BEREZIN-WEYL QUANTIZATION OF HEISENBERG MOTION GROUPS

To the memory of my father, Alfred Cahen

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Abstract. We introduce a Schrödinger model for the generic representations of a Heisenberg motion group and we construct adapted Weyl correspondences for these representations by adapting the method introduced in [B. Cahen, Weyl quantization for semidirect products, Differential Geom. Appl. 25 (2007), 177-190].

1. Introduction

In [12] and [13], we introduced the notion of adapted Weyl correspondence as a direct generalization of the usual Weyl quantization [1], [27].

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\pi$ be a unitary irreducible representation of $G$ on a Hilbert space $\mathcal{H}$. Assume that $\pi$ is associated with a coadjoint orbit $O \subset \mathfrak{g}^*$ of $G$ by the Kirillov-Kostant method of orbits [34], [35]. The following definition for the notion of adapted Weyl correspondence is taken from [15] (see also [30], [2] and [3]).

Definition 1. An adapted Weyl correspondence is an isomorphism $W$ from a vector space $\mathcal{A}$ of complex-valued smooth functions on the orbit $O$ (called symbols) onto a vector space $\mathcal{B}$ of (not necessarily bounded) linear operators on $\mathcal{H}$ satisfying the following properties:

1. the elements of $\mathcal{B}$ preserve a fixed dense domain $\mathcal{D}$ of $\mathcal{H}$;
2. the constant function 1 belongs to $\mathcal{A}$, the identity operator $I_\mathcal{H}$ belongs to $\mathcal{B}$ and $W(1) = I_\mathcal{H}$;
3. $A \in \mathcal{B}$ and $B \in \mathcal{B}$ implies $AB \in \mathcal{B}$;
4. for each $f$ in $\mathcal{A}$ the complex conjugate $\bar{f}$ of $f$ belongs to $\mathcal{A}$ and the adjoint of $W(f)$ is an extension of $W(\bar{f})$;
5. the elements of $\mathcal{D}$ are $C^\infty$-vectors for the representation $\pi$, the functions $\tilde{X}(X \in \mathfrak{g})$ defined on $O$ by $\tilde{X}(\xi) = \langle \xi, X \rangle$ are in $\mathcal{A}$ and we have $W(i\tilde{X})v = d\pi(X)v$ for each $X \in \mathfrak{g}$ and each $v \in \mathcal{D}$.

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We have constructed adapted Weyl correspondences in different situations, especially for unitary representations of semidirect products of the form $V \rtimes K$ where $K$ is a semi-simple Lie group acting linearly on a vector space $V$ [15], [17]. Note that adapted Weyl correspondences have various applications in harmonic analysis and deformation theory as, for instance, the construction of covariant star-products on coadjoint orbits [12] and the study of contractions of Lie group unitary representations [26], [14], [20].

Note also that the notion of adapted Weyl correspondence is close to that of Stratonovich-Weyl correspondence [41], [28], [29]. Roughly speaking, Stratonovich-Weyl correspondences do not require to satisfy (5) of Definition 1 but they have to be unitary and $G$-equivariant [28]. We refer to [18] for a short discussion of the advantages and disadvantages of these two methods of quantization (see also [22]).

Let us consider the typical case of the $(2n + 1)$-dimensional Heisenberg group $G_0$. Each non-degenerate unitary irreducible representation of $G_0$ has two usual realizations: the Schrödinger realization on $L^2(\mathbb{R}^n)$ and the Bargmann-Fock realization on the Fock space [27], [42], an intertwining operator between these realizations being the Segal-Bargmann transform [27], [25]. In the setting of the orbit method, the Schrödinger realization can be obtained from a real polarization of the corresponding coadjoint orbit of $G_0$ and the Bargmann-Fock realization from a totally complex polarization [6], [11]. Moreover, the usual Weyl correspondence provides an adapted Weyl correspondence for the Schrödinger realization [5], [45]. It is also known that this adapted Weyl correspondence is related, by the Segal-Bargmann transform, to the unitary part of the polar decomposition of the Berezin quantization map associated with the Bargmann-Fock realization [37], [36].

In [19] and [23], we made similar considerations for the generic representations of the real diamond group and of Heisenberg motion groups. In these cases, the generic coadjoint orbits of $G$ don’t necessarily admit real polarization and the corresponding representations are usually obtained as holomorphically induced representations on Bargmann-Fock spaces. We can nonetheless obtain ‘Schrödinger realizations’ of these representations from Bargmann-Fock realizations by conjugation with the Segal-Bargmann transform.

Let $G$ be a Heisenberg motion group, that is, the semidirect product of the Heisenberg group $G_0$ by a connected compact subgroup $K$ of the unitary group $U(n)$. Such groups play an important role in the theory of Gelfand pairs, since the study of a Gelfand pair of the form $(K_0, N)$, where $K_0$ is a compact Lie group acting by automorphisms on a nilpotent Lie group $N$, can be reduced to that of the form $(K_0, H_n)$, see in particular [7] and [8].
In the present paper, we exploit some results of [23] in order to construct an adapted Weyl correspondence for each generic representation $\pi$ of $G$. More precisely, we consider a Schrödinger model for $\pi$ as in [23], that is, a realization of $\pi$ in the Hilbert space $L^2(\mathbb{R}^n) \otimes V$ where $V$ is a finite dimensional complex vector space which carries an irreducible unitary representation $\rho$ of $K$. Then we introduce the map $W := W_0 \otimes s^{-1}$ where $W_0$ is the usual Weyl correspondence and $s$ is the Berezin calculus associated with $V$, and we show that $W$ is $G$-equivariant. Moreover, we compute $W^{-1}(d\pi(X))$ for $X \in \mathfrak{g}$ and we conclude that if $K \subset SU(n)$ then $W$ induces an adapted Weyl correspondence for $\pi$.

Note that, in [38], a Schrödinger model and a generalized Segal-Bargmann transform for the scalar highest weight representations of an Hermitian Lie group of tube type were introduced and studied (see also [32]). Then one can hope for further generalizations of our construction to quasi-Hermitian Lie groups.

This paper is organized as follows. In Sections 2-4, we review some facts about the Fock model and the Schrödinger model of the unitary irreducible representations of an Heisenberg group. We follow the presentation of [19] (see also [27] and [25]).

For each $z, w \in \mathbb{C}^n$, we denote $zw := \sum_{k=1}^n z_kw_k$ and we consider the symplectic form $\omega$ on $\mathbb{C}^{2n}$ defined by

$$\omega((z, w), (z', w')) = \frac{i}{2}(zw' - z'w).$$

for $z, w, z', w' \in \mathbb{C}^n$.

Let $G_0$ be the $(2n + 1)$-dimensional Heisenberg group consisting of all elements of the form $((z, \bar{z}), c)$ where $z \in \mathbb{C}^n$ and $c \in \mathbb{R}$. The multiplication of $G_0$ is given by

$$(((z, \bar{z}), c) \cdot ((z', \bar{z'}), c')) = (((z + z', \bar{z} + \bar{z'}), c + c' + \frac{1}{2}\omega((z, \bar{z}), (z', \bar{z'}))))).$$

Let $\mathfrak{g}_0$ be the Lie algebra of $G_0$ and $\mathfrak{g}_0^*$ its complexification. We write the elements of $\mathfrak{g}_0^*$ as $((a, b), c)$ where $a, b \in \mathbb{C}^n$ and $c \in \mathbb{C}$. Then the Lie
brackets of $g_0^c$ are given by
\[ \left[ ((a, b), c), ((a', b'), c') \right] = ((0, 0), \omega((a, b), (a', b'))) \].

Fix a real number $\lambda > 0$ and denote by $O_\lambda$ the orbit of the element $\xi_\lambda : ((a, \bar{a}), c) \rightarrow \lambda c$ of $g_0^c$ under the coadjoint action of $G_0$ (the case $\lambda < 0$ can be treated similarly). By the Stone-von Neumann theorem, there exists a unique (up to unitary equivalence) unitary irreducible representation of $G_0$ whose restriction to the center of $G_0$ is the character $((0, 0), c) \rightarrow e^{i\lambda c}$ [27], [42]. Then this representation is associated with $O_\lambda$ by the Kirillov-Kostant method of orbits [34], [35]. More precisely, if we choose the real polarization at $\xi_\lambda$ to be $\{ ((ib, -ib), c) : b \in \mathbb{R}^n, c \in \mathbb{C} \}$ then we obtain the Schrödinger representation $\sigma_0$ realized on $L^2(\mathbb{R}^n)$ as
\[ \sigma_0((z_0, \bar{z}_0), c_0)f(x) = e^{i\lambda(c_0 - y_0 x + \frac{1}{2}x_0 y_0)} f(x - x_0), \]
where $z_0 = x_o + iy_0, x_0, y_0 \in \mathbb{R}^n$ [27], [42].

On the other hand, if we choose the complex polarization at $\xi_\lambda$ to be $\{ ((0, w), c) : w \in \mathbb{C}^n, c \in \mathbb{C} \}$ then we obtain the Bargmann-Fock representation $\pi_0$ defined as follows [27].

Let $\mathcal{F}_0$ be the Hilbert space of holomorphic functions $F$ on $\mathbb{C}^n$ such that
\[ \|F\|_{\mathcal{F}_0}^2 := \int_{\mathbb{C}^n} |F(z)|^2 e^{-|z|^2/2\lambda} d\mu_\lambda(z) < +\infty \]
where $d\mu_\lambda(z) := (2\pi \lambda)^{-n} dx \, dy$. Here $z = x + iy$ with $x$ and $y$ in $\mathbb{R}^n$. Then $\pi_0$ is the representation of $G_0$ on $\mathcal{F}_0$ given by
\[ (\pi_0(g_0)F)(z) = \exp \left( i\lambda c_0 + \frac{1}{2}i\bar{z}_0 z - \frac{1}{2}|z_0|^2 \right) F(z + i\lambda z_0) \]
where $g = ((z_0, \bar{z}_0), c_0) \in G_0$ and $z \in \mathbb{C}^n$.

We consider the action of $K$ on $G_0$ defined by
\[ k \cdot ((z_0, \bar{z}_0), c_0) := ((kz_0, \bar{kz}_0), c_0). \]
Let $\tau$ be the representation of $K$ on $\mathcal{F}_0$ defined by $\tau(k)F)(z) := F(k^{-1}z)$. Note that, for each $k \in K$ and $g_0 \in G_0$, we have
\[ \pi_0(k \cdot g_0) = \tau(k)\pi_0(g_0)\tau(k)^{-1}. \]

As in [27], Chapter 1, [31], Section 6 or [25], Section 1.3, we can verify that the Segal-Bargmann transform $B_0 : L^2(\mathbb{R}^n) \rightarrow \mathcal{F}_0$ defined by
\[ B_0(f)(z) = (\lambda/\pi)^{n/4} \int_{\mathbb{R}^n} e^{(1/4\lambda)z^2 + ixz - (\lambda/2)x^2} f(x) \, dx \]
is a (unitary) intertwining operator between $\sigma_0$ and $\pi_0$, that is, for each $g_0 \in G_0$, one has $\sigma_0(g_0) = B_0^{-1}\pi_0(g_0)B_0$. 

3. Berezin calculus for Heisenberg groups

We first recall the definition of the Berezin calculus. For each \( z \in \mathbb{C}^n \), we consider the 'coherent state' \( e_z(w) := \exp(\bar{z}w/2\lambda) \). Then we have \( F(z) = \langle F, e_z \rangle_{\mathcal{F}_0} \) for each \( F \in \mathcal{F}_0 \) where \( \langle \cdot, \cdot \rangle_{\mathcal{F}_0} \) denotes the scalar product on \( \mathcal{F}_0 \).

Let \( \mathcal{C}_0 \) be the space of all operators (not necessarily bounded) \( A_0 \) on \( \mathcal{F}_0 \) whose domain contains \( e_z \) for each \( z \in \mathbb{C}^n \). Then the Berezin symbol of \( A_0 \in \mathcal{C}_0 \) is the function \( S^0(A_0) \) defined on \( \mathbb{C}^n \) by

\[
S^0(A_0)(z) := \frac{\langle A_0 e_z, e_z \rangle_{\mathcal{F}_0}}{\langle e_z, e_z \rangle_{\mathcal{F}_0}}.
\]

Let us consider the action of \( G_0 \) on \( \mathbb{C}^n \) defined by \( g_0 \cdot z := z - i\lambda z_0 \) where \( g_0 = ((z_0, z_0), c_0) \). For each function \( F \) on \( \mathbb{C}^n \) (non necessarily in \( \mathcal{F}_0 \)) and each \( g_0 \in G_0 \), we denote by \( L^0_{g_0} F \) the function on \( \mathbb{C}^n \) defined by \( (L^0_{g_0} F)(z) = F(g_0^{-1} \cdot z) \). Then we have the following properties of \( S^0 \), see for instance [19].

**Proposition 3.1.**  
(1) Each \( A_0 \in \mathcal{C}_0 \) is determined by \( S^0(A_0) \);  
(2) For each \( A_0 \in \mathcal{C}_0 \) and each \( z \in \mathbb{C}^n \), we have \( S^0(A_0^*) = S^0(A_0)(z) ; \)  
(3) We have \( S^0(I_{\mathcal{F}_0}) = 1 \);  
(4) The map \( S^0 \) is \( G_0 \)-equivariant with respect to \( G_0 \), that is, for each \( A_0 \in \mathcal{C}_0 \), \( g_0 \in G_0 \) and \( z \in \mathbb{C}^n \), we have \( \pi_0(g_0)^{-1} A_0 \pi_0(g_0) \in \mathcal{C}_0 \) and

\[
S^0(A_0)(g_0 \cdot z) = S^0(\pi_0(g_0)^{-1} A_0 \pi_0(g_0))(z)
\]

or, equivalently,

\[
L^0_{g_0} S^0(A_0) = S^0(\pi_0(g_0)^{-1} A_0 \pi_0(g_0))^{-1} ;
\]

(5) The map \( S^0 \) is a bounded operator from the space \( \mathcal{L}_2(\mathcal{F}_0) \) of all Hilbert-Schmidt operators on \( \mathcal{F}_0 \) (endowed with the Hilbert-Schmidt norm) to \( L^2(\mathbb{C}^n, \mu_\lambda) \) which is one-to-one and has dense range.

Let us recall that the Berezin transform is then the operator \( B^0 \) on \( L^2(\mathbb{C}^n, \mu_\lambda) \) defined by \( B^0 = S^0(S^0)^* \). Thus we can verify that

\[
B^0(F)(z) = \int_{\mathbb{C}^n} F(w) e^{i|z-w|^2/2\lambda} d\mu_\lambda(w) ,
\]

see [9], [10], [43], [40] for instance. Also, it is well-known that we have \( B^0 = \exp(\lambda \Delta/2) \) where \( \Delta = 4\sum_{k=1}^n \partial^2/\partial z_k \partial \bar{z}_k \), see [43], [36].

Let \( U^0 \) be the unitary part in the polar decomposition of \( S^0 \) (seen as a bounded operator from \( \mathcal{L}_2(\mathcal{F}_0) \) to \( L^2(\mathbb{C}^n, \mu_\lambda) \)), that is, \( U^0 := (B^0)^{-1/2} S^0 \). As a particular case of [18], Proposition 6.1, we have the following result.

**Proposition 3.2.** \( U^0 \) is \( G_0 \)-equivariant with respect to \( \pi_0 \), that is, for each \( A_0 \in \mathcal{L}_2(\mathcal{F}_0) \) and \( g_0 \in G_0 \), we have
For each \( k \in K \) and each function \( F \) on \( \mathbb{C}^n \) (not necessarily in \( \mathcal{F}_0 \)) we denote by \( l_k F \) the function on \( \mathbb{C}^n \) defined by \((l_k F)(z) := F(k^{-1}z)\). Then we have the following result.

**Proposition 3.3.** For each \( k \in K \) and \( A_0 \) operator on \( \mathcal{F}_0 \), we have

\[
S^0(\tau(k)A_0 \tau(k)^{-1}) = l_k S^0(A_0)
\]

and, similarly,

\[
U^0(\tau(k)A_0 \tau(k)^{-1}) = l_k U^0(A_0).
\]

**Proof.** For the first assertion, note that for each \( z, w \in \mathbb{C}^n \) and \( k \in K \), we have

\[
(\tau(k)e_z)(w) = e_z(k^{-1}w) = \exp(\bar{z}(k^{-1}w)/2\lambda)
= \exp((\bar{k}z)w/2\lambda) = e_{kz}(w)
\]

hence \( \tau(k)e_z = e_{kz} \). This implies that

\[
S^0(A_0)(k^{-1}z) = \frac{\langle A_0 e_{k^{-1}z}, e_{k^{-1}z} \rangle_{\mathcal{F}_0}}{(e_{k^{-1}z}, e_{k^{-1}z})_{\mathcal{F}_0}}
= \frac{\langle A_0 \tau(k)^{-1}e_z, \tau(k)^{-1}e_z \rangle_{\mathcal{F}_0}}{(e_z, e_z)_{\mathcal{F}_0}}
= \frac{(\tau(k)A_0 \tau(k)^{-1}e_z, e_z)_{\mathcal{F}_0}}{(e_z, e_z)_{\mathcal{F}_0}}
= S^0(\tau(k)A_0 \tau(k)^{-1})(z).
\]

Now we prove the second assertion. First note that, by using the integral formula for \( B_0 \), we see that \( B_0 \)-hence \( B_0^{-1/2} \)-commute with \( l_k \) for each \( k \in K \). Then, denoting by \( \mathcal{I}_r(k) \) the operator \( A_0 \to \tau(k)A_0 \tau(k)^{-1} \) on \( L_2(\mathcal{F}_0) \), we can reformulate the first assertion as \( S^0 \mathcal{I}_r(k) = l_k S^0 \) for each \( k \in K \). Consequently we have

\[
U^0 \mathcal{I}_r(k) = B_0^{-1/2} S^0 \mathcal{I}_r(k) = B_0^{-1/2} l_k S^0 = l_k B_0^{-1/2} S^0 = l_k U^0,
\]

hence the result. \( \Box \)

### 4. Weyl correspondence for Heisenberg groups

In this section, we introduce the usual Weyl correspondence and review some of its properties.
The Weyl correspondence $W_0$ on $\mathbb{R}^{2n}$ is usually defined as follows. For each $f$ in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$, let $W_0(f)$ be the operator on $L^2(\mathbb{R}^n)$ defined by

$$W_0(f)\varphi(p) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{isq} f(p + (1/2)s, q) \varphi(p + s) \, ds \, dq.$$ 

The Weyl calculus can be extended to much larger classes of symbols (see for instance [33]). In particular, if $f(p, q) = u(p)q^\alpha$ where $u \in C^\infty(\mathbb{R}^n)$ then we have

$$W_0(f)\varphi(p) = \left( i\frac{\partial}{\partial s} \right)^\alpha (u(p + (1/2)s) \varphi(p + s)) \bigg|_{s=0},$$

see [44].

Now, we transfer the action of $G_0$ on $\mathbb{C}^n$ introduced in Section 3 to $\mathbb{R}^{2n}$ by means of the map $j : (p, q) \rightarrow q - \lambda ip$, that is, we consider the action of $G_0$ on $\mathbb{R}^{2n}$ defined by

$$g_0 \cdot (p, q) := j^{-1}(g_0 \cdot (p, q)) = (p + x_0, q + \lambda y_0)$$

where $g_0 = ((z_0, \bar{z}_0), c_0)$ and $z_0 = x_0 + iy_0$ with $x_0, y_0 \in \mathbb{R}^n$. Then we have the following result.

**Proposition 4.1.** [19] Let $\Psi_\lambda : \mathbb{R}^{2n} \rightarrow \mathfrak{g}_0^*$ be the map defined by

$$\langle \Psi_\lambda(p, q), X \rangle := \text{Re}((q - \lambda ip)\bar{a}) + \lambda c$$

for each $X = ((a, \bar{a}), c) \in \mathfrak{g}_0$. Then

1. For each $X \in \mathfrak{g}_0$ and each $(p, q) \in \mathbb{R}^{2n}$, we have

$$W_0^{-1}(d\sigma_0(X))(p, q) = i\langle \Psi_\lambda(p, q), X \rangle.$$ 

2. For each $g_0 \in G_0$ and each $(p, q) \in \mathbb{R}^{2n}$, we have $\Psi_\lambda(g_0 \cdot (p, q)) = \text{Ad}^\ast(g_0) \Psi_\lambda(p, q)$.

3. The map $\Psi_\lambda$ is a diffeomorphism from $\mathbb{R}^{2n}$ onto $\mathcal{O}_\lambda$.

Let $\mathcal{L}_2(L^2(\mathbb{R}^n))$ be the Hilbert space of all Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. Then $W_0$ induces a unitary operator from $\mathcal{L}_2(\mathbb{R}^{2n})$ onto $\mathcal{L}_2(L^2(\mathbb{R}^n))$ [27].

For each $g_0 \in G_0$ and each function $f$ on $\mathbb{R}^{2n}$, we denote by $\hat{L}_{g_0}^0 f$ the function on $\mathbb{R}^{2n}$ defined by $\hat{L}_{g_0}^0 f(p, q) = f(g_0^{-1} \cdot (p, q))$. Then we have the following result.

**Proposition 4.2.** [27], [36], [19] $W_0$ is $G_0$-equivariant with respect to $\sigma_0$, that is, for each $g_0 \in G_0$ and each $f \in L^2(\mathbb{R}^{2n})$, we have

$$W_0(\hat{L}_{g_0}^0 f) = \sigma_0(g_0) W_0(f) \sigma_0(g_0)^{-1}.$$
Equivalently, \( W_0^{-1} \) is \( G_0 \)-equivariant with respect to \( \sigma_0 \), that is, for each \( g_0 \in G_0 \) and each \( A_0 \in \mathcal{L}_2(L^2(\mathbb{R}^n)) \), we have

\[
W_0^{-1}(\sigma_0(g_0)A_0\sigma_0(g_0)^{-1}) = \tilde{L}^{0}_{g_0}W_0^{-1}(A_0).
\]

We can then obtain an adapted Weyl correspondence for \( \sigma_0 \) as follows. Let \( A \) be the space of all functions \( f \) on \( \mathcal{O}_\lambda \) such that \( (f \circ \Psi_\lambda)(p, q) \) is a smooth function which is polynomial in the variable \( q \). Let \( \mathcal{B} \) be the space of all differential operators on \( \mathbb{R}^n \) with coefficients in \( C^\infty(\mathbb{R}^n) \). Then from Proposition 4.1 and Proposition 4.2 we can deduce the following result.

**Proposition 4.3.** [19] The map \( W_0 : A \to \mathcal{B} \) defined by \( W_0(f) := W_0(f \circ \Psi_\lambda) \) is an adapted Weyl correspondence which is \( G_0 \)-equivariant with respect to \( \sigma_0 \).

Finally, note that \( W_0 \) (hence \( W_0 \)) can be related to \( U^0 \) (see Section 3) as follows. Let \( I_{B_0} \) be the unitary map from \( \mathcal{L}_2(L^2(\mathbb{R}^n)) \) onto \( \mathcal{L}_2(\mathcal{F}_0) \) defined by \( I_{B_0}(A) = B_0AB_0^{-1} \) and let \( J \) be the map from \( L^2(\mathbb{C}^n, \mu_\lambda) \) onto \( L^2(\mathbb{R}^{2n}) \) defined by \( J(F) = F \circ j \). Then we have the following proposition.

**Proposition 4.4.** [36], [40] We have \( U^0I_{B_0} = (W_0J)^{-1} \).

5. **Heisenberg motion groups**

Let \( K \) be a closed subgroup of \( U(n) \). Recall \( K \) acts on \( G_0 \) by \( k \cdot ((z, \bar{z}), c) = ((kz, \bar{k}z), c) \), see Section 2. Then we can form the semidirect product \( G := G_0 \rtimes K \) with respect to this action. The group \( G \) is called a Heisenberg motion group. The elements of \( G \) can be written as \( ((z, \bar{z}), c, k) \) where \( z \in \mathbb{C}^n, c \in \mathbb{R}, k \in K \) and the multiplication of \( G \) is given by

\[
((z, \bar{z}), c, k) \cdot ((z', \bar{z'}), c', k') = ((z, \bar{z}) + (kz', \bar{k}z'), c + c' + \frac{1}{2}\omega((z, \bar{z}), (kz', \bar{k}z')), kk').
\]

Let \( \mathfrak{k} \) and \( \mathfrak{g} \) be the Lie algebras of \( K \) and \( G \). The Lie brackets of \( \mathfrak{g} \) are given by

\[
[[((w, \bar{w}), c), A], ((w', \bar{w'}), c', A')] = ((Aw' - A'w, A\bar{w'} - A\bar{w}), \omega((w, \bar{w}), (w', \bar{w})), [A, A']).
\]

Now, we give the formulas for the adjoint and coadjoint actions of \( G \).

Let \( g = ((z_0, \bar{z}_0), c_0, k_0) \in G \) where \( z_0 \in \mathbb{C}^n, c_0 \in \mathbb{R}, k_0 \in K \) and \( X = ((w, \bar{w}), c, A) \in \mathfrak{g} \) where \( w \in \mathbb{C}^n, c \in \mathbb{R} \) and \( A \in \mathfrak{t} \). We can easily verify that

\[
\text{Ad}(g)X = \frac{d}{dt}(g \exp(tX)g^{-1})|_{t=0} = ((w', \bar{w'}), c', \text{Ad}(k_0)A)
\]

where \( w' := k_0w - (\text{Ad}(k_0)A)z_0 \) and \( c' := c + \omega((z_0, \bar{z}_0), (k_0w, \bar{k_0}w)) - \frac{1}{2}\omega((z_0, \bar{z}_0), (\text{Ad}(k_0)A)z_0, \text{Ad}(k_0)A\bar{z}_0)) \).
Now, let us denote by \( \xi = ((u, \bar{u}), d, \phi) \), where \( u \in \mathbb{C}^n \), \( d \in \mathbb{R} \) and \( \phi \in \mathfrak{k}^* \), the element of \( \mathfrak{g}^* \) defined by
\[
\langle \xi, ((w, \bar{w}), c, A) \rangle = \omega((u, \bar{u}), (w, \bar{w})) + dc + \langle \phi, A \rangle.
\]
Also, for \( u, v \in \mathbb{C}^n \), we denote by \( (v, \bar{v}) \times (u, \bar{u}) \) the element of \( \mathfrak{k}^* \) defined by
\[
\langle (v, \bar{v}) \times (u, \bar{u}), A \rangle := \omega((u, \bar{u}), (Av, \bar{Av}))
\]
for \( A \in \mathfrak{k} \). Then, from the formula for the adjoint action of \( G \), we deduce that, for each \( \xi = ((u, \bar{u}), d, \phi) \in \mathfrak{g}^* \) and \( g = ((z_0, \bar{z}_0), c_0, k_0) \in G \), we have
\[
\text{Ad}^*(g)\xi = ((k_0u - dz_0, \bar{k}_0u - \bar{d}\bar{z}_0), d, \text{Ad}^*(k_0)\phi + (z_0, \bar{z}_0) \times (k_0u - \frac{d}{2}z_0, \bar{k}_0u - \frac{d}{2}\bar{z}_0)).
\]
From this, we see that if a coadjoint orbit of \( G \) contains a point \( ((u, \bar{u}), d, \phi) \) with \( d \neq 0 \) then it also contains a point of the form \( ((0, 0), d, \phi_0) \). Such an orbit is called \textit{generic}.

We consider the unitary irreducible representations of \( G \) associated with the integral generic orbits. These representations are called \textit{generic} and we can realize them in Fock spaces as holomorphic induced representations by using the general method of [39], Chapter XII.

More precisely, let us consider a unitary irreducible representation \( \rho \) of \( K \) on a (finite-dimensional) complex vector space \( V \) and let us fix an element \( \xi_0 = ((0, 0), d, \phi_0) \) of \( \mathfrak{g}^* \). We assume that \( d \neq 0 \) and that the orbit \( o(\phi_0) \) of \( \phi_0 \) for the coadjoint action of \( K \) is associated with \( \rho \) as in [21] and [46].

Let \( \tilde{K} \) be the subgroup of \( G \) defined by \( \tilde{K} := \{((0, 0), c, k) : c \in \mathbb{R}, k \in K\} \) and let \( \tilde{\rho} \) be the representation of \( \tilde{K} \) on \( V \) defined by \( \tilde{\rho}((0, 0), c, k) = e^{i\lambda c}\rho(k) \) for each \( c \in \mathbb{R} \) and \( k \in K \). Then we can easily verify that the representation \( \pi \) of \( G \) which is holomorphically induced from \( \tilde{\rho} \) can be realized in the Hilbert space \( \mathcal{F} \) of all holomorphic functions \( f : \mathbb{C}^n \to V \) such that
\[
\|f\|^2_{\mathcal{F}} := \int_{\mathbb{C}^n} \|f(z)\|^2_V e^{-|z|^2/2\lambda} d\mu_\lambda(z) < +\infty
\]
as
\[
(\pi(g)f)(z) = \exp(i\lambda c_0 + \frac{1}{2}iz_0z - \frac{1}{2}|z_0|^2) \rho(k) f(k^{-1}(z + i\lambda z_0))
\]
where \( g = ((z_0, \bar{z}_0), c_0, k) \in G \) and \( z \in \mathbb{C}^n \).

Note that we have \( \mathcal{F} = \mathcal{F}_0 \otimes V \). For \( f_0 \in \mathcal{F}_0 \) and \( v \in V \), we denote by \( f_0 \otimes v \) the function \( z \to f_0(z)v \). It is clear that
\[
\langle f_0 \otimes v, f_1 \otimes w \rangle_{\mathcal{F}} = \langle f_0, f_1 \rangle_{\mathcal{F}_0} \langle v, w \rangle_V
\]
for each \( f_0, f_1 \in \mathcal{F}_0 \) and each \( v, w \in V \). Moreover, if \( A_0 \) is an operator of \( \mathcal{F}_0 \) and \( A_1 \) is an operator of \( V \) then we denote by \( A_0 \otimes A_1 \) the operator of
$F$ defined by $(A_0 \otimes A_1)(f_0 \otimes v) = A_0 f_0 \otimes A_1 v$ for each $f_0 \in F_0$ and each $v \in V$. Then we have the decomposition formula
\begin{equation}
\pi((z_0, \overline{z}_0), c_0, k) = \pi_0((z_0, \overline{z}_0), c_0) \tau(k) \otimes \rho(k)
\end{equation}
for each $z_0 \in \mathbb{C}^n$, $c_0 \in \mathbb{R}$ and $k \in K$. This is precisely Formula (3.18) in [7].

Now, we introduce the Schrödinger representations of $G$ by extending $B_0$ to $V$-valued functions. More precisely, we consider the map $B$ from $L^2(\mathbb{R}^n, V) \cong L^2(\mathbb{R}^n) \otimes V$ to $F \cong F_0 \otimes V$ defined by $B := B_0 \otimes I_V$. Then we have the integral formula
\begin{equation}
B(f)(z) = (\lambda/\pi)^n/4 \int_{\mathbb{R}^n} e^{(1/4\lambda)x^2 + ixz - (\lambda/2)x^2} f(x) \, dx
\end{equation}
for each $f \in L^2(\mathbb{R}^n, V)$.

This allows us to imitate the case of the Heisenberg groups and to define the Schrödinger representation $\sigma$ of $G$ on $L^2(\mathbb{R}^n, V)$ by $\sigma(g) := B^{-1} \tau(g) B$.

Similarly, for each $k \in K$ we define the operator $\tilde{\tau}$ of $L^2(\mathbb{R}^n)$ by $\tilde{\tau}(k) := B_0^{-1} \tau(k) B_0$. Then, from Equation 5.1 we immediately obtain the decomposition formula
\begin{equation}
\sigma(g) = \sigma_0(g_0) \tilde{\tau}(k) \otimes \rho(k)
\end{equation}
for each $g_0 \in G_0$, $k \in K$ and $g = (g_0, k) \in G$.

6. Berezin correspondence for Heisenberg motion groups

In this section, we introduced the Berezin correspondence $S$ associated with $\pi$ and show that $S$ is $G$-equivariant.

Recall that the Berezin calculus on $o(\phi_0)$ associates with each operator $A_1$ on $V$ a complex-valued function $s(A_1)$ on the orbit $o(\phi_0)$ which is called the symbol of the operator $A_1$ (see [9]). We denote by $Sy(o(\phi_0))$ the space of all such symbols. Moreover, for each $k \in K$ and each function $u$ on $o(\phi_0)$, we denote by $\tilde{\ell}_k u$ the function on $o(\phi_0)$ defined by $\tilde{\ell}_k u(\phi) = u(\text{Ad}^*(k)^{-1}\phi)$.

The following properties of the Berezin calculus are well-known, see [4], [12], [24] and [46].

**Proposition 6.1.**

1. The map $A_1 \to s(A_1)$ is injective.
2. For each operator $A_1$ on $V$, we have $s(A_1^*) = \overline{s(A_1)}$.
3. For each operator $A_1$ on $V$, $k \in K$ and $\phi \in o(\phi_0)$, we have
\[ s(A_1)(\text{Ad}^*(k)\phi) = s(\rho(k)^{-1}A_1 \rho(k))(\phi) \]
and, equivalently, for each operator $A_1$ on $V$ and each $k \in K$, we have
\[ s(\rho(k)A_1 \rho(k)^{-1}) = \tilde{\ell}_k s(A_1). \]
(4) For $X \in \mathfrak{t}$ and $\phi \in o(\phi_0)$, we have $s(dp(X)))(\phi) = i(\phi, X)$.

In particular, we see that $s$ is an adapted Weyl transform on $o(\phi_0)$ in the sense of Definition 1.

Now, $S$ is defined as follows. For each operator $A_0$ on $\mathcal{F}_0$ and each operator $A_1$ on $V$, we set $S(A_0 \otimes A_1) := S^0(A_0) \otimes s(A_1)$ and then we extend $S$ by linearity to operators on $\mathcal{F}$.

Consider the action of $G$ on $\mathbb{C}^n \times o(\phi_0)$ defined by $g \cdot (z, \phi) = (g \cdot z, \text{Ad}^*(k)^{-1}\phi)$ where $g = (g_0, k) \in G$. Then, for each $g \in G$ and each function $F$ on $\mathbb{C}^n \times o(\phi_0)$, we denote by $L_g F$ the function on $\mathbb{C}^n \times o(\phi_0)$ defined by $L_g F(z, \phi) := F(g^{-1} \cdot (z, \phi))$.

**Proposition 6.2.** The map $S$ is $G$-equivariant with respect to $\pi$, that is, for each operator $A$ on $\mathcal{F}$ and each $g \in G$, we have $S(\pi(g)^{-1}A\pi(g)) = L_g^{-1}S(A)$.

**Proof.** It is sufficient to consider the case where $A = A_0 \otimes A_1$ for $A_0$ operator on $\mathcal{F}_0$ and $A_1$ operator on $V$.

Let $g = (g_0, k) \in G$. By Equation 5.1, we have

$$\pi(g)^{-1}A\pi(g) = \tau(k)^{-1}\pi_0(g_0)^{-1}A_0\pi_0(g_0)\tau(k) \otimes \rho(k)^{-1}A_1\rho(k).$$

Then, by using Proposition 3.1, Proposition 3.3 and Proposition 6.1, we get

$$S(\pi(g)^{-1}A\pi(g)) = S^0(\tau(k)^{-1}\pi_0(g_0)^{-1}A_0\pi_0(g_0)\tau(k)) \otimes s(\rho(k)^{-1}A_1\rho(k))$$

$$= \tilde{l}_{k^{-1}}S^0(\pi_0(g_0)^{-1}A_0\pi_0(g_0)) \otimes \tilde{i}_{k^{-1}}s(A_1)$$

$$= \tilde{l}_{k^{-1}}L_{g_0}^0(S^0(A_0)) \otimes \tilde{i}_{k^{-1}}s(A_1).$$

This implies that

$$S(\pi(g)^{-1}A\pi(g))(z, \phi) = S^0(A_0)(kz - i\lambda z_0)s(A_1)(\text{Ad}^*(k)\phi)$$

$$= (S^0(A_0) \otimes s(A_1))(g \cdot z, \text{Ad}^*(k)\phi)$$

for each $(z, \phi) \in \mathbb{C}^n \times o(\phi_0)$. This gives the desired result. \qed

7. Weyl correspondence for Heisenberg motion groups

In this section, we first introduce the Berezin-Weyl correspondence, in the spirit of [15].

Recall that the Berezin calculus $s$ is an isomorphism from $\text{End}(V)$ onto $Sy(o(\varphi_0))$, see Section 6. We say that a complex-valued smooth function $f : (p, q, \phi) \rightarrow f(p, q, \phi)$ is a symbol on $\mathbb{R}^{2n} \times o(\phi_0)$ if for each $(p, q) \in \mathbb{R}^{2n}$ the function $f(p, q, \cdot) : \phi \rightarrow f(p, q, \phi)$ is in $Sy(o(\phi_0))$. In this case, we denote $\check{f}(p, q) := s^{-1}(f(p, q, \cdot))$. A symbol $f$ on $\mathbb{R}^{2n} \times o(\phi_0)$ is called an $S$-symbol if the function $\check{f}$ belongs to the Schwartz space $S(\mathbb{R}^{2n}, \text{End}(V))$.
of rapidly decreasing smooth functions on \( \mathbb{R}^{2n} \) with values in \( \text{End}(V) \). We define similarly the notion of \( L^2 \)-symbol. For each \( S \)-symbol on \( \mathbb{R}^{2n} \times o(\phi_0) \), we define the operator \( W(f) \) on the Hilbert space \( L^2(\mathbb{R}^n, V) = L^2(\mathbb{R}^n) \otimes V \) by

\[
W(f) \varphi(p) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{itg \cdot f(p + (1/2)t, q)} \varphi(p + t) \, dt \, dq.
\]

Note that \( W \) can be extended to much larger classes of symbols in the same way as \( W_0 \), see Section 4. It is also clear that we have

\[
W(f_0 \otimes f_1) = W_0(f_0) \otimes s^{-1}(f_1)
\]

for each \( f_0 \in S(\mathbb{R}^n) \) and \( f_1 \in Sy(o(\phi_0)) \).

Now, we consider the action of \( G \) on \( \mathbb{R}^{2n} \times o(\phi_0) \) defined by

\[
g \cdot (p, q, \phi) := (g_0 \cdot (p, q), \text{Ad}^*(k)\phi)
\]

for each \( g = (g_0, k) \in G \) and each \( (p, q, \phi) \in \mathbb{R}^{2n} \times o(\phi_0) \). Then, for each function \( f \) on \( \mathbb{R}^{2n} \times o(\phi_0) \), we denote by \( L_g f \) the function defined on \( \mathbb{R}^{2n} \times o(\phi_0) \) by

\[
L_g f(p, q, \phi) := f(g^{-1} \cdot (p, q, \phi)).
\]

**Proposition 7.1.** The map \( W^{-1} \) is \( G \)-equivariant with respect to \( \sigma \), that is, we have \( W^{-1}(\sigma(g)^{-1}A\sigma(g)) = L_g^{-1}(W^{-1}(A)) \) for each \( g \in G \) and each Hilbert-Schmidt operator \( A \) on \( L^2(\mathbb{R}^n, V) \).

Equivalently, \( W \) is \( G \)-equivariant with respect to \( \sigma \), that is, for each \( g \in G \) and each \( L^2 \)-symbol \( f \), we have \( \sigma(g)^{-1}W(f)\sigma(g) = W(L_g^{-1}f) \).

**Proof.** The proof is based on the equivariance of \( s \) and \( U^0 \). As usual, we can assume, without loss of generality, that \( A = A_0 \otimes A_1 \) with \( A_0 \) operator on \( L^2(\mathbb{R}^n) \) and \( A_1 \) operator on \( V \).

Let \( g = (g_0, k) \in G \). Then, by Equation 5.2, we have

\[
W^{-1}(\sigma(g)^{-1}A\sigma(g)) = (W_0^{-1} \otimes s)(\tilde{\tau}(k)^{-1}\sigma_0(g_0)^{-1}A_0\sigma_0(g_0)\tilde{\tau}(k)) \otimes \rho(k)^{-1}A_1\rho(k)
\]

\[
= W_0^{-1}(\tilde{\tau}(k)^{-1}\sigma_0(g_0)^{-1}A_0\sigma_0(g_0)\tilde{\tau}(k)) \otimes s(\rho(k)^{-1}A_1\rho(k)).
\]

But, by using successively Proposition 4.4, the second assertion of Proposition 3.3 and Proposition 3.2, we can write

\[
W_0^{-1}(\tilde{\tau}(k)^{-1}\sigma_0(g_0)^{-1}A_0\sigma_0(g_0)\tilde{\tau}(k)) = (JU^0 I_{B_0})(\tilde{\tau}(k)^{-1}\sigma_0(g_0)^{-1}A_0\sigma_0(g_0)\tilde{\tau}(k))
\]

\[
= JU^0(\tau(k)^{-1}B_0\sigma_0(g_0)^{-1}A_0\sigma_0(g_0)B_0^{-1}\tau(k))
\]

\[
= (Jl_{k^{-1}}U^0)(\pi_0(g_0)^{-1}B_0 A_0 B_0^{-1}\pi_0(g_0))
\]

\[
= (Jl_{k^{-1}}L^0_{g_0^{-1}}U^0 I_{B_0})(A_0)
\]
which implies that
\[
W_0^{-1}(\tilde{\tau}(k)^{-1}\sigma_0(g_0)^{-1}A_0\sigma_0(g_0)\tilde{\tau}(k))(p, q) = (l_{k-1}L_{g_0}^0U^0I_{B_0})(A_0)(j(p, q)) = U^0I_{B_0}(A_0)(g_0 \cdot (k \cdot j(p, q))) = JU^0I_{B_0}(A_0)(j^{-1}(g_0 \cdot (k \cdot j(p, q)))) = W_0^{-1}(A_0)(g \cdot (p, q)).
\]

On the other hand, by (3) of Proposition 6.1, we have
\[
s(\rho(k)^{-1}A_1\rho(k))(\phi) = s(A_1)(\Ad^*(k)\phi).
\]

Then we can conclude that
\[
W^{-1}(\sigma(g)^{-1}A\sigma(g)) = \tilde{L}_g(W_0^{-1}(A_0) \otimes s(A_1)) = \tilde{L}_g(W_0^{-1} \otimes s)(A_0 \otimes A_1) = \tilde{L}_gW^{-1}(A).
\]

Thus we have proved the first assertion of the proposition. The second assertion immediately follows. □

8. Adapted Weyl correspondences

In this section, we first compute $W^{-1}(d\sigma(X))$ for $X \in \mathfrak{g}$. We have the following result.

Proposition 8.1. [23]

(1) For each $X = (X_0, A) \in \mathfrak{g}_0$ and $A \in \mathfrak{k}$, we have
\[
d\sigma(X) = (d\sigma_0(X_0) + d\tilde{\tau}(A)) \otimes I_V + I_{F_0} \otimes d\rho(A).
\]

(2) For each $A = (a_{kl}) \in \mathfrak{k}$, we have
\[
d\tilde{\tau}(A) = \frac{1}{2\lambda} \sum_{k,l} a_{kl} \frac{\partial^2}{\partial p_k \partial p_l} + \frac{1}{2} \sum_{k,l} a_{kl} \left( p_k \frac{\partial}{\partial p_l} - p_l \frac{\partial}{\partial p_k} \right) - \frac{\lambda}{2} \rho(A p) + \frac{1}{2} \text{Tr}(A).
\]

Note that (1) is a simple consequence of Equation 5.2. From this proposition, we can deduce the following result.

Proposition 8.2. For each $X = ((a, \bar{a}), c, A) \in \mathfrak{g}$ and $(p, q, \phi) \in \mathbb{R}^{2n} \times o(\phi_0)$, we have
\[
W^{-1}(d\sigma(X))(p, q, \varphi) = i\lambda c + \frac{1}{2} \text{Tr}(A) + i \left( \bar{a}j(p, q) + a\overline{j(p, q)} \right)
\]
\[
- \frac{1}{2\lambda} j(p, q)(A j(p, q)) + s(d\rho(A))(\phi).
\]
Proof. Let $X = ((a, \bar{a}), c, A) \in \mathfrak{g}$. Consider the following symbols:

$$f_1(p, q, \phi) := \frac{i}{2} \left( \bar{a} j(p, q) + a j(p, q) \right)$$

$$f_2(p, q, \phi) := -\frac{1}{2\lambda} j(p, q) (A j(p, q))$$

$$f_3(p, q, \phi) := s(dp(A))(\phi).$$

Then we have

$$f_1(p, q, \phi) = \frac{i}{2} (a + \bar{a}) q - \frac{\lambda}{2} (a - \bar{a}) p$$

and, by using Equation 4.1, we get

$$W(f_1) = -\frac{\lambda}{2} (a - \bar{a}) p - \frac{1}{2} \sum_{k=1}^{n} (a_k + \bar{a}_k) \frac{\partial}{\partial p_k}.$$  

Similarly, writing

$$f_2(p, q, \phi) = -\frac{1}{2\lambda} (q(A q) + \lambda^2 p(A p) + \lambda q p(A p) - \lambda i q p(A p)),$$

we get

$$W(f_2) = \frac{1}{2\lambda} \sum_{k,l} a_{kl} \frac{\partial^2}{\partial p_k \partial p_l} - \frac{1}{2} \sum_{k,l} a_{kl} \left( p_l \frac{\partial}{\partial p_k} - p_k \frac{\partial}{\partial p_l} \right) - \frac{\lambda}{2} (dp(A) p).$$

On the other hand, by (4) of Proposition 6.1, we have $W(f_3)(\phi)(p) = (dp(A) \phi)(p)$ for each $\phi \in C_0^\infty(\mathbb{R}^n, V)$. The result then follows by Proposition 8.1.

Note that, since $\mathfrak{k} \subset u(n)$, for each $A \in \mathfrak{k}$ and each $(p, q) \in \mathbb{R}^{2n}$, we have $j(p, q) A j(p, q) \in i \mathbb{R}$ and $\text{Tr}(A) \in i \mathbb{R}$. Then, for $(p, q, \phi) \in \mathbb{R}^n \times o(\phi_0)$, the map $X \mapsto -i W^{-1}(d\sigma(X))(p, q, \phi)$ is a real-valued linear map on $\mathfrak{g}$. We denote this map by $\Psi(p, q, \phi)$.

Proposition 8.3. (1) For each $X \in \mathfrak{k}$ and each $(p, q, \phi) \in \mathbb{R}^{2n} \times o(\phi_0)$, we have

$$W^{-1}(d\sigma(X))(p, q, \phi) = i \langle \Psi(p, q, \phi), X \rangle.$$  

Also, for each $(p, q, \phi) \in \mathbb{R}^{2n} \times o(\phi_0)$, we have

$$\Psi(p, q, \phi) = \left( (-i j(p, q), i j(p, q)), \lambda, -\frac{i}{2} \text{Tr} + (j(p, q), j(p, q)) \times (j(p, q), j(p, q)) + \phi \right).$$

(2) For each $g \in G$ and each $(p, q, \phi) \in \mathbb{R}^{2n} \times o(\phi_0)$, we have $\Psi(g \cdot (p, q, \phi)) = \text{Ad}^*(g) \Psi(p, q, \phi)$.

(3) Assume that $K \subset SU(n)$. Then $\Psi$ is a diffeomorphism from $\mathbb{R}^{2n} \times o(\phi_0)$ onto the coadjoint orbit $O(\xi_0) \subset \mathfrak{g}^*$ of $\xi_0$.  

Proof. (1) immediately follows from the definition of $\Psi$ and (2) from the $G$-equivariance of $W^{-1}$. To prove (3), first note that we have $\Psi(0,0,\phi_0) = \zeta_0$ since the hypothesis $K \subset SU(n)$ implies that $\text{Tr}(A) = 0$ for each $A \in \mathfrak{k}$. Then, by (2), we see that $\Psi$ is a surjective map from $\mathbb{R}^n \times o(\phi_0)$ onto $O(\xi_0)$. On the other hand, by (1), $\Psi$ is injective, hence bijective.

It remains to show that $\Psi$ is regular. By (2) again, it is sufficient to verify that $\Psi$ is regular at $(0,0,\phi_0)$. But we have

$$(d\Psi)_{(0,0,\phi_0)}(u,v,\text{ad}^*(A)(\phi_0)) = ((-i(v - i\lambda u), i(v + i\lambda u)), 0, \text{ad}^*(A)(\phi_0))$$

for each $(u,v) \in \mathbb{R}^{2n}$ and $A \in \mathfrak{k}$, hence the result. \hfill $\square$

Finally, we obtain an adapted Weyl correspondence for $\sigma$ by transferring $W$ to $O(\xi_0)$. We say that a smooth function $f$ on $O(\xi_0)$ is a symbol on $O(\xi_0)$ (respectively a $P$-symbol, an $S$-symbol) if $f \circ \Psi$ is a symbol (respectively a $P$-symbol, an $S$-symbol) for $W$. From the properties of $W$, we obtain the following proposition.

**Proposition 8.4.** Let $A$ be the space of $P$-symbols on $O(\xi_0)$ and let $B$ be the space of differential operators on $\mathbb{R}^n$ with coefficients in $C^\infty(\mathbb{R}^n; V)$. Then the map $W : A \to B$ that assigns to each $f \in A$ the operator $W(f \circ \Psi)$ on $L^2(\mathbb{R}^n; V)$ is an adapted Weyl correspondence in the sense of Definition 1. Moreover, $W$ is $G$-equivariant with respect to $\sigma$.

**References**


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