}}\mathbb{E}^{\mathbb{P}_{x_*}}\Big[(\mathcal{K}_r^{(n)})^{4m}(t_{k+1}-t_k)^{4m\alpha}\Big] = \frac{1}{n^{2m-1}\varepsilon^{4m}}\mathbb{E}^{\mathbb{P}_{x_*}}\big[(\mathcal{K}_r^{(n)})^{4m}\big] \longrightarrow 0 \quad (n \to \infty).
\end{aligned}$$

Thus, Slutzky's theorem (cf. Klenke [37, Theorem 13.8]) allows us to obtain the desired convergence $(\mathcal{Y}_s^{(n)}, \mathcal{Y}_t^{(n)}) \xrightarrow{(d)} (Y_s, Y_t)$ as $n \to \infty$. This completes the proof. ∎

4.4 A comment on CLTs of the first kind in the non-centered case

As was already mentioned, the centered condition **(C)** is crucial to establish the FCLT (Theorem 4.1.3). We present a method to reduce the non-centered case $\rho_{\mathbb{R}}(\gamma_p) \neq \mathbf{0}_{\mathfrak{g}}$ to the centered case as a generalization of the measure-change technique in the case of crystal lattices discussed in Section 3.

We consider a *positive* transition probability $p : E \longrightarrow (0,1]$ to avoid several technical difficulties. Then the random walk on X associated with p is automatically irreducible. Let $\Phi_0 : X \longrightarrow G$ be the (p-)modified harmonic realization. We define a function $F = F_x(\lambda) : V_0 \times \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R}) \longrightarrow \mathbb{R}$ by

$$F_x(\lambda) := \sum_{e \in (E_0)_x} p(e) \exp\Big({}_{\mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R})}\big\langle \lambda, \log\big(d\Phi_0(\widetilde{e})\big)\big|_{\mathfrak{g}^{(1)}}\big\rangle_{\mathfrak{g}^{(1)}}\Big) \tag{4.4.1}$$

for $x \in V_0$ and $\lambda \in \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$. Since the lemma below is obtained by following the argument in Lemma 3.1.1, we omit the proof.

Lemma 4.4.1 *For every $x \in V_0$, the function $F_x(\cdot) : \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R}) \longrightarrow (0, \infty)$ has a unique minimizer $\lambda_* = \lambda_*(x) \in \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$.*

We now define a positive function $\mathfrak{p} : E_0 \longrightarrow (0,1]$ by

$$\mathfrak{p}(e) := \frac{\exp\Big({}_{\mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R})}\big\langle \lambda_*\big(o(e)\big), \log\big(d\Phi_0(\widetilde{e})\big)\big|_{\mathfrak{g}^{(1)}}\big\rangle_{\mathfrak{g}^{(1)}}\Big)}{F_{o(e)}\big(\lambda_*\big(o(e)\big)\big)} p(e) \qquad (e \in E_0). \tag{4.4.2}$$

It is straightforward to check that the function $\mathfrak{p}$ also gives a positive transition probability on X_0 and it yields an irreducible Markov chain $(\Omega_x(X), \widehat{\mathbb{P}}_x, \{w_n^{(\mathfrak{p})}\}_{n=0}^{\infty})$ with values in X. We then find a unique positive normalized invariant measure $\mathfrak{m} : V_0 \longrightarrow (0,1]$ by applying the Perron-Frobenius theorem again. We set $\widetilde{\mathfrak{m}}(e) := \mathfrak{p}(e)\mathfrak{m}\big(o(e)\big)$ for $e \in E_0$. We also denote by $\mathfrak{p} : E \longrightarrow (0,1]$ and $\mathfrak{m} : V \longrightarrow (0,1]$ the Γ-invariant lifts of $\mathfrak{p} : E_0 \longrightarrow (0,1]$ and $\mathfrak{m} : V_0 \longrightarrow (0,1]$ to X, respectively. The Albanese metric on $\mathfrak{g}^{(1)}$ associated with the transition probability $\mathfrak{p}$ is denoted by $g_0^{(\mathfrak{p})}$. We write $\{V_1^{(\mathfrak{p})}, V_2^{(\mathfrak{p})}, \dots, V_{d_1}^{(\mathfrak{p})}\}$ for an orthonormal basis of $(\mathfrak{g}^{(1)}, g_0^{(\mathfrak{p})})$.

Let $L_{(\mathfrak{p})} : C_\infty(X) \longrightarrow C_\infty(X)$ be the transition operator associated with the transition probability $\mathfrak{p}$. By virtue of Lemma 4.4.1, we have

$$\sum_{e \in (E_0)_x} p(e) \exp\Big({}_{\mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R})}\big\langle \lambda_*,\, \log\big(d\Phi_0(\widetilde{e})\big)\big|_{\mathfrak{g}^{(1)}}\big\rangle_{\mathfrak{g}^{(1)}}\Big) \log\big(d\Phi_0(\widetilde{e})\big)\big|_{\mathfrak{g}^{(1)}} = \mathbf{0}_{\mathfrak{g}} \qquad (x \in V_0).$$

Hence, we conclude

$$(L_{(\mathfrak{p})} - I)\big(\log \Phi_0\big|_{\mathfrak{g}^{(1)}}\big)(x) = \sum_{e \in E_x} \mathfrak{p}(e) \log\big(d\Phi_0(e)\big)\big|_{\mathfrak{g}^{(1)}} = \mathbf{0}_{\mathfrak{g}} \qquad (x \in V). \tag{4.4.3}$$

This means that the (p-)modified harmonic realization $\Phi_0 : X \longrightarrow G$ in the sense of (2.4.4) is regarded as the ($\mathfrak{p}$-)harmonic realization and $\rho_{\mathbb{R}}(\gamma_{\mathfrak{p}}) = \mathbf{0}_{\mathfrak{g}}$.

We fix a reference point $x_* \in V$ such that $\Phi_0(x_*) = \mathbf{1}_G$ and put

$$\xi_n^{(\mathfrak{p})}(c) := \Phi_0\big(w_n^{(\mathfrak{p})}(c)\big) \qquad \big(n \in \mathbb{N} \cup \{0\},\, c \in \Omega_{x_*}(X)\big).$$

This yields a G-valued random walk $(\Omega_{x_*}(X), \widehat{\mathbb{P}}_{x_*}, \{\xi_n^{(\mathfrak{p})}\}_{n=0}^{\infty})$. We define

$$\mathcal{Y}_{t_k}^{(n;\mathfrak{p})}(c) := \tau_{n^{-1/2}}\big(\xi_{nt_k}^{(\mathfrak{p})}(c)\big) = \tau_{n^{-1/2}}\big(\Phi_0(w_k^{(\mathfrak{p})}(c))\big)$$

for $k = 0, 1, \dots, n$, $t_k \in \mathcal{D}_n$ and $c \in \Omega_{x_*}(X)$. We consider a G-valued stochastic process $(\mathcal{Y}_t^{(n;\mathfrak{p})})_{0 \le t \le 1}$ defined by the d_{CC}-geodesic interpolation of $\{\mathcal{Y}_{t_k}^{(n;\mathfrak{p})}\}_{k=0}^{n}$. Let $(\widetilde{Y}_t)_{0 \le t \le 1}$ be the G-valued diffusion process which solves the SDE

$$d\widetilde{Y}_t = \sum_{i=1}^{d_1} V_{i*}^{(\mathfrak{p})}(\widetilde{Y}_t) \circ dB_t^i + \beta^{(\mathfrak{p})}(\Phi_0)_*(\widetilde{Y}_t)\, dt, \qquad \widetilde{Y}_0 = \mathbf{1}_G,$$

where

$$\beta^{(\mathfrak{p})}(\Phi_0) := \sum_{e \in E_0} \widetilde{\mathfrak{m}}(e) \log\Big(\Phi_0\big(o(\widetilde{e})\big)^{-1} \cdot \Phi_0\big(t(\widetilde{e})\big)\Big)\Big|_{\mathfrak{g}^{(2)}}.$$

The following two theorems are CLTs for non-symmetric random walks associated with the changed transition probability $\mathfrak{p}$. We remark that the proofs of these theorems below are done by combining the ones of Theorems 4.1.2 and 4.1.3 with the argument in Theorem 3.2.1 and Lemma 3.2.3.

Theorem 4.4.2 *Let $P_\varepsilon : C_\infty(G) \longrightarrow C_\infty(X)$ be the approximation operator defined by $P_\varepsilon f(x) := f\big(\tau_\varepsilon\big(\Phi_0(x)\big)\big)$ for $0 \le \varepsilon \le 1$ and $x \in V$. Then we have, for $0 \le s \le t$ and $f \in C_\infty(G)$,*

$$\lim_{n\to\infty} \left\| L_{(\mathfrak{p})}^{[nt]-[ns]} P_{n^{-1/2}} f - P_{n^{-1/2}} \mathrm{e}^{-(t-s)\mathcal{A}_{(\mathfrak{p})}} f \right\|_\infty^X = 0, \tag{4.4.4}$$

where $(\mathrm{e}^{-t\mathcal{A}_{(\mathfrak{p})}})_{t\ge 0}$ is the C_0-semigroup with the infinitesimal generator $\mathcal{A}_{(\mathfrak{p})}$ on $C_0^\infty(G)$ defined by

$$\mathcal{A}_{(\mathfrak{p})} := -\frac{1}{2}\sum_{i=1}^{d_1} (V_{i*}^{(\mathfrak{p})})^2 - \beta^{(\mathfrak{p})}(\Phi_0)_*. \tag{4.4.5}$$

Theorem 4.4.3 *The sequence $(\mathcal{Y}_t^{(n;\mathfrak{p})})_{0\le t\le 1}$ $(n = 1, 2, 3, \dots)$ converges in law to the G-valued diffusion process $(\widetilde{Y}_t)_{0\le t\le 1}$ in $C_{\mathbf{1}_G}^{0,\alpha\text{-Höl}}([0,1]; G)$ as $n \to \infty$ for all $\alpha < 1/2$.*

We emphasize that the transition probability $\mathfrak{p}$ coincides with the given one p under the centered condition **(C)**. Therefore, Theorem 4.4.2 and 4.4.3 are regarded as extensions of Theorems 4.1.2 (under the centered condition **(C)**) and 4.1.3 to the non-centered case. We might prove Theorem 4.1.3 without the centered condition **(C)** via Theorem 4.4.3. We will discuss this problem in the future.

4.5 An explicit representation of the limiting diffusions and a relation with rough path theory

Let us consider an SDE on $\mathbb{R}^N$

$$d\xi_t = \sum_{i=1}^d U_i(\xi_t) \circ dB_t^i + U_0(\xi_t)\,dt, \qquad \xi_0 = x_0 \in \mathbb{R}^N, \tag{4.5.1}$$

where $U_0, U_1, \dots, U_d$ are C^∞-vector fields on $\mathbb{R}^d$ and $(B_t)_{0\le t\le 1} = (B_t^1, B_t^2, \dots, B_t^d)_{0\le t\le 1}$ is a d-dimensional standard Brownian motion. The symbol $\circ$ denotes the usual Stratonovich type stochastic integral. As is well-known, a number of authors have studied explicit representations of the unique solution to (4.5.1) as a functional of Itô/Stratonovich iterated integrals under some assumptions on vector fields $U_0, U_1, \dots, U_d$. In particular, Kunita [46] has obtained the explicit formula by using the CBH formula in the case where the Lie algebra generated by $U_0, U_1, \dots, U_d$ is nilpotent or solvable. Castell [13] gave a universal representation formula, which contains the above results in the nilpotent case and extends the study of Ben Arous [7] to more general diffusions.

We now recall the result in [13] when the Lie algebra generated by $U_0, U_1, \dots, U_d$ is nilpotent of step r. We first introduce several notations of multi-indices. Set $\mathcal{I}^{(k)} = \{0, 1, \dots, d\}^k$ and let $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}^{(k)}$ be a multi-index of length $|I| = k$. For vector fields $U_0, U_1, \dots, U_d$ on $\mathbb{R}^d$ and $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}^{(k)}$, we denote by

U^I the vector field of the form $U^I = [U_{i_1}, [U_{i_2}, \cdots, [U_{i_{k-1}}, U_{i_k}] \cdots]]$. For a multi-index $I = (i_1, i_2, \ldots, i_k) \in \mathcal{I}^{(k)}$, we define the Stratonovich iterated integral B_t^I by

$$B_t^I := \int_{\Delta^{(k)}[0,t]} \circ dB_{t_1}^{i_1} \circ dB_{t_2}^{i_2} \cdots \circ dB_{t_k}^{i_k},$$

where $\Delta^{(k)}[0,t] := \{(t_1, t_2, \ldots, t_k) \in [0,t]^k \,|\, 0 < t_1 < t_2 < \cdots < t_k < t\}$ for $0 \le t \le 1$ and $B_t^0 = t$ for convention. Next we introduce notations of the permutations. Denote by $\mathfrak{S}_k$ be the symmetric group of degree k. For a permutation $\sigma \in \mathfrak{S}_k$, we write $e(\sigma)$ for the cardinality of the set $\{i \in \{1, 2, \ldots, k-1\} \,|\, \sigma(i) > \sigma(i+1)\}$, which we call the number of inversions of σ. For $I = (i_1, i_2, \ldots, i_k) \in \mathcal{I}^{(k)}$ and $\sigma \in \mathfrak{S}_k$, we put $I_\sigma := (i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(k)}) \in \mathcal{I}^{(k)}$.

Proposition 4.5.1 (cf. [13]) *Let $U_0, U_1, \ldots, U_d$ be bounded C^∞-vector fields on $\mathbb{R}^N$ such that the Lie algebra generated by $U_0, U_1, \ldots, U_d$ is nilpotent of step r. We consider the solution $(\xi_t)_{0 \le t \le 1}$ of* (4.5.1). *Then we have*

$$\xi_t = \exp\Big(\sum_{k=1}^{r} \sum_{I \in \mathcal{I}^{(k)}} c_t^I U^I\Big)(x_0) \qquad (0 \le t \le 1) \quad a.s.,$$

where

$$c_t^I := \sum_{\sigma \in \mathfrak{S}_{|I|}} \frac{(-1)^{e(\sigma)}}{|I|^2 \binom{|I|-1}{e(\sigma)}} B_t^{I_{\sigma^{-1}}}.$$

Here we give several concrete computations of $c_t^I U^I$.

• If $I = (i) \in \mathcal{I}^{(1)}$, we see $c_t^I = B_t^i$ for $0 \le t \le 1$ and $i = 0, 1, \ldots, d$. Therefore, we have

$$\sum_{I \in \mathcal{I}^{(1)}} c_t^I U^I = \sum_{i=0}^{d} B_t^i U_i = tU_0 + \sum_{i=1}^{d} B_t^i U_i.$$

• If $I = (i, j) \in \mathcal{I}^{(2)}$ with $i \ne j$, we also see

$$c_t^I = \begin{cases} \dfrac{1}{4} \displaystyle\int_0^t \int_0^u (\circ dB_s^i \circ dB_u^j - \circ dB_s^j \circ dB_u^i) & (i < j), \\ -\dfrac{1}{4} \displaystyle\int_0^t \int_0^u (\circ dB_s^i \circ dB_u^j - \circ dB_s^j \circ dB_u^i) & (i > j). \end{cases}$$

Since $[U_i, U_j] = -[U_j, U_i]$ holds for $i \ne j$, we have

$$\begin{aligned} \sum_{I \in \mathcal{I}^{(2)}} c_t^I U^I &= \sum_{0 \le i < j \le d} \frac{1}{2} \int_0^t \int_0^u (\circ dB_s^i \circ dB_u^j - \circ dB_s^j \circ dB_u^i)[U_i, U_j] \\ &= \sum_{0 \le i < j \le d} \frac{1}{2} \int_0^t (B_s^i dB_s^j - B_s^j dB_s^i)[U_i, U_j]. \end{aligned}$$

The stochastic integral

$$\frac{1}{2}\int_0^t (B_s^i dB_s^j - B_s^j dB_s^i) \qquad (0 \le t \le 1,\ 1 \le i < j \le d)$$

indicates the well-known *Lévy's stochastic area* enclosed by the Brownian curve $\{(B_s^i, B_s^j) \in \mathbb{R}^2 \,|\, 0 \le s \le t\}$ and its chord.

We now provide an explicit representation of $(Y_t)_{0\le t\le 1}$, the solution to the SDE (4.1.7). As mentioned in Section 2.1, since G is identified with $\mathbb{R}^d$ ($d = d_1 + d_2 + \cdots + d_r$), we may apply Proposition 4.5.1 by replacing $U_0, U_1, \dots, U_d$ by $V_0, V_1, \dots, V_{d_1}$, where $V_0 = \beta(\Phi_0)_*$. Then we have

Theorem 4.5.2 *The limiting diffusion process $(Y_t)_{0\le t\le 1}$ is explicitly represented as*

$$\begin{aligned} Y_t = \exp\Big(t\beta(\Phi_0)_* &+ \sum_{i=1}^{d_1} B_t^i V_{i*} \\ &+ \sum_{0\le i<j\le d_1} \frac{1}{2}\int_0^t (B_s^i dB_s^j - B_s^j dB_s^i)[\![V_{i*}, V_{j*}]\!] + \sum_{k=3}^{r}\sum_{I\in\mathcal{I}^{(k)}} c_t^I V_*^I\Big)(\mathbf{1}_G), \end{aligned} \tag{4.5.2}$$

where $V_*^I = [\![V_{i_1*}, [\![V_{i_2*}, \cdots, [\![V_{i_{k-1}*}, V_{i_k*}]\!]\cdots]\!]]\!]$ for $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}^{(k)}$.

We should note that some of $[\![V_{i*}, V_{j*}]\!]$ $(0 \le i < j \le d_1)$ in (4.5.2) may vanish because $\{[\![V_{i*}, V_{j*}]\!]\}_{1\le i<j\le d}$ is not always linearly independent.

In closing this subsection, we prove that the infinitesimal generator of $(Y_t)_{0\le t\le 1}$ coincides with $-\mathcal{A}$ defined by (4.1.4).

Proposition 4.5.3 *The C_0-semigroup $(\mathrm{e}^{-t\mathcal{A}})_{0\le t\le 1}$ coincides with the C_0-semigroup $(T_t)_{0\le t\le 1}$ on $C_\infty(G)$ defined by $T_t f(g) = \mathbb{E}[f(Y_t^g)]$ for $g \in G$, where $(Y_t^g)_{0\le t\le 1}$ is a solution to the stochastic differential equation*

$$dY_t^g = \sum_{i=1}^{d_1} V_{i*}(Y_t^g) \circ dB_t^i + \beta(\Phi_0)_*(Y_t^g)\, dt, \qquad Y_0^g = g \in G. \tag{4.5.3}$$

Proof. By recalling Lemma 4.2.2, the linear operator $\mathcal{A}$ satisfies the maximal dissipativity, that is, $\lambda - \mathcal{A}$ is surjective for some $\lambda > 0$. Therefore, the Lumer–Fillips theorem implies that $(\mathrm{e}^{-t\mathcal{A}})_{0\le t\le 1}$ is the unique Feller semigroup on $C_\infty(G)$ whose infinitesimal generator extends $\big(-\mathcal{A}, C_0^\infty(G)\big)$. By applying Itô's formula to (4.5.3), we easily see that the generator of $(Y_t)_{0\le t\le 1}$ coincides with $-\mathcal{A}$ on $C_0^\infty(G)$. Therefore, it suffices to show that the semigroup $(T_t)_{0\le t\le 1}$ enjoys the Feller property, that is, $T_t\big(C_\infty(G)\big) \subset C_\infty(G)$ for $0 \le t \le 1$.

Suppose $f \in C_\infty(G)$. For any $\varepsilon > 0$, we choose a sufficiently large $R > 0$ such that $|f(g)| < \varepsilon$ for $g \in B_R(\mathbf{1}_G)^c$, where $B_R(\mathbf{1}_G) := \{g \in G \,|\, d_{\mathrm{CC}}(\mathbf{1}_G, g) < R\}$. Then, for $g \in B_{2R}(\mathbf{1}_G)^c$, we have

$$\begin{aligned} |T_t f(g)| &\le \mathbb{E}\big[|f(Y_t^g)| \,:\, d_{\mathrm{CC}}(g, Y_t^g) < R\big] + \mathbb{E}\big[|f(Y_t^g)| \,:\, d_{\mathrm{CC}}(g, Y_t^g) \ge R\big] \\ &\le \varepsilon + \|f\|_\infty^G \mathbb{P}\big(d_{\mathrm{CC}}(g, Y_t^g) \ge R\big). \end{aligned}$$

By combining Proposition 2.3.3 and the Chebyshev inequality with Theorem 4.5.2,

$$\begin{aligned}\mathbb{P}\big(d_{\mathrm{CC}}(g, Y_t^g) \geq R\big) &= \mathbb{P}\big(d_{\mathrm{CC}}(\mathbf{1}_G, Y_t) \geq R\big) \\ &\leq \mathbb{P}\big(C\|Y_t\|_{\mathrm{Hom}} \geq R\big) \\ &\leq \frac{C}{R^2}\mathbb{E}\Big[\Big(\sum_{k=1}^{r}\Big\|\sum_{I\in\mathcal{I}^{(k)}} c_t^I V_*^I\Big\|_{\mathfrak{g}^{(k)}}^{1/k}\Big)^2\Big].\end{aligned}$$

Now we recall the following fact (cf. Friz–Riedel [20, Lemma 2]): For a multi-index $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}^{(k)}$, there exists a constant C depending only on k such that

$$\mathbb{E}\Big[\Big(\int_{\Delta^{(k)}[0,t]} \circ dB_{t_1}^{i_1} \circ dB_{t_2}^{i_2} \cdots \circ dB_{t_k}^{i_k}\Big)^2\Big] \leq Ct^k \qquad (0 \leq t \leq 1).$$

In view of this bound, we obtain

$$\mathbb{P}\big(d_{\mathrm{CC}}(g, Y_t^g) \geq R\big) \leq \frac{C}{R^2}t.$$

Taking a sufficiently large $R > 0$ such that $C\|f\|_\infty^G tR^{-2} < \varepsilon$, we conclude $|T_t f(g)| < 2\varepsilon$ for $g \in B_{2R}(\mathbf{1}_G)^c$. This implies that $T_t\big(C_\infty(G)\big) \subset C_\infty(G)$ for $0 \leq t \leq 1$. ∎

In the end of this section, we discuss the free case and give a relation between Theorem 4.5.2 and rough path theory. Consider the step-r non-commutative tensor algebra $T^{(r)}(\mathbb{R}^d) = \mathbb{R} \oplus \big(\bigoplus_{k=1}^r (\mathbb{R}^d)^{\otimes k}\big)$. The tensor product on $T^{(r)}(\mathbb{R}^d)$ is defined by

$$(g_0, g_1, \dots, g_r) \otimes_r (h_0, h_1, \dots, h_r) = \Big(g_0h_0, g_0h_1 + g_1h_0, \dots, \sum_{k=0}^{r} g_k \otimes h_{r-k}\Big).$$

An element $g = (g_0, g_1, \dots, g_r) \in T^{(r)}(\mathbb{R}^d)$ is occasionally written as $g = g_0 + g_1 + \cdots + g_r$. We define two subsets of $T^{(r)}(\mathbb{R}^d)$ by

$$T_1^{(r)}(\mathbb{R}^d) := \{g \in T^{(r)}(\mathbb{R}^d) \,|\, g_0 = 1\}, \qquad T_0^{(r)}(\mathbb{R}^d) := \{A \in T^{(r)}(\mathbb{R}^d) \,|\, A_0 = 0\},$$

respectively. It is easy to see that $T_1^{(r)}(\mathbb{R}^d)$ is a Lie group under the tensor product $\otimes_r$. In fact, $\mathbf{1} = (1, 0, 0, \dots, 0)$ is the unit element of $T_1^{(r)}(\mathbb{R}^d)$ and the inverse element of $g \in T_1^{(r)}(\mathbb{R}^d)$ is given by $g^{-1} = \sum_{k=1}^r (-1)^k (g - \mathbf{1})^{\otimes_r k}$. The Lie bracket on $T_0^{(r)}(\mathbb{R}^d)$ is defined by $[A, B] = A \otimes_r B - B \otimes_r A$ for $A, B \in T_0^{(r)}(\mathbb{R}^d)$. Note that $T_0^{(r)}(\mathbb{R}^d)$ is the Lie algebra of the Lie group $T_1^{(r)}(\mathbb{R}^d)$, that is, $T_0^{(r)}(\mathbb{R}^d)$ is the tangent space of $T_1^{(r)}(\mathbb{R}^d)$ at $\mathbf{1}$. The diffeomorphism $\exp : T_0^{(r)}(\mathbb{R}^d) \longrightarrow T_1^{(r)}(\mathbb{R}^d)$ is defined by

$$\exp(A) := 1 + \sum_{k=1}^{r} \frac{1}{k!} A^{\otimes_r k} \qquad \big(A \in T_0^{(r)}(\mathbb{R}^d)\big).$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d\}$ be the standard basis of $\mathbb{R}^d$. We introduce a discrete subgroup $\mathfrak{g}^{(r)}(\mathbb{Z}^d) \subset T_0^{(r)}(\mathbb{R}^d)$ by the set of $\mathbb{Z}$-linear combinations of $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ together with $[\mathbf{e}_{i_1}, [\mathbf{e}_{i_2}, \cdots, [\mathbf{e}_{i_{k-1}}, \mathbf{e}_{i_k}] \cdots]]$ for $i_1, i_2, \dots, i_k = 1, 2, \dots, d$ and $k = 2, 3, \dots, r$.

We now set $\Gamma = \mathbb{G}^{(r)}(\mathbb{Z}^d) := \exp\big(\mathfrak{g}^{(r)}(\mathbb{Z}^d)\big)$. We also define $\mathfrak{g}^{(r)}(\mathbb{R}^d)$ and $\mathbb{G}^{(r)}(\mathbb{R}^d)$ analogously. Then we see that $\big(\mathbb{G}^{(r)}(\mathbb{R}^d), \otimes_r\big)$ is the nilpotent Lie group in which Γ is included as its cocompact lattice and the corresponding limit group coincides with $\big(\mathbb{G}^{(r)}(\mathbb{R}^d), \otimes_r\big)$ itself. We call $\big(\mathbb{G}^{(r)}(\mathbb{R}^d), \otimes_r\big)$ the *free nilpotent Lie group of step* r and $\big(\mathfrak{g}^{(r)}(\mathbb{R}^d), [\cdot,\cdot]\big)$ the *free nilpotent Lie algebra of step* r. Let $\mathfrak{g}^{(1)} = \mathbb{R}^d$ and $\mathfrak{g}^{(k)} = [\mathbb{R}^d, [\mathbb{R}^d, \cdots, [\mathbb{R}^d, \mathbb{R}^d]\cdots]]$ (k-times) for $k = 2, 3, \dots, r$. Then we see that the Lie algebra $\mathfrak{g}^{(r)}(\mathbb{R}^d)$ is decomposed into $\mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$. The free nilpotent Lie group $\mathbb{G}^{(r)}(\mathbb{R}^d)$ is highly related to rough path theory, as is seen below (cf. Friz–Victoir [22]). Let $(B_t)_{0\le t\le 1} = (B_t^1, B_t^2, \dots, B_t^d)_{0\le t\le 1}$ be a d-dimensional standard Brownian motion. We give the following two remarks.
(1) Consider the case $r = 2$. Then a $T_1^{(2)}(\mathbb{R}^d)$-valued path $(\mathbf{B}_t)_{0\le t\le 1}$ defined by

$$\begin{aligned}\mathbf{B}_t &:= \exp\Big(\sum_{i=1}^d B_t^i \mathbf{e}_i + \sum_{1\le i<j\le d} \Big(\frac{1}{2}\int_0^t B_s^i \circ dB_s^j - B_s^j \circ dB_s^i\Big)\mathbf{e}_i \otimes \mathbf{e}_j\Big) \\ &= 1 + \sum_{i=1}^d B_t^i \mathbf{e}_i + \sum_{i,j=1}^d \Big(\int_0^t \int_0^s \circ dB_u^i \circ dB_s^j\Big)\mathbf{e}_i \otimes \mathbf{e}_j \qquad (0 \le t \le 1)\end{aligned}$$

is regarded as a $\mathbb{G}^{(2)}(\mathbb{R}^d)$-valued path with probability one. We call it *Stratonovich enhanced Brownian motion* or *standard Brownian rough path*, which is a canonical lift of a sample path of the d-dimensional Brownian motion. We usually identify standard Brownian rough path $(\mathbf{B}_t)_{0\le t\le 1}$ with its increment $(\mathbf{B}_{s,t}) := (\mathbf{B}_s^{-1} \otimes_2 \mathbf{B}_t)_{0\le s\le t\le 1}$.
(2) Consider the case $r \ge 3$. We also see that the $T_1^{(r)}(\mathbb{R}^d)$-valued path $(\mathbf{B}_t)_{0\le t\le 1}$ defined by

$$\mathbf{B}_t := 1 + \sum_{k=1}^r \sum_{i_1, i_2, \dots, i_k \in \{1,2,\dots,d\}} \Big(\int_{\Delta^{(k)}[0,t]} \circ dB_{t_1}^{i_1} \circ dB_{t_2}^{i_2} \cdots \circ dB_{t_k}^{i_k}\Big)\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_k}$$

for $0 \le t \le 1$, is regarded as a $\mathbb{G}^{(r)}(\mathbb{R}^d)$-valued path with probability one, analogously in (1). Note that this path $(\mathbf{B}_t)_{0\le t\le 1}$ is nothing but the *Lyons extension* (or *lift*) of Stratonovich enhanced Brownian motion introduced in (1) to $\mathbb{G}^{(r)}(\mathbb{R}^d)$.

Let $\Gamma = \mathbb{G}^{(r)}(\mathbb{Z}^d)$ and X be a Γ-nilpotent covering graph. Then we see that X is realized into the free nilpotent Lie group $G = \mathbb{G}^{(r)}(\mathbb{R}^d)$ through the modified harmonic realization $\Phi_0 : X \longrightarrow G$, because Γ is a cocompact lattice in G. Then Theorem 4.5.2 reads in terms of rough path theory. Precisely speaking, the $\mathbb{G}^{(r)}(\mathbb{R}^d)$-valued diffusion process $(Y_t)_{0\le t\le 1}$ which solves (4.1.7) is represented as the Lyons extension of the so-called *distorted Brownian rough path* of order r.

Corollary 4.5.4 *Let $\{V_1, V_2, \dots, V_d\}$ be an orthonormal basis of $\mathfrak{g}^{(1)}$ with respect to the Albanese metric g_0. We write*

$$\beta(\Phi_0) = \sum_{1\le i<j\le d} \beta(\Phi_0)^{ij}[V_i, V_j] \in \mathfrak{g}^{(2)},$$

where we note that $\{[V_i, V_j] : 1 \leq i < j \leq d\} \subset \mathfrak{g}^{(2)}$ *forms a basis of* $\mathfrak{g}^{(2)}$. *Let* $\overline{\boldsymbol{\beta}}(\Phi_0) = \big(\overline{\boldsymbol{\beta}}(\Phi_0)^{ij}\big)_{i,j=1}^{d}$ *be an anti-symmetric matrix defined by*

$$\overline{\boldsymbol{\beta}}(\Phi_0)^{ij} := \begin{cases} \beta(\Phi_0)^{ij} & (1 \leq i < j \leq d), \\ -\beta(\Phi_0)^{ij} & (1 \leq j < i \leq d), \\ 0 & (i = j). \end{cases}$$

Then the $\mathbb{G}^{(r)}(\mathbb{R}^d)$*-valued diffusion process* $(Y_t)_{0\leq t\leq 1}$ *coincides with the Lyons extension of the distorted Brownian rough path*

$$\overline{\mathbf{B}}_t = 1 + \overline{\mathbf{B}}_t^1 + \overline{\mathbf{B}}_t^2 \in \mathbb{G}^{(2)}(\mathbb{R}^d) \qquad (0 \leq s \leq t \leq 1)$$

of order r*, where*

$$\overline{\mathbf{B}}_t^1 := \sum_{i=1}^{d} B_t^i V_i \in \mathbb{R}^d, \quad \overline{\mathbf{B}}_t^2 := \int_0^t \int_0^s \circ dB_u \otimes \circ dB_s + t\overline{\boldsymbol{\beta}}(\Phi_0) \in \mathbb{R}^d \otimes \mathbb{R}^d.$$

4.6 FCLTs in the case of non-harmonic realizations

As was discussed, the modified harmonicity of the Γ-equivariant realization $\Phi_0 : X \longrightarrow G$ plays a crucial role to conclude the invariance principle (Theorem 4.1.3). Then one may wonder if the same invariance principle as Theorem 4.1.3 holds or not when we consider a general Γ-equivariant realization $\Phi : X \longrightarrow G$. In this final section, we give an affirmative answer to this problem by employing the notion of so-called "corrector", which is frequently seen in the study of invariance principles on random environments (see e.g., Kumagai [45]).

Let us give a rough overview of the proof in the case of a Γ-crystal lattice X. For simplicity, we consider the centered case, that is, $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$. Let $\Phi_0 : X \longrightarrow \Gamma \otimes \mathbb{R}$ be the harmonic realization and $\Phi : X \longrightarrow \Gamma \otimes \mathbb{R}$ a Γ-periodic realization. We define the *corrector* of Φ by

$$\mathrm{Cor}(x) := \Phi(x) - \Phi_0(x) \qquad (x \in V),$$

which measures the difference between $\Phi(x)$ and $\Phi_0(x)$. Note that the set $\{\mathrm{Cor}(x) \,|\, x \in V\}$ is finite. Because, by the periodicity of Φ and Φ_0, we have $\mathrm{Cor}(\gamma x) = \mathrm{Cor}(x)$ for $\gamma \in \Gamma$ and $x \in V$. Thus, we may write $\{\mathrm{Cor}(x) \,|\, x \in V\} = \{\mathrm{Cor}(x) \,|\, x \in \mathcal{F}\}$, where $\mathcal{F}$ stand for a fundamental domain of X. In particular, there exists a positive constant $C > 0$ such that $\max_{x\in\mathcal{F}} |\mathrm{Cor}(x)|_{\Gamma\otimes\mathbb{R}} \leq C$. According to Ishiwata–Kawabi–Kotani [31, Theorem 2.2], we have already known

$$\Big(\frac{1}{\sqrt{n}}\Phi_0(w_{[nt]})\Big)_{0\leq t\leq 1} \Longrightarrow (B_t)_{0\leq t\leq 1} \quad \text{in law}$$

as $n \to \infty$, where $\{w_n\}_{n=0}^{\infty}$ is a non-symmetric random walk on X and $(B_t)_{0\leq t\leq 1}$ is a d-dimensional standard Brownian motion on $\Gamma \otimes \mathbb{R}$ with respect to the Albanese metric.

On the other hand, we observe

$$\Big|\frac{1}{\sqrt{n}}\Phi(w_{[nt]})-\frac{1}{\sqrt{n}}\Phi_0(w_{[nt]})\Big|=\Big|\frac{1}{\sqrt{n}}\mathrm{Cor}(w_{[nt]})\Big|\le\frac{C}{\sqrt{n}}\longrightarrow 0\quad(n\to\infty)$$

for all $0\le t\le 1$. In fact, we are able to show that $\{n^{-1/2}\Phi(w_{[nt]})\,:\,0\le t\le 1\}$ also converges in law to $(B_t)_{0\le t\le 1}$ as $n\to\infty$.

In what follows, we try to prove this assertion rigorously in the case of a Γ-nilpotent covering graph X. However, we need to notice that the situation quite differs from the case of crystal lattices due to the non-commutativity of Γ. We assume the centered condition **(C)**. Let $\Phi_0:X\longrightarrow G$ be a modified $(\mathfrak{g}^{(1)}$-)harmonic realization and $\Phi:X\longrightarrow G$ a (not necessarily modified harmonic) realization. We define the $(\mathfrak{g}^{(1)}$-)*corrector* $\mathrm{Cor}_{\mathfrak{g}^{(1)}}=\mathrm{Cor}_{\mathfrak{g}^{(1)}}(\Phi):X\longrightarrow\mathfrak{g}^{(1)}$ by

$$\mathrm{Cor}_{\mathfrak{g}^{(1)}}(x):=\log\big(\Phi(x)\big)\big|_{\mathfrak{g}^{(1)}}-\log\big(\Phi_0(x)\big)\big|_{\mathfrak{g}^{(1)}}\qquad(x\in V).$$

This corrector measures the difference between only the $\mathfrak{g}^{(1)}$-components of the harmonic realization and the non-harmonic one. As in the case of crystal lattices, the set $\{\mathrm{Cor}_{\mathfrak{g}^{(1)}}(x)\,|\,x\in V\}$ is finite thanks to $\mathrm{Cor}_{\mathfrak{g}^{(1)}}(\gamma x)=\mathrm{Cor}_{\mathfrak{g}^{(1)}}(x)$ for $\gamma\in\Gamma$ and $x\in V$. We may thus write $\{\mathrm{Cor}(x)\,|\,x\in V\}=\{\mathrm{Cor}(x)\,|\,x\in\mathcal{F}\}$, where $\mathcal{F}$ stand for a fundamental domain of X. The FCLT (Theorem 4.1.3) asserts that the family of stochastic processes $\{\mathcal{Y}^{(n)}_{\cdot}\}_{n=1}^{\infty}$ introduced in Section 4.1 converges in law to the G-valued diffusion process $(Y_t)_{0\le t\le 1}$ which solves (4.1.7) in $C^{0,\alpha\text{-Höl}}_{\mathbf{1}_G}([0,1];G)$ as $n\to\infty$ for $\alpha<1/2$. Since $\beta(\Phi_0)\in\mathfrak{g}^{(2)}$, the drift of the limiting infinitesimal generator $\mathcal{A}$ of $(Y_t)_{0\le t\le 1}$, does not depend on the choice of $\mathfrak{g}^{(2)}$-components of $\Phi_0(x)\,(x\in V)$ by Proposition 4.2.3, we may put $\Phi_0(x)^{(i)}=\Phi(x)^{(i)}$ for $x\in V$ and $i=2,3,\dots,r$ without loss of generality. Let $(\overline{\mathcal{Y}}^{(n)}_t)_{0\le t\le 1}\,(n\in\mathbb{N})$ be the G-valued stochastic processes defined by just replacing Φ_0 by Φ in the definition of $(\mathcal{Y}^{(n)}_t)_{0\le t\le 1}$. We now show that the same pathwise Hölder estimate as Lemma 4.3.3 also holds for the stochastic process $(\overline{\mathcal{Y}}^{(n)}_t)_{0\le t\le 1}\,(n\in\mathbb{N})$.

Lemma 4.6.1 *For $m,n\in\mathbb{N}$ and $\alpha<\frac{2m-1}{4m}$, there exist an $\mathcal{F}_\infty$-measurable set $\overline{\Omega}^{(n)}_r\subset\Omega_{x_*}(X)$ and a non-negative random variable $\overline{\mathcal{K}}^{(n)}_r\in L^{4m}\big(\Omega_{x_*}(X)\to\mathbb{R};\mathbb{P}_{x_*}\big)$ such that*

$$d_{\mathrm{CC}}\big(\overline{\mathcal{Y}}^{(n)}_s(c),\overline{\mathcal{Y}}^{(n)}_t(c)\big)\le\overline{\mathcal{K}}^{(n)}_r(c)(t-s)^{\alpha}\qquad(c\in\overline{\Omega}^{(n)}_r,\ 0\le s<t\le 1).\tag{4.6.1}$$

Proof. Fix $n\in\mathbb{N}$ and $1\le k\le\ell\le n$. By triangular inequality, we have

$$d_{\mathrm{CC}}(\overline{\mathcal{Y}}^{(n)}_{k/n},\overline{\mathcal{Y}}^{(n)}_{\ell/n})\le d_{\mathrm{CC}}(\overline{\mathcal{Y}}^{(n)}_{k/n},\mathcal{Y}^{(n)}_{k/n})+d_{\mathrm{CC}}(\mathcal{Y}^{(n)}_{k/n},\mathcal{Y}^{(n)}_{\ell/n})+d_{\mathrm{CC}}(\mathcal{Y}^{(n)}_{\ell/n},\overline{\mathcal{Y}}^{(n)}_{\ell/n}).$$

We set $\mathcal{Z}^{(n)}_t:=(\mathcal{Y}^{(n)}_t)^{-1}*\overline{\mathcal{Y}}^{(n)}_t$ for $0\le t\le 1$ and $n\in\mathbb{N}$. By definition, we see that

$$\log\big(\mathcal{Z}^{(n)}_{k/n})|_{\mathfrak{g}^{(1)}}=\frac{1}{\sqrt{n}}\mathrm{Cor}_{\mathfrak{g}^{(1)}}(w_k)\qquad(n\in\mathbb{N},\ k=0,1,\dots,n)$$

and there is a constant $C > 0$ such that $\big\|\log\big(\mathcal{Z}^{(n)}_{k/n}\big)|_{\mathfrak{g}^{(1)}}\big\|_{\mathfrak{g}^{(1)}} \le Cn^{-1/2}$ for $n \in \mathbb{N}$ and $k = 0, 1, 2, \dots, n$. Moreover, it follows from the choice of the components of $\Phi_0(x)$ $(x \in V)$ that $\big\|\log\big(\mathcal{Z}^{(n)}_{k/n}\big)|_{\mathfrak{g}^{(i)}}\big\|_{\mathfrak{g}^{(i)}} \le Cn^{-i/2}$ for $n \in \mathbb{N}$ and $k = 0, 1, 2, \dots, n$. By Proposition 2.3.3, we have

$$d_{\mathrm{CC}}(\overline{\mathcal{Y}}^{(n)}_{k/n}, \mathcal{Y}^{(n)}_{k/n}) \le C\big\|\mathcal{Z}^{(n)}_{k/n}\big\|_{\mathrm{Hom}} = C\sum_{i=1}^{r}\big\|\log\big(\mathcal{Z}^{(n)}_{k/n}\big)|_{\mathfrak{g}^{(i)}}\big\|^{1/i}_{\mathfrak{g}^{(i)}} \le \frac{C}{\sqrt{n}} \tag{4.6.2}$$

for $n \in \mathbb{N}$ and $k = 0, 1, 2, \dots, n$. Then Lemma 4.3.3 and (4.6.2) imply the existences of an $\mathcal{F}_\infty$-measurable set $\overline{\Omega}^{(n)}_r \subset \Omega_{x_*}(X)$ and a non-negative random variable $\overline{\mathcal{K}}^{(n)}_r \in L^{4m}\big(\Omega_{x_*}(X) \to \mathbb{R}; \mathbb{P}_{x_*}\big)$ such that $\mathbb{P}_{x_*}(\overline{\Omega}^{(n)}_r) = 1$ and

$$\begin{aligned} d_{\mathrm{CC}}\big(\overline{\mathcal{Y}}^{(n)}_{k/n}(c), \overline{\mathcal{Y}}^{(n)}_{\ell/n}(c)\big) &\le \frac{C}{\sqrt{n}} + \mathcal{K}^{(n)}_r(c)\Big(\frac{\ell-k}{n}\Big)^\alpha + \frac{C}{\sqrt{n}} \\ &\le \overline{\mathcal{K}}^{(n)}_r(c)\Big(\frac{\ell-k}{n}\Big)^\alpha \qquad (c \in \overline{\Omega}^{(n)}_r,\ 0 \le k \le \ell \le n). \end{aligned} \tag{4.6.3}$$

For $0 \le s < t \le 1$, we take $0 \le k \le \ell \le n$ so that $k/n \le s < (k+1)n$ and $\ell/n \le t < (\ell+1)/n$. Since the stochastic process $(\overline{\mathcal{Y}}^{(n)}_t)_{0\le t\le 1}$ is also give by the d_{CC}-geodesic interpolation, we have

$$\begin{aligned} d_{\mathrm{CC}}\big(\overline{\mathcal{Y}}^{(n)}_s, \overline{\mathcal{Y}}^{(n)}_{(k+1)/n}\big) &= (k - ns)d_{\mathrm{CC}}\big(\overline{\mathcal{Y}}^{(n)}_{k/n}, \overline{\mathcal{Y}}^{(n)}_{(k+1)/n}\big), \\ d_{\mathrm{CC}}\big(\overline{\mathcal{Y}}^{(n)}_{\ell/n}, \overline{\mathcal{Y}}^{(n)}_t\big) &= (nt - \ell)d_{\mathrm{CC}}\big(\overline{\mathcal{Y}}^{(n)}_{\ell/n}, \overline{\mathcal{Y}}^{(n)}_{(\ell+1)/n}\big). \end{aligned}$$

Then, by the triangular inequality and (4.6.3), we obtain

$$\begin{aligned} &d_{\mathrm{CC}}\big(\overline{\mathcal{Y}}^{(n)}_s(c), \overline{\mathcal{Y}}^{(n)}_t(c)\big) \\ &\le d_{\mathrm{CC}}\big(\overline{\mathcal{Y}}^{(n)}_s(c), \overline{\mathcal{Y}}^{(n)}_{(k+1)/n}(c)\big) + d_{\mathrm{CC}}\big(\overline{\mathcal{Y}}^{(n)}_{(k+1)/n}(c), \overline{\mathcal{Y}}^{(n)}_{\ell/n}(c)\big) + d_{\mathrm{CC}}\big(\overline{\mathcal{Y}}^{(n)}_{\ell/n}(c), \overline{\mathcal{Y}}^{(n)}_t(c)\big) \\ &\le (k - ns)\overline{\mathcal{K}}^{(n)}_r(c)\Big(\frac{1}{n}\Big)^\alpha + \overline{\mathcal{K}}^{(n)}_r(c)\Big(\frac{\ell-k-1}{n}\Big)^\alpha + (nt-\ell)\overline{\mathcal{K}}^{(n)}_r(c)\Big(\frac{1}{n}\Big)^\alpha \\ &\le \overline{\mathcal{K}}^{(n)}_r(c)\Big\{\Big(\frac{k+1}{n} - s\Big)^\alpha + \Big(\frac{\ell-k-1}{n}\Big)^\alpha + \Big(t - \frac{\ell}{n}\Big)^\alpha\Big\} \\ &\le \overline{\mathcal{K}}^{(n)}_r(c)(t-s)^\alpha \qquad (c \in \overline{\Omega}^{(n)}_r). \end{aligned}$$

This completes the proof. ∎

This Lemma leads to the following invariance principle for the family of stochastic processes $(\overline{\mathcal{Y}}^{(n)}_t)_{0\le t\le 1}$.

Theorem 4.6.2 *The sequence $(\overline{\mathcal{Y}}^{(n)}_t)_{0\le t\le 1}$ $(n = 1, 2, \dots)$ converges in law to the G-valued diffusion process $(Y_t)_{0\le t\le 1}$ in $C^{0,\alpha\text{-Höl}}_{\mathbf{1}_G}([0,1]; G)$ as $n \to \infty$.*

Proof. We split the proof into two steps.

Step 1. We show that the sequence $(\overline{\mathcal{Y}}_t^{(n)})_{0\le t\le 1}$ $(n=1,2,\dots)$ converges in law to $(Y_t)_{0\le t\le 1}$ in $C_{\mathbf{1}_G}([0,1];G)$ as $n\to\infty$. For $0\le t\le 1$, we take an integer $0\le k\le n$ so that $k/n\le t<(k+1)/n$. Then, by the triangular inequality, (4.3.15), (4.6.1) and (4.6.2), we have, $\mathbb{P}_{x_*}$-almost surely,

$$\begin{aligned}
&d_{\mathrm{CC}}(\mathcal{Y}_t^{(n)},\overline{\mathcal{Y}}_t^{(n)})\\
&\le d_{\mathrm{CC}}(\mathcal{Y}_{k/n}^{(n)},\mathcal{Y}_t^{(n)})+d_{\mathrm{CC}}(\mathcal{Y}_{k/n}^{(n)},\overline{\mathcal{Y}}_{k/n}^{(n)})+d_{\mathrm{CC}}(\overline{\mathcal{Y}}_{k/n}^{(n)},\overline{\mathcal{Y}}_t^{(n)})\\
&\le \mathcal{K}_r^{(n)}\Big(t-\frac{k}{n}\Big)^{\alpha}+\frac{C}{\sqrt{n}}+\overline{\mathcal{K}}_r^{(n)}\Big(t-\frac{k}{n}\Big)^{\alpha}\\
&\le \big\{\mathcal{K}_r^{(n)}+\overline{\mathcal{K}}_r^{(n)}+C\big\}\Big(\frac{1}{\sqrt{n}}\Big)^{\alpha}\qquad \Big(m\in\mathbb{N},\,\alpha<\frac{2m-1}{4m}\Big).
\end{aligned}\tag{4.6.4}$$

Let ρ be a metric on $C_{\mathbf{1}_G}([0,1];G)$ defined by

$$\rho(w^{(1)},w^{(2)}):=\max_{0\le t\le 1}d_{\mathrm{CC}}(w_t^{(1)},w_t^{(2)})\qquad \big(w^{(1)},w^{(2)}\in C_{\mathbf{1}_G}([0,1];G)\big).$$

We denote by $\mathbf{1}\in C_{\mathbf{1}_G}([0,1];G)$ the identity map. By applying the Chebyshev inequality and (4.6.4), we have, for $\varepsilon>0$ and $m\in\mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}_{x_*}\Big(\rho(\mathcal{Y}^{(n)},\overline{\mathcal{Y}}^{(n)})>\varepsilon\Big)\\
&\le\Big(\frac{2}{\varepsilon}\Big)^{4m}\mathbb{E}^{\mathbb{P}_{x_*}}\Big[\rho(\mathcal{Y}^{(n)},\overline{\mathcal{Y}}^{(n)})^{4m}\Big]\\
&\le\Big(\frac{2}{\varepsilon}\Big)^{4m}\mathbb{E}^{\mathbb{P}_{x_*}}\Big[\max_{0\le t\le 1}d_{\mathrm{CC}}(\mathcal{Y}_t^{(n)},\overline{\mathcal{Y}}_t^{(n)})^{4m}\Big]\\
&\le 3^{4m-1}\Big(\frac{2}{\varepsilon}\Big)^{4m}\Big(\frac{1}{\sqrt{n}}\Big)^{4m\alpha}\Big\{\mathbb{E}^{\mathbb{P}_{x_*}}\big[(\mathcal{K}_r^{(n)})^{4m}\big]+\mathbb{E}^{\mathbb{P}_{x_*}}\big[(\overline{\mathcal{K}}_r^{(n)})^{4m}\big]+\mathbb{E}^{\mathbb{P}_{x_*}}\big[C^{4m}\big]\Big\}\\
&\to 0\quad (n\to\infty).
\end{aligned}$$

Then, Slutzky's theorem leads to obtain the convergence in law of $\{\overline{\mathcal{Y}}_\cdot^{(n)}\}_{n=1}^{\infty}$ to the diffusion process $(Y_t)_{0\le t\le 1}$ in $C_{\mathbf{1}_G}([0,1];G)$ as $n\to\infty$.

Step 2. The previous step immediately implies the convergence of the finite-dimensional distribution of $(\overline{\mathcal{Y}}_t^{(n)})_{0\le t\le 1}$. On the other hand, we show that the sequence of image probability measures $\{\overline{\mathbf{P}}^{(n)}:=\mathbb{P}_{x_*}\circ(\overline{\mathcal{Y}}_\cdot^{(n)})^{-1}\}_{n=1}^{\infty}$ is tight in $C_{\mathbf{1}_G}^{0,\alpha\text{-Höl}}([0,1];G)$, by noting Lemma 4.6.1 and by following the same argument as the proof of Lemma 4.3.1. Therefore, we conclude the desired convergence in law, by combining these two. This completes the proof. ∎

Chapter 5

CLTs of the second kind for non-symmetric random walks on nilpotent covering graphs

5.1 Settings and statements

As with the previous chapter, suppose that X is a Γ-nilpotent covering graph of a finite graph X_0, where Γ is a torsion free, finitely generated nilpotent group of step r. Let $G = G_\Gamma$ be the connected and simply connected nilpotent Lie group of step r such that Γ is isomorphic to a cocompact lattice in G, and $\mathfrak{g} = \bigoplus_{k=1}^{r} \mathfrak{g}^{(k)}$ the corresponding Lie algebra.

Let us give the settings and statements of CLTs of the second kind in the present section. For the given transition probability p, we introduce a family of Γ-invariant transition probabilities $(p_\varepsilon)_{0 \le \varepsilon \le 1}$ on X by

$$p_\varepsilon(e) := p_0(e) + \varepsilon q(e) \qquad (e \in E), \tag{5.1.1}$$

where

$$p_0(e) := \frac{1}{2}\Big(p(e) + \frac{m\big(t(e)\big)}{m\big(o(e)\big)}p(\overline{e})\Big), \quad q(e) := \frac{1}{2}\Big(p(e) - \frac{m\big(t(e)\big)}{m\big(o(e)\big)}p(\overline{e})\Big).$$

We note that the family $(p_\varepsilon)_{0 \le \varepsilon \le 1}$ is given by the linear interpolation between the transition probability $p = p_1$ and the m-symmetric probability p_0. Moreover, the homological direction γ_{p_ε} equals $\varepsilon\gamma_p$ for every $0 \le \varepsilon \le 1$ (cf. [42, Proposition 2.3]).

Let $L_{(\varepsilon)}$ be the transition operator associated with p_ε for $0 \le \varepsilon \le 1$. We also denote by $g_0^{(\varepsilon)}$ the Albanese metric on $\mathfrak{g}^{(1)}$ associated with p_ε. We write $G_{(\varepsilon)}$ for the nilpotent Lie group of step r whose Lie algebra is $\mathfrak{g} = (\mathfrak{g}^{(1)}, g_0^{(\varepsilon)}) \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$. Let $\Phi_0^{(\varepsilon)} : X \longrightarrow G$ be the (p_ε-)modified harmonic realization for $0 \le \varepsilon \le 1$.

Here we assume

(A1): *For every $0 \le \varepsilon \le 1$, it holds that*

$$\sum_{x \in \mathcal{F}} m(x) \log \big(\Phi_0^{(\varepsilon)}(x)^{-1} \cdot \Phi_0^{(0)}(x)\big)\big|_{\mathfrak{g}^{(1)}} = 0, \tag{5.1.2}$$

where $\mathcal{F}$ denotes a fundamental domain of X.

Since the modified harmonic realizations $(\Phi_0^{(\varepsilon)})_{0 \le \varepsilon \le 1}$ are uniquely determined up to $\mathfrak{g}^{(1)}$-translation, it is always possible to take $(\Phi_0^{(\varepsilon)})_{0 \le \varepsilon \le 1}$ satisfying **(A1)**.

We define an *approximation operator* $P_\varepsilon : C_\infty(G_{(0)}) \longrightarrow C_\infty(X)$ by

$$P_\varepsilon f(x) := f\big(\tau_\varepsilon \Phi_0^{(\varepsilon)}(x)\big) \qquad (0 \le \varepsilon \le 1,\, x \in V).$$

We take an orthonormal basis $\{V_1, V_2, \dots, V_{d_1}\}$ of $(\mathfrak{g}^{(1)}, g_0^{(0)})$. Then the semigroup CLT of the second kind is stated as follows:

Theorem 5.1.1 (1) *For $0 \le s \le t$ and $f \in C_\infty(G_{(0)})$, we have*

$$\lim_{n \to \infty} \Big\| L_{(n^{-1/2})}^{[nt]-[ns]} P_{n^{-1/2}} f - P_{n^{-1/2}} \mathrm{e}^{-(t-s)\mathcal{A}} f \Big\|_\infty^X = 0, \tag{5.1.3}$$

where $(\mathrm{e}^{-t\mathcal{A}})_{t \ge 0}$ is the C^0-semigroup whose infinitesimal generator $\mathcal{A}$ is given by

$$\mathcal{A} = -\frac{1}{2} \sum_{i=1}^{d_1} V_{i*}^2 - \rho_{\mathbb{R}}(\gamma_p)_*. \tag{5.1.4}$$

(2) *Let μ be a Haar measure on $G_{(0)}$. Then, for any $f \in C_\infty(G_{(0)})$ and for any sequence $\{x_n\}_{n=1}^\infty \subset V$ satisfying $\lim_{n \to \infty} \tau_{n^{-1/2}}\big(\Phi_0^{(n^{-1/2})}(x_n)\big) =: g \in G_{(0)}$, we have*

$$\lim_{n \to \infty} L_{(n^{-1/2})}^{[nt]} P_{n^{-1/2}} f(x_n) = \mathrm{e}^{-t\mathcal{A}} f(g) := \int_{G_{(0)}} \mathcal{H}_t(h^{-1} * g) f(h)\, \mu(dh) \qquad (t \ge 0), \tag{5.1.5}$$

where $\mathcal{H}_t(g)$ is a fundamental solution to the heat equation

$$\Big(\frac{\partial}{\partial t} + \mathcal{A}\Big) u(t,g) = 0 \qquad (t > 0,\, g \in G_{(0)}).$$

We now fix a reference point $x_* \in V$ such that $\Phi_0^{(0)}(x_*) = \mathbf{1}_G$ and put

$$\xi_n^{(\varepsilon)}(c) := \Phi_0^{(\varepsilon)}\big(w_n(c)\big) \qquad \big(0 \le \varepsilon \le 1,\, n = 0, 1, 2, \dots,\, c \in \Omega_{x_*}(X)\big).$$

Note that **(A1)** does not imply that $\Phi_0^{(\varepsilon)}(x_*) = \mathbf{1}_G$ for $0 < \varepsilon \le 1$ in general. We then obtain a G-valued random walk $(\Omega_{x_*}(X), \mathbb{P}_{x_*}^{(\varepsilon)}, \{\xi_n^{(\varepsilon)}\}_{n=0}^\infty)$ associated with the transition probability p_ε. For $t \ge 0$, $n = 1, 2, \dots$ and $0 \le \varepsilon \le 1$, let $\mathcal{X}_t^{(\varepsilon,n)}$ be a map from $\Omega_{x_*}(X)$ to G given by

$$\mathcal{X}_t^{(\varepsilon,n)}(c) := \tau_{n^{-1/2}}\big(\xi_{[nt]}^{(\varepsilon)}(c)\big) \qquad \big(c \in \Omega_{x_*}(X)\big).$$

We write $\mathcal{D}_n$ for the partition $\{t_k = k/n \,|\, k = 0, 1, 2, \dots, n\}$ of the time interval $[0, 1]$ for $n \in \mathbb{N}$. We define

$$\mathcal{Y}^{(\varepsilon,n)}_{t_k}(c) := \tau_{n^{-1/2}}\big(\xi^{(\varepsilon)}_{nt_k}(c)\big) = \tau_{n^{-1/2}}\big(\Phi^{(\varepsilon)}_0(w_k(c))\big) \qquad \big(t_k \in \mathcal{D}_n,\, c \in \Omega_{x_*}(X)\big)$$

and consider a G-valued continuous stochastic process $(\mathcal{Y}^{(\varepsilon,n)}_t)_{0\le t\le 1}$ defined by the d_{CC}-geodesic interpolation of $\{\mathcal{Y}^{(\varepsilon,n)}_{t_k}\}_{k=0}^n$. Let $d_1 = \dim_{\mathbb{R}} \mathfrak{g}^{(1)}$. We consider a stochastic differential equation

$$d\widehat{Y}_t = \sum_{i=1}^{d_1} V^{(0)}_{i_*}(\widehat{Y}_t) \circ dB^i_t + \rho_{\mathbb{R}}(\gamma_p)_*(\widehat{Y}_t)\, dt, \qquad \widehat{Y}_0 = \mathbf{1}_G, \tag{5.1.6}$$

where $(B_t)_{0\le t\le 1} = (B^1_t, B^2_t, \dots, B^{d_1}_t)_{0\le t\le 1}$ is a standard Brownian motion with values in $\mathbb{R}^{d_1}$ starting from $B_0 = \mathbf{0}$. We know that the infinitesimal generator of (5.1.6) coincides with $-\mathcal{A}$ defined by (5.1.4) (see Proposition 4.5.3). Let $(\widehat{Y}_t)_{0\le t\le 1}$ be the G-valued diffusion process which is the solution to (5.1.6). We write

$$C^{\alpha\text{-Höl}}([0,1]; G_{(0)}) = \big\{w \in C([0,1]; G_{(0)}) \,:\, \|w\|_{\alpha\text{-Höl}} < \infty\big\} \qquad (\alpha < 1/2)$$

for the set of all α-Hölder continuous paths on $G_{(0)}$, where

$$\|w\|_{\alpha\text{-Höl}} := \sup_{0\le s<t\le 1} \frac{d_{\mathrm{CC}}(w_s, w_t)}{|t-s|^\alpha} + d_{\mathrm{CC}}(\mathbf{1}_G, w_0) \qquad \big(w \in C^{\alpha\text{-Höl}}([0,1]; G_{(0)})\big).$$

Now we define

$$C^{0,\alpha\text{-Höl}}([0,1]; G_{(0)}) := \overline{\mathrm{Lip}([0,1]; G_{(0)})}^{\|\cdot\|_{\alpha\text{-Höl}}}, \tag{5.1.7}$$

which is a Polish space (cf. Friz–Victoir [22, Section 8]). Let $\mathbf{P}^{(\varepsilon,n)}$ be the probability measure on $C^{0,\alpha\text{-Höl}}([0,1]; G_{(0)})$ induced by the stochastic process $\mathcal{Y}^{(\varepsilon,n)}_{\cdot}$ for $0 \le \varepsilon \le 1$ and $n \in \mathbb{N}$.

To present the second result, we need to put an additional assumption.

(A2): *There exists a positive constant C such that, for $k = 2, 3, \dots, r$,*

$$\sup_{0\le\varepsilon\le 1} \max_{x\in\mathcal{F}} \big\| \log\big(\Phi^{(\varepsilon)}_0(x)^{-1} \cdot \Phi^{(0)}_0(x)\big)\big|_{\mathfrak{g}^{(k)}} \big\|_{\mathfrak{g}^{(k)}} \le C, \tag{5.1.8}$$

where $\|\cdot\|_{\mathfrak{g}^{(k)}}$ denotes a Euclidean norm on $\mathfrak{g}^{(k)} \cong \mathbb{R}^{d_k}$ for $k = 2, 3, \dots, r$.

Intuitively speaking, the situations that the distance between $\Phi^{(\varepsilon)}_0$ and $\Phi^{(0)}_0$ tends to be too big as $\varepsilon \searrow 0$ are removed under **(A2)**. By setting

$$\log\big(\Phi^{(\varepsilon)}_0(x)\big)\big|_{\mathfrak{g}^{(k)}} = \log\big(\Phi^{(0)}_0(x)\big)\big|_{\mathfrak{g}^{(k)}} \qquad (x \in \mathcal{F},\, k = 2, 3, \dots, r)$$

for $\Phi^{(\varepsilon)}_0 : X \longrightarrow G$ with (5.1.2), the family $(\Phi^{(\varepsilon)}_0)_{0\le\varepsilon\le 1}$ satisfies **(A2)**. This means that it is always possible to take a family $(\Phi^{(\varepsilon)}_0)_{0\le\varepsilon\le 1}$ satisfying **(A2)** as well as **(A1)**.

Then the following theorem is the functional CLT of the second kind for the family of non-symmetric random walks $\{\xi^{(\varepsilon)}_n\}_{n=0}^\infty$.

Theorem 5.1.2 *We assume* **(A1)** *and* **(A2)**. *Then the sequence* $(\mathcal{Y}_t^{(n^{-1/2},n)})_{0\le t\le 1}$ *converges in law to the diffusion process* $(\widehat{Y}_t)_{0\le t\le 1}$ *in* $C^{0,\alpha\text{-Höl}}([0,1];G_{(0)})$ *as* $n\to\infty$ *for all* $\alpha<1/2$.

In Theorem 4.1.3, we captured a G-valued diffusion process and its infinitesimal generator is the homogenized sub-Laplacian associated with the Albanese metric $g_0=g_0^{(1)}$ with a non-trivial drift $\beta(\Phi_0)\in\mathfrak{g}^{(2)}$. In particular, even in the centered case $\rho_{\mathbb{R}}(\gamma_p)=\mathbf{0}_{\mathfrak{g}}$, the non-trivial drift $\beta(\Phi_0)$ remains in general. On the other hand, in this case, the limiting diffusion $(\widehat{Y}_t)_{0\le t\le 1}$ is generated by the homogenized sub-Laplacian on $G_{(0)}$ associated with the Albanese metric $g_0^{(0)}$ under $\rho_{\mathbb{R}}(\gamma_p)=\mathbf{0}_{\mathfrak{g}}$. See the end of this chapter.

5.2 A one-parameter family of modified harmonic realizations $(\Phi_0^{(\varepsilon)})_{0\le\varepsilon\le 1}$

Let $(p_\varepsilon)_{0\le\varepsilon\le 1}$ be the family of transition probabilities defined by (5.1.1). We easily see $p_1=p$ and $p_\varepsilon(e)>0$ for $e\in E$ if $0\le\varepsilon<1$, by definition. We also observe that the invariant measure of the random walk associated with p_ε coincides with m for $0\le\varepsilon\le 1$. Moreover, p_0 and q are m-symmetric and m-anti-symmetric, respectively. Note that $\gamma_{p_\varepsilon}=\varepsilon\gamma_p$ for all $0\le\varepsilon\le 1$.

For every $0\le\varepsilon\le 1$, we take the modified harmonic realization $\Phi_0^{(\varepsilon)}:X\longrightarrow G$ associated with the transition probability p_ε. Namely, $\Phi_0^{(\varepsilon)}$ is the Γ-equivariant realization of X satisfying

$$\sum_{e\in E_x} p_\varepsilon(e)\log\Big(\Phi_0^{(\varepsilon)}\big(o(e)\big)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(e)\big)\Big)\Big|_{\mathfrak{g}^{(1)}}=\varepsilon\rho_{\mathbb{R}}(\gamma_p)\qquad(x\in V).\tag{5.2.1}$$

We put

$$d\Phi_0^{(\varepsilon)}(e)=\Phi_0^{(\varepsilon)}\big(o(e)\big)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(e)\big)\qquad(0\le\varepsilon\le 1,\, e\in E),$$

The aim of this subsection is to study the quantity

$$\beta_{(\varepsilon)}(\Phi_0^{(\varepsilon)}):=\sum_{e\in E_0}\widetilde{m}_\varepsilon(e)\log\big(d\Phi_0^{(\varepsilon)}(\widetilde{e})\big)\big|_{\mathfrak{g}^{(2)}}\in\mathfrak{g}^{(2)}\qquad(0\le\varepsilon\le 1),$$

where we put $\widetilde{m}_\varepsilon(e)=p_\varepsilon(e)m\big(o(e)\big)$ for $e\in E_0$. Note that, if the transition probability p_0 is m-symmetric, then $\beta_{(0)}(\Phi_0^{(0)})=\mathbf{0}_{\mathfrak{g}}$. Loosely speaking, this quantity will appear as a coefficient of the second order term of the Taylor expansion of $(I-L_{(\varepsilon)}^N)P_\varepsilon f$ in ε, which we have to deal in the proof of Lemma 5.3.1. In particular, we are interested in the short time behavior of $\beta_{(\varepsilon)}(\Phi_0^{(\varepsilon)})$ as $\varepsilon\searrow 0$ for later use. Intuitively there seems to be little hope of knowing such behavior, because of the luck of any information about $\mathfrak{g}^{(2)}$-components of the realizations $\Phi_0^{(\varepsilon)}$ for every $0\le\varepsilon\le 1$. However, the following proposition asserts that $\beta_{(\varepsilon)}(\Phi_0^{(\varepsilon)})$ in fact approaches $\beta_{(0)}(\Phi_0^{(0)})=\mathbf{0}_{\mathfrak{g}}$ as $\varepsilon\searrow 0$ by imposing only **(A1)**.

Proposition 5.2.1 *Under* **(A1)**, *we have*

$$\lim_{\varepsilon \searrow 0} \beta_{(\varepsilon)}(\Phi_0^{(\varepsilon)}) = \beta_{(0)}(\Phi_0^{(0)}) = \mathbf{0}_{\mathfrak{g}}.$$

Fix a fundamental domain $\mathcal{F}$ of X. Set $\Psi^{(\varepsilon)}(x) = \Phi_0^{(\varepsilon)}(x)^{-1} \cdot \Phi_0^{(0)}(x)$ for $0 \leq \varepsilon \leq 1$ and $x \in V$. Note that the map $\Psi^{(\varepsilon)} : V \longrightarrow G$ is Γ-invariant. The following lemma is essential to prove Proposition 5.2.1.

Lemma 5.2.2 *Under* **(A1)**, *we have*

$$\lim_{\varepsilon \searrow 0} \big\| \log\big(\Psi^{(\varepsilon)}(x)\big)\big|_{\mathfrak{g}^{(1)}} \big\|_{\mathfrak{g}^{(1)}} = 0 \qquad (x \in \mathcal{F}). \tag{5.2.2}$$

In particular, there exists a constant C such that

$$\big\| \log\big(\Psi^{(\varepsilon)}(x)\big)\big|_{\mathfrak{g}^{(1)}} \big\|_{\mathfrak{g}^{(1)}} \leq C \qquad (0 \leq \varepsilon \leq 1,\, x \in \mathcal{F}).$$

Proof. We set $\ell^2(\mathcal{F}) := \{f : \mathcal{F} \longrightarrow \mathbb{C}\}$ and equip it with the inner product and the corresponding norm defined by

$$\langle f, g\rangle_{\ell^2(\mathcal{F})} := \sum_{x \in \mathcal{F}} f(x)\overline{g(x)}, \quad \|f\|_{\ell^2(\mathcal{F})} := \Big(\sum_{x\in\mathcal{F}} |f(x)|^2\Big)^{1/2} \qquad \big(f, g \in \ell^2(\mathcal{F})\big),$$

respectively. Since the invariant measure $m|_{\mathcal{F}} : \mathcal{F} \longrightarrow (0, 1]$ is positive on the finite set $\mathcal{F}$, there are positive constants c and C such that

$$c\Big(\sum_{x\in\mathcal{F}} m(x)|f(x)|^2\Big)^{1/2} \leq \|f\|_{\ell^2(\mathcal{F})} \leq C\Big(\sum_{x\in\mathcal{F}} m(x)|f(x)|^2\Big)^{1/2} \qquad \big(f \in \ell^2(\mathcal{F})\big). \tag{5.2.3}$$

It is easy to see that $\ell^2(\mathcal{F})$ is decomposed as $\ell^2(\mathcal{F}) = \langle \phi_0 \rangle \oplus \ell_1(\mathcal{F})$ by virtue of the Perron–Frobenius theorem, where $\phi_0 = |\mathcal{F}|^{-1/2}$ is the normalized right eigenfunction corresponding to the maximal eigenvalue $\alpha_0 = 1$ of L. We define

$$\ell^2_1(\mathcal{F}) := \big\{f \in \ell^2(\mathcal{F}) \,:\, |\mathcal{F}|^{1/2}\langle f, m\rangle_{\ell^2(\mathcal{F})} = 0\big\}.$$

Note that $\ell^2_1(\mathcal{F})$ is preserved by L and the transition operator $L_{(\varepsilon)}$ maps $\ell^2_1(\mathcal{F})$ to itself for all $0 \leq \varepsilon \leq 1$. Moreover, we should emphasize that the inverse operator of $(I - L_{(\varepsilon)})|_{\ell^2_1(\mathcal{F})} : \ell^2_1(\mathcal{F}) \longrightarrow \ell^2_1(\mathcal{F})$ does exists since $L_{(\varepsilon)}$ has a simple eigenvalue $\alpha_0(\varepsilon) = 1$ for $0 \leq \varepsilon \leq 1$. Let $Q : \ell^2(\mathcal{F}) \longrightarrow \ell^2(\mathcal{F})$ be the operator defined by

$$Qf(x) := \sum_{e \in E_x} q(e) f\big(t(e)\big) \qquad \big(f \in \ell^2(\mathcal{F}),\, x \in \mathcal{F}\big).$$

Then we verify that the transition operator $L_{(\varepsilon)}$ has the decomposition of the form $L_{(\varepsilon)} = L_{(0)} + \varepsilon Q$ for every $0 \leq \varepsilon \leq 1$. In order to show (5.2.2), it suffices to show

$$\lim_{\varepsilon \searrow 0} \big\| \log\big(\Psi^{(\varepsilon)}(\cdot)\big)\big|_{X_i^{(1)}} \big\|_{\ell^2(\mathcal{F})} = 0 \qquad (i = 1, 2, \dots, d_1) \tag{5.2.4}$$

by noting (5.2.3). We remark that $\log\big(\Psi^{(\varepsilon)}(\cdot)\big)\big|_{X_i^{(1)}} \in \ell_1^2(\mathcal{F})$ for $i=1,2,\dots,d_1$ thanks to (5.1.2). In the following, we fix $i=1,2,\dots,d_1$. The modified harmonicity of $\Phi_0^{(\varepsilon)}$ gives

$$(I-L_{(\varepsilon)})\Big(\log\big(\Psi^{(\varepsilon)}(x)\big)\big|_{X_i^{(1)}}\Big) = \varepsilon\Big[Q\big(\log\Phi_0^{(0)}(x)\big|_{X_i^{(1)}}\big) - \rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}\Big]$$

for $0\le\varepsilon\le 1$ and $x\in\mathcal{F}$. This identity implies

$$\begin{aligned}
&\big\|\log\big(\Psi^{(\varepsilon)}(\cdot)\big)\big|_{X_i^{(1)}}\big\|_{\ell^2(\mathcal{F})}\\
&\le \varepsilon\big\|(I-L_{(\varepsilon)})\big|_{\ell_1^2(X_0)}^{-1}\big\|\cdot\big\|Q\big(\log\Phi_0^{(0)}(\cdot)\big|_{X_i^{(1)}}\big)-\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}\big\|_{\ell^2(\mathcal{F})}\\
&\le \varepsilon\big\|(I-L_{(\varepsilon)})\big|_{\ell_1^2(X_0)}^{-1}\big\|\cdot\Big\{\big\|\log\Phi_0^{(0)}(\cdot)\big|_{X_i^{(1)}}\big\|_{\ell^2(\mathcal{F})}+\big\|\rho_{\mathbb{R}}(\gamma_p)\big\|_{\mathfrak{g}^{(1)}}\Big\},
\end{aligned}\tag{5.2.5}$$

where we used $\|Q\|\le 1$ for the final line. By combining (5.2.5) with the identity

$$(I-L_{(\varepsilon)})\big|_{\ell_1^2(\mathcal{F})}^{-1} = (I-L_{(0)})\big|_{\ell_1^2(\mathcal{F})}^{-1}\Big[I-\varepsilon Q\big|_{\ell_1^2(\mathcal{F})}(I-L_{(0)})\big|_{\ell_1^2(\mathcal{F})}^{-1}\Big],$$

we obtain

$$\begin{aligned}
\big\|\log\big(\Psi^{(\varepsilon)}(\cdot)\big)\big|_{X_i^{(1)}}\big\|_{\ell^2(\mathcal{F})} &\le \varepsilon\big\|(I-L_{(0)})\big|_{\ell_1^2(\mathcal{F})}^{-1}\big\|\cdot\Big(1-\varepsilon\big\|Q\big|_{\ell_1^2(\mathcal{F})}(I-L_{(0)})\big|_{\ell_1^2(\mathcal{F})}^{-1}\big\|\Big)^{-1}\\
&\qquad\times\Big\{\big\|\log\Phi_0^{(0)}(\cdot)\big|_{X_i^{(1)}}\big\|_{\ell^2(\mathcal{F})}+\big\|\rho_{\mathbb{R}}(\gamma_p)\big\|_{\mathfrak{g}^{(1)}}\Big\}.
\end{aligned}$$

Here we can choose a sufficiently small constant $\varepsilon_0>0$ such that

$$\sup_{0\le\varepsilon\le\varepsilon_0}\Big(1-\varepsilon\big\|Q\big|_{\ell_1^2(\mathcal{F})}(I-L_{(0)})\big|_{\ell_1^2(\mathcal{F})}^{-1}\big\|\Big)^{-1}\le 2.$$

Then we have

$$\big\|\log\big(\Psi^{(\varepsilon)}(\cdot)\big)\big|_{X_i^{(1)}}\big\|_{\ell^2(\mathcal{F})} \le 2\varepsilon\big\|(I-L_{(0)})\big|_{\ell_1^2(\mathcal{F})}^{-1}\big\|\Big\{\big\|\log\Phi_0^{(0)}(\cdot)\big|_{X_i^{(1)}}\big\|_{\ell^2(\mathcal{F})}+\big\|\rho_{\mathbb{R}}(\gamma_p)\big\|_{\mathfrak{g}^{(1)}}\Big\}$$

for sufficiently small $\varepsilon>0$ and this implies (5.2.4). ∎

Proof of Proposition 5.2.1. By recalling (5.1.1) and that p_0 is m-symmetric, we have

$$\begin{aligned}
\beta_{(\varepsilon)}(\Phi_0^{(\varepsilon)}) &= \sum_{e\in E_0}\Big\{\frac{1}{2}\big(\widetilde{m}_0(e)-\widetilde{m}_0(\overline{e})\big)\log\big(d\Phi_0^{(\varepsilon)}(\widetilde{e})\big)\big|_{\mathfrak{g}^{(2)}}+\varepsilon m\big(o(e)\big)q(e)\log\big(d\Phi_0^{(\varepsilon)}(\widetilde{e})\big)\big|_{\mathfrak{g}^{(2)}}\Big\}\\
&= \varepsilon\sum_{e\in E_0} m\big(o(e)\big)q(e)\log\big(d\Phi_0^{(\varepsilon)}(\widetilde{e})\big)\big|_{\mathfrak{g}^{(2)}}.
\end{aligned}$$

Then the identity

$$d\Phi_0^{(\varepsilon)}(e) = \Psi^{(\varepsilon)}\big(o(e)\big)\cdot d\Phi_0^{(0)}(e)\cdot\Psi^{(\varepsilon)}\big(t(e)\big)^{-1}\qquad(0\le\varepsilon\le 1,\ e\in E),$$

and (2.2.2) yield

$$\begin{aligned}\beta_{(\varepsilon)}(\Phi_0^{(\varepsilon)}) &= \varepsilon \sum_{e\in E_0} m\big(o(e)\big)q(e)\Big\{ \log\big(\Psi^{(\varepsilon)}\big(o(\widetilde{e})\big)\big)\big|_{\mathfrak{g}^{(2)}} - \log\big(\Psi^{(\varepsilon)}\big(t(\widetilde{e})\big)\big)\big|_{\mathfrak{g}^{(2)}}\Big\}\\ &\quad + \varepsilon \sum_{e\in E_0} m\big(o(e)\big)q(e)\log\big(d\Phi_0^{(0)}(\widetilde{e})\big)\big|_{X_i^{(2)}}\\ &\quad - \frac{\varepsilon}{2}\sum_{e\in E_0} m\big(o(e)\big)q(e)\Big\{\mathcal{I}_1^{(\varepsilon)}(\widetilde{e}) + \mathcal{I}_2^{(\varepsilon)}(\widetilde{e}) + \mathcal{I}_3^{(\varepsilon)}(\widetilde{e})\Big\}, \end{aligned} \tag{5.2.6}$$

where

$$\begin{aligned}\mathcal{I}_1^{(\varepsilon)}(\widetilde{e}) &= \mathcal{I}_1^{(\varepsilon;\lambda,\nu)}(\widetilde{e}) = \Big[\log\big(\Psi^{(\varepsilon)}(o(\widetilde{e}))\big)\big|_{\mathfrak{g}^{(1)}}, \log\big(d\Phi_0^{(0)}(\widetilde{e})\big)\big|_{\mathfrak{g}^{(1)}}\Big],\\ \mathcal{I}_2^{(\varepsilon)}(\widetilde{e}) &= \mathcal{I}_2^{(\varepsilon;\lambda,\nu)}(\widetilde{e}) = \Big[\log\big(\Psi^{(\varepsilon)}(o(\widetilde{e}))\big)\big|_{\mathfrak{g}^{(1)}}, \log\big(\Psi^{(\varepsilon)}(t(\widetilde{e}))^{-1}\big)\big|_{\mathfrak{g}^{(1)}}\Big],\\ \mathcal{I}_3^{(\varepsilon)}(\widetilde{e}) &= \mathcal{I}_3^{(\varepsilon;\lambda,\nu)}(\widetilde{e}) = \Big[\log\big(d\Phi_0^{(0)}(\widetilde{e})\big)\big|_{\mathfrak{g}^{(1)}}, \log\big(\Psi^{(\varepsilon)}(t(\widetilde{e}))^{-1}\big)\big|_{\mathfrak{g}^{(1)}}\Big].\end{aligned}$$

Let $\{X_1^{(2)}, X_2^{(2)}, \dots, X_{d_2}^{(2)}\}$ be a basis of $\mathfrak{g}^{(2)}$. For $i = 1, 2, \dots, d_2$, we define a function $F_i^{(\varepsilon)} : V \longrightarrow \mathbb{R}$ by $F_i^{(\varepsilon)}(x) := \log\big(\Psi^{(\varepsilon)}(x)\big)\big|_{X_i^{(2)}}$ for $0 \le \varepsilon \le 1$ and $x \in V$. Then we see that the function $F_i^{(\varepsilon)}$ is Γ-invariant. Hence, there exists a function $\widehat{F}^{(\varepsilon)} : V_0 \longrightarrow \mathbb{R}$ such that $\widehat{F}_i^{(\varepsilon)}\big(\pi(x)\big) = F_i^{(\varepsilon)}(x)$ for $0 \le \varepsilon \le 1$ and $x \in V$. Then, by noting $\partial(\gamma_{p_\varepsilon}) = 0$, we have

$$\begin{aligned}&\varepsilon \sum_{e\in E_0} m\big(o(e)\big)q(e)\Big\{ \log\big(\Psi^{(\varepsilon)}\big(o(\widetilde{e})\big)\big)\big|_{\mathfrak{g}^{(2)}} - \log\big(\Psi^{(\varepsilon)}\big(t(\widetilde{e})\big)\big)\big|_{\mathfrak{g}^{(2)}}\Big\}\\ &= \sum_{e\in E_0}\big(\widetilde{m}_\varepsilon(e) - \widetilde{m}_0(e)\big)\Big\{ \log\big(\Psi^{(\varepsilon)}\big(o(\widetilde{e})\big)\big)\big|_{\mathfrak{g}^{(2)}} - \log\big(\Psi^{(\varepsilon)}\big(t(\widetilde{e})\big)\big)\big|_{\mathfrak{g}^{(2)}}\Big\}\\ &= -{}_{C^1(X_0,\mathbb{R})}\big\langle \gamma_{p_\varepsilon}, d\widehat{F}_i^{(\varepsilon)}\big\rangle_{C^1(X_0,\mathbb{R})} + \frac{1}{2}\sum_{e\in E_0}\big(\widetilde{m}_0(e) - \widetilde{m}_0(\overline{e})\big)d\widehat{F}_i^{(\varepsilon)}(e)\\ &= -{}_{C^0(X_0,\mathbb{R})}\big\langle \partial(\gamma_{p_\varepsilon}), \widehat{F}_i^{(\varepsilon)}\big\rangle_{C^0(X_0,\mathbb{R})}\\ &= 0.\end{aligned}$$

By applying Lemma 5.2.2 and the elementary inequality $\|[Z_1, Z_2]\|_{\mathfrak{g}^{(2)}} \le C\|Z_1\|_{\mathfrak{g}^{(1)}}\|Z_2\|_{\mathfrak{g}^{(1)}}$ for $Z_1, Z_2 \in \mathfrak{g}^{(1)}$ and some $C > 0$, we find a sufficiently large $C > 0$ satisfying

$$\sup_{0\le\varepsilon\le 1}\big\|\mathcal{I}_k^{(\varepsilon)}(\widetilde{e})\big\|_{\mathfrak{g}^{(2)}} \le C$$

for $k = 1, 2, 3$. Summing up the all arguments above and letting $\varepsilon \searrow 0$ in both sides of (5.2.6), we obtain the desired convergence. This completes the proof. ∎

We denote $\mathcal{H}^1_{(\varepsilon)}(X_0)$ the set of all modified harmonic 1-forms on X_0. We equip $\mathcal{H}^1_{(\varepsilon)}(X_0)$ with the inner product

$$\langle\!\langle \omega_1, \omega_2 \rangle\!\rangle_{p_\varepsilon} := \sum_{e\in E_0}\widetilde{m}_\varepsilon(e)\omega_1(e)\omega_2(e) - \varepsilon^2\langle\gamma_p, \omega_1\rangle\langle\gamma_p, \omega_2\rangle \qquad \big(\omega_1, \omega_2 \in \mathcal{H}^1_{(\varepsilon)}(X_0)\big).$$

We may identify $\mathrm{H}^1(X_0,\mathbb{R})$ with $\mathcal{H}^1_{(\varepsilon)}(X_0)$ for every $0 \le \varepsilon \le 1$ by applying the discrete Hodge–Kodaira theorem. It should be noted that the identification map depends on the parameter ε and $\mathcal{H}^1_{(1)}(X_0) = \mathcal{H}^1(X_0)$. Moreover, we also identify $\mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R})$ with $\mathrm{Im}({}^t\rho_{\mathbb{R}}) \subset \mathrm{H}^1(X_0,\mathbb{R})$. Therefore, $\mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R})$ may be regarded as a subspace of each $\mathcal{H}^1_{(\varepsilon)}(X_0)$. For an element $\omega \in \mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R})$, we denote ${}^t\rho_{\mathbb{R}}(\omega) \in \mathrm{H}^1(X_0,\mathbb{R}) \cong \mathcal{H}^1_{(\varepsilon)}(X_0)$ by $\omega^{(\varepsilon)}$. Let $g_0^{(\varepsilon)}$ be the Albanese metric on $\mathfrak{g}^{(1)}$ induced by the dual inner product of $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{(\varepsilon)}$ for $0 \le \varepsilon \le 1$.

5.3 Proof of Theorem 5.1.1

We prove Theorem 5.1.1 in this subsection. A key claim to obtain the main theorem is the following Lemma.

Lemma 5.3.1 *For any $f \in C_0^\infty(G_{(0)})$, as $N \to \infty$ and $\varepsilon \searrow 0$ with $N^2\varepsilon \searrow 0$, we have*

$$\Big\| \frac{1}{N\varepsilon^2}(I - L^N_{(\varepsilon)})P_\varepsilon f - P_\varepsilon \mathcal{A} f \Big\|_\infty^X \longrightarrow 0,$$

where $\mathcal{A}$ is the sub-elliptic operator on $C_0^\infty(G_{(0)})$ defined by (5.1.4).

Proof. We apply Taylor's expansion formula (cf. Alexopoulos [2, Lemma 5.3]) for the $(*)$-coordinates of the second kind to $f \in C_0^\infty(G_{(0)})$ at $\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big) \in G_{(0)}$. Then, recalling that $(G_{(0)},*)$ is a stratified Lie group, we have

$$\begin{aligned}
&\frac{1}{N\varepsilon^2}(I - L^N_{(\varepsilon)})P_\varepsilon f(x)\\
&= -\sum_{(i,k)} \frac{\varepsilon^{k-2}}{N} X_{i*}^{(k)} f\Big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\Big) \sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c)\Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i*}^{(k)}\\
&\quad - \Big(\sum_{(i_1,k_1)\ge(i_2,k_2)} \frac{\varepsilon^{k_1+k_2-2}}{2N} X_{i_1*}^{(k_1)} X_{i_2*}^{(k_2)} + \sum_{(i_2,k_2)>(i_1,k_1)} \frac{\varepsilon^{k_1+k_2-2}}{2N} X_{i_2*}^{(k_2)} X_{i_1*}^{(k_1)} \Big) f\Big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\Big)\\
&\qquad \times \sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c)\Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i_1*}^{(k_1)} \Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i_2*}^{(k_2)}\\
&\quad - \sum_{(i_1,k_1),(i_2,k_2),(i_3,k_3)} \frac{\varepsilon^{k_1+k_2+k_3-2}}{6N} \frac{\partial^3 f}{\partial g_{i_1*}^{(k_1)} \partial g_{i_2*}^{(k_2)} \partial g_{i_3*}^{(k_3)}}(\theta)\\
&\qquad \times \sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c)\Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i_1*}^{(k_1)}\\
&\qquad \times \Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i_2*}^{(k_2)} \Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i_3*}^{(k_3)}
\end{aligned} \tag{5.3.1}$$

for $x \in V$ and some $\theta \in G_{(0)}$ satisfying

$$|\theta_{i*}^{(k)}| \le \Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i*}^{(k)} \qquad (i = 1,2,\dots,d_k,\ k = 1,2,\dots r),$$

where the summation $\sum_{(i_1,k_1)\geq(i_2,k_2)}$ runs over all (i_1,k_1) and (i_2,k_2) with $k_1 > k_2$ or $k_1 = k_2$ and $i_1 \geq i_2$. We denote by $\mathrm{Ord}_\varepsilon(k)$ the terms of the right-hand side of (5.3.1) whose order of ε equals just k. Then, (5.3.1) is rewritten as

$$\frac{1}{N\varepsilon^2}(I - L_{(\varepsilon)}^N)P_\varepsilon f(x) = \mathrm{Ord}_\varepsilon(-1) + \mathrm{Ord}_\varepsilon(0) + \sum_{k\geq 1}\mathrm{Ord}_\varepsilon(k) \qquad (x \in V),$$

where

$$\mathrm{Ord}_\varepsilon(-1) = -\frac{1}{N\varepsilon}\sum_{i=1}^{d_1} X_{i*}^{(1)} f\big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\big) \sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i*}^{(1)}$$

and

$$\begin{aligned}
\mathrm{Ord}_\varepsilon(0) = &-\frac{1}{N}\sum_{i=1}^{d_2} X_{i*}^{(2)} f\big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\big) \sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big\{\Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i*}^{(2)} \\
&\quad - \frac{1}{2}\sum_{1\leq\lambda<\nu\leq d_1} [\![X_\lambda^{(1)}, X_\nu^{(1)}]\!]\big|_{X_i^{(2)}} \\
&\quad \times \Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{\lambda*}^{(1)}\Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{\nu*}^{(1)}\Big\} \\
&- \frac{1}{2N}\sum_{1\leq i,j\leq d_1} X_{i*}^{(1)}X_{j*}^{(1)} f\big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\big) \\
&\quad \times \sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i*}^{(1)}\Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{j*}^{(1)}.
\end{aligned}$$

Step 1. We first estimate $\mathrm{Ord}_\varepsilon(-1)$. By recalling (2.2.3) and (5.2.1), we have inductively

$$\begin{aligned}
&\sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i*}^{(1)} \\
&= \sum_{c'\in\Omega_{x,N-1}(X)} p_\varepsilon(c')\sum_{e\in E_{t(c')}} p_\varepsilon(e)\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(c')\big)\cdot\Phi_0^{(\varepsilon)}\big(t(c')\big)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(e)\big)\Big)_i^{(1)} \\
&= \sum_{c'\in\Omega_{x,N-1}(X)} p_\varepsilon(c')\log\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(c')\big)\Big)\Big|_{X_i^{(1)}} + \varepsilon\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}} \\
&= N\varepsilon\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}} \qquad (x\in V,\ i=1,2,\dots,d_1).
\end{aligned} \tag{5.3.2}$$

Step 2. Next we estimate $\mathrm{Ord}_\varepsilon(0)$. Let us consider the coefficient of $X_{i*}^{(2)} f\big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\big)$.

It follows from (5.2.1) and (2.2.3) that

$$\begin{aligned}
&\frac{1}{N}\sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big\{\Big(\Phi_0^{(\varepsilon)}(x)^{-1}*\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)^{(2)}_{i*}\\
&\qquad -\frac{1}{2}\sum_{1\le\lambda<\nu\le d_1}\Big(\Phi_0^{(\varepsilon)}(x)^{-1}*\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)^{(1)}_{\lambda*}\Big(\Phi_0^{(\varepsilon)}(x)^{-1}*\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)^{(1)}_{\nu*}[\![X_\lambda^{(1)},X_\nu^{(1)}]\!]\big|_{X_i^{(2)}}\Big\}\\
&=\frac{1}{N}\sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\log\Big(\Phi_0^{(\varepsilon)}(x)^{-1}*\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)\Big|_{X_i^{(2)}}\\
&=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{c'\in\Omega_{x,k}(X)} p_\varepsilon(c')\sum_{e\in E_{t(c')}} p_\varepsilon(e)\log\big(d\Phi_0^{(\varepsilon)}(e)\big)\big|_{X_i^{(2)}} \qquad (x\in V).
\end{aligned}\tag{5.3.3}$$

Since the function

$$M_i^{(\varepsilon)}(x):=\sum_{e\in E_x} p_\varepsilon(e)\log\big(d\Phi_0^{(\varepsilon)}(e)\big)\big|_{X_i^{(2)}} \qquad (0\le\varepsilon\le1,\ i=1,2,\dots,d_2,\ x\in V)$$

satisfies $M_i^{(\varepsilon)}(\gamma x)=M_i^{(\varepsilon)}(x)$ for $\gamma\in\Gamma$ and $x\in V$ due to the Γ-invariance of p and the Γ-equivariance of Φ_0, there exists a function $\mathcal{M}_i^{(\varepsilon)}: V_0\longrightarrow\mathbb{R}$ such that $\mathcal{M}_i^{(\varepsilon)}\big(\pi(x)\big)=M_i^{(\varepsilon)}(x)$ for $0\le\varepsilon\le1$, $i=1,2,\dots,d_2$ and $x\in V$. Moreover, we have

$$L_{(\varepsilon)}^k\mathcal{M}_i^{(\varepsilon)}\big(\pi(x)\big)=L_{(\varepsilon)}^k M_i(x) \qquad (k\in\mathbb{N},\ 0\le\varepsilon\le1,\ i=1,2,\dots,d_2,\ x\in V)$$

by the Γ-invariance of p. We then find a sufficiently small $\varepsilon_0>0$ such that

$$\begin{aligned}
&\frac{1}{N}\sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big\{\Big(\Phi_0^{(\varepsilon)}(x)^{-1}*\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)^{(2)}_{i*}\\
&\qquad -\frac{1}{2}\sum_{1\le\lambda<\nu\le d_1}\Big(\Phi_0^{(\varepsilon)}(x)^{-1}*\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)^{(1)}_{\lambda*}\Big(\Phi_0^{(\varepsilon)}(x)^{-1}*\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)^{(1)}_{\nu*}[\![X_\lambda^{(1)},X_\nu^{(1)}]\!]\big|_{X_i^{(2)}}\Big\}\\
&=\frac{1}{N}\sum_{k=0}^{N-1}L_{(\varepsilon)}^k M_i^{(\varepsilon)}(x)\\
&=\frac{1}{N}\sum_{k=0}^{N-1}L_{(\varepsilon)}^k \mathcal{M}_i^{(\varepsilon)}\big(\pi(x)\big)\\
&=\sum_{x\in V_0} m(x)\mathcal{M}_i^{(\varepsilon)}(x)+O_{\varepsilon_0}\Big(\frac{1}{N}\Big)\\
&=\beta_{(\varepsilon)}(\Phi_0^{(\varepsilon)})\big|_{X_i^{(2)}}+O_{\varepsilon_0}\Big(\frac{1}{N}\Big) \qquad (0\le\varepsilon\le\varepsilon_0,\ i=1,2,\dots,d_2)
\end{aligned}$$

by applying the ergodic theorem (cf. [31, Theorem 3.4]) for the transition operator $L_{(\varepsilon)}$. Combining this calculation with Proposition 5.2.1 implies that the coefficient of $X_{i_*}^{(2)}f\big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\big)$ vanishes as $N\to\infty$ and $\varepsilon\searrow0$ with $N^2\varepsilon\searrow0$.

We also consider the coefficient of $X_{i*}^{(1)}X_{j*}^{(1)}f\big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\big)$. We have

$$\begin{aligned}
&\frac{1}{2N}\sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big(\Phi_0^{(\varepsilon)}(x)^{-1}*\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i*}^{(1)}\Big(\Phi_0^{(\varepsilon)}(x)^{-1}*\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{j*}^{(1)}\\
&=\frac{1}{2N}\Big\{\sum_{c'\in\Omega_{x,N-1(X)}} p_\varepsilon(c')\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(c')\big)\Big)_{i}^{(1)}\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(c')\big)\Big)_{j}^{(1)}\\
&\qquad+\sum_{e\in E_{t(c')}} p_\varepsilon(e)\log\big(d\Phi_0^{(\varepsilon)}(e)\big)\big|_{X_i^{(1)}}\log\big(d\Phi_0^{(\varepsilon)}(e)\big)\big|_{X_j^{(1)}}\\
&\qquad+2(N-1)\rho_{\mathbb{R}}(\gamma_{p_\varepsilon})\big|_{X_i^{(1)}}\rho_{\mathbb{R}}(\gamma_{p_\varepsilon})\big|_{X_j^{(1)}}\Big\}\\
&=\frac{1}{2N}\sum_{k=0}^{N-1}\sum_{c'\in\Omega_{x,k(X)}} p_\varepsilon(c')\sum_{e\in E_{t(c')}} p_\varepsilon(e)\log\big(d\Phi_0^{(\varepsilon)}(e)\big)\big|_{X_i^{(1)}}\log\big(d\Phi_0^{(\varepsilon)}(e)\big)\big|_{X_j^{(1)}}\\
&\qquad+\frac{1}{2}(N-1)\varepsilon^2\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}\rho_{\mathbb{R}}(\gamma_p)\big|_{X_j^{(1)}}\\
&=\frac{1}{2N}\sum_{k=0}^{N-1}L_{(\varepsilon)}^kN_{ij}^{(\varepsilon)}(x)+\frac{1}{2}(N-1)\varepsilon^2\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}\rho_{\mathbb{R}}(\gamma_p)\big|_{X_j^{(1)}}\quad(x\in V)
\end{aligned}\tag{5.3.4}$$

by using (5.2.1) and (2.2.4), where the function $N_{ij}^{(\varepsilon)}: V\longrightarrow\mathbb{R}$ is defined by

$$N_{ij}^{(\varepsilon)}(x):=\sum_{e\in E_x} p_\varepsilon(e)\log\big(d\Phi_0^{(\varepsilon)}(e)\big)\big|_{X_i^{(1)}}\log\big(d\Phi_0^{(\varepsilon)}(e)\big)\big|_{X_j^{(1)}}.$$

for $0\le\varepsilon\le1$, $i,j=1,2,\dots,d_1$ and $x\in V$. In the same argument as above, $N_{ij}^{(\varepsilon)}$ is Γ-invariant and there exists a function $\mathcal{N}_{ij}^{(\varepsilon)}: V_0\longrightarrow\mathbb{R}$ such that $\mathcal{N}_{ij}^{(\varepsilon)}\big(\pi(x)\big)=N_{ij}^{(\varepsilon)}(x)$ for $x\in V$. We also have

$$L_{(\varepsilon)}^k\mathcal{N}_{ij}^{(\varepsilon)}\big(\pi(x)\big)=L_{(\varepsilon)}^kN_{ij}^{(\varepsilon)}(x)\qquad(k\in\mathbb{N},\,0\le\varepsilon\le1,\,i,j=1,2,\dots,d_2,\,x\in V)$$

by the Γ-invariance of p. Thus, we choose a sufficiently small $\varepsilon_0'>0$ such that

$$\begin{aligned}
\frac{1}{2N}\sum_{k=0}^{N-1}L_{(\varepsilon)}^kN_{ij}^{(\varepsilon)}(x)&=\frac{1}{2N}\sum_{k=0}^{N-1}L_{(\varepsilon)}^k\mathcal{N}_{ij}^{(\varepsilon)}\big(\pi(x)\big)\\
&=\frac{1}{2}\sum_{x\in V_0}m(x)\big(\mathcal{N}(\Phi_0^{(\varepsilon)})_{ij}\big)(x)+O_{\varepsilon_0'}\Big(\frac{1}{N}\Big)\\
&=\frac{1}{2}\sum_{e\in E_0}\widetilde{m}_\varepsilon(e)\log\big(d\Phi_0^{(\varepsilon)}(e)\big)\big|_{X_i^{(1)}}\log\big(d\Phi_0^{(\varepsilon)}(e)\big)\big|_{X_j^{(1)}}\\
&\qquad+O_{\varepsilon_0'}\Big(\frac{1}{N}\Big)\qquad(0\le\varepsilon\le\varepsilon_0',\,i,j=1,2,\dots,d_1)
\end{aligned}\tag{5.3.5}$$

by the ergodic theorem. Recall that $\{V_1,V_2,\dots,V_{d_1}\}$ denotes the orthonormal basis in $(\mathfrak{g}^{(1)},g_0^{(0)})$. In particular, put $X_i^{(1)}=V_i$ for $i=1,2,\dots,d_1$ and let $\{\omega_1,\omega_2,\dots,\omega_{d_1}\}$ be the dual basis of $\{V_1, V_2, \dots, V_{d_1}\}$. Then we have

$$\begin{aligned}
&\frac{1}{2}\sum_{e\in E_0} \widetilde{m}_\varepsilon(e) \log\big(d\Phi_0^{(\varepsilon)}(\widetilde{e})\big)\big|_{V_i} \log\big(d\Phi_0^{(\varepsilon)}(\widetilde{e})\big)\big|_{V_j} \\
&= \frac{1}{2}\Big(\sum_{e\in E_0} \widetilde{m}_\varepsilon(e)\omega_i^{(\varepsilon)}(e)\omega_j^{(\varepsilon)}(e) - \langle \gamma_{p_\varepsilon}, \omega_i\rangle\langle \gamma_{p_\varepsilon}, \omega_j\rangle\Big) + \frac{1}{2}\varepsilon^2\langle \gamma_p, \omega_i\rangle\langle\gamma_p, \omega_j\rangle \\
&= \frac{1}{2}\langle\!\langle \omega_i^{(\varepsilon)}, \omega_j^{(\varepsilon)}\rangle\!\rangle_{(\varepsilon)} + \frac{1}{2}\varepsilon^2 \rho_{\mathbb{R}}(\gamma_p)\big|_{V_i}\rho_{\mathbb{R}}(\gamma_p)\big|_{V_j} \qquad (i,j=1,2,\dots,d_1). \qquad (5.3.6)
\end{aligned}$$

The coefficient of $X_{i*}^{(1)}X_{j*}^{(1)}f\big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\big)$ equals

$$-\Big(\frac{1}{2}\langle\!\langle \omega_i^{(\varepsilon)}, \omega_j^{(\varepsilon)}\rangle\!\rangle_{(\varepsilon)} + \frac{1}{2}N\varepsilon^2\rho_{\mathbb{R}}(\gamma_p)\big|_{V_i}\rho_{\mathbb{R}}(\gamma_p)\big|_{V_j}\Big) + O_{\varepsilon_0'}\Big(\frac{1}{N}\Big) \qquad (i,j=1,2,\dots,d_1) \quad (5.3.7)$$

by combining (5.3.4) with (5.3.5) and (5.3.6). Therefore, (5.3.7) and the continuity of $\langle\!\langle \cdot, \cdot\rangle\!\rangle_{(\varepsilon)}$ as $\varepsilon \searrow 0$ (cf. [31, Lemma 5.2]) imply

$$\mathrm{Ord}_\varepsilon(0) \longrightarrow -\frac{1}{2}\sum_{i=1}^{d_1} V_{i*}^2 f\Big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\Big) \qquad (5.3.8)$$

as $N\to\infty$ and $\varepsilon\searrow 0$ with $N^2\varepsilon \searrow 0$.

We finally discuss the estimate of $\sum_{k\geq 1}\mathrm{Ord}_\varepsilon(k)$. At the beginning, we show that the coefficient of $X_{i*}^{(k)}f\big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\big)$ vanishes as $N\to\infty$ and $\varepsilon\searrow 0$ with $N^2\varepsilon\searrow 0$. Thanks to

$$\Big|\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_i^{(k)}\Big| \leq CN^k \qquad (0\leq\varepsilon\leq 1,\ x\in V),$$

(5.2.1) and (2.2.7), we have

$$\begin{aligned}
&\frac{\varepsilon^{k-2}}{N}\sum_{x\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big(\Phi_0^{(\varepsilon)}(x)^{-1} * \Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i*}^{(k)} \\
&= \frac{\varepsilon^{k-2}}{N}\sum_{x\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big\{\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_i^{(k)} \\
&\qquad + \sum_{\substack{|K_1|+|K_2|\leq k-1\\ |K_2|>0}} C_{K_1,K_2}\mathcal{P}_*^{K_1}\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\Big)\mathcal{P}^{K_2}\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)\Big\} \\
&\leq CM_i^{(k)}\Big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\Big)\Big(\varepsilon^{k-2}N^{k-1} + \sum_{|K_1|\leq k-2}\varepsilon^{k-1-|K_1|} + \sum_{\substack{|K_1|+|K_2|\leq k-1\\ |K_2|\geq 2}}\varepsilon^{k-2-|K_1|}N^{|K_2|-1}\Big)
\end{aligned}$$

for $i=1,2,\dots,d_k$ and some continuous function $M_i^{(k)}: G\longrightarrow\mathbb{R}$. This converges to zero as $N\to\infty$ and $\varepsilon\searrow 0$ with $N^2\varepsilon\searrow 0$. We also observe that the coefficient of $X_{i_1*}^{(k_1)}X_{i_2*}^{(k_2)}f\big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\big)$ converges to zero as $N\to\infty$ and $\varepsilon\searrow 0$ with $N^2\varepsilon\searrow 0$ by following the same argument as above.

We also consider the coefficient of $(\partial^3 f/\partial g_{i_1*}^{(k_1)}\partial g_{i_2*}^{(k_2)}\partial g_{i_3*}^{(k_3)})(\theta)$. Since f is compactly supported, it is sufficient to show by induction on $k=1,2,\dots,r$ that, if $\varepsilon N<1$, then

$$\varepsilon^k\Big(\Phi_0^{(\varepsilon)}(x)^{-1}*\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i*}^{(k)} \le M_i^{(k)}\Big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)*\theta\big)\Big)\times\varepsilon N \tag{5.3.9}$$

for $i=1,2,\dots,d_k$ and some continuous function $M_i^{(k)}:G\longrightarrow\mathbb{R}$. The cases $k=1$ and $k=2$ are clear. Suppose that (5.3.9) holds for less than k. We have

$$\begin{aligned}\varepsilon^k\Big(\Phi_0^{(\varepsilon)}(x)^{-1}*\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i*}^{(k)} &= \varepsilon^k\bigg\{\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_i^{(k)} + \sum_{\substack{|K_1|+|K_2|\le k-1\\|K_2|>0}} C_{K_1,K_2}\\ &\qquad\times\mathcal{P}_*^{K_1}\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\Big)\mathcal{P}^{K_2}\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)\bigg\}.\end{aligned}$$

by using (2.2.2) and (2.2.7). Then we see that

$$\begin{aligned}\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\Big)_{i_1*}^{(k_1)} &= \Big(\theta*\big(\Phi_0^{(\varepsilon)}(x)*\theta\big)^{-1}\Big)_{i_1*}^{(k_1)}\\ &= \theta_{i_1*}^{(k_1)} + \Big(\big(\Phi_0^{(\varepsilon)}(x)*\theta\big)^{-1}\Big)_{i_1*}^{(k_1)}\\ &\qquad+\sum_{\substack{|L_1|+|L_2|=k_1\\|L_1|,|L_2|>0}} C_{L_1,L_2}\mathcal{P}_*^{L_1}(\theta)\mathcal{P}_*^{L_2}\Big(\big(\Phi_0^{(\varepsilon)}(x)*\theta\big)^{-1}\Big).\end{aligned}$$

Thus, we have inductively

$$\left|\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\Big)_{i_1*}^{(k_1)}\right| \le M\Big(\Phi_0^{(\varepsilon)}(x)*\theta\Big)$$

for a continuous function $M:G\longrightarrow\mathbb{R}$ and $k_1\le k-1$. We then conclude

$$\begin{aligned}&\varepsilon^k\Big(\Phi_0^{(\varepsilon)}(x)^{-1}*\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)_{i*}^{(k)}\\ &\le C\bigg(\varepsilon^k N^k + \sum_{\substack{|K_1|+|K_2|\le k-1\\|K_2|>0}} M\Big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)*\theta\big)\Big)\varepsilon^{k-|K_1|}N^{|K_2|}\bigg)\\ &\le M_i^{(k)}\Big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)*\theta\big)\Big)\times\varepsilon N.\end{aligned}$$

for some continuous function $M_i^{(k)}:G\longrightarrow\mathbb{R}$. These estimates implies that $\sum_{k\ge1}\mathrm{Ord}_\varepsilon(k)$ converges to zero as $N\to\infty$ and $\varepsilon\searrow 0$ with $N^2\varepsilon\searrow 0$.

Consequently, we obtain

$$\left\|\frac{1}{N\varepsilon^2}\big(I-L_{(\varepsilon)}^N\big)P_\varepsilon f(x) - P_\varepsilon\mathcal{A}f(x)\right\|_\infty^X \longrightarrow 0$$

as $N\to\infty$ and $\varepsilon\searrow 0$ with $N^2\varepsilon\searrow 0$ by combining (5.3.1) with (5.3.2) and (5.3.8). This completes the proof. ∎

Proof of Theorem 5.1.1. We basically follow the argument by Kotani [38, Theorem 4]. Let $N = N(n)$ be the integer satisfying $n^{1/5} \le N < n^{1/5} + 1$ and let k_N and r_N be the quotient and the remainder of $([nt] - [ns])/N(n)$, respectively. We put $\varepsilon_N := n^{-1/2}$ and $h_N := N\varepsilon_N^2$. Then we have $k_N h_N = \big([nt] - [ns] - r_N\big)\varepsilon_N^2 \to t - s \quad (n \to \infty)$.

Since $C_0^\infty(G_{(0)}) \subset \mathrm{Dom}(\mathcal{A}) \subset C_\infty(G_{(0)})$ and $C_0^\infty(G_{(0)})$ is dense in $C_\infty(G_{(0)})$, the linear operator $\mathcal{A}$ is densely defined in $C_\infty(G_{(0)})$. Furthermore, $(\lambda - \mathcal{A})\big(C_0^\infty(G_{(0)})\big)$ is dense in $C_\infty(G_{(0)})$ for some $\lambda > 0$ (cf. Robinson [64, p.304]). Hence, by combining Lemma 5.3.1 and Trotter's approximation theorem (cf. [74]), we obtain

$$\lim_{n\to\infty} \Big\| L_{(n^{-1/2})}^{Nk_N} P_{n^{-1/2}} f - P_{n^{-1/2}} \mathrm{e}^{-(t-s)\mathcal{A}} f \Big\|_\infty^X = 0 \qquad \big(f \in C_0^\infty(G_{(0)})\big). \tag{5.3.10}$$

On the other hand, Lemma 5.3.1 implies

$$\lim_{n\to\infty} \Big\| \frac{1}{r_N \varepsilon_N^2}\big(I - L_{(n^{-1/2})}^{r_N}\big) P_{n^{-1/2}} f - P_{n^{-1/2}} \mathcal{A} f \Big\|_\infty^X = 0 \qquad \big(f \in C_0^\infty(G_{(0)})\big). \tag{5.3.11}$$

Here we have

$$\begin{aligned}
&\Big\| L_{(n^{-1/2})}^{[nt]-[ns]} P_{n^{-1/2}} f - P_{n^{-1/2}} \mathrm{e}^{-(t-s)\mathcal{A}} f \Big\|_\infty^X \\
&\le \Big\| \big(I - L_{(n^{-1/2})}^{r_N}\big) P_{n^{-1/2}} f \Big\|_\infty^X + \Big\| L_{(n^{-1/2})}^{Nk_N} P_{n^{-1/2}} f - P_{n^{-1/2}} \mathrm{e}^{-(t-s)\mathcal{A}} f \Big\|_\infty^X.
\end{aligned} \tag{5.3.12}$$

It follows from $\|P_{n^{-1/2}}\| \le 1$ that

$$\begin{aligned}
&\Big\| \big(I - L_{(n^{-1/2})}^{r_N}\big) P_{n^{-1/2}} f \Big\|_\infty^X \\
&\le r_N \varepsilon_N^2 \Big\| \frac{1}{r_N \varepsilon_N^2}\big(I - L_{(n^{-1/2})}^{r_N}\big) P_{n^{-1/2}} f - P_{n^{-1/2}} \mathcal{A} f \Big\|_\infty^X + r_N \varepsilon_N^2 \big\| P_{n^{-1/2}} \mathcal{A} f \big\|_\infty^X \\
&\le r_N \varepsilon_N^2 \Big\| \frac{1}{r_N \varepsilon_N^2}\big(I - L_{(n^{-1/2})}^{r_N}\big) P_{n^{-1/2}} f - P_{n^{-1/2}} \mathcal{A} f \Big\|_\infty^X + r_N \varepsilon_N^2 \big\| \mathcal{A} f \big\|_\infty^G.
\end{aligned} \tag{5.3.13}$$

Then, we obtain (5.1.3) for $f \in C_0^\infty(G_{(0)})$ by combining (5.3.10), (5.3.11), (5.3.12) and (5.3.13) with $r_N \varepsilon_N^2 \to 0 \, (n \to \infty)$. For $f \in C_\infty(G_{(0)})$, we also obtain (5.1.3) by following the same argument as [31, Theorem 2.1]. we complete the proof of Theorem 5.1.1. ∎

5.4 Proof of Theorem 5.1.2

In what follows, we assume **(A2)** as well as **(A1)**. Put

$$\|d\Phi_0^{(\varepsilon)}\|_\infty = \max_{e \in E_0} \max_{k=1,2,\dots,r} \big\| \log\big(d\Phi_0^{(\varepsilon)}(\widetilde{e})\big)\big|_{\mathfrak{g}^{(k)}} \big\|_{\mathfrak{g}^{(k)}}^{1/k} \qquad (0 \le \varepsilon \le 1).$$

We describe a relation between $\|d\Phi_0^{(\varepsilon)}\|_\infty$ and $\|d\Phi_0^{(0)}\|_\infty$ for every $0 \le \varepsilon \le 1$. Thanks to the identity

$$d\Phi_0^{(\varepsilon)}(e) = \Psi^{(\varepsilon)}\big(o(e)\big) \cdot d\Phi_0^{(0)}(e) \cdot \Psi^{(\varepsilon)}\big(t(e)\big)^{-1} \qquad (0 \le \varepsilon \le 1,\, e \in E),$$

[31, Lemma 5.3 (3)] and **(A2)**, we obtain the following:

Lemma 5.4.1 *Under* **(A2)***, there exists a positive constant* C *such that*

$$\sup_{0\le\varepsilon\le1}\|d\Phi_0^{(\varepsilon)}\|_\infty \le C\|d\Phi_0^{(0)}\|_\infty.$$

We denote by $(G_{(0)}^{(k)},\cdot)$ and $(G_{(0)}^{(k)},*)$ the connected and simply connected nilpotent Lie group of step k and the corresponding limit group whose Lie algebras are

$$\big((\mathfrak{g}^{(1)},g_0^{(0)})\oplus\mathfrak{g}^{(2)}\oplus\cdots\oplus\mathfrak{g}^{(k)},[\cdot,\cdot]\big),\quad \big((\mathfrak{g}^{(1)},g_0^{(0)})\oplus\mathfrak{g}^{(2)}\oplus\cdots\oplus\mathfrak{g}^{(k)},[\![\cdot,\cdot]\!]\big),$$

respectively. For the piecewise smooth stochastic process $(\mathcal{Y}_t^{(\varepsilon,n)})_{0\le t\le1}$, we define its truncated process by

$$\mathcal{Y}_t^{(\varepsilon,n;\,k)}=\big(\mathcal{Y}_t^{(\varepsilon,n),1},\mathcal{Y}_t^{(\varepsilon,n),2},\dots,\mathcal{Y}_t^{(\varepsilon,n),k}\big)\in G_{(0)}^{(k)}\qquad(k=1,2,\dots,r)$$

in the $(\cdot)$-coordinate system. We may put

$$\sup_{0\le\varepsilon\le1}\Big\{\|d\Phi_0^{(\varepsilon)}\|_\infty+\|\rho_{\mathbb{R}}(\gamma_p)\|_{\mathfrak{g}^{(1)}}\Big\}\le C\|d\Phi_0^{(0)}\|_\infty+\|\rho_{\mathbb{R}}(\gamma_p)\|_{\mathfrak{g}^{(1)}}=:M,$$

by recalling Lemma 5.4.1.

As is well-known in probability theory, it suffices to show the tightness of $\{\mathbf{P}^{(n^{-1/2},n)}\}_{n=1}^\infty$ and the convergence of the finite dimensional distribution of $\{\mathcal{Y}_\cdot^{(n^{-1/2},n)}\}_{n=1}^\infty$ to obtain Theorem 5.1.2. In the former part of this section, we aim to show the following.

Lemma 5.4.2 $\{\mathbf{P}^{(n^{-1/2},n)}\}_{n=1}^\infty$ *is tight in* $C^{0,\alpha\text{-Höl}}([0,1];G_{(0)})$*, where* $\alpha<1/2$.

As the first step of the proof of Lemma 5.4.2, we prepare the following lemma.

Lemma 5.4.3 *Let* m,n *be positive integers. Then there exists a constant* $C>0$ *independent of* n *(however, it may depend on* m*) such that*

$$\mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\Big[d_{\mathrm{CC}}(\mathcal{Y}_s^{(n^{-1/2},n;\,2)},\mathcal{Y}_t^{(n^{-1/2},n;\,2)})^{4m}\Big]\le C(t-s)^{2m}\qquad(0\le s\le t\le1).\tag{5.4.1}$$

Proof. Our argument is partially based on Bayer–Friz [6, Proposition 4.3]. We split the proof into several steps.

Step 1. First we show

$$\mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\Big[d_{\mathrm{CC}}\big(\mathcal{Y}_{t_k}^{(n^{-1/2},n;\,2)},\mathcal{Y}_{t_\ell}^{(n^{-1/2},n;\,2)}\big)^{4m}\Big]\le C\Big(\frac{\ell-k}{n}\Big)^{2m}$$
$$\big(n,m\in\mathbb{N},\ t_k,t_\ell\in\mathcal{D}_n\,(k\le\ell)\big)\tag{5.4.2}$$

for some $C>0$ which is independent of n (depending on m). By noting the equivalence of two homogeneous norms $\|\cdot\|_{\mathrm{CC}}$ and $\|\cdot\|_{\mathrm{Hom}}$ (cf. [32, Proposition 3.1]), we know that

(5.4.2) is equivalent to the existence of positive constants $C^{(1)}$ and $C^{(2)}$ independent of n such that

$$\mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\Big[\big\|\log\big((\mathcal{Y}_{t_k}^{(n^{-1/2},n)})^{-1}\cdot\mathcal{Y}_{t_\ell}^{(n^{-1/2},n)}\big)\big|_{\mathfrak{g}^{(1)}}\big\|_{\mathfrak{g}^{(1)}}^{4m}\Big]\le C^{(1)}\Big(\frac{\ell-k}{n}\Big)^{2m}, \tag{5.4.3}$$

$$\mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\Big[\big\|\log\big((\mathcal{Y}_{t_k}^{(n^{-1/2},n)})^{-1}\cdot\mathcal{Y}_{t_\ell}^{(n^{-1/2},n)}\big)\big|_{\mathfrak{g}^{(2)}}\big\|_{\mathfrak{g}^{(2)}}^{2m}\Big]\le C^{(2)}\Big(\frac{\ell-k}{n}\Big)^{2m}. \tag{5.4.4}$$

Step 2. We here prove (5.4.3). We have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_{x_*}^{(\varepsilon)}}\Big[\big\|\log\big((\mathcal{Y}_{t_k}^{(\varepsilon,n)})^{-1}\cdot\mathcal{Y}_{t_\ell}^{(\varepsilon,n)}\big)\big|_{\mathfrak{g}^{(1)}}\big\|_{\mathfrak{g}^{(1)}}^{4m}\Big]\\
&=\Big(\frac{1}{\sqrt{n}}\Big)^{4m}\mathbb{E}^{\mathbb{P}_{x_*}^{(\varepsilon)}}\Big[\Big(\sum_{i=1}^{d_1}\log\big((\xi_k^{(\varepsilon)})^{-1}\cdot\xi_\ell^{(\varepsilon)}\big)\big|_{X_i^{(1)}}^2\Big)^{2m}\Big]\\
&\le\Big(\frac{1}{\sqrt{n}}\Big)^{4m}\cdot d_1^{2m}\max_{i=1,2,\dots,d_1}\max_{x\in\mathcal{F}}\Big\{\sum_{c\in\Omega_{x,\ell-k}(X)}p_\varepsilon(c)\\
&\qquad\times\log\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)\Big|_{X_i^{(1)}}^{4m}\Big\}\qquad(0\le\varepsilon\le1),
\end{aligned}\tag{5.4.5}$$

where $\mathcal{F}$ stands for the fundamental domain in X containing the reference point $x_*\in V$. For $i=1,2,\dots,d_1$, $x\in\mathcal{F}$, $N\in\mathbb{N}$, $0\le\varepsilon\le1$ and $c=(e_1,e_2,\dots,e_N)\in\Omega_{x_*,N}(X)$, put

$$\mathcal{J}_i^{(\varepsilon)}(j):=\log\big(d\Phi_0^{(\varepsilon)}(e_j)\big)\big|_{X_i^{(1)}}-\varepsilon\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}$$

and

$$\mathcal{N}_N^{(i,x)}(\Phi_0^{(\varepsilon)};c):=\log\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)\Big|_{X_i^{(1)}}-N\rho_{\mathbb{R}}(\gamma_{p_\varepsilon})\big|_{X_i^{(1)}}=\sum_{j=1}^{N}\mathcal{J}_i^{(\varepsilon)}(j).$$

We note that

$$|\mathcal{J}_i^{(\varepsilon)}(j)|\le\|d\Phi_0^{(\varepsilon)}\|_\infty+\|\rho_{\mathbb{R}}(\gamma_p)\|_{\mathfrak{g}^{(1)}}\le M\qquad(0\le\varepsilon\le1,\ i=1,2,\dots,d_1,j=1,2,\dots,N).$$

Then we know that $\{\mathcal{N}_N^{(i,x)}\}_{N=1}^\infty$ is a martingale for every $i=1,2,\dots,d_1$ and $x\in\mathcal{F}$ (see Lemma 2.5.3). Hence, we apply the Burkholder–Davis–Gundy inequality with the exponent $4m$. By the elementary inequality $(a+b)^{4m}\le2^{4m-1}(a^{4m}+b^{4m})$ for $m\in\mathbb{N}$, we have

$$\begin{aligned}
&\sum_{c\in\Omega_{x,N}(X)}p_\varepsilon(c)\log\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)\Big|_{X_i^{(1)}}^{4m}\\
&\le2^{4m-1}\sum_{c\in\Omega_{x,N}(X)}p_\varepsilon(c)\Big\{\big(\mathcal{N}_N^{(i,x)}(c)\big)^{4m}+\Big(N\varepsilon\rho_{\mathbb{R}}(\gamma_p)\big|_{X_i^{(1)}}\Big)^{4m}\Big\}\\
&\le2^{4m-1}\mathcal{C}_{(4m)}^{4m}\sum_{c\in\Omega_{x,N}(X)}p_\varepsilon(c)\Big\{\sum_{j=1}^{N}\mathcal{J}_i^{(\varepsilon)}(j)^2\Big\}^{2m}+2^{4m-1}\varepsilon^{4m}N^{4m}\big\|\rho_{\mathbb{R}}(\gamma_p)\big\|_{\mathfrak{g}^{(1)}}^{4m}\\
&\le2^{4m}\mathcal{C}_{(4m)}^{4m}M^{2m}N^{2m}+2^{4m-1}M^{4m}\varepsilon^{4m}N^{4m}\\
&\qquad\qquad(x\in\mathcal{F},\ i=1,2,\dots,d_1,\ 0\le\varepsilon\le1,\ N\in\mathbb{N}).
\end{aligned}\tag{5.4.6}$$

In particular, (5.4.6) implies

$$\begin{aligned}&\sum_{c\in\Omega_{x,\ell-k}(X)} p_{n^{-1/2}}(c)\log\Big(\Phi_0^{(n^{-1/2})}(x)^{-1}\cdot\Phi_0^{(n^{-1/2})}\big(t(c)\big)\Big)\Big|_{X_i^{(1)}}^{4m}\\ &\le\Big\{2^{4m}\mathcal{C}_{(4m)}^{4m}M^{2m}+2^{4m-1}M^{4m}\Big\}(\ell-k)^{2m} \qquad (5.4.7)\end{aligned}$$

by putting $\varepsilon=n^{-1/2}$ and $N=\ell-k$, where we should note that $(\ell-k)/n<1$ since $1\le k\le\ell\le n$. We then obtain

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\Big[\big\|\log\big((\widetilde{\mathcal{Y}}_{t_k}^{(n^{-1/2},n)})^{-1}\cdot\widetilde{\mathcal{Y}}_{t_\ell}^{(n^{-1/2},n)}\big)\big|_{\mathfrak{g}^{(1)}}\big\|_{\mathfrak{g}^{(1)}}^{4m}\Big]\\ &\le d_1^{2m}\Big\{2^{4m}\mathcal{C}_{(4m)}^{4m}M^{2m}+2^{4m-1}M^{4m}\Big\}\Big(\frac{\ell-k}{n}\Big)^{2m}=:C^{(1)}\Big(\frac{\ell-k}{n}\Big)^{2m}\end{aligned}$$

by combining (5.4.5) with (5.4.7), which leads to (5.4.3).

Step 3. We show (5.4.4) at this step. We also see

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}_{x_*}^{(\varepsilon)}}\Big[\big\|\log\big((\mathcal{Y}_{t_k}^{(\varepsilon,n)})^{-1}\cdot\mathcal{Y}_{t_\ell}^{(\varepsilon,n)}\big)\big|_{\mathfrak{g}^{(2)}}\big\|_{\mathfrak{g}^{(2)}}^{2m}\Big]\\ &\le\Big(\frac{1}{n}\Big)^{2m}\cdot d_2^{2m}\max_{i=1,2,\dots,d_2}\max_{x\in\mathcal{F}}\Big\{\sum_{c\in\Omega_{x,\ell-k}(X)}p_\varepsilon(c)\\ &\qquad\times\log\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)\Big|_{X_i^{(2)}}^{2m}\Big\}\qquad(0\le\varepsilon\le1). \qquad (5.4.8)\end{aligned}$$

in the similar way to (5.4.5). Then it follows from (2.2.2) that

$$\begin{aligned}&\log\Big(\Phi_0^{(\varepsilon)}(x)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(c)\big)\Big)\Big|_{X_i^{(2)}}^{2m}\\ &=\log\Big(\Phi_0^{(\varepsilon)}\big(o(e_1)\big)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(e_1)\big)\cdot\dots\cdot\Phi_0^{(\varepsilon)}\big(o(e_{\ell-k})\big)^{-1}\cdot\Phi_0^{(\varepsilon)}\big(t(e_{\ell-k})\big)\Big)\Big|_{X_i^{(2)}}^{2m}\\ &=\Big(\sum_{j=1}^{\ell-k}\log\big(d\Phi_0^{(\varepsilon)}(e_j)\big)\big|_{X_i^{(2)}}-\frac{1}{2}\sum_{1\le j_1<j_2\le\ell-k}\sum_{1\le\lambda<\nu\le d_1}[\![X_\lambda^{(1)},X_\nu^{(1)}]\!]\big|_{X_i^{(2)}}\\ &\quad\times\Big\{\log\big(d\Phi_0^{(\varepsilon)}(e_{j_1})\big)\big|_{X_\lambda^{(1)}}\log\big(d\Phi_0^{(\varepsilon)}(e_{j_2})\big)\big|_{X_\nu^{(1)}}\\ &\qquad-\log\big(d\Phi_0^{(\varepsilon)}(e_{j_1})\big)\big|_{X_\nu^{(1)}}\log\big(d\Phi_0^{(\varepsilon)}(e_{j_2})\big)\big|_{X_\lambda^{(1)}}\Big\}\Big)^{2m}\\ &\le3^{2m-1}\Big\{\Big(\sum_{j=1}^{\ell-k}\log\big(d\Phi_0^{(\varepsilon)}(e_j)\big)\big|_{X_i^{(2)}}\Big)^{2m}\\ &\quad+L\max_{1\le\lambda<\nu\le d_1}\Big(\sum_{1\le j_1<j_2\le\ell-k}\log\big(d\Phi_0^{(\varepsilon)}(e_{j_1})\big)\big|_{X_\lambda^{(1)}}\log\big(d\Phi_0^{(\varepsilon)}(e_{j_2})\big)\big|_{X_\nu^{(1)}}\Big)^{2m}\\ &\quad+L\max_{1\le\lambda<\nu\le d_1}\Big(\sum_{1\le j_1<j_2\le\ell-k}\log\big(d\Phi_0^{(\varepsilon)}(e_{j_1})\big)\big|_{X_\nu^{(1)}}\log\big(d\Phi_0^{(\varepsilon)}(e_{j_2})\big)\big|_{X_\lambda^{(1)}}\Big)^{2m}\Big\}, \qquad (5.4.9)\end{aligned}$$

where we put

$$L := \frac{1}{2} \max_{i=1,2,\dots,d_2} \max_{1 \le \lambda < \nu \le d_1} \Big| [\![X_\lambda^{(1)}, X_\nu^{(1)}]\!] \big|_{X_i^{(2)}} \Big|.$$

We fix $i = 1, 2, \dots, d_2$. Then we have

$$\begin{aligned}
\Big(\sum_{j=1}^{\ell-k} \log \big(d\Phi_0^{(\varepsilon)}(e_j) \big) \big|_{X_i^{(2)}} \Big)^{2m} &= (\ell - k)^{2m} \Big(\sum_{j=1}^{\ell-k} \frac{1}{\ell - k} \log \big(d\Phi_0^{(\varepsilon)}(e_j) \big) \big|_{X_i^{(2)}} \Big)^{2m} \\
&\le (\ell - k)^{2m} \sum_{j=1}^{\ell-k} \frac{1}{\ell - k} \log \big(d\Phi_0^{(\varepsilon)}(e_j) \big) \big|_{X_i^{(2)}}^{2m} \\
&\le \| d\Phi_0^{(\varepsilon)} \|_\infty^{4m} (\ell - k)^{2m} \le M^{4m} (\ell - k)^{2m}. \qquad (5.4.10)
\end{aligned}$$

by applying the Jensen inequality. For $1 \le \lambda < \nu \le d_1$, $x \in \mathcal{F}$, $0 \le \varepsilon \le 1$, $N \in \mathbb{N}$ and $c = (e_1, e_2, \dots, e_N) \in \Omega_{x,N}(X)$, we set

$$\widetilde{\mathcal{N}}_N^{(\lambda,\nu,x)}(\Phi_0^{(\varepsilon)}; c) := \sum_{1 \le j_1 < j_2 \le N} \mathcal{J}_\lambda^{(\varepsilon)}(j_1) \mathcal{J}_\nu^{(\varepsilon)}(j_2) = \sum_{j_2=2}^{N} \mathcal{J}_\nu^{(\varepsilon)}(j_2) \sum_{j_1=1}^{j_2-1} \mathcal{J}_\lambda^{(\varepsilon)}(j_1).$$

Then we also see that $\{\widetilde{\mathcal{N}}_N^{(\lambda,\nu,x)}\}_{N=1}^\infty$ is an $\mathbb{R}$-valued martingale for every $1 \le \lambda < \nu \le d$ and $x \in \mathcal{F}$. By applying the Burkholder–Davis–Gundy inequality with the exponent $2m$, we have

$$\begin{aligned}
&\sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c) \big(\widetilde{\mathcal{N}}_N^{(\lambda,\nu,x)}(c) \big)^{2m} \\
&\le \mathcal{C}_{(2m)}^{2m} \sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c) \Big\{ \sum_{j_2=2}^{N} \mathcal{J}_\nu^{(\varepsilon)}(j_2)^2 \times \Big(\sum_{j_1=1}^{j_2-1} \mathcal{J}_\lambda^{(\varepsilon)}(j_1) \Big)^2 \Big\}^m \\
&\le \mathcal{C}_{(2m)}^{2m} \sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c) (N-1)^m \sum_{j_2=2}^{N} \frac{1}{N-1} \mathcal{J}_\nu^{(\varepsilon)}(j_2)^{2m} \Big(\sum_{j_1=1}^{j_2-1} \mathcal{J}_\lambda^{(\varepsilon)}(j_1) \Big)^{2m} \\
&\le \mathcal{C}_{(2m)}^{2m} N^m \sum_{j_2=2}^{N} \frac{1}{N-1} \Big(\sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c) \mathcal{J}_\nu^{(\varepsilon)}(j_2)^{4m} \Big)^{1/2} \\
&\qquad \times \Big\{ \sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c) \Big(\sum_{j_1=1}^{j_2-1} \mathcal{J}_\lambda^{(\varepsilon)}(j_1) \Big)^{4m} \Big\}^{1/2} \\
&\le \mathcal{C}_{(2m)}^{2m} M^{2m} N^m \sum_{j_2=2}^{N} \frac{1}{N-1} \Big\{ \sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c) \Big(\sum_{j_1=1}^{j_2-1} \mathcal{J}_\lambda^{(\varepsilon)}(j_1) \Big)^{4m} \Big\}^{1/2}, \qquad (5.4.11)
\end{aligned}$$

where we used Jensen's inequality for the third line and Schwarz' inequality for the final line. Then the again use of the Burkholder–Davis–Gundy inequality with the exponent

$4m$ gives

$$
\begin{aligned}
&\sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big(\sum_{j_1=1}^{j_2-1}\mathcal{J}_\lambda^{(\varepsilon)}(j_1)\Big)^{4m}\\
&\le \mathcal{C}_{(4m)}^{4m}\sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big(\sum_{j_1=1}^{j_2-1}\mathcal{J}_\lambda^{(\varepsilon)}(j_1)^2\Big)^{2m}\\
&= \mathcal{C}_{(4m)}^{4m}(j_2-1)^{2m}\sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big(\sum_{j_1=1}^{j_2-1}\frac{1}{j_2-1}\mathcal{J}_\lambda^{(\varepsilon)}(j_1)^2\Big)^{2m}\\
&\le \mathcal{C}_{(4m)}^{4m}j_2^{2m}\sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\sum_{j_1=1}^{j_2-1}\frac{1}{j_2-1}\mathcal{J}_\lambda^{(\varepsilon)}(j_1)^{4m} \le \mathcal{C}_{(4m)}^{4m}M^{4m}j_2^{2m}. \qquad (5.4.12)
\end{aligned}
$$

It follows from (5.4.11) and (5.4.12) that

$$
\begin{aligned}
\sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\big(\widetilde{\mathcal{N}}_N^{(\lambda,\nu,x)}(c)\big)^{2m} &\le \mathcal{C}_{(2m)}^{2m}M^{2m}N^m\sum_{j_2=2}^{N}\frac{1}{N-1}(\mathcal{C}_{(4m)}^{4m}M^{4m}j_2^{2m})^{1/2}\\
&\le \mathcal{C}_{(2m)}^{2m}\mathcal{C}_{(4m)}^{2m}M^{4m}N^{2m}. \qquad (5.4.13)
\end{aligned}
$$

Hence, (5.4.13) implies

$$
\begin{aligned}
&\sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big(\sum_{1\le j_1<j_2\le N}\log\big(d\Phi_0^{(\varepsilon)}(e_{j_1})\big)\big|_{X_\lambda^{(1)}}\log\big(d\Phi_0^{(\varepsilon)}(e_{j_2})\big)\big|_{X_\nu^{(1)}}\Big)^{2m}\\
&\le 4^{2m-1}\sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big\{\big(\widetilde{\mathcal{N}}_N^{(\lambda,\nu,x)}(c)\big)^{2m}+\Big(\varepsilon^2\rho_{\mathbb{R}}(\gamma_p)\big|_{X_\lambda^{(1)}}\rho_{\mathbb{R}}(\gamma_p)\big|_{X_\nu^{(1)}}\cdot\frac{N(N-1)}{2}\Big)^{2m}\\
&\quad+\Big(\varepsilon\rho_{\mathbb{R}}(\gamma_p)\big|_{X_\nu^{(1)}}\sum_{1\le j_1<j_2\le N}\mathcal{J}_\lambda^{(\varepsilon)}(j_1)\Big)^{2m}+\Big(\varepsilon\rho_{\mathbb{R}}(\gamma_p)\big|_{X_\lambda^{(1)}}\sum_{1\le j_1<j_2\le N}\mathcal{J}_\nu^{(\varepsilon)}(j_2)\Big)^{2m}\Big\}\\
&\le 4^{2m-1}\Big\{\mathcal{C}_{(2m)}^{2m}\mathcal{C}_{(4m)}^{2m}M^{4m}N^{2m}+2^{-2m}M^{4m}\varepsilon^{4m}N^{4m}\\
&\qquad+2M^{2m}\varepsilon^{2m}N^{2m}\max_{1\le i\le d_1}\sum_{c\in\Omega_{x,N}(X)} p_\varepsilon(c)\Big(\sum_{j=1}^{N}\mathcal{J}_i^{(\varepsilon)}(j)\Big)^{2m}\Big\}\\
&\le 4^{2m-1}\Big\{\mathcal{C}_{(2m)}^{2m}\mathcal{C}_{(4m)}^{2m}M^{4m}N^{2m}+2^{-2m}M^{4m}\varepsilon^{4m}N^{4m}\\
&\qquad+2M^{2m}\varepsilon^{2m}N^{2m}\Big(2^{2m}\mathcal{C}_{(2m)}^{2m}M^mN^m+2^{2m-1}M^{2m}\varepsilon^{2m}N^{2m}\Big)\Big\}, \qquad (5.4.14)
\end{aligned}
$$

where we used (5.4.6) for the final line.

We now put $\varepsilon=n^{-1/2}$ and $N=\ell-k$. Then we have, for $1\le\lambda<\nu\le d_1$,

$$
\begin{aligned}
&\sum_{c\in\Omega_{x,\ell-k}(X)} p_\varepsilon(c)\Big(\sum_{1\le j_1<j_2\le \ell-k}\log\big(d\Phi_0^{(\varepsilon)}(e_{j_1})\big)\big|_{X_\lambda^{(1)}}\log\big(d\Phi_0^{(\varepsilon)}(e_{j_2})\big)\big|_{X_\nu^{(1)}}\Big)^{2m}\\
&\le 4^{2m-1}M^{4m}\Big(\mathcal{C}_{(2m)}^{2m}\mathcal{C}_{(2m)}^{4m}+2^{-2m}+2^{2m+1}\mathcal{C}_{(2m)}^{2m}M^{-m}+2^{2m}\Big)(\ell-k)^{2m} \qquad (5.4.15)
\end{aligned}
$$

due to (5.4.14) and $(\ell-k)/n<1$. We obtain

$$\mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\Big[\big\|\log\big((\widetilde{\mathcal{Y}}_{t_k}^{(n^{-1/2},n)})^{-1}\cdot\widetilde{\mathcal{Y}}_{t_\ell}^{(n^{-1/2},n)}\big)\big|_{\mathfrak{g}^{(2)}}\big\|_{\mathfrak{g}^{(2)}}^{2m}\Big]\le C^{(2)}\Big(\frac{\ell-k}{n}\Big)^{2m}.$$

by combining (5.4.8) with (5.4.9), (5.4.10) and (5.4.15), where

$$C^{(2)}:=d_2^{2m}3^{2m-1}\Big\{M^{4m}+2L\cdot 4^{2m-1}M^{4m}\Big(\mathcal{C}_{(2m)}^{2m}\mathcal{C}_{(2m)}^{4m}+2^{-2m}+2^{2m+1}\mathcal{C}_{(2m)}^{2m}M^{-m}+2^{2m}\Big)\Big\}.$$

This means (5.4.4) and we thus obtain (5.4.2).

Step 4. We show (5.4.1) at the last step. Suppose that $t_k\le s\le t_{k+1}$ and $t_\ell\le t\le t_{\ell+1}$ for some $1\le k\le\ell\le n$. Then we have

$$\begin{aligned}
d_{\mathrm{CC}}(\mathcal{Y}_s^{(n^{-1/2},n;\,2)},\mathcal{Y}_{t_{k+1}}^{(n^{-1/2},n;\,2)})&=(k-ns)d_{\mathrm{CC}}(\mathcal{Y}_{t_k}^{(n^{-1/2},n;\,2)},\mathcal{Y}_{t_{k+1}}^{(n^{-1/2},n;\,2)}),\\
d_{\mathrm{CC}}(\mathcal{Y}_{t_\ell}^{(n^{-1/2},n;\,2)},\mathcal{Y}_t^{(n^{-1/2},n;\,2)})&=(nt-\ell)d_{\mathrm{CC}}(\mathcal{Y}_{t_\ell}^{(n^{-1/2},n;\,2)},\mathcal{Y}_{t_{\ell+1}}^{(n^{-1/2},n;\,2)})
\end{aligned}$$

by noting that the piecewise smooth stochastic process $\mathcal{Y}_\cdot^{(n^{-1/2},n)}$ is given by the d_{CC}-geodesic interpolation. Hence, (5.4.2) and the triangle inequality yield

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\Big[d_{\mathrm{CC}}(\mathcal{Y}_s^{(n^{-1/2},n;\,2)},\mathcal{Y}_t^{(n^{-1/2},n;\,2)})^{4m}\Big]\\
&\le 3^{4m-1}\Big\{(k+1-ns)^{4m}\cdot C\Big(\frac{1}{n}\Big)^{2m}+C\Big(\frac{\ell-k-1}{n}\Big)^{2m}+(nt-\ell)^{4m}\cdot C\Big(\frac{1}{n}\Big)^{2m}\Big\}\\
&\le C\Big\{(t_{k+1}-s)^{2m}+(t_\ell-t_{k+1})^{2m}+(t-t_\ell)^{2m}\Big\}\le C(t-s)^{2m}.
\end{aligned}$$

This completes the proof of Lemma 5.4.3. ∎

In what follows, we write

$$d\mathcal{Y}_{s,t}^{(\varepsilon,n)*}:=(\mathcal{Y}_s^{(\varepsilon,n)})^{-1}*\mathcal{Y}_t^{(\varepsilon,n)}\qquad(0\le\varepsilon\le1,\ n\in\mathbb{N},\ 0\le s\le t\le1)$$

for brevity. We now show the following lemma by using Lemma 5.4.3.

Lemma 5.4.4 *For $m,n\in\mathbb{N}$, $k=1,2,\dots,r$ and $\alpha<\frac{2m-1}{4m}$, there exist an $\mathcal{F}_\infty$-measurable set $\Omega_k^{(n)}\subset\Omega_{x_*}(X)$, a non-negative random variable $\mathcal{K}_k^{(n)}\in L^{4m}(\Omega_{x_*}(X)\to\mathbb{R};\mathbb{P}_{x_*}^{(n^{-1/2})})$ such that $\mathbb{P}_{x_*}^{(n^{-1/2})}(\Omega_k^{(n)})=1$ and*

$$d_{\mathrm{CC}}\big(\mathcal{Y}_s^{(n^{-1/2},n;\,k)}(c),\mathcal{Y}_t^{(n^{-1/2},n;\,k)}(c)\big)\le\mathcal{K}_k^{(n)}(c)(t-s)^\alpha\quad(c\in\Omega_k^{(n)},\ 0\le s\le t\le1).\ (5.4.16)$$

Proof. As in the proof of Lemma 4.3.3, we partially apply Lyons' original proof (cf. [54, Theorem 2.2.1]) for the extension theorem in rough path theory to the proof. We prove (4.3.15) by induction on the step number $k=1,2,\dots,r$.

Step 1. In the cases $k = 1, 2$, we have already obtained (5.4.16) in Lemma 5.4.3. In fact, (5.4.16) for $k = 1, 2$ are obtained by a simple application of the Kolmogorov–Chentsov criterion with the bound

$$\|\mathcal{K}_k^{(n)}\|_{L^{4m}(\mathbb{P}_{x_*}^{(n^{-1/2})})} \le \frac{5C}{(1-2^{-\theta})(1-2^{\alpha-\theta})} \qquad (n, m \in \mathbb{N},\ k = 1, 2), \tag{5.4.17}$$

where $\theta = (2m-1)/4m$ and C is a constant independent of n which appears in the right-hand side of (5.4.1).

Step 2. We now fix $n \in \mathbb{N}$. Assume that (5.4.16) holds up to step k. We note that this assumption is equivalent to the existences of measurable sets $\{\widehat{\Omega}_j^{(n)}\}_{j=1}^k$ and non-negative random variables $\{\widehat{\mathcal{K}}_j^{(n)}\}_{j=1}^k$ such that $\mathbb{P}_{x_*}^{(n^{-1/2})}(\widehat{\Omega}_j^{(n)}) = 1$ and

$$\big\|\big(d\mathcal{Y}_{s,t}^{(n^{-1/2},n)*}(c)\big)^{(j)}\big\|_{\mathbb{R}^{d_j}} \le \widehat{\mathcal{K}}_j^{(n)}(c)(t-s)^{j\alpha} \qquad (c \in \widehat{\Omega}_j^{(n)},\ 0 \le s \le t \le 1) \tag{5.4.18}$$

with $\widehat{\mathcal{K}}_j^{(n)} \in L^{4m/j}(\Omega_{x_*}(X) \to \mathbb{R}; \mathbb{P}_{x_*}^{(n^{-1/2})})$ for $n, m \in \mathbb{N}$ and $j = 1, 2, \dots, k$.

We fix $0 \le s \le t \le 1$, $n \in \mathbb{N}$ and write $\widehat{\Omega}_{k+1}^{(n)} = \bigcap_{j=1}^k \widehat{\Omega}_j^{(n)}$. We denote by Δ the partition $\{s = t_0 < t_1 < \cdots < t_N = t\}$ of the time interval $[s, t]$ independent of $n \in \mathbb{N}$. We now define two $G_{(0)}^{(k+1)}$-valued random variables $\mathcal{Z}_{s,t}^{(n)}$ and $\mathcal{Z}(\Delta)_{s,t}^{(n)}$ by

$$\begin{aligned}
\big(\mathcal{Z}_{s,t}^{(n)}\big)^{(j)} &:= \begin{cases} \big(d\mathcal{Y}_{s,t}^{(n^{-1/2},n)*}\big)^{(j)}, & (j = 1, 2, \dots, k), \\ \mathbf{0} & (j = k+1), \end{cases} \\
\mathcal{Z}(\Delta)_{s,t}^{(n)} &:= \mathcal{Z}_{t_0,t_1}^{(n)} * \mathcal{Z}_{t_1,t_2}^{(n)} * \cdots * \mathcal{Z}_{t_{N-1},t_N}^{(n)},
\end{aligned}$$

respectively. For $i = 1, 2, \dots, d_{k+1}$, (2.2.2) and (5.4.18) implies

$$\begin{aligned}
&\Big|\big(\mathcal{Z}(\Delta)_{s,t}^{(n)}(c)\big)_{i*}^{(k+1)} - \big(\mathcal{Z}(\Delta \setminus \{t_{N-1}\})_{s,t}^{(n)}(c)\big)_{i*}^{(k+1)}\Big| \\
&= \Big|\big(\mathcal{Z}_{t_{N-2},t_{N-1}}^{(n)}(c) * \mathcal{Z}_{t_{N-1},t_N}^{(n)}(c)\big)_{i*}^{(k+1)} - \big(\mathcal{Z}_{t_{N-2},t_N}^{(n)}(c)\big)_{i*}^{(k+1)}\Big| \\
&= \Bigg|\sum_{\substack{|K_1|+|K_2|=k+1 \\ |K_1|,|K_2|>0}} C_{K_1,K_2}\mathcal{P}_*^{K_1}\big(\mathcal{Z}_{t_{N-2},t_{N-1}}^{(n)}(c)\big)\mathcal{P}_*^{K_2}\big(\mathcal{Z}_{t_{N-1},t_N}^{(n)}(c)\big)\Bigg| \\
&\le C \sum_{\substack{|K_1|+|K_2|=k+1 \\ |K_1|,|K_2|>0}} \Big|\mathcal{P}_*^{K_1}\big(d\mathcal{Y}_{t_{N-2},t_{N-1}}^{(n^{-1/2},n)*}(c)\big)\Big|\Big|\mathcal{P}_*^{K_2}\big(d\mathcal{Y}_{t_{N-1},t_N}^{(n^{-1/2},n)*}(c)\big)\Big| \\
&\le \widehat{\mathcal{K}}_{k+1}^{(n)}(c)(t_N - t_{N-2})^{(k+1)\alpha} \le \widehat{\mathcal{K}}_{k+1}^{(n)}(c)\Big(\frac{2}{N-1}(t-s)\Big)^{(k+1)\alpha} \qquad (c \in \widehat{\Omega}_{k+1}^{(n)}),
\end{aligned}$$

where the random variable $\widehat{\mathcal{K}}_{k+1}^{(n)} : \Omega_{x_*}(X) \longrightarrow \mathbb{R}$ is given by

$$\begin{aligned}
\widehat{\mathcal{K}}_{k+1}^{(n)}(c) &:= C \sum_{\substack{|K_1|+|K_2|=k+1 \\ |K_1|,|K_2|>0}} \mathcal{Q}^{(n,K_1)}(c)\mathcal{Q}^{(n,K_2)}(c), \\
\mathcal{Q}^{(n,K)}(c) &:= \widehat{\mathcal{K}}_{k_1}^{(n)}(c) \cdot \cdots \cdot \widehat{\mathcal{K}}_{k_\ell}^{(n)}(c) \qquad \big(K = \big((i_1, k_1), (i_2, k_2), \dots, (i_\ell, k_\ell)\big)\big).
\end{aligned}$$

We emphasize that $\widehat{\mathcal{K}}^{(n)}_{k+1}$ is non-negative and has the following integrability:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^{(n^{-1/2})}_{x_*}}\big[(\widehat{\mathcal{K}}^{(n)}_{k+1})^{4m/(k+1)}\big] &\le C \sum_{\substack{k_1,\dots,k_\ell>0\\ k_1+\cdots+k_\ell=k+1}} \mathbb{E}^{\mathbb{P}^{(n^{-1/2})}_{x_*}}\Big[\big(\widehat{\mathcal{K}}^{(n)}_{k_1}\cdot\cdots\cdot\widehat{\mathcal{K}}^{(n)}_{k_\ell}\big)^{4m/(k+1)}\Big] \\
&\le C \sum_{\substack{k_1,\dots,k_\ell>0\\ k_1+\cdots+k_\ell=k+1}} \prod_{\lambda=1}^{\ell} \mathbb{E}^{\mathbb{P}^{(n^{-1/2})}_{x_*}}\Big[\big(\widehat{\mathcal{K}}^{(n)}_{k_\lambda}\big)^{4m/k_\lambda}\Big]^{k_\lambda/(k+1)} < \infty,
\end{aligned}$$

where we used the generalized Hölder inequality for the second line. We then have

$$\begin{aligned}
&\Big|\big(\mathcal{Z}(\Delta)^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| \\
&\le \Big|\big(\mathcal{Z}(\Delta\setminus\{t_{N-1}\})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| + \widehat{\mathcal{K}}^{(n)}_{k+1}(c)\Big(\frac{2}{N-1}(t-s)\Big)^{(k+1)\alpha} \\
&\le \Big|\big(\mathcal{Z}(\{s,t\})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| + \sum_{\ell=1}^{N-2}\widehat{\mathcal{K}}^{(n)}_{k+1}(c)\Big(\frac{2}{N-\ell}\Big)^{(k+1)\alpha}(t-s)^{(k+1)\alpha} \\
&\le \Big|\big(\mathcal{Z}^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| + \widehat{\mathcal{K}}^{(n)}_{k+1}(c)2^{(k+1)\alpha}\zeta\big((k+1)\alpha\big)(t-s)^{(k+1)\alpha} \\
&\le \widehat{\mathcal{K}}^{(n)}_{k+1}(c)(t-s)^{(k+1)\alpha} \qquad (i=1,2,\dots,d_{k+1},\ c\in\widehat{\Omega}^{(n)}_{k+1})
\end{aligned} \tag{5.4.19}$$

by successively removing points until the partition Δ coincides with $\{s,t\}$, where $\zeta(z)$ denotes the Riemann zeta function $\zeta(z) := \sum_{n=1}^{\infty}(1/n^z)$ for $z\in\mathbb{R}$.

We now show that the family $\{\mathcal{Z}(\Delta)^{(n)}_{s,t}\}$ satisfies the Cauchy convergence principle. Let $\delta>0$ and we take two partitions $\Delta=\{s=t_0<t_1\cdots<t_N=t\}$ and Δ' of $[s,t]$ independent of $n\in\mathbb{N}$ satisfying $|\Delta|,|\Delta'|<\delta$. We set $\widehat{\Delta} := \Delta\cup\Delta'$ and write

$$\widehat{\Delta}_\ell = \widehat{\Delta}\cap[t_\ell,t_{\ell+1}] = \{t_\ell = s_{\ell 0} < s_{\ell 1} < \cdots < s_{\ell L_\ell} = t_{\ell+1}\} \qquad (\ell=0,1,\dots,N-1).$$

Then (2.2.2) and (5.4.19) give

$$\begin{aligned}
&\Big|\big(\mathcal{Z}(\Delta)^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*} - \big(\mathcal{Z}(\widehat{\Delta})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| \\
&= \Big|\big(\mathcal{Z}^{(n)}_{t_0,t_1}(c)*\cdots*\mathcal{Z}^{(n)}_{t_{N-1},t_N}(c)\big)^{(k+1)}_{i_*} - \big(\mathcal{Z}(\widehat{\Delta}_0)^{(n)}_{t_0,t_1}(c)*\cdots*\mathcal{Z}(\widehat{\Delta}_{N-1})^{(n)}_{t_{N-1},t_N}(c)\big)^{(k+1)}_{i_*}\Big| \\
&= \Big|\big(\mathcal{Z}^{(n)}_{t_0,t_1}(c)\big)^{(k+1)}_{i_*} + \big(\mathcal{Z}^{(n)}_{t_1,t_2}(c)*\cdots*\mathcal{Z}^{(n)}_{t_{N-1},t_N}(c)\big)^{(k+1)}_{i_*} \\
&\qquad - \big(\mathcal{Z}(\widehat{\Delta}_0)^{(n)}_{t_0,t_1}(c)\big)^{(k+1)}_{i_*} - \big(\mathcal{Z}(\widehat{\Delta}_1)^{(n)}_{t_1,t_2}(c)*\cdots*\mathcal{Z}(\widehat{\Delta}_{N-1})^{(n)}_{t_{N-1},t_N}(c)\big)^{(k+1)}_{i_*}\Big| \\
&\le \widehat{\mathcal{K}}^{(n)}_{k+1}(c)(t_1-t_0)^{(k+1)\alpha} + \Big|\big(\mathcal{Z}^{(n)}_{t_1,t_2}(c)*\cdots*\mathcal{Z}^{(n)}_{t_{N-1},t_N}(c)\big)^{(k+1)}_{i_*} \\
&\qquad - \big(\mathcal{Z}(\widehat{\Delta}_0)^{(n)}_{t_1,t_2}(c)*\cdots*\mathcal{Z}(\widehat{\Delta}_{N-1})^{(n)}_{t_{N-1},t_N}(c)\big)^{(k+1)}_{i_*}\Big| \qquad (i=1,2,\dots,d_{k+1},\ c\in\widehat{\Omega}^{(n)}_{k+1}).
\end{aligned}$$

By repeating this kind of estimate and noting $(k+1)\alpha > 1$, we obtain

$$\begin{aligned}
&\Big|\big(\mathcal{Z}(\Delta)^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*} - \big(\mathcal{Z}(\widehat{\Delta})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| \\
&\le \sum_{\ell=0}^{N-1} \widehat{\mathcal{K}}^{(n)}_{k+1}(c)(t_{\ell+1}-t_\ell)^{(k+1)\alpha} \\
&\le \widehat{\mathcal{K}}^{(n)}_{k+1}(c)\Big(\max_{\Delta}(t_{\ell+1}-t_\ell)^{(k+1)\alpha-1}\Big)\sum_{\ell=0}^{N-1}(t_{\ell+1}-t_\ell) \\
&\le \widehat{\mathcal{K}}^{(n)}_{k+1}(c)(t-s)\times\delta^{(k+1)\alpha-1} \qquad (i=1,2,\dots,d_{k+1},\, c\in\widehat{\Omega}^{(n)}_{k+1}).
\end{aligned} \tag{5.4.20}$$

Thus, (5.4.20) leads to

$$\begin{aligned}
&\Big|\big(\mathcal{Z}(\Delta)^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*} - \big(\mathcal{Z}(\Delta')^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| \\
&\le \Big|\big(\mathcal{Z}(\Delta)^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*} - \big(\mathcal{Z}(\widehat{\Delta})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| + \Big|\big(\mathcal{Z}(\widehat{\Delta})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*} - \big(\mathcal{Z}(\widetilde{\Delta})^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*}\Big| \\
&\le 2\widehat{\mathcal{K}}^{(n)}_{k+1}(c)(t-s)\times\delta^{(k+1)\alpha-1} \longrightarrow 0 \qquad (i=1,2,\dots,d_{k+1},\, c\in\widehat{\Omega}^{(n)}_{k+1})
\end{aligned}$$

as $\delta\searrow 0$ uniformly in $0\le s\le t\le 1$. Therefore, there exists, for $0\le s\le t\le 1$,

$$\overline{\mathcal{Z}}^{(n)}_{s,t}(c) := \begin{cases} \lim\limits_{|\Delta|\searrow 0}\mathcal{Z}(\Delta)^{(n)}_{s,t}(c) & (c\in\widehat{\Omega}^{(n)}_{k+1}), \\ \mathbf{1}_G & (c\in\Omega_{x_*}(X)\setminus\widehat{\Omega}^{(n)}_{k+1}). \end{cases}$$

satisfying

$$\big\|\big(\overline{\mathcal{Z}}^{(n)}_{s,t}(c)\big)^{(k+1)}\big\|_{\mathbb{R}^{d_{k+1}}} \le \widehat{\mathcal{K}}^{(n)}_{k+1}(c)(t-s)^{(k+1)\alpha} \qquad (c\in\widehat{\Omega}^{(n)}_{k+1}),$$

due to (5.4.19). We will show

$$\overline{\mathcal{Z}}^{(n)}_{s,t}(c) = \mathcal{Y}^{(n^{-1/2},n;\,k+1)}_s(c)^{-1} * \mathcal{Y}^{(n^{-1/2},n;\,k+1)}_t(c) \qquad (0\le s\le t\le 1,\, c\in\widehat{\Omega}^{(n)}_{k+1})$$

as the last step. For this, it is sufficient to check that

$$\big(\overline{\mathcal{Z}}^{(n)}_{s,t}(c)\big)^{(k+1)} = \big(d\mathcal{Y}^{(n^{-1/2},n)*}_{s,t}(c)\big)^{(k+1)} \qquad (0\le s\le t\le 1,\, c\in\widehat{\Omega}^{(n)}_{k+1}) \tag{5.4.21}$$

by the definition of $\overline{\mathcal{Z}}^{(n)}_{s,t}$. We fix $i=1,2,\dots,d_{k+1}$ and $c\in\widehat{\Omega}^{(n)}_{k+1}$. Put

$$\Psi^i_{s,t}(c) := \big(d\mathcal{Y}^{(n^{-1/2},n)*}_{s,t}(c)\big)^{(k+1)}_{i*} - \big(\overline{\mathcal{Z}}^{(n)}_{s,t}(c)\big)^{(k+1)}_{i_*} \quad (0\le s\le t\le 1).$$

Then we easily see that $\Psi^i_{s,t}(c)$ is additive in the sense that

$$\Psi^i_{s,t}(c) = \Psi^i_{s,u}(c) + \Psi^i_{u,t}(c) \qquad (0\le s\le u\le t\le 1). \tag{5.4.22}$$

Since the piecewise smooth stochastic process $(\mathcal{Y}^{(n^{-1/2},n)}_t)_{0\le t\le 1}$ is given by the d_{CC}-geodesic interpolation of $\{\mathcal{X}^{(n^{-1/2},n)}_{t_k}\}_{k=0}^n$, we have

$$\big\|\big(d\mathcal{Y}^{(n^{-1/2},n)*}_{s,t}(c)\big)^{(k+1)}\big\|_{\mathbb{R}^{d_{k+1}}} \le \widetilde{\mathcal{K}}^{(n)}_{k+1}(c)(t-s)^{(k+1)\alpha} \qquad (c\in\widetilde{\Omega}^{(n)}_{k+1})$$

for some set $\widetilde{\Omega}^{(n)}_{k+1}$ with $\mathbb{P}^{(n^{-1/2})}_{x_*}(\widetilde{\Omega}^{(n)}_{k+1}) = 1$ and random variable $\widetilde{\mathcal{K}}^{(n)}_{k+1} : \Omega_{x_*}(X) \longrightarrow \mathbb{R}$. Thus, we have

$$\big|\Psi^i_{s,t}(c)\big| \le \big(\widetilde{\mathcal{K}}^{(n)}_{k+1}(c) + \widehat{\mathcal{K}}^{(n)}_{k+1}(c)\big)(t-s)^{(k+1)\alpha} \qquad (0 \le s \le t \le 1,\ c \in \widetilde{\Omega}^{(n)}_{k+1} \cap \widehat{\Omega}^{(n)}_{k+1}).$$

We may write $\widehat{\Omega}^{(n)}_{k+1}$ instead of $\widetilde{\Omega}^{(n)}_{k+1} \cap \widehat{\Omega}^{(n)}_{k+1}$ by abuse of notation. Because its probability equals one. For any small $\varepsilon > 0$, there is a sufficiently large $N \in \mathbb{N}$ such that $1/N < \varepsilon$. We then obtain as $\varepsilon \searrow 0$,

$$\begin{aligned}
\Big|\Psi^i_{0,t}(c)\Big| &= \Big|\Psi^i_{0,1/N}(c) + \Psi^i_{1/N,2/N}(c) + \cdots + \Psi^i_{[Nt]/N,t}(c)\Big| \\
&\le \big(\widetilde{\mathcal{K}}^{(n)}_{k+1}(c) + \widehat{\mathcal{K}}^{(n)}_{k+1}(c)\big)\varepsilon^{(k+1)\alpha-1}\Big\{\underbrace{\frac{1}{N} + \cdots + \frac{1}{N}}_{[Nt]\text{-times}} + \Big(t - \frac{[Nt]}{N}\Big)\Big\} \\
&= \big(\widetilde{\mathcal{K}}^{(n)}_{k+1}(c) + \widehat{\mathcal{K}}^{(n)}_{k+1}(c)\big)\varepsilon^{(k+1)\alpha-1} t \longrightarrow 0 \qquad (0 \le t \le 1,\ c \in \widehat{\Omega}^{(n)}_{k+1})
\end{aligned}$$

by (5.4.22) and $(k+1)\alpha - 1 > 0$. This implies that $\Psi^i_{0,t}(c) = 0$ for $0 \le t \le 1$ and $c \in \widehat{\Omega}^{(n)}_{k+1}$. Hence, it follows from (5.4.22) that

$$\Psi^i_{s,t}(c) = \Psi^i_{0,t}(c) - \Psi^i_{0,s}(c) = 0 \qquad (0 \le s \le t \le 1,\ c \in \widehat{\Omega}^{(n)}_{k+1}),$$

which leads to (5.4.21). Consequently, we know that there are a measurable set $\Omega^{(n)}_{k+1} \subset \Omega_{x_*}(X)$ with probability one and a non-negative random variable $\mathcal{K}^{(n)}_{k+1} \in L^{4m}(\Omega_{x_*}(X) \to \mathbb{R};\, \mathbb{P}^{(n^{-1/2})}_{x_*})$ satisfying

$$d_{\mathrm{CC}}\big(\mathcal{Y}^{(n^{-1/2},n;\,k+1)}_s(c), \mathcal{Y}^{(n^{-1/2},n;\,k+1)}_t(c)\big) \le \mathcal{K}^{(n)}_{k+1}(c)(t-s)^{\alpha} \quad (c \in \Omega^{(n)}_{k+1},\ 0 \le s \le t \le 1).$$

This completes the proof of Lemma 5.4.4. ∎

Proof of Lemma 5.4.2. For $m, n \in \mathbb{N}$ and $\widehat{\alpha} < \frac{2m-1}{4m}$, it follows from (4.3.15) that

$$\mathbb{E}^{\mathbb{P}^{(n^{-1/2})}_{x_*}}\Big[d_{\mathrm{CC}}\big(\mathcal{Y}^{(n^{-1/2},n;\,r)}_s, \mathcal{Y}^{(n^{-1/2},n;\,r)}_t\big)^{4m}\Big] \le \mathbb{E}^{\mathbb{P}^{(n^{-1/2})}_{x_*}}\big[\big(\mathcal{K}^{(n)}_r\big)^{4m}\big](t-s)^{4m\widehat{\alpha}}$$

for $0 \le s \le t \le 1$. We thus have, by (5.4.17),

$$\mathbb{E}^{\mathbb{P}^{(n^{-1/2})}_{x_*}}\Big[d_{\mathrm{CC}}\big(\mathcal{Y}^{(n^{-1/2},n;\,r)}_s, \mathcal{Y}^{(n^{-1/2},n;\,r)}_t\big)^{4m}\Big] \le C(t-s)^{4m\widehat{\alpha}} \qquad (0 \le s \le t \le 1).$$

for a positive constant $C > 0$ independent of $n \in \mathbb{N}$. Furthermore, thanks to **(A-2)** and $\Phi^{(0)}_0(x_*) = \mathbf{1}_G$, there is a sufficiently large constant $C > 0$ such that

$$\sup_{n \in \mathbb{N}} \big\| \log\big(\Phi^{(n^{-1/2})}_0(x_*)\big)\big|_{\mathfrak{g}^{(k)}}\big\|_{\mathfrak{g}^{(k)}} \le C \qquad (k = 1, 2, \ldots, r).$$

Thanks to the Kolmogorov tightness criterion, we know that the family $\{\mathbf{P}^{(n^{-1/2},n)}\}_{n=1}^{\infty}$ is tight in $C^{0,\alpha\text{-Höl}}([0,1]; G_{(0)})$ for $\alpha < \frac{4m\widehat{\alpha}-1}{4m} < \frac{1}{2} - \frac{1}{2m}$. By letting $m \to \infty$, we complete the proof. ∎

By using Lemma 5.4.4, we easily obtain the convergence of finite dimensional distribution of $(\mathcal{Y}^{(n^{-1/2},n)})_{0 \le t \le 1}$.

Lemma 5.4.5 *Let $\ell \in \mathbb{N}$. For fixed $0 \le s_1 < s_2 < \cdots < s_\ell \le 1$, we have*

$$(\mathcal{Y}_{s_1}^{(n^{-1/2},n)}, \mathcal{Y}_{s_2}^{(n^{-1/2},n)}, \dots, \mathcal{Y}_{s_\ell}^{(n^{-1/2},n)}) \xrightarrow{(d)} (Y_{s_1}, Y_{s_2}, \dots, Y_{s_\ell})$$

as $n \to \infty$.

Proof. We only show that case of $\ell = 2$. General cases ($\ell \ge 3$) can be also proved by repeating the same argument. For simplicity, we put $s = s_1, t = s_2$. We obtain $(\mathcal{X}_s^{(n^{-1/2},n)}, \mathcal{X}_t^{(n^{-1/2},n)}) \xrightarrow{(d)} (Y_s, Y_t)$ as $n \to \infty$ in the same way as [31, Lemma 5.5]. On the other hand, there exists a non-negative random variable $\mathcal{K}_r^{(n)} \in L^{4m}(\Omega_{x_*}(X) \to \mathbb{R}; \mathbb{P}_{x_*}^{(n^{-1/2})})$ satisfying

$$d_{\mathrm{CC}}\big(\mathcal{Y}_s^{(n^{-1/2},n)}(c), \mathcal{Y}_t^{(n^{-1/2},n)}(c)\big) \le \mathcal{K}_r^{(n)}(c)(t-s)^\alpha \quad \mathbb{P}_{x_*}^{(n^{-1/2})}\text{-a.s.} \qquad (0 \le s \le t \le 1)$$

by Lemma 5.4.4. Suppose $t_k \le t \le t_{k+1}$ for some $k = 0, 1, \dots, n-1$. Then we have, for all $\varepsilon > 0$ and sufficiently large $m \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}_{x_*}^{(n^{-1/2})}\Big(d_{\mathrm{CC}}\big(\mathcal{X}_t^{(n^{-1/2},n)}, \mathcal{Y}_t^{(n^{-1/2},n)}\big) > \varepsilon\Big) \\
&\le \frac{1}{\varepsilon^{4m}} \mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\Big[d_{\mathrm{CC}}\big(\mathcal{X}_t^{(n^{-1/2},n)}, \mathcal{Y}_t^{(n^{-1/2},n)}\big)^{4m}\Big] \\
&\le \frac{1}{\varepsilon^{4m}} \mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\Big[d_{\mathrm{CC}}\big(\mathcal{Y}_{t_k}^{(n^{-1/2},n)}, \mathcal{Y}_{t_{k+1}}^{(n^{-1/2},n)}\big)^{4m}\Big] \\
&\le \frac{1}{\varepsilon^{4m}} \mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\Big[(\mathcal{K}_r^{(n)})^{4m}(t_{k+1}-t_k)^{4m\alpha}\Big] = \frac{1}{n^{2m-1}\varepsilon^{4m}} \mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\big[(\mathcal{K}_r^{(n)})^{4m}\big] \longrightarrow 0
\end{aligned}$$

as $n \to \infty$, where we used Chebyshev's inequality for the second line and (5.4.17) for the final line. Hence, Slutzky's theorem (cf. Klenke [37, Theorem 13.8]) tells us that the desired convergence

$$(\mathcal{Y}_s^{(n^{-1/2},n)}, \mathcal{Y}_t^{(n^{-1/2},n)}) \xrightarrow{(d)} (Y_s, Y_t)$$

holds as $n \to \infty$. This completes the proof. ∎

We complete the proof of Theorem 5.1.2, by combining Lemma 5.4.2 and Lemma 5.4.5.

As in Theorem 4.6.2, we can also extend Theorem 5.1.2 to non-harmonic cases. Let $(\Phi_0^{(\varepsilon)})_{0 \le \varepsilon \le 1}$ be the family of modified harmonic realizations associated with $(p_\varepsilon)_{0 \le \varepsilon \le 1}$ and we take a family of realizations $(\Phi^{(\varepsilon)})_{0 \le \varepsilon \le 1}$, which is not necessary to be that of harmonic ones. In particular, we may put $\Phi_0^{(0)}(x_*) = \Phi^{(0)}(x_*) = \mathbf{1}_G$ for some reference point $x_* \in V$ and $\Phi_0^{(\varepsilon)}(x)^{(i)} = \Phi^{(\varepsilon)}(x)^{(i)}$ for $x \in V$, $0 \le \varepsilon \le 1$ and $i = 2, 3, \dots, r$ without loss of generality. Define $\mathrm{Cor}_{\mathfrak{g}^{(1)}}^{(\varepsilon)} : X \longrightarrow \mathfrak{g}^{(1)}$ by

$$\mathrm{Cor}_{\mathfrak{g}^{(1)}}^{(\varepsilon)}(x) := \log\big(\Phi^{(\varepsilon)}(x)\big)\Big|_{\mathfrak{g}^{(1)}} - \log\big(\Phi_0^{(\varepsilon)}(x)\big)\Big|_{\mathfrak{g}^{(1)}} \qquad (x \in V,\ 0 \le \varepsilon \le 1).$$

Instead of **(A1)** and **(A2)**, we impose the following assumptions on $(\Phi^{(\varepsilon)})_{0 \le \varepsilon \le 1}$.

(A1)′: *For every* $0 \le \varepsilon \le 1$, *it holds that*

$$\sum_{x \in \mathcal{F}} m(x) \log \big(\Phi^{(\varepsilon)}(x)^{-1} \cdot \Phi^{(0)}(x)\big)\big|_{\mathfrak{g}^{(1)}} = 0, \tag{5.4.23}$$

where $\mathcal{F}$ *denotes a fundamental domain of* X.

(A2)′: *There exists a positive constant* C *such that, for* $k = 2, 3, \ldots, r$,

$$\sup_{0 \le \varepsilon \le 1} \max_{x \in \mathcal{F}} \big\| \log \big(\Phi^{(\varepsilon)}(x)^{-1} \cdot \Phi^{(0)}(x)\big)\big|_{\mathfrak{g}^{(k)}} \big\|_{\mathfrak{g}^{(k)}} \le C, \tag{5.4.24}$$

where $\| \cdot \|_{\mathfrak{g}^{(k)}}$ *denotes a Euclidean norm on* $\mathfrak{g}^{(k)} \cong \mathbb{R}^{d_k}$ *for* $k = 2, 3, \ldots, r$.

We note that, thanks to **(A1)′**, we have

$$\sum_{x \in \mathcal{F}} m(x)\mathrm{Cor}^{(\varepsilon)}_{\mathfrak{g}^{(1)}}(x) = \sum_{x \in \mathcal{F}} m(x)\mathrm{Cor}^{(0)}_{\mathfrak{g}^{(1)}}(x) \qquad (0 \le \varepsilon \le 1). \tag{5.4.25}$$

In particular, there exists a positive constant $M > 0$ independent of $\varepsilon \in [0, 1]$ such that $\max_{x \in \mathcal{F}} \|\mathrm{Cor}^{(\varepsilon)}_{\mathfrak{g}^{(1)}}(x)\|_{\mathfrak{g}^{(1)}} \le M$ for $0 \le \varepsilon \le 1$.

Remark 5.4.6 We show that **(A1)′** and **(A2)′** imply that the family $(\Phi_0^{(\varepsilon)})_{0 \le \varepsilon \le 1}$ satisfies **(A1)** and **(A2)**, respectively. Indeed, by combining (5.4.25) and **(A1)′**, we see that

$$\begin{aligned}
&\sum_{x \in \mathcal{F}} m(x) \log \big(\Phi_0^{(\varepsilon)}(x)^{-1} \cdot \Phi_0^{(0)}(x)\big)\big|_{\mathfrak{g}^{(1)}} \\
&= \sum_{x \in \mathcal{F}} m(x)\mathrm{Cor}^{(\varepsilon)}_{\mathfrak{g}^{(1)}}(x) - \sum_{x \in \mathcal{F}} m(x)\mathrm{Cor}^{(0)}_{\mathfrak{g}^{(1)}}(x) + \sum_{x \in \mathcal{F}} m(x) \log \big(\Phi^{(\varepsilon)}(x)^{-1} \cdot \Phi^{(0)}(x)\big)\big|_{\mathfrak{g}^{(1)}} \\
&= 0 \qquad (0 \le \varepsilon \le 1),
\end{aligned}$$

which means that that the family $(\Phi_0^{(\varepsilon)})_{0 \le \varepsilon \le 1}$ enjoys the assumption **(A1)**. Furthermore, by using **(A2)′** and $\Phi_0^{(\varepsilon)}(x)^{(i)} = \Phi^{(\varepsilon)}(x)^{(i)}$ for $x \in V$, $0 \le \varepsilon \le 1$ and $i = 2, 3, \ldots, r$, we have

$$\sup_{0 \le \varepsilon \le 1} \max_{x \in \mathcal{F}} \big\| \log \big(\Phi_0^{(\varepsilon)}(x)^{-1} \cdot \Phi_0^{(0)}(x)\big)\big|_{\mathfrak{g}^{(k)}} \big\|_{\mathfrak{g}^{(k)}} \le C$$

for some $C > 0$, which implies that the family $(\Phi_0^{(\varepsilon)})_{0 \le \varepsilon \le 1}$ satisfies the assumption **(A2)**.

Let $(\overline{\mathcal{Y}}_t^{(\varepsilon,n)})_{0 \le t \le 1}$ $(0 \le \varepsilon \le 1,\ n \in \mathbb{N})$ be the $G_{(0)}$-valued stochastic processes defined by just replacing $\Phi_0^{(\varepsilon)}$ by $\Phi^{(\varepsilon)}$ in the definition of $(\mathcal{Y}_t^{(\varepsilon,n)})_{0 \le t \le 1}$. Recall that $(\widehat{Y}_t)_{0 \le t \le 1}$ is the G-valued diffusion process which is the solution to the SDE (5.1.6).

Then we can show the following FCLT of the second kind as in the same way as Theorem 4.6.2.

Theorem 5.4.7 *The sequence* $(\overline{\mathcal{Y}}_t^{(n^{-1/2},n)})_{0 \le t \le 1}$ $(n = 1, 2, \ldots)$ *converges in law to the* G*-valued diffusion process* $(\widehat{Y}_t)_{0 \le t \le 1}$ *in* $C^{0,\alpha\text{-Höl}}_{\mathbf{1}_G}([0, 1]; G_{(0)})$ *as* $n \to \infty$.

We have captured in Chapters 4 and 5 two kinds of limiting infinitesimal generators and limiting diffusions by applying the scheme to "delete" the diverging drift (Scheme 1) and the scheme to "weaken" it (Scheme 2). Before closing this chapter, we summarize them, as well as the case of crystal lattices obtained in Ishiwata–Kawabi–Kotani [31].

First we summarize the case of a Γ-crystal lattice X. We denote by $\{\omega_1, \omega_2, \dots, \omega_d\}$ an orthonormal basis of $\mathrm{Hom}(\Gamma, \mathbb{R})$ and put $x_i = \omega_i[\mathbf{x}]_{\boldsymbol{\Gamma}\otimes\mathbb{R}}$ for $i = 1, 2, \dots, d$ and $\mathbf{x} \in \Gamma \otimes \mathbb{R}$. For simplicity, we write $\Delta_{g_0} := \sum_{i=1}^d (\partial^2/\partial x_i^2)$ for the homogenized Laplacian on $\Gamma \otimes \mathbb{R}$ with respect to the Albanese metric g_0. For $\mathbf{x} \in \Gamma \otimes \mathbb{R}$, we put $\langle \mathbf{x}, \nabla \rangle_{g_0} := \sum_{i=1}^d x_i (\partial/\partial x_i)$. Recall that $(g_0^{(\varepsilon)} = g_0(\varepsilon))_{0 \le \varepsilon \le 1}$ stand for the family of Albanese metrics associated with a family of transition probability $(p_\varepsilon)_{0 \le \varepsilon \le 1}$. We note that $g_0(1) = g_0$. Then the limiting infinitesimal generators on X are summarized as follows:

symmetric ($\gamma_p = 0$)
$\Delta_{g_0}/2$

	non-symmetric ($\gamma_p \neq 0$)	
	centered ($\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$)	non-centered ($\rho_{\mathbb{R}}(\gamma_p) \neq \mathbf{0}$)
Scheme 1	$\Delta_{g_0(1)}/2$	$\Delta_{g_0(1)}/2$
Scheme 2	$\Delta_{g_0(0)}/2$	$\Delta_{g_0(0)}/2 + \langle \rho_{\mathbb{R}}(\gamma_p), \nabla \rangle_{g_0(0)}$

Table 5.1: Limiting infinitesimal generators in the case of a crystal lattice X

Let $(B_t^{(g_0)})_{0 \le t \le 1}$ be a standard Brownian motion on $(\Gamma \otimes \mathbb{R}, g_0)$. Then the limiting diffusions are also summarized as follows:

symmetric ($\gamma_p = 0$)
$(B_t^{(g_0)})_{0 \le t \le 1}$

	non-symmetric ($\gamma_p \neq 0$)	
	centered ($\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$)	non-centered ($\rho_{\mathbb{R}}(\gamma_p) \neq \mathbf{0}$)
Scheme 1	$(B_t^{(g_0(1))})_{0 \le t \le 1}$	$(B_t^{(g_0(1))})_{0 \le t \le 1}$
Scheme 2	$(B_t^{(g_0(0))})_{0 \le t \le 1}$	$(B_t^{(g_0(0))} + t\rho_{\mathbb{R}}(\gamma_p))_{0 \le t \le 1}$

Table 5.2: Limiting diffusion processes in the case of a crystal lattice X

Next we summarize the case of a Γ-nilpotent covering graph X. Let $\{V_1, V_2, \dots, V_{d_1}\}$ be an orthonormal basis of $(\mathfrak{g}^{(1)}, g_0)$ and write $\Delta_{g_0} := \sum_{i=1}^{d_1} V_i^2$ for the homogenized sub-Laplacian on $G = G_\Gamma$. Then the limiting infinitesimal generators on X are summarized as follows:

symmetric ($\gamma_p = 0$)
$\Delta_{g_0}/2$

	non-symmetric ($\gamma_p \neq 0$)	
	centered ($\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}_{\mathfrak{g}}$)	non-centered ($\rho_{\mathbb{R}}(\gamma_p) \neq \mathbf{0}_{\mathfrak{g}}$)
Scheme 1	$\Delta_{g_0(1)}/2 + \beta(\Phi_0)_*$	$\Delta_{g_0(1)}/2 + \beta(\Phi_0)_*$
Scheme 2	$\Delta_{g_0(0)}/2$	$\Delta_{g_0(0)}/2 + \rho_{\mathbb{R}}(\gamma_p)_*$

Table 5.3: Limiting infinitesimal generators in the case of a nilpotent covering graph X

We emphasize that, in the centered case, the limiting generator of Scheme 2 is noting but the sub-Laplacian on G, while the drift $\beta(\Phi_0)$ arising the non-symmetry of the random walk on X still remains in the one of Scheme 1. As for the corresponding limiting diffusions, we write down them only in the non-centered case.

• Scheme 1: We put $V_0 = \beta(\Phi_0)_*$. Then,

$$Y_t = \exp\Big(t\beta(\Phi_0)_* + \sum_{i=1}^{d_1} B_t^i V_{i*} + \sum_{0\le i<j\le d_1} \frac{1}{2}\int_0^t (B_s^i dB_s^j - B_s^j dB_s^i)[\![V_{i*}, V_{j*}]\!] + \cdots\Big)(\mathbf{1}_G),$$

where $(B_t^1, B_t^2, \dots, B_t^{d_1})_{0\le t\le 1}$ is a standard Brownian motion on $(\mathfrak{g}^{(1)}, g_0^{(1)}) \cong (\mathbb{R}^{d_1}, g_0^{(1)})$.

• Scheme 2: We put $V_0 = \rho_{\mathbb{R}}(\gamma_p)_*$. Then,

$$\widehat{Y}_t = \exp\Big(t\rho_{\mathbb{R}}(\gamma_p)_* + \sum_{i=1}^{d_1} B_t^i V_{i*} + \sum_{0\le i<j\le d_1} \frac{1}{2}\int_0^t (B_s^i dB_s^j - B_s^j dB_s^i)[\![V_{i*}, V_{j*}]\!] + \cdots\Big)(\mathbf{1}_G),$$

where $(B_t^1, B_t^2, \dots, B_t^{d_1})_{0\le t\le 1}$ is a standard Brownian motion on $(\mathfrak{g}^{(1)}, g_0^{(0)}) \cong (\mathbb{R}^{d_1}, g_0^{(0)})$.

When we see Table 5.3 again, one may wonder if a G-valued diffusion process whose drift term belongs to $\mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)}$ can be captured or not through our schemes. To our best knowledge, there seems to be no results which capture such a limiting diffusion process in any nilpotent frameworks. As a further problem, we suggest a hybrid scheme of our two ones and discuss a CLT corresponding to it in order to capture such a limiting diffusion.

For $q > 1$, we define the transition-shift operator $\widehat{\mathcal{L}}_{p,\varepsilon} : C_{\infty,q}(X \times \mathbb{Z}) \longrightarrow C_{\infty,q}(X \times \mathbb{Z})$ associated with p_ε by

$$\widehat{\mathcal{L}}_{p,\varepsilon} f(x,z) := \sum_{e\in E_x} p_\varepsilon(e) f\big(t(e), z+1\big) \qquad (x \in X,\ z \in \mathbb{Z}).$$

Let us fix $\mathfrak{b} \in \mathfrak{g}^{(2)}$ and define, for $0 \le \varepsilon \le 1$, the scaling operator $\widehat{\mathcal{P}}_\varepsilon : C_\infty(G) \longrightarrow C_{\infty,q}(X \times \mathbb{Z})$ by

$$\widehat{\mathcal{P}}_\varepsilon f(x,z) := f\Big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x) * \exp(z\mathfrak{b})\big)\Big) \qquad (x \in X,\ z \in \mathbb{Z}).$$

This new scheme is based on our two schemes. Namely, it provides an effect which not only weakens the diverging drift term by introducing the family $(p_\varepsilon)_{0\le\varepsilon\le1}$ but creates an arbitrary $\mathfrak{g}^{(2)}$-drift $\mathfrak{b} \in \mathfrak{g}^{(2)}$ in the limiting infinitesimal generator. We still assume **(A1)** and **(A2)**. Then, thanks to $\mathfrak{b} \in \mathfrak{g}^{(2)}$, we might prove the followings as in the proof of Theorems 4.1.2 and 5.1.1.

Conjecture 5.4.8 *For $q > 4r + 1$, $0 \le s \le t$ and $f \in C_\infty(G)$, we have*

$$\lim_{n\to\infty} \Big\| \widehat{\mathcal{L}}_{p,n^{-1/2}}^{[nt]-[ns]} \widehat{\mathcal{P}}_{n^{-1/2}} f - \widehat{\mathcal{P}}_{n^{-1/2}} \mathrm{e}^{-(t-s)\mathfrak{A}} f \Big\|_{\infty,q} = 0,$$

where $(\mathrm{e}^{-t\mathfrak{A}})_{t\geq 0}$ *is the* C_0*-semigroup with the infinitesimal generator* $\mathfrak{A}$ *on* $C_0^\infty(G)$ *defined by*

$$\mathfrak{A} := -\frac{1}{2}\sum_{i=1}^{d_1} V_{i*}^2 - \underbrace{(\rho_{\mathbb{R}}(\gamma_p)_* + \mathfrak{b}_*)}_{\in \mathfrak{g}^{(1)}\oplus\mathfrak{g}^{(2)}},$$

where $\{V_1, V_2, \dots, V_{d_1}\}$ *denotes an orthonormal basis of* $(\mathfrak{g}^{(1)}, g_0)$*.*

Conjecture 5.4.9 *Let* $(\widetilde{\mathcal{Y}}_t^{(\varepsilon,n)})_{0\leq t\leq 1}$ *be the* G*-valued stochastic process given by the* d_{CC}*-geodesic interpolation of*

$$\widetilde{\mathcal{Y}}_{k/n}^{(\varepsilon,n)}(c) := \tau_{n^{-1/2}}\Big(\Phi_0^{(\varepsilon)}(w_k(c)) * \exp(k^2\mathfrak{b})\Big) \qquad \big(k = 0, 1, \dots, n,\, c \in \Omega_{x_*}(X)\big)$$

for $n \in \mathbb{N}$ *and* $0 \leq \varepsilon \leq 1$*. Then the sequence* $\{\widetilde{\mathcal{Y}}^{(n^{-1/2},n)}\}_{n=1}^\infty$ *converges in law to the* G*-valued diffusion process* $\mathfrak{Y}$ *in* $C^{0,\alpha\text{-Höl}}([0,1]; G_{(0)})$ *which solves the SDE*

$$d\mathfrak{Y}_t = \sum_{i=1}^{d_1} V_{i*}(\mathfrak{Y}_t) \circ dB_t^i + \rho_{\mathbb{R}}(\gamma_p)_*(\mathfrak{Y}_t)\, dt - \mathfrak{b}(\mathfrak{Y}_t)\, dt, \qquad \mathfrak{Y}_0 = \mathbf{1}_G.$$

Chapter 6

Examples

6.1 The 3D Heisenberg group

It goes without saying that the most typical but non-trivial example of nilpotent Lie groups of step 2 is the 3-*dimensional Heisenberg group* defined by

$$G = \mathbb{H}^3(\mathbb{R}) := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} = (\mathbb{R}^3, \star),$$

where the product $\star$ on $\mathbb{R}^3$ is given by

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' + xy').$$

This Lie group naturally appears in a lot of parts of mathematics including Fourier analysis, geometry, topology and so on. First of all, we give a quick review of the basics of $G = \mathbb{H}^3(\mathbb{R})$. Let $\Gamma = \mathbb{H}^3(\mathbb{Z})$ be the 3-dimensional discrete Heisenberg group. Then, $G = \mathbb{H}^3(\mathbb{R})$ is the corresponding connected and simply connected nilpotent Lie group of step 2 such that Γ is isomorphic to a cocompact lattice in G. Furthermore, the corresponding Lie algebra $\mathfrak{g}$ is given by

$$\mathfrak{g} = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}.$$

Let $\{X_1, X_2, X_3\}$ be the standard basis of $\mathfrak{g}$, that is,

$$X_1 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_3 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We then see that the Lie algebra $\mathfrak{g}$ is decomposed as $\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)}$, where $\mathfrak{g}^{(1)} := \mathrm{span}_{\mathbb{R}}\{X_1, X_2\}$ and $\mathfrak{g}^{(2)} := \mathrm{span}_{\mathbb{R}}\{X_3\}$, due to the algebraic relations $[X_1, X_2] = X_3$ and $[X_1, X_3] = [X_2, X_3] = \mathbf{0}_{\mathfrak{g}}$ under the matrix bracket $[X, Y] := XY - YX$ for $X, Y \in \mathfrak{g}$.

6.2 The 3D Heisenberg triangular lattice

Let Γ be generated by $\gamma_1 = (1,0,0)$, $\gamma_2 = (0,1,0)$ and $\gamma_3 = (-1,1,0)$. We consider the *Cayley graph* $X = (V, E)$ of Γ with the *generating set* $\mathcal{S} := \{\gamma_1, \gamma_2, \gamma_3, \gamma_1^{-1}, \gamma_2^{-1}, \gamma_3^{-1}\}$. Namely, $V = \mathbb{Z}^3$ and $E = \{(g,h) \in V \times V \,|\, h \cdot g^{-1} \in \mathcal{S}\}$ (see Figure 6.1). If $e \in E$ is represented as (g,h) for some $g, h \in V$, then its inverse edge $\overline{e}$ is equal to (h, g). Moreover, the left action Γ on the Cayley graph X is given by

$$\begin{aligned} \gamma_1 g &= (x+1, y, z+y), & \gamma_2 g &= (x, y+1, z), & \gamma_3 g &= (x-1, y+1, z-y), \\ \gamma_1^{-1} g &= (x-1, y, z-y), & \gamma_2^{-1} g &= (x, y-1, z), & \gamma_3^{-1} g &= (x+1, y-1, z+y-1), \end{aligned}$$

for $g = (x,y,z) \in G$. In view of the algebraic relation $\gamma_3 \star \gamma_1 = \gamma_2$, we may call this Cayley graph X a 3-*dimensional Heisenberg triangular lattice*. The quotient graph of X by the action Γ is the 3-bouquet graph $X_0 = (V_0, E_0)$, where $V_0 = \{\mathbf{x}\}$ and $E_0 = \{e_1, e_2, e_3\} \cup \{\overline{e}_1, \overline{e}_2, \overline{e}_3\}$ (see Figure 6.2).

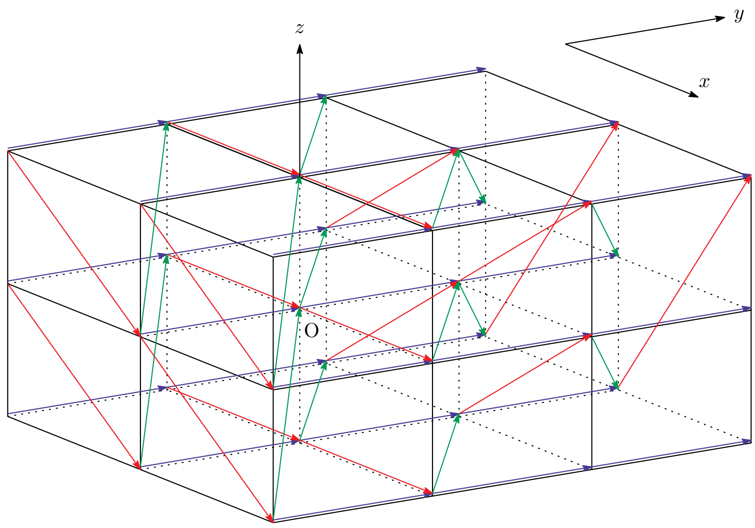

Figure 6.1: A part of the 3-dimensional Heisenberg triangular lattice

Now we define a non-symmetric random walk on X. We introduce a transition probability $p : E \longrightarrow (0,1]$ on X by setting

$$\begin{aligned} p\big((g, \gamma_1 g)\big) &:= \xi, & p\big((g, \gamma_2 g)\big) &:= \eta', & p\big((g, \gamma_3 g)\big) &:= \zeta, \\ p\big((g, \gamma_1^{-1} g)\big) &:= \xi', & p\big((g, \gamma_2^{-1} g)\big) &:= \eta, & p\big((g, \gamma_3^{-1} g)\big) &:= \zeta', \end{aligned}$$

where $\xi, \xi', \eta, \eta', \zeta, \zeta' > 0$, $\xi + \xi' + \eta + \eta' + \zeta + \zeta' = 1$ and

$$\xi - \xi' = \eta - \eta' = \zeta - \zeta' =: \varepsilon \geq 0. \tag{6.2.1}$$

In what follows, we write

$$\hat{\xi} := \xi + \xi', \quad \check{\xi} := \xi - \xi', \quad \hat{\eta} := \eta + \eta', \quad \check{\eta} := \eta - \eta', \quad \hat{\zeta} := \zeta + \zeta', \quad \check{\zeta} := \zeta - \zeta'$$

for brevity. The invariant measure on $V_0 = \{\mathbf{x}\}$ is given by $m(\mathbf{x}) = 1$. The quantity ε in (6.2.1) indicates the intensity of the non-symmetry of this random walk and it is clear that the random walk is m-symmetric if and only if $\varepsilon = 0$.

The first homology group of X_0 is given by $\mathrm{H}_1(X_0, \mathbb{R}) = \{[e_1], [e_2], [e_3]\}$. Since X_0 is a bouquet graph, the difference operator $d : C^0(X_0, \mathbb{R}) \longrightarrow C^1(X_0, \mathbb{R})$ is the zero-map. Then we have $\mathrm{H}^1(X_0, \mathbb{R}) \cong \big(\mathcal{H}^1(X_0), \langle\!\langle \cdot, \cdot \rangle\!\rangle_p\big) = C^1(X_0, \mathbb{R})$. Moreover, we obtain

$$\gamma_p = \sum_{e \in E_0} p(e)[e] = \varepsilon\big([e_1] - [e_2] + [e_3]\big) \in \mathrm{H}_1(X_0, \mathbb{R}) \tag{6.2.2}$$

by definition. The canonical surjective linear map $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{R}) \longrightarrow \mathfrak{g}^{(1)}$ is given by

$$\rho_{\mathbb{R}}([e_1]) = X_1, \quad \rho_{\mathbb{R}}([e_2]) = X_2, \quad \rho_{\mathbb{R}}([e_3]) = X_2 - X_1.$$

Then we easily see that $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}_{\mathfrak{g}}$. We introduce a basis $\{u_1, u_2\}$ in $\mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$ by

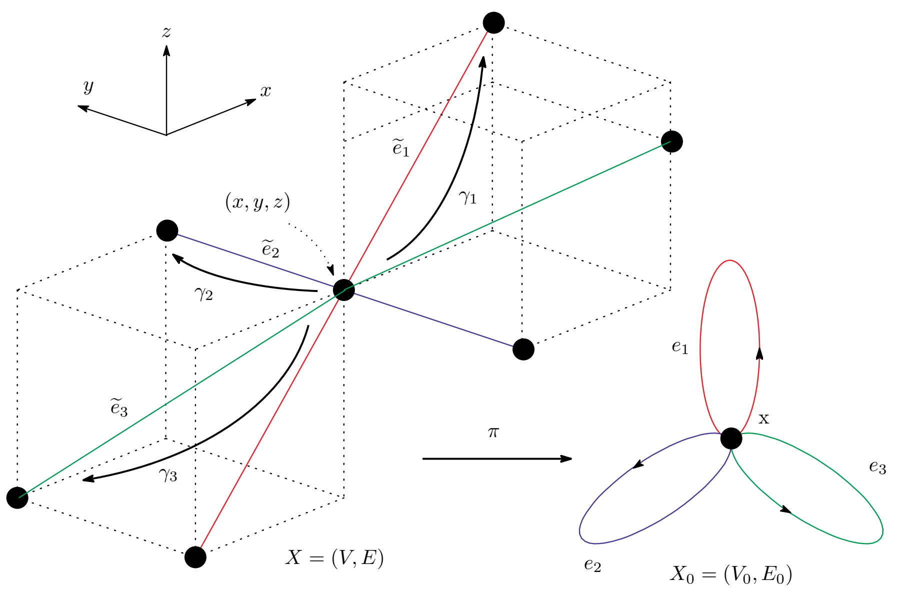

Figure 6.2: The quotient $X_0 = (V_0, E_0) = \Gamma \backslash X$ and the nearest neighbor vertices of (x, y, z)

$$u_1(X) = x, \quad u_2(X) = y \quad \big(X = xX_1 + yX_2 \in \mathfrak{g}^{(1)},\ x, y \in \mathbb{Z}\big).$$

It should be noted that $\{u_1, u_2\}$ is the dual basis of $\{X_1, X_2\}$ in $\mathfrak{g}^{(1)}$. We write $\{\omega_1, \omega_2, \omega_3\} \subset \big(\mathrm{H}^1(X_0, \mathbb{R}), \langle\!\langle \cdot, \cdot \rangle\!\rangle_p\big)$ for the dual basis of $\{[e_1], [e_2], [e_3]\} \subset \mathrm{H}_1(X_0, \mathbb{R})$. By direct computation, we obtain

$$\begin{aligned}
&\langle\!\langle \omega_1, \omega_1 \rangle\!\rangle_p = \hat{\xi} - \check{\xi}^2 = \hat{\xi} - \varepsilon^2, \qquad && \langle\!\langle \omega_1, \omega_2 \rangle\!\rangle_p = \check{\xi}\check{\eta} = \varepsilon^2, \\
&\langle\!\langle \omega_2, \omega_2 \rangle\!\rangle_p = \hat{\eta} - \check{\eta}^2 = \hat{\eta} - \varepsilon^2, && \langle\!\langle \omega_2, \omega_3 \rangle\!\rangle_p = \check{\eta}\check{\zeta} = \varepsilon^2, \\
&\langle\!\langle \omega_3, \omega_3 \rangle\!\rangle_p = \hat{\zeta} - \check{\zeta}^2 = \hat{\zeta} - \varepsilon^2, && \langle\!\langle \omega_1, \omega_3 \rangle\!\rangle_p = -\check{\xi}\check{\zeta} = -\varepsilon^2.
\end{aligned} \tag{6.2.3}$$

We know that $u_1 = {}^t\rho_{\mathbb{R}}(u_1) = \omega_1 - \omega_3$, $u_2 = {}^t\rho_{\mathbb{R}}(u_2) = \omega_2 + \omega_3$ form a $\mathbb{Z}$-basis in $\mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$ by noting that $\mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$ is regarded as a 2-dimensional subspace of $\mathrm{H}^1(X_0, \mathbb{R})$ through the injective map ${}^t\rho_{\mathbb{R}}$. It follows from (6.2.3) that

$$\langle\!\langle u_1, u_1 \rangle\!\rangle_p = \hat{\xi} + \hat{\zeta}, \quad \langle\!\langle u_1, u_2 \rangle\!\rangle_p = -\hat{\zeta}, \quad \langle\!\langle u_2, u_2 \rangle\!\rangle_p = \hat{\eta} + \hat{\zeta}. \tag{6.2.4}$$

Then the volume of the Albanese torus is computed as

$$\mathrm{vol}(\mathrm{Alb}^{\Gamma})^{-1} := \sqrt{\det\big(\langle\!\langle u_i, u_j \rangle\!\rangle_p\big)_{i,j=1}^{2}} = (\hat{\xi}\hat{\eta} + \hat{\eta}\hat{\zeta} + \hat{\zeta}\hat{\xi})^{1/2}.$$

Moreover, the Albanese metric g_0 on $\mathfrak{g}^{(1)}$ is given by the following:

$$\begin{aligned}
\langle X_1, X_1 \rangle_{g_0} &= \frac{\hat{\eta} + \hat{\zeta}}{\hat{\xi}\hat{\eta} + \hat{\eta}\hat{\zeta} + \hat{\zeta}\hat{\xi}} = (\hat{\eta} + \hat{\zeta})\mathrm{vol}(\mathrm{Alb}^{\Gamma})^2, \\
\langle X_1, X_2 \rangle_{g_0} &= \frac{\hat{\zeta}}{\hat{\xi}\hat{\eta} + \hat{\eta}\hat{\zeta} + \hat{\zeta}\hat{\xi}} = \hat{\zeta}\mathrm{vol}(\mathrm{Alb}^{\Gamma})^2, \\
\langle X_2, X_2 \rangle_{g_0} &= \frac{\hat{\xi} + \hat{\zeta}}{\hat{\xi}\hat{\eta} + \hat{\eta}\hat{\zeta} + \hat{\zeta}\hat{\xi}} = (\hat{\xi} + \hat{\zeta})\mathrm{vol}(\mathrm{Alb}^{\Gamma})^2.
\end{aligned}$$

We are now in a position to determine the modified standard realization $\Phi_0 : X \longrightarrow G$. Let $\widetilde{e}_i$ $(i = 1, 2, 3)$ be a lift of $e_i \in E_0$ to X and put $\Phi_0\big(o(\widetilde{e}_i)\big) = \mathbf{1}_G = (0, 0, 0)$. Then we easily see that the realization satisfying

$$\Phi_0\big(t(\widetilde{e}_1)\big) = \gamma_1, \quad \Phi_0\big(t(\widetilde{e}_2)\big) = \gamma_2, \quad \Phi_0\big(t(\widetilde{e}_3)\big) = \gamma_3$$

is the modified harmonic realization. Let $\{v_1, v_2\}$ be the Gram–Schmidt orthonormalization of $\{u_1, u_2\}$, and $\{V_1, V_2\}$ be the dual basis of $\{v_1, v_2\}$ in $\mathfrak{g}^{(1)}$. We put $V_3 := [V_1, V_2] = V_1V_2 - V_2V_1$. We then have

$$v_1 = (\hat{\xi} + \hat{\zeta})^{-1/2}u_1, \quad v_2 = (\hat{\xi} + \hat{\zeta})^{1/2}\mathrm{vol}(\mathrm{Alb}^{\Gamma})\Big(\frac{\hat{\zeta}}{\hat{\xi} + \hat{\zeta}}u_1 + u_2\Big)$$

by (6.2.4) and hence we obtain

$$\begin{aligned}
V_1 &= (\hat{\xi} + \hat{\zeta})^{1/2}X_1 - \hat{\zeta}(\hat{\xi} + \hat{\zeta})^{-1/2}X_2, \\
V_2 &= (\hat{\xi} + \hat{\zeta})^{-1/2}\mathrm{vol}(\mathrm{Alb}^{\Gamma})^{-1}X_2, \\
V_3 &= \mathrm{vol}(\mathrm{Alb}^{\Gamma})^{-1}X_3.
\end{aligned}$$

Finally, $\beta(\Phi_0) \in \mathfrak{g}^{(2)}$ and the infinitesimal generator $\mathcal{A}$ in Theorem 4.1.2 are calculated as

$$\beta(\Phi_0) = \frac{\varepsilon}{2}\mathrm{vol}(\mathrm{Alb}^{\Gamma})V_3, \qquad \mathcal{A} = -\frac{1}{2}(V_1^2 + V_2^2) - \frac{\varepsilon}{2}\mathrm{vol}(\mathrm{Alb}^{\Gamma})V_3,$$

respectively.

6.3 The 3D Heisenberg dice lattice

As another example of nilpotent covering graphs, we introduce the 3-*dimensional Heisenberg dice lattice*. This graph is defined by a covering graph of a finite graph consisting of three vertices with a covering transformation group $\Gamma = \mathbb{H}^3(\mathbb{Z})$ (see Figure 6.3). We emphasize that it is regarded as an extension of the *dice graph* discussed in [58] to the nilpotent case.

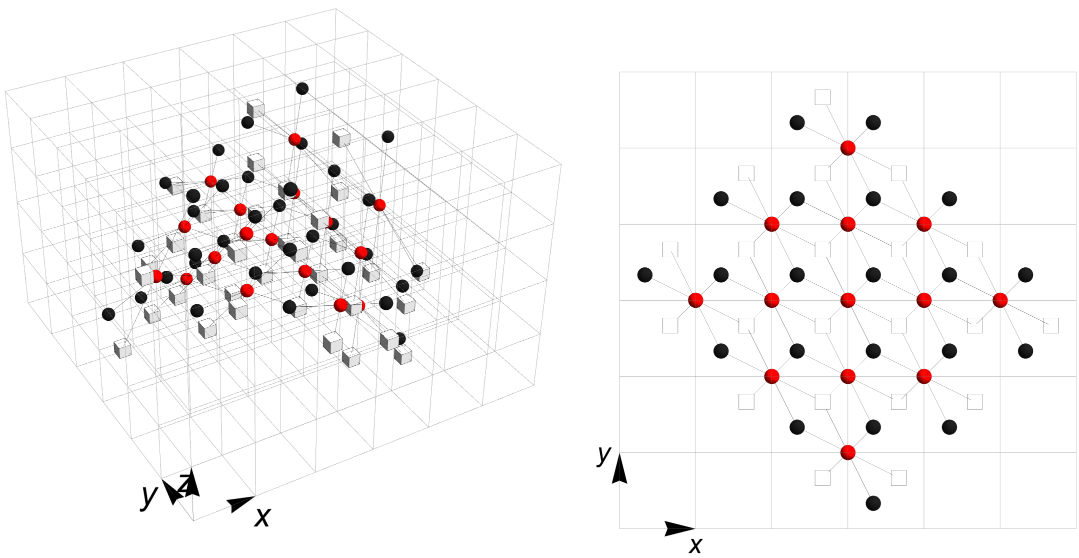

Figure 6.3: A part of 3-dimensional Heisenberg dice lattice and the projection of it on the xy-plane

Suppose that $\Gamma = \mathbb{H}^3(\mathbb{Z})$ is generated by two elements $\gamma_1 = (1,0,0)$ and $\gamma_2 = (0,1,0)$. We also set two elements $\mathbf{g}_1 := (1/3, 1/3, 1/3)$, $\mathbf{g}_2 := (-1/3, -1/3, -1/3)$ in $G = \mathbb{H}^3(\mathbb{R})$. We put

$$\begin{aligned}
V_1 &:= \big\{ g = \gamma_{i_1}^{\varepsilon_1} \star \cdots \star \gamma_{i_\ell}^{\varepsilon_\ell} \star \mathbf{1}_G \,\big|\, i_k \in \{1,2\},\, \varepsilon_k = \pm 1 \,(1 \le k \le \ell),\, \ell \in \mathbb{N} \cup \{0\} \big\}, \\
V_2 &:= \big\{ g = \gamma_{i_1}^{\varepsilon_1} \star \cdots \star \gamma_{i_\ell}^{\varepsilon_\ell} \star \mathbf{g}_1 \,\big|\, i_k \in \{1,2\},\, \varepsilon_k = \pm 1 \,(1 \le k \le \ell),\, \ell \in \mathbb{N} \cup \{0\} \big\}, \\
V_3 &:= \big\{ g = \gamma_{i_1}^{\varepsilon_1} \star \cdots \star \gamma_{i_\ell}^{\varepsilon_\ell} \star \mathbf{g}_2 \,\big|\, i_k \in \{1,2\},\, \varepsilon_k = \pm 1 \,(1 \le k \le \ell),\, \ell \in \mathbb{N} \cup \{0\} \big\}.
\end{aligned}$$

We consider a $\mathbb{H}^3(\mathbb{Z})$-nilpotent covering graph $X = (V, E)$ defined by $V = V_1 \sqcup V_2 \sqcup V_3$ and $E = E_1 \sqcup E_2$, where

$$\begin{aligned}
E_1 &:= \big\{ (g,h) \in V_1 \times V_2 \,|\, g^{-1} \star h = \mathbf{g}_1,\, \gamma_1^{-1} \star \mathbf{g}_1,\, \gamma_2^{-1} \star \mathbf{g}_1 \big\}, \\
E_2 &:= \big\{ (g,h) \in V_1 \times V_3 \,|\, g^{-1} \star h = \mathbf{g}_2,\, \gamma_1 \star \mathbf{g}_2,\, \gamma_2 \star \mathbf{g}_2 \big\}.
\end{aligned}$$

We note that X is invariant under the actions γ_1 and γ_2. Its quotient graph $X_0 = (V_0, E_0) = \Gamma \backslash X$ is given by $V_0 = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ and $E_0 = \{e_i, \overline{e}_i \,|\, 1 \le i \le 6\}$ (cf. Figure 6.4).

From now on we define a non-symmetric random walk on X. We define the transition probability $p : E \longrightarrow (0,1]$ by

$$
\begin{aligned}
&p\big((g, g\star \mathbf{g}_1)\big) = \xi, && p\big((g, g\star \gamma_1^{-1} \star \mathbf{g}_1)\big) = \eta, && p\big((g, g\star \gamma_2^{-1} \star \mathbf{g}_1)\big) = \zeta,\\
&p\big((g, g\star \mathbf{g}_2)\big) = \zeta, && p\big((g, g\star \gamma_1 \star \mathbf{g}_2)\big) = \eta, && p\big((g, g\star \gamma_2 \star \mathbf{g}_2)\big) = \xi,\\
&p\big(\overline{(g, g\star \mathbf{g}_1)}\big) = \gamma, && p\big(\overline{(g, g\star \gamma_1^{-1} \star \mathbf{g}_1)}\big) = \beta, && p\big(\overline{(g, g\star \gamma_2^{-1} \star \mathbf{g}_1)}\big) = \alpha,\\
&p\big(\overline{(g, g\star \mathbf{g}_2)}\big) = \alpha, && p\big(\overline{(g, g\star \gamma_1 \star \mathbf{g}_2)}\big) = \beta, && p\big(\overline{(g, g\star \gamma_2 \star \mathbf{g}_2)}\big) = \gamma,
\end{aligned}
$$

for every $g \in V_1$, where $\xi, \eta, \zeta, \alpha, \beta, \gamma > 0$, $2(\xi+\eta+\zeta) = 1$ and $\alpha+\beta+\gamma = 1$. The invariant measure $m : V_0 = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\} \longrightarrow (0,1]$ is given by $m(\mathbf{x}) = 1/2$ and $m(\mathbf{y}) = m(\mathbf{z}) = 1/4$. Note that this random walk is (m-)symmetric if and only if $\alpha = 2\zeta$, $\beta = 2\eta$ and $\gamma = 2\xi$.

The first homology group $\mathrm{H}_1(X_0, \mathbb{R})$ is spanned by the four 1-cycles

$$
[c_1] := [e_1 * \overline{e}_2], \quad [c_2] := [e_1 * \overline{e}_3], \quad [c_3] := [e_4 * \overline{e}_5], \quad [c_4] := [e_4 * \overline{e}_6].
$$

Then the homological direction is calculated as

$$
\gamma_p = \frac{\beta - 2\eta}{4}[c_1] + \frac{\alpha - 2\zeta}{4}[c_2] + \frac{\beta - 2\eta}{4}[c_3] + \frac{\gamma - 2\xi}{4}[c_4].
$$

The canonical surjective linear map $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{R}) \longrightarrow \mathfrak{g}^{(1)}$ is given by

$$
\rho_{\mathbb{R}}([c_1]) = X_1, \quad \rho_{\mathbb{R}}([c_2]) = X_2, \quad \rho_{\mathbb{R}}([c_3]) = -X_1, \quad \rho_{\mathbb{R}}([c_4]) = -X_2.
$$

Then we obtain

$$
\rho_{\mathbb{R}}(\gamma_p) = \frac{(\alpha - \gamma) - 2(\zeta - \xi)}{4} X_2. \tag{6.3.1}
$$

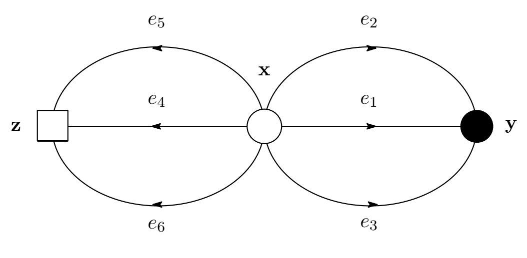

Figure 6.4: The quotient $X_0 = (V_0, E_0)$ of the 3D-Heisenberg dice graph $X = (V, E)$

Let $\{u_1, u_2\} \subset \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$ be the dual basis of $\{X_1, X_2\} \subset \mathfrak{g}^{(1)}$. We also denote by $\{\omega_1, \omega_2, \omega_3, \omega_4\} \subset \big(\mathrm{H}^1(X_0, \mathbb{R}), \langle\!\langle \cdot, \cdot \rangle\!\rangle_p\big)$ the dual basis of $\{[c_1], [c_2], [c_3], [c_4]\} \subset \mathrm{H}_1(X_0, \mathbb{R})$.

Namely, $\omega_i([c_j]) = \delta_{ij}$ for $1 \leq i, j \leq 4$. Then the modified harmonicity (2.4.1) yields

$$
\begin{aligned}
&\omega_1(e_1) = \beta - \frac{\beta - 2\eta}{4}, && \omega_1(e_2) = -(1-\beta) - \frac{\beta - 2\eta}{4}, && \omega_1(e_3) = \beta - \frac{\beta - 2\eta}{4},\\
&\omega_1(e_4) = -\frac{\beta - 2\eta}{4}, && \omega_1(e_5) = -\frac{\beta - 2\eta}{4}, && \omega_1(e_6) = -\frac{\beta - 2\eta}{4},\\
&\omega_2(e_1) = \alpha - \frac{\alpha - 2\zeta}{4}, && \omega_2(e_2) = \alpha - \frac{\alpha - 2\zeta}{4}, && \omega_2(e_3) = -(1-\alpha) - \frac{\alpha - 2\zeta}{4},\\
&\omega_2(e_4) = -\frac{\alpha - 2\zeta}{4}, && \omega_2(e_5) = -\frac{\alpha - 2\zeta}{4}, && \omega_2(e_6) = -\frac{\alpha - 2\zeta}{4},\\
&\omega_3(e_1) = -\frac{\beta - 2\eta}{4}, && \omega_3(e_2) = -\frac{\beta - 2\eta}{4}, && \omega_3(e_3) = -\frac{\beta - 2\eta}{4},\\
&\omega_3(e_4) = \beta - \frac{\beta - 2\eta}{4}, && \omega_3(e_5) = -(1-\beta) - \frac{\beta - 2\eta}{4}, && \omega_3(e_6) = \beta - \frac{\beta - 2\eta}{4},\\
&\omega_4(e_1) = -\frac{\gamma - 2\xi}{4}, && \omega_4(e_2) = -\frac{\gamma - 2\xi}{4}, && \omega_4(e_3) = -\frac{\gamma - 2\xi}{4},\\
&\omega_4(e_4) = \gamma - \frac{\gamma - 2\xi}{4}, && \omega_4(e_5) = \gamma - \frac{\gamma - 2\xi}{4}, && \omega_4(e_6) = -(1-\gamma) - \frac{\gamma - 2\xi}{4}.
\end{aligned}
$$

By direct computation, we have

$$
\begin{aligned}
&\langle\!\langle \omega_1, \omega_1 \rangle\!\rangle_p = \frac{\beta + 2\eta}{4} - \frac{(\beta + 2\eta)^2}{8}, && \langle\!\langle \omega_1, \omega_2 \rangle\!\rangle_p = -\frac{(\alpha + 2\zeta)(\beta + 2\eta)}{8},\\
&\langle\!\langle \omega_1, \omega_3 \rangle\!\rangle_p = -\frac{(\beta - 2\eta)^2}{8}, && \langle\!\langle \omega_1, \omega_4 \rangle\!\rangle_p = -\frac{(\beta - 2\eta)(\gamma - 2\xi)}{8},\\
&\langle\!\langle \omega_2, \omega_2 \rangle\!\rangle_p = \frac{\alpha + 2\zeta}{4} - \frac{(\alpha + 2\zeta)^2}{8}, && \langle\!\langle \omega_2, \omega_3 \rangle\!\rangle_p = -\frac{(\alpha - 2\zeta)(\beta - 2\eta)}{8},\\
&\langle\!\langle \omega_2, \omega_4 \rangle\!\rangle_p = -\frac{(\alpha - 2\zeta)(\gamma - 2\xi)}{8}, && \langle\!\langle \omega_3, \omega_3 \rangle\!\rangle_p = \frac{\beta + 2\eta}{4} - \frac{(\beta + 2\eta)^2}{8},\\
&\langle\!\langle \omega_3, \omega_4 \rangle\!\rangle_p = -\frac{(\beta + 2\eta)(\gamma + 2\xi)}{8}, && \langle\!\langle \omega_4, \omega_4 \rangle\!\rangle_p = \frac{\gamma + 2\xi}{4} - \frac{(\gamma + 2\xi)^2}{8}.
\end{aligned}
\tag{6.3.2}
$$

Since the linear space $\mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$ can be seen as a 2-dimensional subspace of $\mathrm{H}^1(X_0, \mathbb{R})$ through the injection ${}^t\rho_{\mathbb{R}}$, we see that $u_1 = {}^t\rho_{\mathbb{R}}(u_1) = \omega_1 - \omega_3$ and $u_2 = {}^t\rho_{\mathbb{R}}(u_2) = \omega_2 - \omega_4$ form a $\mathbb{Z}$-basis in $\mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$. We then obtain

$$
\langle\!\langle u_1, u_1 \rangle\!\rangle_p = \frac{\beta + 2\eta - 4\beta\eta}{2}, \quad \langle\!\langle u_1, u_2 \rangle\!\rangle_p = -\frac{\beta + 2\eta - 4\beta\eta}{4},
$$
$$
\langle\!\langle u_2, u_2 \rangle\!\rangle_p = \frac{(\beta + 2\eta)(2 - \beta - 2\eta) + 4\alpha\gamma + 16\xi\zeta}{8}.
$$

by (6.3.2). Thus the volume of the Albanese torus is computed as

$$
\mathrm{vol}(\mathrm{Alb}^{\Gamma})^{-1} = \frac{1}{4}\sqrt{(\beta + 2\eta - 4\beta\eta)\big\{(\beta + 2\eta) - (\beta^2 + 4\eta^2) + 4\alpha\gamma + 16\xi\zeta\big\}}.
$$

Furthermore, the Albanese metric g_0 on $\mathfrak{g}^{(1)}$ is given by

$$
\langle X_1, X_1 \rangle_{g_0} = \frac{(\beta + 2\eta)(2 - \beta - 2\eta) + 4\alpha\gamma + 16\xi\zeta}{8}\mathrm{vol}(\mathrm{Alb}^{\Gamma}),
$$

$$\langle X_1, X_2\rangle_{g_0} = \frac{\beta+2\eta-4\beta\eta}{4}\mathrm{vol}(\mathrm{Alb}^{\Gamma}), \quad \langle X_2, X_2\rangle_{g_0} = \frac{\beta+2\eta-4\beta\eta}{2}\mathrm{vol}(\mathrm{Alb}^{\Gamma}).$$

We now determine the modified standard realization $\Phi_0 : X \longrightarrow G = \mathbb{H}^3(\mathbb{R})$. Let $\widetilde{e}_i\,(i=1,2,3,4,5,6)$ be a lift of $e_i \in E_0$ to X and put $\Phi_0\big(o(\widetilde{e}_i)\big) = \mathbf{1}_G$. Then it follows from (2.4.4) and (6.3.1) that the Γ-equivariant realization $\Phi_0 : X \longrightarrow G$ satisfying

$$\begin{aligned}
\Phi_0\big(t(\widetilde{e}_1)\big) &= \Big(\beta, \frac{(3\alpha+\gamma)+2(\zeta-\xi)}{4}, \kappa_1\Big),\\
\Phi_0\big(t(\widetilde{e}_2)\big) &= \Big(\beta-1, \frac{(3\alpha+\gamma)+2(\zeta-\xi)}{4}, \kappa_1 - \frac{(3\alpha+\gamma)+2(\zeta-\xi)}{4}\Big),\\
\Phi_0\big(t(\widetilde{e}_3)\big) &= \Big(\beta, \frac{(3\alpha+\gamma)+2(\zeta-\xi)}{4} - 1, \kappa_1\Big),\\
\Phi_0\big(t(\widetilde{e}_4)\big) &= \Big(-\beta, \frac{-(\alpha+3\gamma)+2(\zeta-\xi)}{4}, -\kappa_2\Big),\\
\Phi_0\big(t(\widetilde{e}_5)\big) &= \Big(1-\beta, \frac{-(\alpha+3\gamma)+2(\zeta-\xi)}{4}, -\kappa_2 + \frac{-(\alpha+3\gamma)+2(\zeta-\xi)}{4}\Big),\\
\Phi_0\big(t(\widetilde{e}_6)\big) &= \Big(-\beta, \frac{-(\alpha+3\gamma)+2(\zeta-\xi)}{4} + 1, -\kappa_2\Big)
\end{aligned}$$

is the modified harmonic realization, where κ_1, κ_2 is two real parameters which indicates the ambiguity of the realization corresponding to $\mathfrak{g}^{(2)}$. Let $\{v_1, v_2\}$ be the Gram–Schmidt orthonormalization of the basis $\{u_1, u_2\}$, that is,

$$v_1 = \langle\!\langle u_1, u_1\rangle\!\rangle_p^{-1/2} u_1, \quad v_2 = \langle\!\langle u_1, u_1\rangle\!\rangle_p^{1/2}\mathrm{vol}(\mathrm{Alb}^{\Gamma})\Big(u_2 - \frac{\langle\!\langle u_1, u_2\rangle\!\rangle_p}{\langle\!\langle u_1, u_1\rangle\!\rangle_p}u_1\Big),$$

and $\{V_1, V_2\} \subset \mathfrak{g}^{(1)}$ its dual basis. We write $V_3 := [V_1, V_2] = V_1V_2 - V_2V_1$. Then we obtain

$$v_1 = \Big(\frac{\beta+2\eta-4\beta\eta}{2}\Big)^{-1/2} u_1, \quad v_2 = \Big(\frac{\beta+2\eta-4\beta\eta}{2}\Big)^{1/2}\mathrm{vol}(\mathrm{Alb}^{\Gamma})\Big(u_2 + \frac{1}{2}u_1\Big)$$

by (6.3.2). Moreover, we have

$$\begin{aligned}
V_1 &= \Big(\frac{\beta+2\eta-4\beta\eta}{2}\Big)^{1/2} X_1 - \frac{1}{2}\Big(\frac{\beta+2\eta-4\beta\eta}{2}\Big)^{1/2} X_2,\\
V_2 &= \Big(\frac{\beta+2\eta-4\beta\eta}{2}\Big)^{-1/2}\mathrm{vol}(\mathrm{Alb}^{\Gamma})^{-1} X_2,\\
V_3 &= \mathrm{vol}(\mathrm{Alb}^{\Gamma})^{-1} X_3.
\end{aligned}$$

Finally, we see that $\beta(\Phi_0) \in \mathfrak{g}^{(2)}$ and the infinitesimal generator $\mathcal{A}$ are calculated as

$$\begin{aligned}
\beta(\Phi_0) &= \sum_{i=1}^{6} \big(\widetilde{m}(e_i) - \widetilde{m}(\overline{e}_i)\big)\log\Big(d\Phi_0(\widetilde{e}_i)\cdot\exp\big(-\rho_{\mathbb{R}}(\gamma_p)\big)\Big)\Big|_{\mathfrak{g}^{(2)}} = \frac{\beta-2\eta}{8}\mathrm{vol}(\mathrm{Alb}^{\Gamma})V_3,\\
\mathcal{A} &= -\frac{1}{2}(V_1^2 + V_2^2) - \frac{\beta-2\eta}{8}\mathrm{vol}(\mathrm{Alb}^{\Gamma})V_3.
\end{aligned}$$

We should observe that the coefficient of $\beta(\Phi_0)$ does not include the parameters κ_1 and κ_2, though the realization Φ_0 has the ambiguity of $\mathfrak{g}^{(2)}$-components.

Finally, we find the transition probability $\mathfrak{p} : E \longrightarrow (0,1]$ defined by (4.4.2), when

$$\xi = \frac{1}{12}, \quad \eta = \frac{1}{6}, \quad \zeta = \frac{1}{4}, \quad \alpha = \frac{1}{6}, \quad \beta = \frac{1}{3}, \quad \gamma = \frac{1}{2}.$$

Then we have

$$\gamma_p = -\frac{1}{12}[c_2] + \frac{1}{12}[c_4], \quad \rho_{\mathbb{R}}(\gamma_p) = -\frac{1}{6}X_2, \quad \mathrm{vol}(\mathrm{Alb}^{\Gamma}) = \frac{9\sqrt{10}}{5}.$$

Now consider the function $F = F_x(\lambda) : V_0 \times \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R}) \longrightarrow (0,\infty)$ defined by (4.4.1). It is useful to write each $\log\big(d\Phi_0(\widetilde{e}_i)\big)\big|_{\mathfrak{g}^{(1)}}$ $(i = 1,2,3,4,5,6)$ in terms of the Albanese basis $\{V_1, V_2\}$, that is,

$$\begin{aligned}
&\log\big(d\Phi_0(\widetilde{e}_1)\big)\big|_{\mathfrak{g}^{(1)}} = \frac{\sqrt{2}}{2}V_1 + \frac{3\sqrt{5}}{5}V_2, &&\log\big(d\Phi_0(\widetilde{e}_2)\big)\big|_{\mathfrak{g}^{(1)}} = -\sqrt{2}V_1,\\
&\log\big(d\Phi_0(\widetilde{e}_3)\big)\big|_{\mathfrak{g}^{(1)}} = \frac{\sqrt{2}}{2}V_1 - \frac{3\sqrt{5}}{5}V_2, &&\log\big(d\Phi_0(\widetilde{e}_4)\big)\big|_{\mathfrak{g}^{(1)}} = -\frac{\sqrt{2}}{2}V_1 - \frac{3\sqrt{5}}{5}V_2,\\
&\log\big(d\Phi_0(\widetilde{e}_5)\big)\big|_{\mathfrak{g}^{(1)}} = \sqrt{2}V_1, &&\log\big(d\Phi_0(\widetilde{e}_6)\big)\big|_{\mathfrak{g}^{(1)}} = -\frac{\sqrt{2}}{2}V_1 + \frac{3\sqrt{5}}{5}V_2.
\end{aligned}$$

We write $\lambda = \lambda_1 v_1 + \lambda_2 v_2 \in \mathrm{Hom}(\mathfrak{g}^{(1)}, \mathbb{R})$. Then one has

$$\begin{aligned}
F_{\mathbf{x}}(\lambda) &= \frac{1}{12}\exp\Big(\frac{\sqrt{2}}{2}\lambda_1 + \frac{3\sqrt{5}}{5}\lambda_2\Big) + \frac{1}{6}\exp\Big(-\sqrt{2}\lambda_1\Big) + \frac{1}{4}\exp\Big(\frac{\sqrt{2}}{2}\lambda_1 - \frac{3\sqrt{5}}{5}\lambda_2\Big)\\
&\quad + \frac{1}{4}\exp\Big(-\frac{\sqrt{2}}{2}\lambda_1 - \frac{3\sqrt{5}}{5}\lambda_2\Big) + \frac{1}{6}\exp\Big(\sqrt{2}\lambda_1\Big) + \frac{1}{12}\exp\Big(-\frac{\sqrt{2}}{2}\lambda_1 + \frac{3\sqrt{5}}{5}\lambda_2\Big),\\
F_{\mathbf{y}}(\lambda) &= \frac{1}{2}\exp\Big(-\frac{\sqrt{2}}{2}\lambda_1 - \frac{3\sqrt{5}}{5}\lambda_2\Big) + \frac{1}{3}\exp\Big(\sqrt{2}\lambda_1\Big) + \frac{1}{6}\exp\Big(-\frac{\sqrt{2}}{2}\lambda_1 + \frac{3\sqrt{5}}{5}\lambda_2\Big),\\
F_{\mathbf{z}}(\lambda) &= \frac{1}{6}\exp\Big(\frac{\sqrt{2}}{2}\lambda_1 + \frac{3\sqrt{5}}{5}\lambda_2\Big) + \frac{1}{3}\exp\Big(-\sqrt{2}\lambda_1\Big) + \frac{1}{2}\exp\Big(\frac{\sqrt{2}}{2}\lambda_1 - \frac{3\sqrt{5}}{5}\lambda_2\Big).
\end{aligned}$$

To find minimizers of the functions $F_{\mathbf{x}}(\cdot)$, $F_{\mathbf{y}}(\cdot)$ and $F_{\mathbf{z}}(\cdot)$, we solve the following equations

$$\begin{cases}(\partial/\partial\lambda_1)F_{\mathbf{x}}(\lambda_1,\lambda_2) = 0,\\ (\partial/\partial\lambda_2)F_{\mathbf{x}}(\lambda_1,\lambda_2) = 0,\end{cases} \quad \begin{cases}(\partial/\partial\lambda_1)F_{\mathbf{y}}(\lambda_1,\lambda_2) = 0,\\ (\partial/\partial\lambda_2)F_{\mathbf{y}}(\lambda_1,\lambda_2) = 0,\end{cases} \quad \begin{cases}(\partial/\partial\lambda_1)F_{\mathbf{z}}(\lambda_1,\lambda_2) = 0,\\ (\partial/\partial\lambda_2)F_{\mathbf{z}}(\lambda_1,\lambda_2) = 0.\end{cases}$$

Then we obtain

$$\lambda_*(\mathbf{x}) = \Big(0, \frac{\sqrt{5}}{6}\log 3\Big),$$

$$\lambda_*(\mathbf{y}) = \Big(-\frac{\sqrt{2}}{3}\log\frac{2\sqrt{3}}{3}, \frac{\sqrt{5}}{6}\log 3\Big), \quad \lambda_*(\mathbf{z}) = \Big(\frac{\sqrt{2}}{3}\log\frac{2\sqrt{3}}{3}, \frac{\sqrt{5}}{6}\log 3\Big).$$

Hence we find

$$F_{\mathbf{x}}\big(\lambda_*(\mathbf{x})\big) = \frac{\sqrt{3}+1}{3}, \quad F_{\mathbf{y}}\big(\lambda_*(\mathbf{y})\big) = F_{\mathbf{z}}\big(\lambda_*(\mathbf{z})\big) = 3 \cdot 6^{-2/3}$$

and the transition probability $\mathfrak{p} : E_0 \longrightarrow (0,1]$ is given by

$$\mathfrak{p}(e_1) = \frac{3-\sqrt{3}}{8}, \qquad \mathfrak{p}(e_2) = \frac{\sqrt{3}-1}{4}, \qquad \mathfrak{p}(e_3) = \frac{3-\sqrt{3}}{8},$$
$$\mathfrak{p}(e_4) = \frac{3-\sqrt{3}}{8}, \qquad \mathfrak{p}(e_5) = \frac{\sqrt{3}-1}{4}, \qquad \mathfrak{p}(e_6) = \frac{3-\sqrt{3}}{8},$$
$$\mathfrak{p}(\overline{e}_1) = \mathfrak{p}(\overline{e}_2) = \mathfrak{p}(\overline{e}_3) = \mathfrak{p}(\overline{e}_4) = \mathfrak{p}(\overline{e}_5) = \mathfrak{p}(\overline{e}_6) = \frac{1}{3}.$$

Furthermore, we also obtain $\mathfrak{m}(\mathbf{x}) = 1/2$, $\mathfrak{m}(\mathbf{y}) = \mathfrak{m}(\mathbf{z}) = 1/4$. Therefore, the homological direction $\gamma_{\mathfrak{p}}$ is computed as

$$\gamma_{\mathfrak{p}} = \sum_{i=1}^{6} \big(\widetilde{\mathfrak{m}}(e_i) - \widetilde{\mathfrak{m}}(\overline{e}_i)\big) e_i = \frac{5-3\sqrt{3}}{48}\big(2[c_1] - [c_2] + 2[c_3] - [c_4]\big).$$

This implies

$$\rho_{\mathbb{R}}(\gamma_{\mathfrak{p}}) = \frac{5-3\sqrt{3}}{48}\big\{2X_1 - X_2 + 2 \times (-X_1) - (-X_2)\big\} = \mathbf{0}_{\mathfrak{g}}.$$

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