Math. J. Okayama Univ. **61** (2019), 85–98

ON THE STRUCTURE OF THE PROFILE OF FINITE CONNECTED QUANDLES

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Abstract. We verify some cases of a conjecture by C. Hayashi on the structure of the profile of a finite connected quandle.

1. Introduction

Definition 1.1. A quandle is a set Q with a binary operation $*: Q\times Q \to Q$ satisfying the following three axioms.

(Q1) For any $a \in Q$, $a * a = a$.

(Q2) For any pair $a, b \in Q$, there exists a unique $c \in Q$ such that $c*a = b$.

(Q3) For any triple $a, b, c \in Q$, $(a * b) * c = (a * c) * (b * c)$.

Example 1.2. Let A be a finite abelian group, and $T \in \mathrm{Aut}(A)$. We endow A with a quandle structure $x*y = T(x)+(1-T)(y)$ for $x, y \in A$. We denote this quandle $\mathrm{Aff}(A, T)$, which is called an affine quandle.

Let $(Q, *)$ and $(Q', *')$ be two quandles. A map $f : Q \to Q'$ is said to be a homomorphism if $f(a*b) = f(a)*' f(b)$ for any $a, b \in Q$. If a homomorphism is bijective as a map, then it is said to be an isomorphism. An isomorphism from a quandle Q to Q itself is said to be an automorphism of Q.

The map $r_c : Q \to Q; x \mapsto x * c$ is a bijection for any $c \in Q$ by axiom (Q2) and we have $r_c(a * b) = (a * b) * c = (a * c) * (b * c) = r_c(a) * r_c(b)$ for any pair $a, b \in Q$ by axiom (Q3), so r_c is an automorphism.

Let $\mathrm{Aut}(Q)$ be the group of all automorphisms of Q. $\mathrm{Aut}(Q)$ is called the quandle automorphism group of Q. The inner group of a quandle Q is the subgroup of $\mathrm{Aut}(Q)$ generated by the maps r_c for all $c \in Q$. We write $\mathrm{Inn}(Q)$ for the inner group of Q. A quandle Q is said to be connected if $\mathrm{Inn}(Q)$ acts transitively on Q.

Let Q be a finite quandle of order n. We write its elements as $1, 2, \ldots, n$. Since the map r_c is a bijection, it can be regarded as a permutation on the set $\{1, 2, \ldots, n\}$.

Mathematics Subject Classification. Primary 20N02; Secondary 57M27.
Key words and phrases. connected quandle, finite quandle.

In general, when a permutation σ on $\{1, 2, \ldots, n\}$ can be written as the product of disjoint cycles $(i_{1,1} \cdots i_{1,\ell_1})(i_{2,1} \cdots i_{2,\ell_2}) \cdots (i_{k,1} \cdots i_{k,\ell_k})$, we call the multiple set of the length of the cycles $\{\ell_1, \ell_2, \ldots, \ell_k\}$ the pattern of σ. In [4], P. Lopes and D. Roseman defined the profile of a quandle with n elements to be the sequence of the patterns of $r_1, r_2, \ldots, r_n$. In the case of a connected quandle Q of order n, it is easily seen that r_i and r_j are mutually conjugate for any pair i, j with $1 \leq i < j \leq n$. Therefore, r_i and r_j have the same pattern. In this paper, we call this common pattern the profile of Q for short.

In [2], C. Hayashi conjectured the following, which related to the structure of the profile of a connected quandle.

Conjecture 1.3. For a connected quandle with the profile $\{1, \ell_1, \ldots, \ell_k\}$, where $1 \leq \ell_1 \leq \cdots \leq \ell_k$, ℓ_i divides ℓ_k, where $1 \leq i \leq k-1$.

In this paper, we study this conjecture in the case of $k = 2$, by using the method of non-trivial orbits, and the case where the order of a quandle is less than or equal to 47, by listing up the profile of it by Rig which is made by L. Vendramin.

2. In the case of $k = 2$

In this section, we prove that Conjecture 1.3 is true in the case of $k = 2$ by using a nontrivial orbit. We review the definition of a nontrivial orbit by the action of an element of a quandle in [6] Definition 3.3. For a group G acting on a set X, we denote by G_x the stabilizer of $x \in X$ in the group G, that is, G_x is the set of the elements $g \in G$ satisfying $g(x) = x$.

Definition 2.1. (cf. Definition 3.3 in [6]) Let Q be a connected quandle with the profile $\{1, \ell_1, \ldots, \ell_k\}$ and $x \in Q$. By the description $r_x = (x_{1,1} \cdots x_{1,\ell_1})(x_{2,1} \cdots x_{2,\ell_2}) \cdots (x_{k,1} \cdots x_{k,\ell_k})(x)$, $Q \setminus \{x\}$ is divided into k sets of the size $\ell_1, \cdots, \ell_k$. We call these sets the nontrivial orbits by the action of x.

The following proposition proves Conjecture 1.3 in the case of $k = 2$.

Proposition 2.2. *Let Q be a connected quandle with the profile $\{1, \ell_1, \ell_2\}$, where $1 \leq \ell_1 \leq \ell_2$. Then, ℓ_1 divides ℓ_2.*

Proof. Suppose that ℓ_1 does not divide ℓ_2. Let n be the order of Q, G be $\mathrm{Inn}(Q)$ and $O(x)$ be the nontrivial orbit by the action of $x \in Q$ such that its order $|O(x)|$ is equal to ℓ_1. We write $C(x) = Q \setminus O(x)$ which includes x and its cardinality is $1 + \ell_2$.

Claim. *For any pair $x, y \in Q$, $C(x) = C(y)$ or $C(x) \cap C(y) = \emptyset$ holds.*

We show that this claim implies the proposition. We define the equivalence relation $\sim$ on Q such that $a \sim b$ if and only if $C(a) = C(b)$ $(a, b \in Q)$. The claim shows that $C(x)$ is the equivalence class including x by $\sim$. Thus $1 + \ell_2$ divides the order $1 + \ell_1 + \ell_2$ of Q, and hence, $1 + \ell_2$ divides ℓ_1, which contradicts to the assumption $\ell_1 < \ell_2$. Therefore, ℓ_1 divides ℓ_2 and the proposition follows. The rest is to prove the claim.

Proof of Claim. For $x \in Q$, we write $r_x = (x_{1,1} \cdots x_{1,\ell_1})(x_{2,1} \cdots x_{2,\ell_2})(x)$ and consider the element $g = r_x^{\ell_2}$ of G. By the assumption that ℓ_1 does not divide ℓ_2, we see that g is not trivial. For any $y \in \{x_{2,1}, \ldots, x_{2,\ell_2}\}$, we assume $r_y = (y_{1,1} \cdots y_{1,\ell_1})(y_{2,1} \cdots y_{2,\ell_2})(y)$. Similarly as Lemma 3.2 in [6], we have $G_y = \{f \in S_n |\ f \text{ is of the form below}\} \cap G$. Here,

$$f = \begin{pmatrix} y_{1,1} & \cdots & y_{1,\ell_1} & y_{2,1} & \cdots & y_{2,\ell_2} & y \\ y_{1,i+1} & \cdots & y_{1,i+\ell_1} & y_{2,j+1} & \cdots & y_{2,j+\ell_2} & y \end{pmatrix},$$

where $0 \leq i \leq \ell_1 - 1$ and $i + k$ is to be the representative (modulo ℓ_1) from $[1, \ell_1]$ and $0 \leq j \leq \ell_2 - 1$ and $j + k$ is to be the representative (modulo ℓ_2) from $[1, \ell_2]$. The set of the non-fixed points of such an f is either empty, $O(y)$, $C(y) \setminus \{y\}$, or $Q \setminus \{y\}$, whose order is 0, ℓ_1, ℓ_2, $\ell_1 + \ell_2$, respectively.

Now, since $g = (x_{1,1} \cdots x_{1,\ell_1})^{\ell_2} \neq e$, the set of the non-fixed points of g coincides with $O(x)$. On the other hand, since $g = (x_{1,1} \cdots x_{1,\ell_1})^{\ell_2}$ fixes y, it belongs to G_y, and, by the above and by $|O(x)| = \ell_1$, the set of non-fixed points of it has to be $O(y)$, that is, $O(x) = O(y)$ and $C(x) = C(y)$ for any $y \in \{x_{2,1}, \ldots, x_{2,\ell_2}\}$. If $y = x$, then trivially $O(x) = O(y)$ and $C(x) = C(y)$. Therefore, we have $C(x) = C(y)$ for $y \in C(x)$. We suppose that $C(x) \cap C(y) \neq \emptyset$, that is, there exists an element $z \in C(x) \cap C(y)$. Then, $C(x) = C(z)$ by $z \in C(x)$ and $C(y) = C(z)$ by $z \in C(y)$, so $C(x) = C(y)$. The proof is complete. ☐

Remark 2.3. For x and $y \in \{x_{2,1}, \ldots, x_{2,\ell_2}\}$ in the proof of Claim, an element of G_x which fixes y also fixes $x_{2,1}, \ldots, x_{2,\ell_2}$. Therefore, it is written by $(x_{1,1} \cdots x_{1,\ell_1})^i$ for some i with $0 \leq i \leq \ell_1 - 1$, so we have $G_x \cap G_y \subset \{(x_{1,1} \cdots x_{1,\ell_1})^i \mid 0 \leq i \leq \ell_1 - 1\} \cap G$. Furthermore, any element of G whose form is $(x_{1,1} \cdots x_{1,\ell_1})^i$ with $0 \leq i \leq \ell_1 - 1$ fixes x and y, so we have $G_x \cap G_y \supset \{(x_{1,1} \cdots x_{1,\ell_1})^i \mid 0 \leq i \leq \ell_1 - 1\} \cap G$. Therefore,

$$G_x \cap G_y = \{(x_{1,1} \cdots x_{1,\ell_1})^i \mid 0 \leq i \leq \ell_1 - 1\} \cap G.$$

Since $C(x) = C(y)$, we have $x \in C(y) \setminus \{y\} = \{y_{2,1}, \ldots, y_{2,\ell_2}\}$, and by interchanging the roles of x and y, we also have

$$G_x \cap G_y = \{(y_{1,1} \cdots y_{1,\ell_1})^j \mid 0 \leq j \leq \ell_1 - 1\} \cap G.$$

3. The case where the order is less than or equal to 47

In this section, we prove Conjecture 1.3 in the case where the order of Q is less than or equal to 47 by listing up the profile of it.

Proposition 3.1. *In the case where the order of Q is less than or equal to 47, Conjecture* 1.3 *is true.*

Proof. We have the complete list of non-isomorphic connected quandles whose order is less than or equal to 47 by Rig which is available at `https://github.com/vendramin/rig` and is made by L. Vendramin. By the list [5], we can distinguish the affine quandles and the non-affine ones up to the order 35. In [3], T. Kajiwara and C. Nakayama proved that Conjecture 1.3 is true in the case of an affine quandle. By [1], it is known that a connected quandle whose order is a prime or the square of a prime is affine. In Table 1, we enumerate non-affine quandles whose order is less than or equal to 35. In Table 2, we enumerate all quandles whose order n is neither a prime nor the square of a prime, and satisfies $36 \leq n \leq 47$. By these tables, the proof is complete. □

Table 1: Profile of non-affine quandles whose order is less than or equal to 35

Order	$Q_{n,m}$	Profile
6	$Q_{6,1}$	$\{1,1,2,2\}$
	$Q_{6,2}$	$\{1,1,4\}$
8	$Q_{8,1}$	$\{1,1,3,3\}$
10	$Q_{10,1}$	$\{1,1,1,1,2,2,2\}$
12	$Q_{12,1}$ $Q_{12,6}$	$\{1,1,2,2,2,2,2\}$
	$Q_{12,2}$ $Q_{12,5}$	$\{1,1,2,4,4\}$
	$Q_{12,3}$	$\{1,1,5,5\}$
	$Q_{12,7}$	$\{1,1,2,8\}$
	$Q_{12,8}$	$\{1,1,1,1,2,2,2,2\}$
	$Q_{12,9}$	$\{1,1,1,1,4,4\}$
	$Q_{12,10}$	$\{1,1,1,3,3,3\}$
15	$Q_{15,2}$	$\{1,1,1,2,2,2,2,2,2\}$
	$Q_{15,5}$	$\{1,2,2,10\}$
	$Q_{15,6}$	$\{1,2,2,2,2,2,2,2\}$
	$Q_{15,7}$	$\{1,1,1,1,1,1,1,2,2,2,2\}$
18	$Q_{18,1}$ $Q_{18,2}$	$\{1,1,2,2,2,2,2,2,2,2\}$

Order	$Q_{n,m}$	Profile
	$Q_{18,3}$ $Q_{18,4}$	$\{1,1,2,2,4,4,4\}$
	$Q_{18,5}$ $Q_{18,6}$ $Q_{18,7}$	$\{1,1,2,2,12\}$
	$Q_{18,8}$ $Q_{18,9}$ $Q_{18,10}$	$\{1,1,2,2,6,6\}$
	$Q_{18,11}$	$\{1,1,1,1,2,6,6\}$
	$Q_{18,12}$	$\{1,1,1,1,2,2,2,2,2,2,2\}$
20	$Q_{20,1}$	$\{1,1,3,3,3,3,3,3\}$
	$Q_{20,2}$	$\{1,1,6,6,6\}$
	$Q_{20,3}$	$\{1,1,2,2,2,2,2,2,2,2,2\}$
	$Q_{20,5}$	$\{1,1,1,1,2,2,2,2,2,2,2,2\}$
	$Q_{20,6}$	$\{1,1,1,1,4,4,4,4\}$
	$Q_{20,9}$	$\{1,1,1,1,1,1,1,1,2,2,2,2,2,2\}$
	$Q_{20,10}$	$\{1,1,1,1,1,1,1,1,4,4,4\}$
21	$Q_{21,6}$	$\{1,1,1,1,1,2,2,2,2,2,2,2,2\}$
	$Q_{21,7}$	$\{1,2,2,2,14\}$
	$Q_{21,8}$	$\{1,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{21,9}$	$\{1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2\}$
24	$Q_{24,1}$ $Q_{24,17}$ $Q_{24,27}$ $Q_{24,37}$	$\{1,1,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{24,2}$	$\{1,1,1,1,4,4,4,4,4\}$
	$Q_{24,3}$ $Q_{24,5}$ $Q_{24,30}$	$\{1,1,1,1,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{24,4}$ $Q_{24,6}$ $Q_{24,19}$ $Q_{24,29}$ $Q_{24,31}$	$\{1,1,1,1,2,2,4,4,4,4\}$
	$Q_{24,7}$	$\{1,1,1,1,5,5,5,5\}$
	$Q_{24,8}$ $Q_{24,22}$	$\{1,1,2,2,3,3,6,6\}$
	$Q_{24,9}$	$\{1,1,1,7,7,7\}$
	$Q_{24,10}$ $Q_{24,11}$	$\{1,1,2,2,2,2,2,4,4,4\}$

Order	$Q_{n,m}$	Profile
	$Q_{24,12}$ $Q_{24,14}$ $Q_{24,15}$	$\{1,1,2,4,8,8\}$
	$Q_{24,13}$	$\{1,1,2,4,4,4,4,4\}$
	$Q_{24,16}$ $Q_{24,34}$	$\{1,1,1,1,1,1,2,2,2,2,2,2,2,2,2\}$
	$Q_{24,18}$ $Q_{24,33}$	$\{1,1,1,1,2,2,8,8\}$
	$Q_{24,20}$	$\{1,1,1,1,1,1,3,3,3,3,3,3\}$
	$Q_{24,21}$	$\{1,1,1,3,3,3,3,3,3,3\}$
	$Q_{24,23}$	$\{1,1,3,3,4,12\}$
	$Q_{24,26}$ $Q_{24,40}$	$\{1,1,2,2,2,8,8\}$
	$Q_{24,28}$ $Q_{24,35}$	$\{1,1,2,2,2,4,4,4,4\}$
	$Q_{24,32}$	$\{1,1,2,2,2,16\}$
	$Q_{24,36}$ $Q_{24,42}$	$\{1,1,1,1,1,1,2,4,4,4,4\}$
	$Q_{24,38}$	$\{1,1,1,1,1,1,1,1,4,4,4,4\}$
	$Q_{24,39}$	$\{1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2\}$
	$Q_{24,41}$	$\{1,1,1,1,1,1,2,8,8\}$
27	$Q_{27,1}$	$\{1,1,1,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{27,2}$	$\{1,1,1,4,4,4,4,4,4\}$
	$Q_{27,7}$ $Q_{27,9}$ $Q_{27,11}$ $Q_{27,12}$ $Q_{27,16}$ $Q_{27,35}$ $Q_{27,36}$ $Q_{27,41}$ $\vdots$ $Q_{27,46}$ $Q_{27,56}$ $\vdots$ $Q_{27,59}$	$\{1,2,2,2,2,6,6,6\}$

Order	$Q_{n,m}$	Profile
	$Q_{27,8}$, $Q_{27,10}$, $Q_{27,13}$, $Q_{27,15}$	$\{1,2,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{27,14}$	$\{1,1,1,2,2,2,6,6,6\}$
	$Q_{27,27}$, $Q_{27,28}$	$\{1,2,8,8,8\}$
	$Q_{27,37}$, $\vdots$, $Q_{27,40}$, $Q_{27,60}$, $Q_{27,61}$	$\{1,2,6,18\}$
	$Q_{27,53}$, $Q_{27,54}$, $Q_{27,55}$	$\{1,2,2,2,2,18\}$
28	$Q_{28,3}$, $Q_{28,4}$, $Q_{28,11}$, $Q_{28,12}$	$\{1,3,3,3,3,3,3,3,3,3\}$
	$Q_{28,5}$, $Q_{28,6}$	$\{1,3,6,6,6,6\}$
	$Q_{28,10}$	$\{1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{28,13}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2\}$
30	$Q_{30,1}$, $Q_{30,2}$, $Q_{30,13}$, $Q_{30,16}$	$\{1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{30,3}$, $Q_{30,9}$, $Q_{30,10}$	$\{1,1,4,4,4,4,4,4,4\}$
	$Q_{30,4}$, $Q_{30,20}$	$\{1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{30,5}$, $Q_{30,11}$	$\{1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{30,6}$, $Q_{30,15}$	$\{1,1,2,2,2,2,4,4,4,4,4\}$
	$Q_{30,7}$, $Q_{30,8}$	$\{1,1,2,2,4,4,4,4,4,4\}$
	$Q_{30,12}$	$\{1,1,2,2,2,2,10,10\}$

Order	$Q_{n,m}$	Profile
	$Q_{30,14}$	$\{1,1,2,2,2,2,20\}$
	$Q_{30,17}$ $Q_{30,18}$	$\{1,1,1,1,1,1,4,4,4,4,4,4\}$
	$Q_{30,19}$	$\{1,1,1,1,1,1,2,2,10,10\}$
	$Q_{30,21}$	$\{1,1,1,1,1,1,1,1,1,1,2,6,6,6\}$
	$Q_{30,22}$	$\{1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{30,23}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,4,4,4,4\}$
	$Q_{30,24}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2\}$
32	$Q_{32,1}$ $\vdots$ $Q_{32,3}$	$\{1,1,3,3,3,3,3,3,3,3,3,3\}$
	$Q_{32,4}$	$\{1,1,5,5,5,5,5,5\}$
	$Q_{32,5}$ $\vdots$ $Q_{32,9}$	$\{1,1,3,3,6,6,6,6\}$
33	$Q_{33,10}$	$\{1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{33,11}$	$\{1,2,2,2,2,2,22\}$

Table 2: Profile of all quandles whose order n is neither a prime nor the square of a prime, and satisfies $36 \leq n \leq 47$

Order	$Q_{n,m}$	Profile
36	$Q_{36,1}$ $Q_{36,3}$ $Q_{36,4}$ $Q_{36,5}$ $Q_{36,28}$ $Q_{36,33}$	$\{1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{36,2}$ $Q_{36,6}$ $Q_{36,7}$ $Q_{36,9}$ $Q_{36,35}$	$\{1,1,2,2,2,2,2,4,4,4,4,4,4\}$
	$Q_{36,8}$ $Q_{36,10}$	$\{1,1,2,2,2,2,2,8,8,8\}$

Order	$Q_{n,m}$	Profile
	$Q_{36,11}$ $Q_{36,13}$ $Q_{36,14}$ $Q_{36,18}$ $Q_{36,24}$ $Q_{36,36}$	$\{1, 1, 2, 2, 2, 2, 2, 12, 12\}$
	$Q_{36,12}$ $Q_{36,15}$ $Q_{36,16}$	$\{1, 1, 2, 2, 2, 2, 2, 24\}$
	$Q_{36,17}$ $Q_{36,19}$ $\vdots$ $Q_{36,22}$ $Q_{36,25}$ $Q_{36,30}$ $Q_{36,34}$	$\{1, 1, 2, 2, 2, 2, 2, 6, 6, 6, 6\}$
	$Q_{36,26}$	$\{1, 1, 2, 2, 5, 5, 10, 10\}$
	$Q_{36,27}$	$\{1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 6, 6, 6\}$
	$Q_{36,29}$	$\{1, 1, 2, 2, 6, 6, 6, 6, 6\}$
	$Q_{36,31}$	$\{1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2\}$
	$Q_{36,32}$	$\{1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 6, 6, 6, 6\}$
	$Q_{36,37}$	$\{1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 12, 12\}$
	$Q_{36,38}$	$\{1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 4, 4, 4, 4, 4, 4\}$
	$Q_{36,39}$ $Q_{36,42}$ $Q_{36,43}$	$\{1, 1, 1, 1, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4\}$
	$Q_{36,40}$ $Q_{36,41}$ $Q_{36,44}$ $Q_{36,63}$ $Q_{36,67}$	$\{1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2\}$
	$Q_{36,45}$	$\{1, 1, 1, 1, 4, 4, 4, 4, 4, 4, 4, 4\}$
	$Q_{36,46}$ $Q_{36,47}$ $Q_{36,48}$	$\{1, 1, 1, 1, 2, 2, 2, 2, 12, 12\}$
	$Q_{36,49}$ $Q_{36,50}$ $Q_{36,51}$	$\{1, 1, 1, 1, 2, 2, 2, 2, 6, 6, 6, 6\}$

Order	$Q_{n,m}$	Profile
	$Q_{36,52}$ $Q_{36,54}$ $Q_{36,55}$ $Q_{36,59}$	$\{1, 2, 3, 6, 6, 6, 6, 6\}$
	$Q_{36,53}$ $Q_{36,65}$	$\{1, 2, 2, 2, 2, 3, 6, 6, 6, 6\}$
	$Q_{36,56}$	$\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2\}$
	$Q_{36,57}$ $Q_{36,58}$ $Q_{36,60}$ $Q_{36,62}$	$\{1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3\}$
	$Q_{36,61}$	$\{1, 1, 1, 3, 3, 9, 9, 9\}$
	$Q_{36,64}$	$\{1, 1, 1, 2, 2, 2, 3, 3, 3, 6, 6, 6\}$
	$Q_{36,66}$	$\{1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 3, 3\}$
	$Q_{36,68}$	$\{1, 3, 4, 4, 12, 12\}$
	$Q_{36,69}$ $Q_{36,70}$	$\{1, 3, 8, 24\}$
	$Q_{36,71}$	$\{1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2\}$
	$Q_{36,72}$	$\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2\}$
	$Q_{36,73}$	$\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2\}$
39	$Q_{39,1}$ $Q_{39,13}$	$\{1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2\}$
	$Q_{39,2}$ $Q_{39,3}$	$\{1, 2, 3, 3, 3, 3, 6, 6, 6, 6\}$
	$Q_{39,4}$ $Q_{39,5}$	$\{1, 2, 4, 4, 4, 4, 4, 4, 4, 4, 4\}$
	$Q_{39,6}$ $Q_{39,7}$	$\{1, 2, 6, 6, 6, 6, 6, 6\}$
	$Q_{39,8}$ $Q_{39,9}$ $Q_{39,10}$ $Q_{39,11}$	$\{1, 2, 12, 12, 12\}$
	$Q_{39,12}$	$\{1, 2, 2, 2, 2, 2, 2, 26\}$
40	$Q_{40,1}$ $Q_{40,11}$	$\{1, 1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3\}$
	$Q_{40,2}$ $Q_{40,12}$	$\{1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2\}$
	$Q_{40,3}$	$\{1, 1, 2, 6, 6, 6, 6, 6, 6\}$

Order	$Q_{n,m}$	Profile
	$Q_{40,4}$	$\{1,1,2,2,2,2,3,3,6,6,6,6\}$
	$Q_{40,5}$ $Q_{40,6}$	$\{1,1,3,3,4,4,12,12\}$
	$Q_{40,7}$ $Q_{40,8}$	$\{1,1,2,4,8,8,8,8\}$
	$Q_{40,9}$ $Q_{40,10}$	$\{1,1,2,4,4,4,4,4,4,4,4,4\}$
	$Q_{40,13}$ $Q_{40,14}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,$ $2,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{40,15}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,4,4,4,4,4,4\}$
	$Q_{40,16}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,8,8,8\}$
	$Q_{40,17}$	$\{1,1,1,1,2,2,2,2,2,2,4,4,4,4,4,4\}$
	$Q_{40,18}$ $Q_{40,20}$	$\{1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{40,19}$ $Q_{40,21}$	$\{1,1,1,1,2,2,4,4,4,4,4,4,4,4\}$
	$Q_{40,22}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,$ $1,1,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{40,23}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,4,4,4,4,4,4\}$
	$Q_{40,24}$	$\{1,1,1,1,1,1,1,1,1,1,2,2,2,2,$ $2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{40,25}$	$\{1,1,1,1,1,1,1,1,1,1,2,2,2,4,4,4,4,4,4\}$
	$Q_{40,26}$	$\{1,1,1,1,2,2,2,3,3,3,3,6,6,6\}$
	$Q_{40,27}$ $Q_{40,28}$	$\{1,2,2,7,14,14\}$
	$Q_{40,29}$ ⋮ $Q_{40,32}$	$\{1,4,7,28\}$
	$Q_{40,33}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,3,3,3,3,3,3,3,3,3\}$
42	$Q_{42,1}$ $Q_{42,12}$ $Q_{42,15}$ $Q_{42,21}$	$\{1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{42,2}$ $Q_{42,18}$	$\{1,1,2,2,2,2,2,2,4,4,4,4,4,4,4\}$
	$Q_{42,3}$ $Q_{42,4}$	$\{1,1,2,2,6,6,6,6,6,6\}$
	$Q_{42,5}$ $Q_{42,6}$	$\{1,1,4,6,6,12,12\}$

Order	$Q_{n,m}$	Profile
	$Q_{42,7}$ $Q_{42,8}$	$\{1,1,2,2,3,3,3,3,6,6,6,6\}$
	$Q_{42,9}$ $Q_{42,10}$	$\{1,1,3,3,3,3,4,12,12\}$
	$Q_{42,11}$	$\{1,1,4,4,4,4,4,4,4,4,4,4\}$
	$Q_{42,13}$	$\{1,1,1,1,1,1,2,2,2,2,2,2,4,4,4,4,4,4\}$
	$Q_{42,14}$	$\{1,1,1,1,1,1,1,1,1,1,2,2,2,2,$ $2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{42,16}$	$\{1,1,2,2,2,2,2,2,14,14\}$
	$Q_{42,17}$	$\{1,1,2,2,2,2,2,2,28\}$
	$Q_{42,19}$ $Q_{42,20}$	$\{1,1,8,8,8,8,8\}$
	$Q_{42,22}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,$ $1,1,1,1,1,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{42,23}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,$ $1,1,1,1,1,1,1,4,4,4,4,4\}$
	$Q_{42,24}$	$\{1,1,1,1,1,1,2,2,2,2,2,2,2,2,$ $2,2,2,2,2,2,2,2,2,2\}$
	$Q_{42,25}$	$\{1,1,1,1,1,1,1,1,2,2,2,14,14\}$
	$Q_{42,26}$	$\{1,1,1,1,1,1,1,1,2,2,2,2,2,$ $2,2,2,2,2,2,2,2,2,2,2,2\}$
44	$Q_{44,1}$	$\{1,2,2,2,2,2,3,6,6,6,6,6\}$
	$Q_{44,2}$ $\vdots$ $Q_{44,5}$	$\{1,3,5,5,15,15\}$
	$Q_{44,6}$ $\vdots$ $Q_{44,9}$	$\{1,3,10,30\}$
45	$Q_{45,1}$ $Q_{45,13}$ $Q_{45,15}$ $Q_{45,30}$ $Q_{45,33}$	$\{1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{45,2}$ $Q_{45,3}$ $Q_{45,4}$ $Q_{45,17}$ $Q_{45,19}$ $Q_{45,22}$	$\{1,2,2,2,2,2,2,2,6,6,6,6,6\}$

Order	$Q_{n,m}$	Profile
	$Q_{45,5}$ $Q_{45,6}$ $Q_{45,31}$ $Q_{45,32}$	$\{1,2,2,2,2,4,4,4,4,4,4,4,4,4\}$
	$Q_{45,7}$ $\vdots$ $Q_{45,12}$	$\{1,2,4,4,4,6,12,12\}$
	$Q_{45,14}$ $Q_{45,16}$	$\{1,2,2,2,2,2,2,2,10,10,10\}$
	$Q_{45,18}$ $Q_{45,20}$ $Q_{45,21}$	$\{1,2,2,2,2,2,2,2,30\}$
	$Q_{45,23}$ $Q_{45,24}$	$\{1,1,1,1,1,1,1,2,2,2,2,2,2,2,$ $2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{45,25}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,$ $2,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{45,26}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,$ $1,1,1,1,1,1,2,6,6,6,6\}$
	$Q_{45,27}$	$\{1,1,1,2,2,2,2,2,2,2,2,2,2,$ $2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{45,28}$	$\{1,1,1,1,1,2,2,2,2,2,2,2,2,$ $2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{45,29}$	$\{1,1,1,1,1,1,1,1,1,2,2,2,2,2,$ $2,2,2,2,2,2,2,2,2,2,2,2,2.\}$
	$Q_{45,34}$	$\{1,2,2,2,2,6,6,6,6,6,6\}$
	$Q_{45,35}$	$\{1,2,2,4,4,4,4,4,4,4,4,4,4\}$
	$Q_{45,36}$ $Q_{45,37}$	$\{1,4,4,4,4,4,4,4,4,4,4,4\}$
	$Q_{45,38}$ $Q_{45,39}$	$\{1,2,2,8,8,8,8,8\}$
	$Q_{45,40}$ $\vdots$ $Q_{45,43}$	$\{1,4,8,8,8,8,8\}$
	$Q_{45,44}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,$ $2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\}$
	$Q_{45,45}$	$\{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,$ $1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2\}$

ACKNOWLEDGMENT. The author would like to thank Professor Chikara Nakayama for his valuable suggestions and many helpful comments in writing this paper. He also would like to thank the referee for careful reading and valuable comments.

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(*Received April 1, 2017*)
(*Accepted August 25, 2017*)