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COMPLEX INTERPOLATION OF SMOOTHNESS TRIEBEL-LIZORKIN-MORREY SPACES

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Abstract. This paper extends the result in [8] to Triebel-Lizorkin-Morrey spaces which contains 4 parameters p, q, r, s. This paper reinforces our earlier paper [8] by Nakamura, the first and the third authors in two different directions. First, we include the smoothness parameter s and the second smoothness parameter r. In [8] we assumed $s = 0$ and $r = 2$. Here we relax the conditions on s and r to $s \in \mathbb{R}$ and $1 < r \leq \infty$. Second, we apply a formula obtained by Bergh in 1978 to prove our main theorem without using the underlying sequence spaces.

1. Introduction

In [38], Yuan, Sickel and Yang defined the diamond subspace of the smoothness Morrey spaces. The smoothness Morrey spaces are (recent) generic names of Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces. We aim to decribe the complex interpolation of a class of subspaces of smoothness Morrey spaces defined in [38], which extend the results in [8]. Let $1 < q \leq p < \infty$. For an $L^q_{\rm loc}$-function f, its Morrey norm is defined by:

$$\|f\|_{\mathcal{M}^p_q} := \sup_{x\in\mathbb{R}^n,\, R>0} |B(x,R)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,R)} |f(y)|^q \, dy\right)^{\frac{1}{q}}, \tag{1.1}$$

where $B(x,R)$ denotes the ball centered at $x \in \mathbb{R}^n$ of radius $R > 0$. The Morrey space $\mathcal{M}^p_q$ is the set of all L^q-locally integrable functions f for which the norm $\|f\|_{\mathcal{M}^p_q}$ is finite. We recall the definition of Triebel–Lizorkin–Morrey spaces as follows. Let $1 < q \leq p < \infty$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$. Choose $\varphi \in \mathcal{S}$ so that $\chi_{B(2)} \leq \varphi \leq \chi_{B(3)}$ holds. Set

$$\varphi_0 := \varphi \tag{1.2}$$

and

$$\varphi_j := \varphi(2^{-j}\cdot) - \varphi(2^{-j+1}\cdot) \tag{1.3}$$

for $j \in \mathbb{N}$. Note that φ_j satisfies

$$\sum_{j=0}^{\infty} \varphi_j = 1. \tag{1.4}$$

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Next, we write

$$\varphi_j(D)g = \mathcal{F}^{-1}(\varphi_j \mathcal{F}g)$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, defined by

$$\begin{cases} \mathcal{F}g(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(x)e^{-ix\cdot\xi}dx & (\xi \in \mathbb{R}^n) \\ \mathcal{F}^{-1}g(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(\xi)e^{ix\cdot\xi}d\xi & (x \in \mathbb{R}^n), \end{cases}$$

for $g \in L^1(\mathbb{R}^n)$. Now, for $f \in \mathcal{S}'$, we define

$$\|f\|_{\mathcal{E}^s_{pqr}} := \|\varphi_0(D)f\|_{\mathcal{M}^p_q} + \left\| \left(\sum_{j=1}^{\infty} 2^{jrs}|\varphi_j(D)f|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q}. \tag{1.5}$$

The *Triebel–Lizorkin–Morrey space* $\mathcal{E}^s_{pqr}$ is the set of all $f \in \mathcal{S}'$ for which the norm $\|f\|_{\mathcal{E}^s_{pqr}}$ is finite. The parameters r and s are sometimes called the second smoothness parameter and the smoothness parameter, respectively. Remark that the definition of $\mathcal{E}^s_{pqr}$ does not depend on the choice of the function φ (see [20, Theorem 1.4] or [29]).

We are interested in the following closed subspace of $\mathcal{E}^s_{pqr}$:

Definition 1. [38, Definition 2.23](smoothness space) Let $1 < q \le p < \infty$ and $1 \le r \le \infty$. The space $\overset{\diamond}{\mathcal{E}}{}^s_{pqr}$ denotes the closure with respect to $\mathcal{E}^s_{pqr}$ of the set of all smooth functions f such that $\partial^\alpha f \in \mathcal{E}^s_{pqr}$ for all multi-indices α.

We characterize $\overset{\diamond}{\mathcal{E}}{}^s_{pqr}$ in terms of the Littlewood-Paley decomposition, which is a starting point of this paper.

Theorem 1.1. *Let $1 < q \le p < \infty$, $1 \le r \le \infty$, and $f \in \mathcal{E}^s_{pqr}$. Then f is in $\overset{\diamond}{\mathcal{E}}{}^s_{pqr}$, if and only if $\sum_{j=0}^N \varphi_j(D)f$ converges to f as $N \to \infty$ in $\mathcal{E}^s_{pqr}$.*

We seek to describe the first and the second complex interpolation spaces of $\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}$ and $\overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}$, where the parameters $p_0, p_1, q_0, q_1, r_0, r_1$ satisfy

$$\begin{aligned} p_0 > p_1, \quad 1 < q_0 \le p_0 < \infty, \quad 1 < q_1 \le p_1 < \infty, \\ 1 < r_0, r_1 < \infty, \quad \frac{p_0}{q_0} = \frac{p_1}{q_1}. \end{aligned} \tag{1.6}$$

Here, we may assume $p_0 > p_1$ due to symmetry between p_0 and p_1. To state our main result, we need the following notation. Let (X_0, X_1) be a compatible couple of Banach spaces. Let $[X_0, X_1]_\theta$ and $[X_0, X_1]^\theta$ be the first and second Calderón complex interpolation space whose definition we recall in Section 2. For $\theta \in (0,1)$, define p, q, r and s by:

(1.7)
$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} := \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad s := (1-\theta)s_0 + \theta s_1.$$

A direct consequence of (1.6) and (1.7) is

$$\frac{p}{q} = \frac{p_0}{q_0} = \frac{p_1}{q_1}. \tag{1.8}$$

For $f \in \mathcal{S}'$, we define

$$S(f;r,s) := \left(\sum_{j=0}^{\infty} |2^{js} \varphi_j(D) f|^r \right)^{\frac{1}{r}},$$

$$S(f;a,J,r,s) := \chi_{[a,a^{-1}]}(S(f;r,s)) \left(\sum_{j=J}^{\infty} |2^{js} \varphi_j(D) f|^r \right)^{\frac{1}{r}}.$$

Based on this notation, we state our main results as follows:

Theorem 1.2. *Suppose that we have* 13 *parameters* $p_0, p_1, p, q_0, q_1, q, r, r_0, r_1, s, s_0, s_1$, *and* θ *satisfying* (1.6) *and* (1.7).

(1) *We have*

$$[\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0 q_0 r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1 q_1 r_1}]_\theta = \overset{\diamond}{\mathcal{E}}{}^{s}_{pqr} \cap [\mathcal{E}^{s_0}_{p_0 q_0 r_0}, \mathcal{E}^{s_1}_{p_1 q_1 r_1}]_\theta. \tag{1.9}$$

(2) *If* $r_0 = r_1$ *and* $s_0 = s_1$, *then*

(1.10)
$$[\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0 q_0 r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1 q_1 r_1}]^\theta = \bigcap_{0<a<1} \left\{ f \in \mathcal{E}^s_{pqr} \,:\, \lim_{J \to \infty} \|S(f;a,J,r,s)\|_{\mathcal{M}^p_q} = 0 \right\}.$$

Theorem 1.3. *Suppose that we have* 13 *parameters* $p_0, p_1, p, q_0, q_1, q, r, r_0, r_1, s, s_0, s_1$, *and* θ *satisfying* (1.6) *and* (1.7). *Then we have*

$$[\mathcal{E}^{s_0}_{p_0 q_0 r_0}, \mathcal{E}^{s_1}_{p_1 q_1 r_1}]^\theta = \mathcal{E}^s_{pqr} \tag{1.11}$$

with equivalence of norms.

Having stated the main result in this paper, let us investigate its relation with the existing results. The corresponding results for the first complex interpolation of Triebel–Lizorkin–Morrey spaces was obtained by Yang, Yuan and Zhuo (see [35, Corollary 1.11]). They proved the following theorem.

Theorem 1.4. [35, Corollary 1.11] *Suppose that we have* 13 *parameters* p_0, p_1, p, q_0, q_1, q, r, r_0, r_1, s, s_0, s_1, *and* θ *satisfying* (1.6) *and* (1.7). *Then*

$$[\mathcal{E}^{s_0}_{p_0 q_0 r_0}, \mathcal{E}^{s_1}_{p_1 q_1 r_1}]_\theta \subseteq \mathcal{E}^s_{pqr}. \tag{1.12}$$

Remark that (1.12) will be used in the proof of Theorem 1.3. As a corollary of (2.4) to follow and Theorem 1.3, we have the corresponding result for the first complex interpolation of Tribel–Lizorkin–Morrey spaces which refines (1.12).

Theorem 1.5. *Suppose that we have* 13 *parameters* p_0, p_1, p, q_0, q_1, q, r, r_0, r_1, s, s_0, s_1, *and* θ *satisfying* (1.6) *and* (1.7)*. Then*

$$[\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1}]_\theta = \overline{\mathcal{E}^{s_0}_{p_0q_0r_0} \cap \mathcal{E}^{s_1}_{p_1q_1r_1}}^{\mathcal{E}^s_{pqr}}. \tag{1.13}$$

Meanwhile, Nakamura, the first and the third authors obtained the description of the interpolation of diamond Morrey spaces in [8], which we describe below. Let $1 < q \le p < \infty$. The space $\overset{\diamond}{\mathcal{M}}{}^p_q$ denotes the closure with respect to $\mathcal{M}^p_q$ of the set of all smooth functions f such that $\partial^\alpha f \in \mathcal{M}^p_q$ for all multi-indices α [38].

Due to the result by Mazzucato [17, Proposition 4.1], we see that

$$\mathcal{M}^p_q = \mathcal{E}^0_{pq2}.$$

Thus, $\overset{\diamond}{\mathcal{E}}{}^0_{pq2} = \overset{\diamond}{\mathcal{M}}{}^p_q$ with norm equivalence and Theorem 1.2 recaptures the interpolation of $\overset{\diamond}{\mathcal{M}}{}^{p_0}_{q_0}$ and $\overset{\diamond}{\mathcal{M}}{}^{p_1}_{q_1}$ as the special case of $r_0 = r_1 = r = 2$ and $s_0 = s_1 = s = 0$. Thus, we see that Theorem 1.2 extends [8, Theorem 1.9]

One of the difficulties in dealing with the space $\overset{\diamond}{\mathcal{M}}{}^p_q$ with $1 < q < p < \infty$ is that this closed subspace does not enjoy the lattice property unlike many other important subspaces defined in [6, 27, 38].

Let us now recall some progress in interpolation theory of Morrey spaces. The earlier result about the interpolation of Morrey spaces can be traced back in [28]. In [7, p. 35] Cobos, Peetre, and Persson pointed out that

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subset \mathcal{M}^p_q,$$

whenever $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$, and $1 \le q \le p < \infty$ satisfy

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{1.14}$$

A counterexample by Blasco, Ruiz, and Vega [3, 22], shows that if we assume (1.14) only, then there exists a bounded linear operator T from $\mathcal{M}^{p_k}_{q_k}(\mathbb{R}^n)$ $(k = 0, 1)$ to $L^1(\mathbb{R}^n)$, but T is unbounded from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. By using the counterexample by Ruiz and Vega in [22], Lemarié-Rieusset [14, Theorem 3(ii)] showed that if an interpolation functor F satisfies $F[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}] = \mathcal{M}^p_q$ under the condition (1.14), then

$$\frac{q_0}{p_0} = \frac{q_1}{p_1} \tag{1.15}$$

holds. Lemarié-Rieusset [14, 15] also showed that Morrey space is closed under the second complex interpolation method, namely,

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]^\theta = \mathcal{M}^p_q. \tag{1.16}$$

Meanwhile, as for the interpolation result under (1.14) and (1.15) by using the first Calderón complex interpolation functor, Lu, Yang, and Yuan obtained the following description:

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q} \tag{1.17}$$

in [16, Theorem 1.2]. Their result is in the setting of a metric measure space. The generalization of the result of Lu et. al and Lemarié-Rieusset in the setting of generalized Morrey spaces and generalized Orlicz–Morrey spaces can be seen in [9]. The first and third authors [10] also obtain a refinement of (1.17) as follows:

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \left\{ f \in \mathcal{M}^p_q \,:\, \lim_{a \to 0^+} \|(1 - \chi_{[a,a^{-1}]}(|f|))f\|_{\mathcal{M}^p_q} = 0 \right\}. \tag{1.18}$$

The complex interpolation of variable exponent Morrey spaces can be found in [18]. As for the real interpolation results, Burenkov and Nursultanov obtained an interpolation result in local Morrey spaces [4] and their results are generalized by Nakai and Sobukawa to B^u_w setting [19]. In [35], Yang, Yuan, and Zhuo considered the interpolation of smoothness Morrey spaces considered in [11, 12, 13, 17, 20, 23, 26, 29, 32, 33, 36, 37, 38].

If we compare this paper with the work, we believe that the main tool is Lemma 2.9, where the function "log" plays the key role. An experience obtained in [9] shows that the function "log" is essential when we consider the complex interpolation functor.

Let us explain why the interpolation of Morrey spaces are complicated unlike Lebesgue spaces. From (1.16) and (1.18) we learn that the first complex interpolation functor behaves differently from Lebesgue spaces. This problem comes basically from the fact that the Morrey norm $\mathcal{M}^p_q$ involves the supremum over all balls $B(x, R)$. Due to this fact, we have many difficulties when $1 < q < p < \infty$, namely:

(1) The Morrey space $\mathcal{M}^p_q$ is not included in $L^1 + L^\infty$; see [10, Section 6].
(2) The Morrey space $\mathcal{M}^p_q$ is not reflexive; see [27, Example 5.2] and [34, Theorem 1.3].
(3) Let p_0, p_1, p, q_0, q_1, q satisfy (1.6). Let $q < \tilde{q} < p$. The spaces C^∞_c, $\mathcal{M}^p_{\tilde{q}}$, $\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}$ are not dense in $\mathcal{M}^p_q$; see [31, Proposition 2.16], [24] and [9, 38], respectively.
(4) The Morrey space $\mathcal{M}^p_q$ is not separable; see [31, Proposition 2.16].

These facts prevent us from using many theorems in the textbook in [1].

We organize the remaining part of this paper as follows: Section 2 collects some preliminary facts such as the property of the complex interpolation and the maximal inequalities for Morrey spaces. We prove Theorems 1.2 and 1.3 in Section 3 except a key fact on G defined in Section 3. This fact will be proved in Section 4.

2. Preliminaries

2.1. **Complex interpolation functors.** Let E be a subset of $\mathbb{C}$ and X be a Banach space, and define

$$\begin{aligned} &\mathrm{BC}(E,X) \\ &:= \left\{ f : E \to X \,:\, f \text{ is continous and satisfies } \sup_{z\in E}\|f(z)\|_X < \infty \right\}. \end{aligned} \tag{2.1}$$

If E is an open set in $\mathbb{C}$, then $\mathcal{O}(E,X)$ denotes the set of all holomorphic functions on E whose value assumes X.

Definition 2. Let $U := \{z \in \mathbb{C} \,:\, 0 < \mathrm{Re}\,(z) < 1\}$ and $\overline{U}$ be its closure.

We recall the definition of the complex interpolation functors as follows:

Definition 3 (Calderón's first complex interpolation space, [1, 5])**.** Let (X_0, X_1) be a compatible couple of Banach spaces.

(1) The space $\mathcal{F}(X_0, X_1)$ is defined as the set of all functions $F : \bar{U} \to X_0 + X_1$ such that
  (a) $F \in \mathrm{BC}(\overline{U}, X_0 + X_1)$,
  (b) $F \in \mathcal{O}(U, X_0 + X_1)$,
  (c) the functions $t \in \mathbb{R} \mapsto F(j + it) \in X_j$ are bounded and continuous on $\mathbb{R}$ for $j = 0, 1$.

  The space $\mathcal{F}(X_0, X_1)$ is equipped with the norm
$$\|F\|_{\mathcal{F}(X_0,X_1)} := \max\left\{ \sup_{t\in\mathbb{R}} \|F(it)\|_{X_0},\ \sup_{t\in\mathbb{R}} \|F(1+it)\|_{X_1} \right\}.$$

(2) Let $\theta \in (0, 1)$. Define the complex interpolation space $[X_0, X_1]_\theta$ with respect to (X_0, X_1) to be the set of all functions $f \in X_0 + X_1$ such that $f = F(\theta)$ for some $F \in \mathcal{F}(X_0, X_1)$. The norm on $[X_0, X_1]_\theta$ is defined by
$$\|f\|_{[X_0,X_1]_\theta} := \inf\{\|F\|_{\mathcal{F}(X_0,X_1)} : f = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\}.$$

According to [5], $[X_0, X_1]_\theta$ is a Banach space. See also [1, Theorem 4.1.2].

Now, we recall the definition of Calderón's second complex interpolation space. Let X be a Banach space. The space $\mathrm{Lip}(\mathbb{R}, X)$ is defined to be the set of all functions $F:\mathbb{R}\to X$ for which

$$\|F\|_{\mathrm{Lip}(\mathbb{R},X)} := \sup_{-\infty<s<t<\infty} \frac{\|F(t)-F(s)\|_X}{|t-s|} < \infty.$$

Definition 4 (Calderón's second complex interpolation space, [1, 5])**.** Suppose that we have a pair (X_0, X_1) is a compatible couple of Banach spaces.

(1) Define $\mathcal{G}(X_0, X_1)$ as the set of all functions $G:\overline{U}\to X_0+X_1$ such that

(a) G is continuous on $\overline{U}$ and $\sup\limits_{z\in\overline{U}} \left\| \frac{G(z)}{1+|z|} \right\|_{X_0+X_1} < \infty$,

(b) G is holomorphic in U,

(c) the functions

$$t\in\mathbb{R}\mapsto G(j+it)-G(j)\in X_j$$

are Lipschitz continuous on $\mathbb{R}$ for $j=0,1$.

The space $\mathcal{G}(X_0, X_1)$ is equipped with the norm

$$\|G\|_{\mathcal{G}(X_0,X_1)} := \max\left\{\|G(i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_0)},\ \|G(1+i\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_1)}\right\}. \tag{2.2}$$

(2) Let $\theta\in(0,1)$. Define the complex interpolation space $[X_0,X_1]^\theta$ with respect to (X_0,X_1) to be the set of all functions $f\in X_0+X_1$ such that

$$f = G'(\theta) = \lim_{h\to 0}\frac{G(\theta+h)-G(\theta)}{h} \tag{2.3}$$

for some $G\in\mathcal{G}(X_0,X_1)$. The norm on $[X_0,X_1]^\theta$ is defined by

$$\|f\|_{[X_0,X_1]^\theta} := \inf\{\|G\|_{\mathcal{G}(X_0,X_1)} : f=G'(\theta) \text{ for some } G\in\mathcal{G}(X_0,X_1)\}.$$

The space $[X_0,X_1]^\theta$ is called Calderón's second complex interpolation space, or the upper complex interpolation space of (X_0,X_1).

One of the fundamental relations between the first and the second complex interpolation functors is as follows:

$$[X_0,X_1]_\theta = \overline{X_0\cap X_1}^{[X_0,X_1]^\theta} \tag{2.4}$$

according to the result by Bergh [2]. This relation explains why we start by calculating the second interpolation in the proof of Theorems 1.3 and 1.5.

If we combine Lemmas 2.1 and 2.2 below, we see that (2.4) follows.

Lemma 2.1. [2] *Let $x\in X_0\cap X_1$. Then* $\|x\|_{[X_0,X_1]^\theta} = \|x\|_{[X_0,X_1]_\theta}$.

Lemma 2.2. [1, Theorem 4.22 (a)] *The space $X_0\cap X_1$ is dense in $[X_0,X_1]_\theta$.*

A direct consequence of Lemma 2.2 is:

Lemma 2.3. $[X_0,X_1]^\theta \subseteq \overline{X_0\cap X_1}^{X_0+X_1}$.

Proof. We observe that $[X_0, X_1]^\theta \subset \overline{[X_0, X_1]_\theta}^{X_0+X_1}$ from the definition of $[X_0, X_1]^\theta$; see (2.3). In fact, for $f \in [X_0, X_1]^\theta$, there exists $G \in \mathcal{G}(X_0, X_1)$ such that $f = G'(\theta)$. We define

$$F_j(z) := \frac{G(z + ij^{-1}) - G(z)}{ij^{-1}}$$

for $j \in \mathbb{N}$ and $z \in \overline{S}$. Then, $F_j(\theta) \in [X_0, X_1]_\theta$ and according to (2.3), we have

$$f \in \overline{[X_0, X_1]_\theta}^{X_0+X_1}.$$

Since

$$[X_0, X_1]_\theta = \overline{X_0 \cap X_1}^{[X_0,X_1]_\theta} \subset \overline{X_0 \cap X_1}^{X_0+X_1}$$

from Lemma 2.2, it follows that $\overline{[X_0, X_1]_\theta}^{X_0+X_1} \subset \overline{X_0 \cap X_1}^{X_0+X_1}$. Putting together these observations, we obtain the desired result. □

2.2. **Operators on Morrey spaces.** Let $\mathcal{B}$ denote the set of all balls in $\mathbb{R}^n$. We recall the definition and the fundamental property of the Hardy-Littlewood maximal operator M.

Definition 5 (Hardy-Littlewood maximal operator)**.** For a measurable function f, define a function Mf by:

$$Mf(x) := \sup_{B \in \mathcal{B}} \frac{\chi_B(x)}{|B|} \int_B |f(y)|\, dy. \tag{2.5}$$

The mapping $M : f \mapsto Mf$ is called the Hardy-Littlewood maximal operator.

Theorem 2.4. [25, Theorem 2.4], [29, Lemma 2.5] *Suppose that the parameters p, q, r satisfy*

$$1 < q \le p < \infty \text{ and } 1 < r \le \infty.$$

Then

$$\left\| \left(\sum_{j=1}^{\infty} (Mf_j)^r \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} \tag{2.6}$$

for every sequence of measurable functions $\{f_j\}_{j=0}^{\infty}$. When $r = \infty$, then (2.6) *reads;*

$$\left\| \sup_{j \in \mathbb{Z}} Mf_j \right\|_{\mathcal{M}^p_q} \lesssim \left\| \sup_{j \in \mathbb{Z}} |f_j| \right\|_{\mathcal{M}^p_q}. \tag{2.7}$$

As a direct consequence of Theorem 2.4, we have the following lemma.

Lemma 2.5. *Let $1 < q \le p < \infty$, $1 < r \le \infty$, and $J \in \mathbb{N}$. Let $\{g_j\}_{j=J}^{\infty}$ be a sequence of measurable functions such that*

$$\left\| \left(\sum_{j=J}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^p} < \infty.$$

Then

$$\left\| \left(\sum_{l=1}^{\infty} \left| \varphi_l(D) \left(\sum_{j=J}^{\infty} \varphi_j(D) g_j \right) \right|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^p} \lesssim \left\| \left(\sum_{j=J}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^p}. \tag{2.8}$$

Proof. Note that, for $f \in L^1_{\rm loc}(\mathbb{R}^n)$, we have

$$|\varphi_l(D) f| \lesssim Mf. \tag{2.9}$$

We use (2.9) and the fact that $\varphi_l \varphi_j = 0$ whenever $|l-j| \ge 2$ to obtain

$$\begin{aligned} \sum_{l=1}^{\infty} \left| \varphi_l(D) \left(\sum_{j=J}^{\infty} \varphi_j(D) g_j \right) \right|^r &\le \sum_{l=J-1}^{\infty} \left| \sum_{j=\max(l-1,J)}^{l+1} \varphi_l(D) \varphi_j(D) g_j \right|^r \\ &\lesssim \sum_{l=J-1}^{\infty} \sum_{j=\max(l-1,J)}^{l+1} |\varphi_l(D)[\varphi_j(D) g_j]|^r \\ &\lesssim \sum_{j=J}^{\infty} M(\varphi_j(D) g_j)^r. \end{aligned} \tag{2.10}$$

By combining (2.9), (2.10), and Theorem 2.4 , we have

$$\begin{aligned} &\left\| \left(\sum_{l=1}^{\infty} \left| \varphi_l(D) \left(\sum_{j=J}^{\infty} \varphi_j(D) g_j \right) \right|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^p} \lesssim \left\| \left(\sum_{j=J}^{\infty} M(\varphi_j(D) g_j)^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^p} \\ &\lesssim \left\| \left(\sum_{j=J}^{\infty} |\varphi_j(D) g_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^p} \lesssim \left\| \left(\sum_{j=J}^{\infty} |M g_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^p} \lesssim \left\| \left(\sum_{j=J}^{\infty} |g_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^p}. \end{aligned}$$

□

2.3. **Some inequalities.** We use the following inequality which improves slightly the one in [30].

Lemma 2.6. [21, Lemma 2.17] *Fix $J \in \mathbb{Z} \cup \{-\infty\}$. Let $\{a_j\}_{j=J}^{\infty}$ be a non-negative sequence and $\kappa > 0$. Then*

$$\sum_{j=J}^{\infty} a_j \left(\sum_{k=J}^{j} a_k \right)^{\kappa-1} \le \frac{1}{\min(\kappa, 1)} \left(\sum_{j=J}^{\infty} a_j \right)^{\kappa}.$$

Here, we assume there is a non-zero a_j.

When we consider the complex interpolation of the second kind of classical Morrey spaces, we are faced with the function $|\log|f||^{-1}$ in the proof; see [9]. To take an advantage of this "log" factor fully, we will use the following series of lemmas:

Lemma 2.7. *Let $1 \le r < \infty$ and $z \in \mathbb{C}$ be such that $\mathrm{Re}(z) \ge 0$. Then there exists a constant $C = C_z > 0$ such that*

$$\left| \frac{s^z - 1}{\log(s^r)} \right| \le \frac{C}{r} \left(\log\left(s + \frac{1}{s} \right) \right)^{-1} \tag{2.11}$$

for every $s \in (0,1)$ and

$$\left| \frac{s^{-z} - 1}{\log(s^r)} \right| \le \frac{C}{r} \left(\log\left(s + \frac{1}{s} \right) \right)^{-1} \tag{2.12}$$

for every $s > 1$.

Lemma 2.8. *Let $1 \le r < \infty$ and fix $t \in \mathbb{R}$. Then there exists a constant $C_t > 0$ such that*

$$\left| \frac{s^{it} - 1}{\log(s^r)} \right| \le \frac{C_t}{r} \left(\log\left(s + \frac{1}{s} \right) \right)^{-1}, \tag{2.13}$$

for all $s > 0$ with $s \neq 1$.

As we have mentioned, the function of the form $|\log|f||^{-1}$ plays a crucial role for later considerations. We will need some variant including the logarithm. We use the functions defined by

$$\Phi_\kappa(t) := t^{\kappa-1} \left(\log\left(t + \frac{1}{t} \right) \right)^{-1}, \quad \Psi_\kappa(t) := \int_0^t \Phi_\kappa(\sqrt[r]{s})^r \, ds \tag{2.14}$$

for $t, \kappa > 0$ and $1 \le r < \infty$.

Lemma 2.9. *Let $1 \le r < \infty$ and $\kappa > 0$. Then we have*

$$\sum_{j=0}^{\infty} \left[a_j \Phi_\kappa \left(\sqrt[r]{\sum_{k=0}^{j} {a_k}^r} \right) \right]^r \lesssim \Psi_\kappa \left(\sum_{j=0}^{\infty} {a_j}^r \right) \tag{2.15}$$

for all nonnegative square summable sequences $\{a_j\}_{j=0}^{\infty}$.

Proof. Assume first that $\kappa \in (0,1)$. In this case,

$$\Phi_\kappa(t_1) \gtrsim \Phi_\kappa(t_2) \tag{2.16}$$

for every $t_1 \le t_2$. We observe

$$\begin{aligned}
\Psi_\kappa\left(\sum_{j=0}^\infty {a_j}^r\right) &= \int_0^{\sum_{j=0}^\infty {a_j}^r} \Phi_\kappa(\sqrt[r]{s})^r \, ds \\
&= \int_0^{{a_0}^r} \Phi_\kappa(\sqrt[r]{s})^r \, ds + \sum_{j=0}^\infty \int_{\sum_{k=0}^{j} {a_k}^r}^{\sum_{k=0}^{j+1} {a_k}^r} \Phi_\kappa(\sqrt[r]{s})^r \, ds.
\end{aligned}$$

Using (2.16), we have

$$\begin{aligned}
\Psi_\kappa\left(\sum_{j=0}^\infty {a_j}^r\right) &\gtrsim {a_0}^r \Phi_\kappa\left(\sqrt[r]{{a_0}^r}\right)^r + \sum_{j=0}^\infty {a_{j+1}}^r \Phi_\kappa\left(\sqrt[r]{\sum_{k=0}^{j+1} {a_k}^r}\right)^r \\
&= \sum_{j=0}^\infty {a_j}^r \Phi_\kappa\left(\sqrt[r]{\sum_{k=0}^{j} {a_k}^r}\right)^r.
\end{aligned}$$

For the case $\kappa \ge 1$, observe that $\Phi_\kappa(t)$ satisfies

$$\Phi_\kappa(2t) \le 2^\kappa \Phi_\kappa(t) \tag{2.17}$$

for all $t > 0$. In addition, we also can choose $C_2 > 0$ such that

$$\Phi_\kappa(t_1) \le C_2 \Phi_\kappa(t_2) \tag{2.18}$$

for every $t_1 \le t_2$. Write $R := \sum_{j=0}^\infty {a_j}^r$. By combining (2.17) and (2.18), we get

$$\begin{aligned}
\sum_{j=0}^\infty \left[a_j \Phi_\kappa\left(\sqrt[r]{\sum_{k=0}^{j} {a_k}^r}\right)\right]^r &\lesssim R\Phi_\kappa(\sqrt[r]{R})^r \\
&\lesssim 2^{2\kappa} R \Phi_\kappa\left(\frac{1}{2}\sqrt[r]{R}\right)^r \\
&\lesssim \int_{R/4}^{R} \Phi_\kappa(\sqrt[r]{s})^r \, ds \\
&\le \Psi_\kappa\left(\sum_{j=0}^\infty {a_j}^r\right),
\end{aligned}$$

as desired. □

Lemma 2.10. *Let* $1 \le r < \infty$, $\kappa > 0$ *and* $a \in (0,1)$. *Then, we have*

$$\Psi_\kappa(t^r) \le \frac{1}{\kappa(\log 2)^r}\left(a^{(r-1)\kappa} + \left(\log\left(\sqrt[r]{a} + \frac{1}{\sqrt[r]{a}}\right)\right)^{-r}\right) t^{r\kappa}, \tag{2.19}$$

for every $t \in (0,a) \cup (a^{-1},\infty)$.

Proof. By the fundamental theorem on calculus, we have

$$\begin{aligned}\Psi_\kappa(t^r) = \int_0^t s^{\kappa-1}\left(\log\left(\sqrt[r]{s} + \frac{1}{\sqrt[r]{s}}\right)\right)^{-r} ds \\ + \int_t^{t^r} s^{\kappa-1}\left(\log\left(\sqrt[r]{s} + \frac{1}{\sqrt[r]{s}}\right)\right)^{-r} ds.\end{aligned}$$

For $t > a^{-1}$, we have

$$\begin{aligned}\Psi_\kappa(t^r) &\le \frac{1}{(\log 2)^r}\int_0^t s^{\kappa-1}\, ds + \left(\log\left(\sqrt[r]{t} + \frac{1}{\sqrt[r]{t}}\right)\right)^{-r}\int_t^{t^r} s^{\kappa-1}\, ds \\ &\le \frac{1}{\kappa(\log 2)^r} t^\kappa + \frac{1}{\kappa}\left(\log\left(\sqrt[r]{a} + \frac{1}{\sqrt[r]{a}}\right)\right)^{-r} t^{r\kappa} \\ &= \frac{1}{\kappa(\log 2)^r}\left(\frac{1}{t^{(r-1)\kappa}} + \left(\log 2 \cdot \left(\log\left(\sqrt[r]{a} + \frac{1}{\sqrt[r]{a}}\right)\right)^{-1}\right)^r\right) t^{r\kappa} \\ &\le \frac{1}{\kappa(\log 2)^r}\left(a^{(r-1)\kappa} + \left(\log\left(\sqrt[r]{a} + \frac{1}{\sqrt[r]{a}}\right)\right)^{-r}\right) t^{r\kappa}.\end{aligned}$$

Meanwhile, using

$$\Psi_\kappa(t^r) = \int_0^{t^r} \Phi_\kappa(\sqrt[r]{s})^r\, ds = \int_0^{t^r} s^{\kappa-1}\left(\log\left(\sqrt[r]{s} + \frac{1}{\sqrt[r]{s}}\right)\right)^{-r} ds,$$

we have for $0 < t < a$, we have

$$\begin{aligned}\Psi_\kappa(t^r) &\le \left(\log\left(\sqrt[r]{t} + \frac{1}{\sqrt[r]{t}}\right)\right)^{-r}\int_0^{t^r} s^{\kappa-1}\, ds \\ &\le \frac{1}{\kappa(\log 2)^r} t^{r\kappa}\left(\log\left(\sqrt[r]{a} + \frac{1}{\sqrt[r]{a}}\right)\right)^{-r} \\ &\le \frac{1}{\kappa(\log 2)^r}\left(a^{(r-1)\kappa} + \left(\log\left(\sqrt[r]{a} + \frac{1}{\sqrt[r]{a}}\right)\right)^{-r}\right) t^{r\kappa},\end{aligned}$$

as desired. □

For checking the holomorphicity of the second complex interpolation functor, we invoke the following lemma:

Lemma 2.11. [9, Lemma 3] *Let $h \in \mathbb{C}$ and $\varepsilon > 0$. Assume that $\varepsilon > 2|h|$. Then, there exists $C_\varepsilon > 0$ such that*

$$\sup_{0<t\le 1} t^{\varepsilon}\left|\frac{\exp(h\log t)-1}{h\log t}-1\right| \le C_\varepsilon |h| \tag{2.20}$$

and

$$\sup_{t>1} t^{-\varepsilon}\left|\frac{\exp(h\log t)-1}{h\log t}-1\right| \le C_\varepsilon |h|. \tag{2.21}$$

Lemma 2.12. *Suppose that we have* 13 *parameters*

$$p_0, p_1, p, q_0, q_1, q, r, r_0, r_1, s, s_0, s_1, \theta$$

satisfying (1.6) *and* (1.7). *Then* $\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0} \cap \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1} \subset \overset{\diamond}{\mathcal{E}}{}^{s}_{pqr}$.

Proof. We take $f \in \overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0} \cap \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}$. Theorem 1.1 implies that

$$\left\| f - \sum_{j=0}^{N} \varphi_j(D) f \right\|_{\mathcal{E}^{s_k}_{p_kq_kr_k}} \to 0 \tag{2.22}$$

as $N \to \infty$ for $k = 0, 1$. By the Hölder inequality, we have

(2.23)

$$\left\| f - \sum_{j=0}^{N} \varphi_j(D) f \right\|_{\mathcal{E}^{s}_{pqr}} \le \left\| f - \sum_{j=0}^{N} \varphi_j(D) f \right\|_{\mathcal{E}^{s_0}_{p_0q_0r_0}}^{1-\theta} \left\| f - \sum_{j=0}^{N} \varphi_j(D) f \right\|_{\mathcal{E}^{s_1}_{p_1q_1r_1}}^{\theta}.$$

Combining (2.22) and (2.23), we obtain the desired result. □

3. Proofs

3.1. **Proof of Theorem 1.3.** According to [35, Corollary 1.11], we have

$$[\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1}]_\theta \subset \mathcal{E}^{s}_{pqr} \tag{3.1}$$

with equivalence of norms. Based on (3.1), we shall establish (1.11) as follows: First, if $G \in \mathcal{G}(\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1})$. Then

$$F_j(z) := -i2^j(G(z+2^{-j}i) - G(z))$$

belongs to $\mathcal{F}(\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1})$ and the norm is less than or equal to $\|G\|_{\mathcal{G}(\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1})}$. According to (3.1), we have

$$\|F_j\|_{\mathcal{E}^{s}_{pqr}} \lesssim \|G\|_{\mathcal{G}(\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1})}.$$

Since $F_j \to G(\theta)$ as $j \to \infty$ in $\mathcal{E}^{s_0}_{p_0q_0r_0} + \mathcal{E}^{s_1}_{p_1q_1r_1}$, and hence in $\mathcal{S}'(\mathbb{R}^n)$, by the Fatou property $\|G(\theta)\|_{\mathcal{E}^{s}_{pqr}} \lesssim \|G\|_{\mathcal{G}(\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1})}$.

Conversely, let $f \in \mathcal{E}^s_{pqr}$ with norm 1. Define linear functions $\rho_1, \rho_2, \rho_3, \rho_4$ of the variable $z \in \mathbb{C}$ uniquely by

$$\rho_1(l) := s\frac{r}{r_l} - s_l, \quad \rho_2(l) := \frac{p}{p_l} - \frac{r}{r_l}, \quad \rho_3(l) := 1 - \frac{p}{p_l}, \quad \rho_4(l) := \frac{r}{r_l},$$

for $l = 0, 1$. Since $\rho_k(\theta) = (1-\theta)\rho_k(0) + \theta\,\rho_k(1)$, $\rho_k(\theta) = 0$ for $k = 1, 2, 3$ and $\rho_4(\theta) = 1$. Define

$$F_\nu(z) := \varphi_\nu(D) \times \left[2^{\nu\rho_1(z)} \left(\sum_{j=1}^{\nu} |2^{js}\varphi_j(D)|^r \right)^{\frac{\rho_2(z)}{r}} \|f\|^{\rho_3(z)}_{\mathcal{E}^s_{pqr}} \operatorname{sgn}(\varphi_\nu(D)f)|\varphi_\nu(D)f|^{\rho_4(z)} \right],$$

$$F(z) := \sum_{\nu=0}^{\infty} F_\nu(z),$$

and

$$G(z) := \int_\theta^z F(w)\, dw.$$

In Section 4, we prove

$$G \in \mathcal{G}(\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1}). \tag{3.2}$$

So,

$$\|f\|_{[\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1}]^\theta} \le \|G'\|_{\mathcal{G}(\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1})} \lesssim 1.$$

3.2. **Proof of Theorem 1.1.** Suppose that $\sum_{j=0}^N \varphi_j(D)f \to f$ in $\mathcal{E}^s_{pqr}$ as $N \to \infty$. Let α be a multiindex. Then since

$$\sum_{j=0}^{N} \varphi_j(D)f = \sum_{j=0}^{N} \mathcal{F}^{-1}\left[\sum_{k=0}^{N+1} \varphi_k \cdot \varphi_j \mathcal{F}f\right] = c_n \sum_{j=0}^{N}\sum_{k=0}^{N+1} \mathcal{F}^{-1}\varphi_k * \varphi_j(D)f$$

for some constant $c_n > 0$, it follows that

$$\frac{\partial^\alpha}{\partial x^\alpha} \sum_{j=0}^{N} \varphi_j(D)f = c_{n,\alpha} \sum_{j=0}^{N}\sum_{k=0}^{N+1} \mathcal{F}^{-1}[\xi^\alpha\varphi_k] * \varphi_j(D)f.$$

Since $\mathcal{F}^{-1}[\xi^\alpha\varphi_k] \in \mathcal{S} \subset L^1$ and $p, q, r > 1$, we have

$$\frac{\partial^\alpha}{\partial x^\alpha} \sum_{j=0}^{N} \varphi_j(D)f \in \mathcal{E}^s_{pqr}.$$

Thus $f \in \overset{\diamond}{\mathcal{E}}{}^s_{pqr}$.

Suppose instead that $f \in \overset{\diamond}{\mathcal{E}}{}^s_{pqr}$. Let $\varepsilon > 0$ be arbitrary. Then by the definition of $\overset{\diamond}{\mathcal{E}}{}^s_{pqr}$, we can find $g \in \mathcal{E}^s_{pqr}$ such that $\partial^\alpha g \in \mathcal{E}^s_{pqr}$ and that $\|g - f\|_{\mathcal{E}^s_{pqr}} < \varepsilon$. Then for $N \geq 3$, we have

$$\left\| g - \sum_{j=0}^{N} \varphi_j(D) g \right\|_{\mathcal{E}^s_{pqr}} \leq 2 \left\| \left(\sum_{k=0}^{\infty} 2^{ksr} \left| \varphi_k(D) \left[g - \sum_{j=0}^{N} \varphi_j(D) g \right] \right|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q}.$$

From the size of the support condition, we obtain

$$\left\| g - \sum_{j=0}^{N} \varphi_j(D) g \right\|_{\mathcal{E}^s_{pqr}} \leq 2 \left\| \left(\sum_{k=N}^{\infty} 2^{ksr} \left| \varphi_k(D) \left[g - \sum_{j=0}^{N} \varphi_j(D) g \right] \right|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q}.$$

Since $\mathcal{F}^{-1}\varphi_k(\xi) = 2^{kn}\mathcal{F}^{-1}\varphi(2^{-k}\xi)$, we can use the Hardy-Littlewood maximal operator to have

$$\left\| g - \sum_{j=0}^{N} \varphi_j(D) g \right\|_{\mathcal{E}^s_{pqr}} \leq C \left\| \left(\sum_{k=N}^{\infty} 2^{ksr} M[\varphi_k(D) g]^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q}.$$

By Theorem 2.4, we have

$$\left\| g - \sum_{j=0}^{N} \varphi_j(D) g \right\|_{\mathcal{E}^s_{pqr}} \leq C \left\| \left(\sum_{k=N}^{\infty} 2^{ksr} |\varphi_k(D) g|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q}. \tag{3.3}$$

Let us set

$$\varphi_k^*(\xi) := \frac{\varphi_{k-1}(\xi) + \varphi_k(\xi) + \varphi_{k+1}(\xi)}{|2^{-k}\xi|^2}.$$

Then we have $\varphi_k(D)g = -2^{-2k}\varphi_k^*(D)\varphi_k(D)[\Delta g]$. Inserting this relation into (3.3), we obtain

$$\left\| g - \sum_{j=0}^{N} \varphi_j(D) g \right\|_{\mathcal{E}^s_{pqr}} \leq C \left\| \left(\sum_{k=N}^{\infty} 2^{k(s-2)r} |\varphi_k^*(D)\varphi_k(D)[\Delta g]|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q}.$$

Again by using the convolution and the maximal operator, we obtain

$$\left\| g - \sum_{j=0}^{N} \varphi_j(D) g \right\|_{\mathcal{E}^s_{pqr}} \leq C 2^{-2N} \left\| \left(\sum_{k=N}^{\infty} 2^{ksr} M\left[\varphi_k(D)[\Delta g]\right]^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q}.$$

Using Theorem 2.4 once more, we have

$$\begin{aligned}\left\|g-\sum_{j=0}^{N}\varphi_j(D)g\right\|_{\mathcal{E}^s_{pqr}} &\le C2^{-2N}\left\|\left(\sum_{k=N}^{\infty}2^{ksr}|\varphi_k(D)[\Delta g]|^r\right)^{\frac{1}{r}}\right\|_{\mathcal{M}^p_q}\\ &\le C2^{-2N}\|\Delta g\|_{\mathcal{E}^s_{pqr}}.\end{aligned}$$

Since $\Delta g \in \mathcal{E}^s_{pqr}$, there exists $N_0 \in \mathbb{N}$ such that

$$\left\|g-\sum_{j=0}^{N}\varphi_j(D)g\right\|_{\mathcal{E}^s_{pqr}} < \varepsilon$$

as long as $N \ge N_0$.

If we use Theorem 2.4, we obtain

$$\left\|\sum_{j=0}^{N}\varphi_j(D)g-\sum_{j=0}^{N}\varphi_j(D)f\right\|_{\mathcal{E}^s_{pqr}} \le C\|f-g\|_{\mathcal{E}^s_{pqr}} \le C\varepsilon.$$

Thus, if $N \ge N_0$, then we have

$$\begin{aligned}&\left\|f-\sum_{j=0}^{N}\varphi_j(D)f\right\|_{\mathcal{E}^s_{pqr}}\\ &\le \|f-g\|_{\mathcal{E}^s_{pqr}}+\left\|g-\sum_{j=0}^{N}\varphi_j(D)g\right\|_{\mathcal{E}^s_{pqr}}+\left\|\sum_{j=0}^{N}\varphi_j(D)g-\sum_{j=0}^{N}\varphi_j(D)f\right\|_{\mathcal{E}^s_{pqr}}\\ &\le (2+C)\varepsilon,\end{aligned}$$

as required.

3.3. **The description of the second interpolation spaces.** We prove

$$[\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}]^\theta \subset \bigcap_{0<a<1}\left\{f \in \mathcal{E}^s_{pqr} \,:\, \lim_{J\to\infty}\|S(f;a,J,r,s)\|_{\mathcal{M}^p_q}=0\right\}$$

in (1.10).

Let $f \in [\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}]^\theta$. By Lemmas 2.3 and 2.12, we have

$$f \in \mathcal{E}^s_{pqr} \cap \overline{\overset{\diamond}{\mathcal{E}}{}^s_{pqr}}^{\mathcal{E}^{s_0}_{p_0q_0r_0}+\mathcal{E}^{s_1}_{p_1q_1r_1}}.$$

Therefore,

$$f = f_k + f_{k,0} + f_{k,1},$$

where $f_k \in \overset{\diamond}{\mathcal{E}}{}^s_{pqr}$, $f_{k,0} \in \mathcal{E}^{s_0}_{p_0q_0r_0}$, $f_{k,1} \in \mathcal{E}^{s_1}_{p_1q_1r_1}$ for each $k \in \mathbb{N}$ and

$$\|f_{k,0}\|_{\mathcal{E}^{s_0}_{p_0q_0r_0}} + \|f_{k,1}\|_{\mathcal{E}^{s_1}_{p_1q_1r_1}} \le k^{-1}.$$

For $0 < a < 1$ and $b > 0$, we see that

$$\begin{aligned}
&S(f;a,J,r,s)\\
&= \chi_{[0,bS(f;r,s)]}\left(\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}\right) S(f;a,J,r,s)\\
&\quad + \chi_{(bS(f;r,s),\infty)}\left(\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}\right) S(f;a,J,r,s)\\
&\le bS(f;r,s) + \chi_{(ab,a^{-1}]}\left(\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}\right)\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\limsup_{J\to\infty}\|S(f;a,J,r,s)\|_{\mathcal{M}^p_q}\\
&\lesssim b\|f\|_{\mathcal{E}^s_{pqr}}\\
&\quad + \limsup_{J\to\infty}\left\|\chi_{(ab,a^{-1}]}\left(\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}\right)\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}\right\|_{\mathcal{M}^p_q}.
\end{aligned}$$

Once we show that

$$\lim_{J\to\infty}\left\|\chi_{(ab,a^{-1}]}\left(\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}\right)\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}\right\|_{\mathcal{M}^p_q} = 0, \tag{3.4}$$

then we have the desired result. By setting

$$\Phi_A(t) := \max(0,(t-A)(A^{-1}-t))^r \quad (t \in \mathbb{R}),$$

where $A > 0$, we have only to show that

$$\lim_{J\to\infty}\left\|\Phi_A\left(\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}\right)\right\|_{\mathcal{M}^p_q} = 0 \tag{3.5}$$

in $\mathcal{M}^p_q$ for all $A > 0$.

By the mean-value theorem, we have

$$\begin{aligned}
&\left|\Phi_A\left(\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}\right)-\Phi_A\left(\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f_k|^r}\right)\right|\\
&\lesssim_A \min\left(1,\left|\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}-\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f_k|^r}\right|\right)\\
&\le \min\left(1,\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)(f-f_k)|^r}\right)\\
&= \min\left(1,\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)(f_{k,0}+f_{k,1})|^r}\right).
\end{aligned}$$

We let

$$\mathfrak{A} := \chi_{[A,A^{-1}]}(S(f;r,s))\Phi_A\left(\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}\right),$$

$$\mathfrak{B} := \chi_{[A,A^{-1}]}(S(f;r,s))\Phi_A\left(\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f_k|^r}\right).$$

So, we have

$$\begin{aligned}
&\|\mathfrak{A}-\mathfrak{B}\|_{\mathcal{M}^p_q}\\
&\lesssim_A \left\|\chi_{[A,A^{-1}]}(S(f;r,s))\min\left\{\left(\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f_{k,0}|^r\right)^{\frac{1}{r}},1\right\}\right\|_{\mathcal{M}^p_q}\\
&\quad+\left\|\chi_{[A,A^{-1}]}(S(f;r,s))\min\left\{\left(\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f_{k,1}|^r\right)^{\frac{1}{r}},1\right\}\right\|_{\mathcal{M}^p_q}.
\end{aligned}\tag{3.6}$$

Recall that we are assuming $r_0 = r_1 = r$ and $s_0 = s_1 = s$. By using $q_0 > q > q_1$, $\frac{p}{q} = \frac{p_0}{q_0} = \frac{p_1}{q_1}$, and the Hölder inequality, we get

$$\begin{aligned}
&\|\mathfrak{A}-\mathfrak{B}\|_{\mathcal{M}^p_q}\\
&\lesssim_A \left\|\left(\sum_{j=0}^{\infty}|2^{js_0}\varphi_j(D)f_{k,0}|^{r_0}\right)^{\frac{1}{r_0}}\right\|_{\mathcal{M}^{p_0}_{q_0}}\left\|\left(\sum_{j=0}^{\infty}|2^{js}\varphi_j(D)f|^r\right)^{\frac{1}{r}}\right\|_{\mathcal{M}^p_q}^{1-\frac{q}{q_0}}
\end{aligned}$$

$$+\left\|\left(\sum_{j=0}^{\infty}|2^{js_1}\varphi_j(D)f_{k,1}|^{r_1}\right)^{\frac{1}{r_1}}\right\|_{\mathcal{M}^{p_1}_{q_1}}^{\frac{q_1}{q}}$$

$$\lesssim_A \|f_{k,0}\|_{\mathcal{E}^{s_0}_{p_0q_0r_0}}\|f\|_{\mathcal{E}^{s}_{pqr}}^{1-\frac{q}{q_0}}+\|f_{k,1}\|_{\mathcal{E}^{s_1}_{p_1q_1r_1}}^{\frac{q_1}{q}}. \tag{3.7}$$

Consequently,

$$\left\|\chi_{[A,A^{-1}]}(S(f;r,s))\Phi_A\left(\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}\right)\right\|_{\mathcal{M}^p_q}$$

$$\lesssim_A \|f_{k,0}\|_{\mathcal{E}^{s_0}_{p_0q_0r_0}}\|f\|_{\mathcal{E}^{s}_{pqr}}^{1-\frac{q}{q_0}}+\|f_{k,1}\|_{\mathcal{E}^{s_1}_{p_1q_1r_1}}^{\frac{q_1}{q}}+\left\|\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f_k|^r}\right\|_{\mathcal{M}^p_q}.$$

By letting $J\to\infty$, we obtain

$$\limsup_{J\to\infty}\left\|\Phi_A\left(\sqrt[r]{\sum_{j=J}^{\infty}|2^{js}\varphi_j(D)f|^r}\right)\right\|_{\mathcal{M}^p_q}$$

$$\lesssim_A \|f_{k,0}\|_{\mathcal{E}^{s_0}_{p_0q_0r_0}}\|f\|_{\mathcal{E}^{s}_{pqr}}^{1-\frac{q}{q_0}}+\|f_{k,1}\|_{\mathcal{E}^{s_1}_{p_1q_1r_1}}^{\frac{q_1}{q}}.$$

Finally letting $k\to\infty$, we obtain (3.4).

We prove the reverse inclusion in (1.10), namely,

$$[\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0},\overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}]^\theta\supset\bigcap_{0<a<1}\left\{f\in\mathcal{E}^s_{pqr}\ :\ \lim_{J\to\infty}\|S(f;a,J,r,s)\|_{\mathcal{M}^p_q}=0\right\}.$$

Let $f\in\mathcal{E}^s_{pqr}$ be such that $\lim_{J\to\infty}\|S(f;a,J,r,s)\|_{\mathcal{M}^p_q}=0$ for all $0<a<1$. We suppose that f has $\mathcal{E}^s_{pqr}$-norm 1. Choose $\varphi\in\mathcal{S}$ so that $\varphi\ge 0$ and $\chi_{B(2)}\le\varphi\le\chi_{B(3)}$. Write $\varphi_0:=\varphi$ and $\varphi_j:=\sqrt{\varphi_0(2^{-j}\cdot)^2-\varphi_0(2^{-j+1}\cdot)^2}$ for $j\in\mathbb{N}$. We may assume each φ_j is smooth if we choose φ suitably. Then, $\{\varphi_j\}_{j=0}^\infty$ satisfies

$$\sum_{j=0}^{\infty}\varphi_j^2=1.$$

For each $\nu\in\mathbb{N}\cup\{0\}$, define

$$V_\nu(f):=\left(\sum_{j=0}^{\nu}|2^{js}\varphi_j(D)f|^r\right)^{\frac{1}{r}}.$$

For $z \in \overline{U}$, we define

$$F(z) := \sum_{\nu=0}^{\infty} \varphi_\nu(D) \left(V_\nu(f)^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)-1} \cdot \varphi_\nu(D) f \right) \tag{3.8}$$

and

$$G(z) := \int_\theta^z F(w)\,dw.$$

We prove in Section 4 that

$$G \in \mathcal{G}(\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}), \quad \|G\|_{\mathcal{G}(\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1})} \lesssim 1. \tag{3.9}$$

From (3.9) and $f = G'(\theta)$, we conclude that $f \in [\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}]^\theta$, as desired.

3.4. **The description of the first interpolation.** We readily obtain the inclusion by combining (1.10), (2.4), Lemma 2.12 and

$$[\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}]_\theta \subset [\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1}]_\theta.$$

We thus obtain one of the inclusions in (1.9). We concentrate on the opposite direction. We choose ψ so that

$$\mathrm{supp}(\psi) \subset B(8)$$

and define

$$\varphi_j := \psi(2^{-j}\cdot) - \psi(2^{-j+1}\cdot).$$

Write $\varphi_j = \varphi(2^{-j}\cdot)$ as before.

Let $f \in \overset{\diamond}{\mathcal{E}}{}^s_{pqr} \cap [\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1}]_\theta$. Then since $f \in \overset{\diamond}{\mathcal{E}}{}^s_{pqr}$, we have

$$f = \psi(D)f + \sum_{j=1}^{\infty} \varphi_j(D) f$$

in $\mathcal{E}^s_{pqr}$. We write $f_J := \psi(D)f + \sum_{j=1}^{J} \varphi_j(D) f$. By virtue of Theorem 1.3 and $f_J - f_{J'} \in [\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1}]_\theta$ for any $J, J' \in \mathbb{N}$, we can use (1.11) and (2.4) to have

$$\|f_J - f_{J'}\|_{[\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1}]_\theta} = \|f_J - f_{J'}\|_{[\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1}]^\theta} \sim \|f_J - f_{J'}\|_{\mathcal{E}^s_{pqr}}.$$

Since $f_J - f_{J'} \in [\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1}]_\theta$ and $\mathrm{supp}(\mathcal{F}(f_J - f_{J'}))$ is a compact set in $\mathbb{R}^n \setminus \{0\}$, we can find $F_{J,J'} \in \mathcal{F}(\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1})$ such that

$$F_{J,J'}(\theta) = f_J - f_{J'}, \quad \|F_{J,J'}\|_{\mathcal{F}(\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1})} \lesssim \|f_J - f_{J'}\|_{[\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1}]_\theta}.$$

Thus, it follows that

$$\begin{aligned}\|f_J - f_{J'}\|_{[\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}]_\theta} &\le \|F_{J,J'}\|_{\mathcal{F}(\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1})} \\ &\lesssim \|F_{J,J'}(\theta)\|_{[\mathcal{E}^{s_0}_{p_0q_0r_0}, \mathcal{E}^{s_1}_{p_1q_1r_1}]_\theta} \\ &\lesssim \|f_J - f_{J'}\|_{\mathcal{E}^s_{pqr}}.\end{aligned}$$

Here we used [35, Corollary 1.11] for the last inequality. Hence $\{f_J\}_{J=1}^\infty$ is a Cauchy sequence in $[\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}]_\theta$. Since $\{f_J\}_{J=1}^\infty$ converges to f in $\mathcal{E}^{s_0}_{p_0q_0r_0} + \mathcal{E}^{s_1}_{p_1q_1r_1}$. We see that $\{f_J\}_{J=1}^\infty$ converges to f in $[\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}]_\theta$.

4. Proof of (3.2) and (3.9)

Let $p_0, p_1, p, \ldots$ be the same parameters as before. We check the conditions of membership of $\mathcal{G}(\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}, \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1})$ by proving the following lemmas.

Lemma 4.1.

(1) *Let $f \in \mathcal{E}^s_{pqr}$. For $z \in \overline{U}$, we have $G(z) \in \mathcal{E}^{s_0}_{p_0q_0r_0} + \mathcal{E}^{s_1}_{p_1q_1r_1}$. Moreover,*

$$\sup_{z\in\overline{U}} \left\| \frac{G(z)}{1+|z|} \right\|_{\mathcal{E}^{s_0}_{p_0q_0r_0} + \mathcal{E}^{s_1}_{p_1q_1r_1}} < \infty. \tag{4.1}$$

(2) *Let $f \in \overset{\diamond}{\mathcal{E}}{}^s_{pqr}$. For $z \in \overline{U}$, we have $G(z) \in \overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0} + \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}$. Moreover,*

$$\sup_{z\in\overline{U}} \left\| \frac{G(z)}{1+|z|} \right\|_{\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0} + \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}} < \infty. \tag{4.2}$$

Proof. We concentrate on (4.2); the proof of (4.1) being simpler. For each $z \in \overline{U}$, we define

$$F_0(z) := \sum_{\nu=0}^\infty \varphi_\nu(D) \left(V_\nu(f)^{p\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right) - 1} \cdot \varphi_\nu(D) f \cdot \chi_{\{V_\nu(f) \le 1\}} \right),$$

$F_1(z) := F(z) - F_0(z)$, $G_0(z) := \int_\theta^z F_0(w)\, dw$, and $G_1(z) := \int_\theta^z F_1(w)\, dw$.

We shall show that

$$G_0(z) \in \overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0} \tag{4.3}$$

and that

$$G_1(z) \in \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}. \tag{4.4}$$

Let $J \in \mathbb{N} \cap [5, \infty)$. We use (2.6), (2.9), and the fact that $\varphi_l \varphi_j = 0$ whenever $|l - j| \geq 2$ to obtain

$$\begin{aligned}
&\left\| \sum_{\ell=J}^{\infty} \varphi_\ell(D)(G_0(z)) \right\|_{\mathcal{E}^{s_0}_{p_0 q_0 r_0}} \\
&= \left\| \left(\sum_{j=J-1}^{\infty} \left| \varphi_j(D) \left[\sum_{\ell=J}^{\infty} \varphi_\ell(D)(2^{js_0} G_0(z)) \right] \right|^{r_0} \right)^{\frac{1}{r_0}} \right\|_{\mathcal{M}^{p_0}_{q_0}} \\
&\lesssim \left\| \left(\sum_{j=J-1}^{\infty} |\varphi_j(D)(2^{js_0} G_0(z))|^{r_0} \right)^{\frac{1}{r_0}} \right\|_{\mathcal{M}^{p_0}_{q_0}}.
\end{aligned} \tag{4.5}$$

Let $Q := \frac{p}{p_1} - \frac{p}{p_0}$. Combining

$$\begin{aligned}
&\sum_{j=J-1}^{\infty} |\varphi_j(D)[2^{js_0} G_0(z)]|^{r_0} \\
&\lesssim \sum_{j=J-1}^{\infty} 2^{js_0 r_0} M \left(\varphi_j(D) \left[\chi_{\{V_j(f) \leq 1\}} \varphi_j(D) f \cdot V_j(f)^{\frac{p}{p_0} - 1} \int_\theta^z V_j(f)^{Qw} \, dw \right] \right)^{r_0},
\end{aligned}$$

(2.6), (2.9), and (4.5), we get

$$\left\| \sum_{\ell=J}^{\infty} \varphi_\ell(D)(G_0(z)) \right\|_{\mathcal{E}^{s_0}_{p_0 q_0 r_0}} \lesssim \|I_1\|_{\mathcal{M}^{p_0}_{q_0}} + \|I_2\|_{\mathcal{M}^{p_0}_{q_0}} \tag{4.6}$$

where

$$\begin{aligned}
&I_1 \\
&:= \chi_{[a, a^{-1}]}(S(f; r, s)) \left(\sum_{j=J-1}^{\infty} |2^{js} \varphi_j(D) f|^r V_j(f)^{\frac{pr}{p_0} - r} \left| \chi_{\{V_j(f) \leq 1\}} \int_\theta^z V_j(f)^{Qw} \, dw \right|^r \right)^{\frac{1}{r}}
\end{aligned}$$

and

$$\begin{aligned}
&I_2 := (1 - \chi_{[a, a^{-1}]}(S(f; r, s))) \\
&\quad \times \left(\sum_{j=J-1}^{\infty} |2^{js} \varphi_j(D) f|^r V_j(f)^{\frac{pr}{p_0} - r} \left| \frac{V_j(f)^{Qz} - V_j(f)^{Q\theta}}{\log(V_j(f))} \chi_{\{V_j(f) \leq 1\}} \right|^r \right)^{\frac{1}{r}}.
\end{aligned}$$

By virtue of Lemma 2.6, we get

$$\begin{aligned}\|I_1\|_{\mathcal{M}^{p_0}_{q_0}} &\lesssim (1+|z|)\left\|\chi_{[a,a^{-1}]}(S(f;r,s))\left(\sum_{j=J-1}^{\infty}|2^{js}\varphi_j(D)f|^r V_j(f)^{\frac{pr}{p_0}-r}\right)^{\frac{1}{r}}\right\|_{\mathcal{M}^{p_0}_{q_0}} \\ &\lesssim (1+|z|)\left\|\chi_{[a,a^{-1}]}(S(f;r,s))\left(\sum_{\ell=J-2}^{\infty}|2^{\ell s}\varphi_\ell(D)f|^r\right)^{\frac{p}{rp_0}}\right\|_{\mathcal{M}^{p_0}_{q_0}} \\ &\le (1+|z|)\|S(f;a,J-1,r,s)\|_{\mathcal{M}^p_q}^{\frac{p}{p_0}}.\end{aligned} \tag{4.7}$$

We combine Lemmas 2.7, 2.9, and 2.10 to obtain

$$\begin{aligned}I_2 &\lesssim (1-\chi_{[a,a^{-1}]}(S(f;r,s)))\left(\sum_{j=0}^{\infty}|2^{js}\varphi_j(D)f|^r\Phi_{\frac{p}{p_0}}(V_j(f))^r\right)^{\frac{1}{r}} \\ &\lesssim (1-\chi_{[a,a^{-1}]}(S(f;r,s)))\left(\Psi_{\frac{p}{p_0}}(S(f;r,s)^r)^{\frac{1}{r}}\right) \\ &\lesssim \left(a^{\frac{p}{p_0}(r-1)}+\left(\log\left(\sqrt{a}+\frac{1}{\sqrt{a}}\right)\right)^{-r}\right)S(f;r,s)^{\frac{p}{p_0}}.\end{aligned}$$

Consequently,

$$\|I_2\|_{\mathcal{M}^{p_0}_{q_0}} \lesssim \left(a^{\frac{p}{p_0}(r-1)}+\left(\log\left(\sqrt{a}+\frac{1}{\sqrt{a}}\right)\right)^{-r}\right)\|f\|_{\mathcal{E}^s_{pqr}}^{\frac{p}{p_0}}. \tag{4.8}$$

Therefore, by combing (4.6), (4.7) and (4.8), and then taking $J \to \infty$ and $a \to 0^+$, we have

$$\lim_{J\to\infty}\left\|\sum_{\ell=J}^{\infty}\varphi_\ell(D)(G_0(z))\right\|_{\mathcal{E}^{s_0}_{p_0q_0r_0}} = 0.$$

Thus, $G_0(z) \in \overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}$. By a similar argument, we also have

$$\lim_{J\to\infty}\left\|\sum_{\ell=J}^{\infty}\varphi_\ell(D)(G_1(z))\right\|_{\mathcal{E}^{s_1}_{p_1q_1r_1}} = 0,$$

which implies (4.4). Since $G(z) = G_0(z)+G_1(z)$, we have $G(z) \in \overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0} + \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}$.

The proof of (4.2) goes as follows. By virtue of (2.9) and Theorem 2.4, we have

$$
\begin{aligned}
&\|G_0(z)\|_{\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}} \\
&= \|G_0(z)\|_{\mathcal{E}^{s_0}_{p_0q_0r_0}} \\
&\lesssim \left\| \varphi_0(D)f \cdot V_0(f)^{\frac{p}{p_0}-1} \int_\theta^z V_0(f)^{Qw}\, dw \chi_{\{V_0(f)\le 1\}} \right\|_{\mathcal{M}^{p_0}_{q_0}} \\
&\quad + \left\| \varphi_1(D)f \cdot V_1(f)^{\frac{p}{p_0}-1} \int_\theta^z V_1(f)^{Qw}\, dw \chi_{\{V_1(f)\le 1\}} \right\|_{\mathcal{M}^{p_0}_{q_0}} \\
&\quad + \left\| \left(\sum_{l=1}^\infty \left| 2^{ls}\varphi_l(D)f \cdot V_l(f)^{\frac{p}{p_0}-1} \int_\theta^z V_l(f)^{Qw}\, dw \chi_{\{V_l(f)\le 1\}} \right|^{r_0} \right)^{\frac{1}{r_0}} \right\|_{\mathcal{M}^{p_0}_{q_0}}.
\end{aligned} \tag{4.9}
$$

Combining (4.9) and Lemma 2.6, we have

$$
\begin{aligned}
\|G_0(z)\|_{\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}} &\lesssim (1+|z|)\|\varphi(D)f\|_{\mathcal{M}^p_q}^{\frac{p}{p_0}} + (1+|z|)\|\varphi_1(D)f\|_{\mathcal{M}^p_q}^{\frac{p}{p_0}} \\
&\quad + (1+|z|) \left\| \left(\sum_{l=1}^\infty |2^{ls}\varphi_l(D)f|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q}^{\frac{p}{p_0}} \\
&\lesssim (1+|z|)\|f\|_{\mathcal{E}^s_{pqr}}^{\frac{p}{p_0}}.
\end{aligned} \tag{4.10}
$$

By a similar argument

$$
\|G_0(z)\|_{\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}} \lesssim (1+|z|)\|f\|_{\mathcal{E}^s_{pqr}}^{\frac{p}{p_1}}. \tag{4.11}
$$

Thus, (4.2) follows from (4.10) and (4.11). □

Lemma 4.2.

(1) *Let* $f \in \mathcal{E}^s_{pqr}$. *Then the function* $G : \overline{U} \to \mathcal{E}^{s_0}_{p_0q_0r_0} + \mathcal{E}^{s_1}_{p_1q_1r_1}$ *is continuous and* $G : U \to \mathcal{E}^{s_0}_{p_0q_0r_0} + \mathcal{E}^{s_1}_{p_1q_1r_1}$ *is holomorphic.*

(2) *Let* $f \in \overset{\diamond}{\mathcal{E}}{}^s_{pqr}$. *Then the function* $G : \overline{U} \to \overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0} + \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}$ *is continuous and* $G : U \to \overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0} + \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}$ *is holomorphic.*

Proof. We suppose $f \in \overset{\diamond}{\mathcal{E}}{}^s_{pqr}$. The case of $f \in \mathcal{E}^s_{pqr}$ can be handled similarly. Let $z_1, z_2 \in \overline{U}$. By virtue of (2.9) and Theorem 2.4, we have

$$
\|G_0(z_1) - G_0(z_2)\|_{\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}}
$$

$$
\begin{aligned}
&= \|G_0(z_1) - G_0(z_2)\|_{\mathcal{E}^{s_0}_{p_0q_0r_0}} \\
&\lesssim \left\| \varphi_0(D)f \cdot V_0(f)^{\frac{p}{p_0}-1} \int_{z_2}^{z_1} V_0(f)^{Qw}\, dw \chi_{\{V_0(f)\le 1\}} \right\|_{\mathcal{M}^{p_0}_{q_0}} \\
&\quad + \left\| \varphi_1(D)f \cdot V_1(f)^{\frac{p}{p_0}-1} \int_{z_2}^{z_1} V_1(f)^{Qw}\, dw \chi_{\{V_1(f)\le 1\}} \right\|_{\mathcal{M}^{p_0}_{q_0}}
\end{aligned}
$$

(4.12)

$$
\quad + \left\| \left(\sum_{l=1}^{\infty} \left| 2^{ls} \varphi_l(D)f \cdot V_l(f)^{\frac{p}{p_0}-1} \int_{z_2}^{z_1} V_l(f)^{Qw}\, dw \chi_{\{V_l(f)\le 1\}} \right|^{r_0} \right)^{\frac{1}{r_0}} \right\|_{\mathcal{M}^{p_0}_{q_0}}.
$$

Combining (4.12) and Lemma 2.6, we get

$$
\begin{aligned}
\|G_0(z_1) - G_0(z_2)\|_{\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}} &\lesssim |z_1 - z_2| \|\varphi(D)f\|_{\mathcal{M}^p_q}^{\frac{p}{p_0}} + |z_1 - z_2| \|\varphi_1(D)f\|_{\mathcal{M}^p_q}^{\frac{p}{p_0}} \\
&\quad + |z_1 - z_2| \left\| \left(\sum_{l=1}^{\infty} |2^{ls}\varphi_l(D)f|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q}^{\frac{p}{p_0}} \\
&\lesssim |z_1 - z_2| \|f\|_{\mathcal{E}^s_{pqr}}^{\frac{p}{p_0}}. \qquad (4.13)
\end{aligned}
$$

Likewise,

$$
\|G_1(z_1) - G_1(z_2)\|_{\overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}} \lesssim |z_1 - z_2| \|f\|_{\mathcal{E}^s_{pqr}}^{\frac{p}{p_1}}. \qquad (4.14)
$$

Therefore, (4.13) and (4.14) yield

$$
\|G(z_1) - G(z_2)\|_{\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0} + \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}} \lesssim |z_1 - z_2| \left(\|f\|_{\mathcal{E}^s_{pqr}}^{\frac{p}{p_0}} + \|f\|_{\mathcal{E}^s_{pqr}}^{\frac{p}{p_1}} \right). \qquad (4.15)
$$

This implies the continuity of G. Furthermore, for every $z \in U$, we have $G'(z) = F(z)$ and $F(z) \in \overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0} + \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}$. □

Let $k = 0, 1$ and $t_1, t_2 \in \mathbb{R}$. By a similar argument for obtaining (4.6), we have

$$
G(k + it_1) - G(k + it_2) \in \mathcal{E}^{s_k}_{p_kq_kr_k} \qquad (4.16)
$$

if $f \in \mathcal{E}^s_{pqr}$ and we have

$$
G(k + it_1) - G(k + it_2) \in \overset{\diamond}{\mathcal{E}}{}^{s_k}_{p_kq_kr_k} \qquad (4.17)
$$

if $f \in \overset{\diamond}{\mathcal{E}}{}^s_{pqr}$.

Lemma 4.3. *Let $k = 0, 1$. Let $f \in \mathcal{E}^s_{pqr}$ with norm 1.*

(1) *Then the function* $t \in \mathbb{R} \mapsto G(k+it) - G(k) \in \mathcal{E}^{s_k}_{p_k q_k r_k}$ *is Lipschitz continuous.*

(2) *Assume* $r_0 = r_1 = r$ *and* $s_0 = s_1 = s$. *Let* $f \in \overset{\diamond}{\mathcal{E}}{}^s_{pqr}$. *Then the function* $t \in \mathbb{R} \mapsto G(k+it) - G(k) \in \overset{\diamond}{\mathcal{E}}{}^{s_k}_{p_k q_k r_k}$ *is Lipschitz continuous.*

Proof. We suppose $f \in \overset{\diamond}{\mathcal{E}}{}^s_{pqr}$. The case of $f \in \mathcal{E}^s_{pqr}$ can be handled similarly. Let $t_1, t_2 \in \mathbb{R}$ and let $J \in \mathbb{N} \cap [5, \infty)$. By a similar argument for obtaining (4.6), we have

$$\left\| \sum_{j=J}^{\infty} \varphi_j(D)\left(G(it_1) - G(it_2)\right) \right\|_{\mathcal{E}^{s_0}_{p_0 q_0 r_0}} \lesssim \|I_1\|_{\mathcal{M}^{p_0}_{q_0}} + \|I_2\|_{\mathcal{M}^{p_0}_{q_0}}, \tag{4.18}$$

where

$$I_1 := \chi_{[a,a^{-1}]}(S(f;r,s)) \times \left(\sum_{j=J-1}^{\infty} |2^{js}\varphi_j(D)f|^r V_j(f)^{\frac{pr}{p_0}-r} \left| \int_{t_2}^{t_1} V_j(f)^{Qit}\, dt \right|^r \right)^{\frac{1}{r}}$$

and

$$I_2 := (1 - \chi_{[a,a^{-1}]}(S(f;r,s))) \times \left(\sum_{j=J-1}^{\infty} |2^{js}\varphi_j(D)f|^r V_j(f)^{\frac{pr}{p_0}-r} \left| \frac{V_j(f)^{Qit_1} - V_j(f)^{Qit_2}}{\log(V_j(f))} \right|^r \right)^{\frac{1}{r}}.$$

By virtue of Lemma 2.6, we get

$$\begin{aligned} \|I_1\|_{\mathcal{M}^{p_0}_{q_0}} &\lesssim |t_1 - t_2| \left\| \chi_{[a,a^{-1}]}(S(f;r,s)) \left(\sum_{j=J-1}^{\infty} |2^{js}\varphi_j(D)f|^r V_j(f)^{\frac{pr}{p_0}-r} \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^{p_0}_{q_0}} \\ &\lesssim |t_1 - t_2| \left\| \chi_{[a,a^{-1}]}(S(f;r,s)) \left(\sum_{\ell=J-1}^{\infty} |2^{\ell s}\varphi_\ell(D)f|^r \right)^{\frac{p}{rp_0}} \right\|_{\mathcal{M}^{p_0}_{q_0}} \\ &\le |t_1 - t_2| \|S(f;a,J-1,r,s)\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q}. \end{aligned} \tag{4.19}$$

We combine Lemmas 2.8, 2.9, and 2.10 to obtain

$$\begin{aligned} I_2 &\lesssim (1-\chi_{[a,a^{-1}]}(S(f;r,s)))\left(\sum_{j=0}^{\infty}|2^{js}\varphi_j(D)f|^r\Phi_{\frac{p}{p_0}}(V_j(f))^r\right)^{\frac{1}{r}} \\ &\lesssim (1-\chi_{[a,a^{-1}]}(S(f;r,s)))\Psi_{\frac{p}{p_0}}(S(f;r,s)^r)^{\frac{1}{r}} \\ &\lesssim \left(a^{\frac{p}{p_0}(r-1)}+\left(\log\left(\sqrt{a}+\frac{1}{\sqrt{a}}\right)\right)^{-r}\right)S(f;r,s)^{\frac{p}{p_0}}. \end{aligned}$$

Consequently,

$$\|I_2\|_{\mathcal{M}^{p_0}_{q_0}} \lesssim \left(a^{\frac{p}{p_0}(r-1)}+\left(\log\left(\sqrt{a}+\frac{1}{\sqrt{a}}\right)\right)^{-r}\right)\|f\|^{\frac{p}{p_0}}_{\mathcal{E}^s_{pqr}}. \tag{4.20}$$

Therefore, by combining (4.18), (4.19) and (4.20), and then taking $J\to\infty$ and $a\to 0^+$, we have

$$\lim_{J\to\infty}\left\|\sum_{\ell=J}^{\infty}\varphi_\ell(D)(G(it_1)-G(it_2))\right\|_{\mathcal{E}^{s_0}_{p_0q_0r_0}}=0.$$

Thus, $G(it_1)-G(it_2)\in \overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}$. The proof of $G(1+it_1)-G(1+it_2)\in \overset{\diamond}{\mathcal{E}}{}^{s_1}_{p_1q_1r_1}$ is similar.

Now we prove the second part of this lemma. From (2.9) and Theorem 2.4, it follows that

$$\begin{aligned} &\|G(it_1)-G(it_2)\|_{\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}} \\ &=\|G(it_1)-G(it_2)\|_{\mathcal{E}^{s_0}_{p_0q_0r_0}} \\ &\lesssim \left\|\varphi_0(D)f\cdot V_0(f)^{\frac{p}{p_0}-1}\int_{t_2}^{t_1}V_0(f)^{Qit}\,dt\right\|_{\mathcal{M}^{p_0}_{q_0}} \\ &\quad+\left\|\varphi_1(D)f\cdot V_1(f)^{\frac{p}{p_0}-1}\int_{t_2}^{t_1}V_1(f)^{Qit}\,dt\right\|_{\mathcal{M}^{p_0}_{q_0}} \\ &\quad+\left\|\left(\sum_{l=1}^{\infty}\left|2^{ls}\varphi_l(D)f\cdot V_l(f)^{\frac{p}{p_0}-1}\int_{t_2}^{t_1}V_l(f)^{Qit}\,dt\right|^{r_0}\right)^{\frac{1}{r_0}}\right\|_{\mathcal{M}^{p_0}_{q_0}}. \end{aligned} \tag{4.21}$$

By virtue of Lemma 2.6, we get

$$\begin{aligned} &\|G(it_1)-G_0(it_2)\|_{\overset{\diamond}{\mathcal{E}}{}^{s_0}_{p_0q_0r_0}} \\ &\lesssim |t_1-t_2|\|\varphi(D)f\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q}+|t_1-t_2|\|\varphi_1(D)f\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q} \end{aligned}$$

$$+|t_1-t_2|\left\|\left(\sum_{l=1}^{\infty}|2^{ls}\varphi_l(D)f|^r\right)^{\frac{1}{r}}\right\|_{\mathcal{M}_q^p}^{\frac{p}{p_0}} \lesssim |t_1-t_2|\|f\|_{\mathcal{E}_{pqr}^s}^{\frac{p}{p_0}}.$$

By a similar argument, we also have

$$\|G(1+it_1)-G_0(1+it_2)\|_{\mathring{\mathcal{E}}_{p_1q_1r_1}^{s_1}} \lesssim |t_1-t_2|\|f\|_{\mathcal{E}_{pqr}^s}^{\frac{p}{p_1}},$$

as desired. □

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