Math. J. Okayama Univ. **61** (2019), 19–35

BEREZIN-WEYL QUANTIZATION OF HEISENBERG MOTION GROUPS

To the memory of my father, Alfred Cahen

Benjamin Cahen

Abstract. We introduce a Schrödinger model for the generic representations of a Heisenberg motion group and we construct adapted Weyl correspondences for these representations by adapting the method introduced in [B. Cahen, *Weyl quantization for semidirect products,* Differential Geom. Appl. 25 (2007), 177-190].

1. Introduction

In [12] and [13], we introduced the notion of adapted Weyl correspondence as a direct generalization of the usual Weyl quantization [1], [27].

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let π be a unitary irreducible representation of G on a Hilbert space $\mathcal{H}$. Assume that π is associated with a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ of G by the Kirillov-Kostant method of orbits [34], [35]. The following definition for the notion of adapted Weyl correspondence is taken from [15] (see also [30], [2] and [3]).

Definition 1. An adapted Weyl correspondence is an isomorphism $\mathcal{W}$ from a vector space $\mathcal{A}$ of complex-valued smooth functions on the orbit $\mathcal{O}$ (called symbols) onto a vector space $\mathcal{B}$ of (not necessarily bounded) linear operators on $\mathcal{H}$ satisfying the following properties:

(1) the elements of $\mathcal{B}$ preserve a fixed dense domain $\mathcal{D}$ of $\mathcal{H}$;
(2) the constant function 1 belongs to $\mathcal{A}$, the identity operator $I_{\mathcal{H}}$ belongs to $\mathcal{B}$ and $\mathcal{W}(1) = I_{\mathcal{H}}$;
(3) $A \in \mathcal{B}$ and $B \in \mathcal{B}$ implies $AB \in \mathcal{B}$;
(4) for each f in $\mathcal{A}$ the complex conjugate $\bar{f}$ of f belongs to $\mathcal{A}$ and the adjoint of $\mathcal{W}(f)$ is an extension of $\mathcal{W}(\bar{f})$;
(5) the elements of $\mathcal{D}$ are C^∞-vectors for the representation π, the functions $\tilde{X}$ $(X \in \mathfrak{g})$ defined on $\mathcal{O}$ by $\tilde{X}(\xi) = \langle \xi, X \rangle$ are in $\mathcal{A}$ and we have $\mathcal{W}(i\tilde{X})\, v = d\pi(X)v$ for each $X \in \mathfrak{g}$ and each $v \in \mathcal{D}$.

Mathematics Subject Classification. Primary 22E45; Secondary 22E70, 22E20, 81S10, 81R30.

Key words and phrases. Weyl correspondence; Berezin quantization; Heisenberg motion group; Schrödinger representation; Bargmann-Fock representation; Segal-Bargmann transform; unitary representation; coadjoint orbit.

We have constructed adapted Weyl correspondences in different situations, especially for unitary representations of semidirect products of the form $V \rtimes K$ where K is a semi-simple Lie group acting linearly on a vector space V [15], [17]. Note that adapted Weyl correspondences have various applications in harmonic analysis and deformation theory as, for instance, the construction of covariant star-products on coadjoint orbits [12] and the study of contractions of Lie group unitary representations [26], [14], [20].

Note also that the notion of adapted Weyl correspondence is close to that of Stratonovich-Weyl correspondence [41], [28], [29]. Roughly speaking, Stratonovich-Weyl correspondences do not require to satisfy (5) of Definition 1 but they have to be unitary and G-equivariant [28]. We refer to [18] for a short discussion of the advantages and disadvantages of these two methods of quantization (see also [22]).

Let us consider the typical case of the $(2n+1)$-dimensional Heisenberg group G_0. Each non-degenerate unitary irreducible representation of G_0 has two usual realizations: the Schrödinger realization on $L^2(\mathbb{R}^n)$ and the Bargmann-Fock realization on the Fock space [27], [42], an intertwining operator between these realizations being the Segal-Bargmann transform [27], [25]. In the setting of the orbit method, the Schrödinger realization can be obtained from a real polarization of the corresponding coadjoint orbit of G_0 and the Bargmann-Fock realization from a totally complex polarization [6], [11]. Moreover, the usual Weyl correspondence provides an adapted Weyl correspondence for the Schrödinger realization [5], [45]. It is also known that this adapted Weyl correspondence is related, by the Segal-Bargmann transform, to the unitary part of the polar decomposition of the Berezin quantization map associated with the Bargmann-Fock realization [37], [36].

In [19] and [23], we made similar considerations for the generic representations of the real diamond group and of Heisenberg motion groups. In these cases, the generic coadjoint orbits of G don't necessarily admit real polarization and the corresponding representations are usually obtained as holomorphically induced representations on Bargmann-Fock spaces. We can nonetheless obtain 'Schrödinger realizations' of these representations from Bargmann-Fock realizations by conjugation with the Segal-Bargmann transform.

Let G be a Heisenberg motion group, that is, the semidirect product of the Heisenberg group G_0 by a connected compact subgroup K of the unitary group $U(n)$. Such groups play an important role in the theory of Gelfand pairs, since the study of a Gelfand pair of the form (K_0, N), where K_0 is a compact Lie group acting by automorphisms on a nilpotent Lie group N, can be reduced to that of the form (K_0, H_n), see in particular [7] and [8].

In the present paper, we exploit some results of [23] in order to construct an adapted Weyl correspondence for each generic representation π of G. More precisely, we consider a Schrödinger model for π as in [23], that is, a realization of π in the Hilbert space $L^2(\mathbb{R}^n)\otimes V$ where V is a finite dimensional complex vector space which carries an irreductible unitary representation ρ of K. Then we introduce the map $W := W_0 \otimes s^{-1}$ where W_0 is the usual Weyl correspondence and s is the Berezin calculus associated with V, and we show that W is G-equivariant. Moreover, we compute $W^{-1}(d\pi(X))$ for $X\in\mathfrak{g}$ and we conclude that if $K\subset SU(n)$ then W induces an adapted Weyl correspondence for π.

Note that, in [38], a Schrödinger model and a generalized Segal-Bargmann transform for the scalar highest weight representations of an Hermitian Lie group of tube type were introduced and studied (see also [32]). Then one can hope for futher generalizations of our construction to quasi-Hermitian Lie groups.

This paper is organized as follows. In Sections 2-4, we review some facts about the Fock model and the Schrödinger model of the unitary irreducible representations of an Heisenberg group and about the Weyl correspondence. In Section 5, we introduce the Heisenberg motion groups and their unitary irreducible representations in the Fock model and in the Schrödinger model. Section 6 is devoted to the Berezin calculus corresponding to the Fock model of these representations. The construction of the adapted Weyl correspondence for π is done in Sections 7-8, as described above.

2. Representations of the Heisenberg group

In this section, we review some known facts about the the Schrödinger model and the Fock model of the unitary irreducible (non-degenerated) representations of the Heisenberg group. We follow the presentation of [19] (see also [27] and [25]).

For each $z,\, w\in\mathbb{C}^n$, we denote $zw := \sum_{k=1}^n z_k w_k$ and we consider the symplectic form ω on $\mathbb{C}^{2n}$ defined by

$$\omega((z,w),(z',w')) = \frac{i}{2}(zw' - z'w).$$

for $z,w,z',w'\in\mathbb{C}^n$.

Let G_0 be the $(2n+1)$-dimensional Heisenberg group consisting of all elements of the form $((z,\bar{z}),c)$ where $z\in\mathbb{C}^n$ and $c\in\mathbb{R}$. The multiplication of G_0 is given by

$$((z,\bar{z}),c)\cdot((z',\bar{z}'),c') = ((z+z',\bar{z}+\bar{z}'),c+c'+\tfrac{1}{2}\omega((z,\bar{z}),(z',\bar{z}'))).$$

Let $\mathfrak{g}_0$ be the Lie algebra of G_0 and $\mathfrak{g}_0^c$ its complexification. We write the elements of $\mathfrak{g}_0^c$ as $((a,b),c)$ where $a,b\in\mathbb{C}^n$ and $c\in\mathbb{C}$. Then the Lie brackets of $\mathfrak{g}_0^c$ are given by

$$[((a,b),c),((a',b'),c')]=((0,0),\omega((a,b),(a',b'))).$$

Fix a real number $\lambda>0$ and denote by $\mathcal{O}_\lambda$ the orbit of the element $\xi_\lambda:((a,\bar{a}),c)\to\lambda c$ of $\mathfrak{g}_0^*$ under the coadjoint action of G_0 (the case $\lambda<0$ can be treated similarly). By the Stone-von Neumann theorem, there exists a unique (up to unitary equivalence) unitary irreducible representation of G_0 whose restriction to the center of G_0 is the character $((0,0),c)\to e^{i\lambda c}$ [27], [42]. Then this representation is associated with $\mathcal{O}_\lambda$ by the Kirillov-Kostant method of orbits [34], [35]. More precisely, if we choose the real polarization at ξ_λ to be $\{((ib,-ib),c)\,:\,b\in\mathbb{R}^n,c\in\mathbb{R}\}$ then we obtain the Schrödinger representation σ_0 realized on $L^2(\mathbb{R}^n)$ as

$$(\sigma_0((z_0,\bar{z}_0),c_0)f)(x)=e^{i\lambda(c_0-y_0x+\frac{1}{2}x_0y_0)}f(x-x_0),$$

where $z_0=x_0+iy_0$, $x_0,y_0\in\mathbb{R}^n$ [27], [42].

On the other hand, if we choose the complex polarization at ξ_λ to be $\{((0,w),c)\,:\,w\in\mathbb{C}^n,c\in\mathbb{C}\}$ then we obtain the Bargmann-Fock representation π_0 defined as follows [27].

Let $\mathcal{F}_0$ be the Hilbert space of holomorphic functions F on $\mathbb{C}^n$ such that

$$\|F\|^2_{\mathcal{F}_0}:=\int_{\mathbb{C}^n}|F(z)|^2\,e^{-|z|^2/2\lambda}\,d\mu_\lambda(z)<+\infty$$

where $d\mu_\lambda(z):=(2\pi\lambda)^{-n}\,dx\,dy$. Here $z=x+iy$ with x and y in $\mathbb{R}^n$. Then π_0 is the representation of G_0 on $\mathcal{F}_0$ given by

$$(\pi_0(g_0)F)(z)=\exp\left(i\lambda c_0+\tfrac{1}{2}i\bar{z}_0z-\tfrac{\lambda}{4}|z_0|^2\right)\,F(z+i\lambda z_0)$$

where $g=((z_0,\bar{z}_0),c_0)\in G_0$ and $z\in\mathbb{C}^n$.

We consider the action of K on G_0 defined by

$$k\cdot((z_0,\bar{z}_0),c_0):=((kz_0,\overline{kz_0}),c_0).$$

Let τ be the representation of K on $\mathcal{F}_0$ defined by $(\tau(k)F)(z):=F(k^{-1}z)$. Note that, for each $k\in K$ and $g_0\in G_0$, we have

$$\pi_0(k\cdot g_0)=\tau(k)\pi_0(g_0)\tau(k)^{-1}.$$

As in [27], Chapter 1, [31], Section 6 or [25], Section 1.3, we can verify that the Segal-Bargmann transform $B_0:L^2(\mathbb{R}^n)\to\mathcal{F}_0$ defined by

$$B_0(f)(z)=(\lambda/\pi)^{n/4}\int_{\mathbb{R}^n}e^{(1/4\lambda)z^2+ixz-(\lambda/2)x^2}\,f(x)\,dx$$

is a (unitary) intertwining operator between σ_0 and π_0, that is, for each $g_0\in G_0$, one has $\sigma_0(g_0)=B_0^{-1}\pi_0(g_0)B_0$.

3. Berezin calculus for Heisenberg groups

We first recall the definition of the Berezin calculus. For each $z \in \mathbb{C}^n$, we consider the 'coherent state' $e_z(w) := \exp(\bar{z}w/2\lambda)$. Then we have $F(z) = \langle F, e_z\rangle_{\mathcal{F}_0}$ for each $F \in \mathcal{F}_0$ where $\langle\cdot,\cdot\rangle_{\mathcal{F}_0}$ denotes the scalar product on $\mathcal{F}_0$.

Let $\mathcal{C}_0$ be the space of all operators (not necessarily bounded) A_0 on $\mathcal{F}_0$ whose domain contains e_z for each $z \in \mathbb{C}^n$. Then the Berezin symbol of $A_0 \in \mathcal{C}_0$ is the function $S^0(A_0)$ defined on $\mathbb{C}^n$ by

$$S^0(A_0)(z) := \frac{\langle A_0\, e_z\,,\, e_z\rangle_{\mathcal{F}_0}}{\langle e_z\,,\, e_z\rangle_{\mathcal{F}_0}}.$$

Let us consider the action of G_0 on $\mathbb{C}^n$ defined by $g_0 \cdot z := z - i\lambda z_0$ where $g_0 = ((z_0, \bar{z}_0), c_0)$. For each function F on $\mathbb{C}^n$ (non necessarily in $\mathcal{F}_0$) and each $g_0 \in G_0$, we denote by $L^0_{g_0}F$ the function on $\mathbb{C}^n$ defined by $(L^0_{g_0}F)(z) = F(g_0^{-1}\cdot z)$. Then we have the following properties of S^0, see for instance [19].

Proposition 3.1. (1) *Each $A_0 \in \mathcal{C}_0$ is determined by $S^0(A_0)$;*

(2) *For each $A_0 \in \mathcal{C}_0$ and each $z \in \mathbb{C}^n$, we have $S^0(A_0^*)(z) = \overline{S^0(A_0)(z)}$;*

(3) *We have $S^0(I_{\mathcal{F}_0}) = 1$;*

(4) *The map S^0 is G_0-equivariant with respect to G_0, that is, for each $A_0 \in \mathcal{C}_0$, $g_0 \in G_0$ and $z \in \mathbb{C}^n$, we have $\pi_0(g_0)^{-1}A_0\pi_0(g_0) \in \mathcal{C}_0$ and*

$$S^0(A_0)(g_0 \cdot z) = S^0(\pi_0(g_0)^{-1}A_0\pi_0(g_0))(z)$$

or, equivalently,

$$L^0_{g_0}S^0(A_0) = S^0(\pi_0(g_0)A_0\pi_0(g_0)^{-1});$$

(5) *The map S^0 is a bounded operator from the space $\mathcal{L}_2(\mathcal{F}_0)$ of all Hilbert-Schmidt operators on $\mathcal{F}_0$ (endowed with the Hilbert-Schmidt norm) to $L^2(\mathbb{C}^n, \mu_\lambda)$ which is one-to-one and has dense range.*

Let us recall that the Berezin transform is then the operator $\mathcal{B}^0$ on $L^2(\mathbb{C}^n, \mu_\lambda)$ defined by $\mathcal{B}^0 = S^0(S^0)^*$. Thus we can verify that

$$\mathcal{B}^0(F)(z) = \int_{\mathbb{C}^n} F(w)\, e^{|z-w|^2/2\lambda}\, d\mu_\lambda(w),$$

see [9], [10], [43], [40] for instance. Also, it is well-known that we have $\mathcal{B}^0 = \exp(\lambda\Delta/2)$ where $\Delta = 4\sum_{k=1}^n \partial^2/\partial z_k\partial\bar{z}_k$, see [43], [36].

Let U^0 be the unitary part in the polar decomposition of S^0 (seen as a bounded operator from $\mathcal{L}_2(\mathcal{F}_0)$ to $L^2(\mathbb{C}^n, \mu_\lambda)$), that is, $U^0 := (\mathcal{B}^0)^{-1/2}S^0$. As a particular case of [18], Proposition 6.1, we have the following result.

Proposition 3.2. *U^0 is G_0-equivariant with respect to π_0, that is, for each $A_0 \in \mathcal{L}_2(\mathcal{F}_0)$ and $g_0 \in G_0$, we have*

$$L^0_{g_0}U^0(A_0)=U^0(\pi_0(g_0)A_0\pi_0(g_0)^{-1}).$$

For each $k\in K$ and each function F on $\mathbb{C}^n$ (not necessarily in $\mathcal{F}_0$) we denote by l_kF the function on $\mathbb{C}^n$ defined by $(l_kF)(z):=F(k^{-1}z)$. Then we have the following result.

Proposition 3.3. *For each $k\in K$ and A_0 operator on $\mathcal{F}_0$, we have*

$$S^0(\tau(k)A_0\tau(k)^{-1})=l_kS^0(A_0)$$

and, similarly,

$$U^0(\tau(k)A_0\tau(k)^{-1})=l_kU^0(A_0).$$

Proof. For the first assertion, note that for each $z,w\in\mathbb{C}^n$ and $k\in K$, we have

$$\begin{aligned}(\tau(k)e_z)(w)&=e_z(k^{-1}w)=\exp(\bar z(k^{-1}w)/2\lambda)\\&=\exp((\overline{kz})w/2\lambda)=e_{kz}(w)\end{aligned}$$

hence $\tau(k)e_z=e_{kz}$. This implies that

$$\begin{aligned}S^0(A_0)(k^{-1}z)&=\frac{\langle A_0\,e_{k^{-1}z}\,,\,e_{k^{-1}z}\rangle_{\mathcal{F}_0}}{\langle e_{k^{-1}z}\,,\,e_{k^{-1}z}\rangle_{\mathcal{F}_0}}\\&=\frac{\langle A_0\,\tau(k)^{-1}e_z\,,\,\tau(k)^{-1}e_z\rangle_{\mathcal{F}_0}}{\langle e_z\,,\,e_z\rangle_{\mathcal{F}_0}}\\&=\frac{\langle \tau(k)A_0\,\tau(k)^{-1}e_z\,,\,e_z\rangle_{\mathcal{F}_0}}{\langle e_z\,,\,e_z\rangle_{\mathcal{F}_0}}\\&=S^0(\tau(k)A_0\tau(k)^{-1})(z).\end{aligned}$$

Now we prove the second assertion. First note that, by using the integral formula for $\mathcal{B}_0$, we see that $\mathcal{B}_0$-hence $\mathcal{B}_0^{-1/2}$- commute with l_k for each $k\in K$. Then, denoting by $\mathcal{I}_{\tau(k)}$ the operator $A_0\to\tau(k)A_0\tau(k)^{-1}$ on $\mathcal{L}_2(\mathcal{F}_0)$, we can reformulate the first assertion as $S^0\mathcal{I}_{\tau(k)}=l_kS^0$ for each $k\in K$. Consequently we have

$$U^0\mathcal{I}_{\tau(k)}=\mathcal{B}_0^{-1/2}S^0\mathcal{I}_{\tau(k)}=\mathcal{B}_0^{-1/2}l_kS^0=l_k\mathcal{B}_0^{-1/2}S^0=l_kU^0,$$

hence the result. $\square$

4. Weyl correspondence for Heisenberg groups

In this section, we introduce the usual Weyl correspondence and review some of its properties.

The Weyl correspondence W_0 on $\mathbb{R}^{2n}$ is usually defined as follows. For each f in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$, let $W_0(f)$ be the operator on $L^2(\mathbb{R}^n)$ defined by

$$W_0(f)\varphi(p) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{isq}\, f(p+(1/2)s,q)\,\varphi(p+s)\,ds\,dq.$$

The Weyl calculus can be extended to much larger classes of symbols (see for instance [33]). In particular, if $f(p,q) = u(p)q^\alpha$ where $u \in C^\infty(\mathbb{R}^n)$ then we have

$$W_0(f)\varphi(p) = \left(i\frac{\partial}{\partial s}\right)^\alpha (u(p+(1/2)s)\,\varphi(p+s))\Big|_{s=0}, \tag{4.1}$$

see [44].

Now, we transfer the action of G_0 on $\mathbb{C}^n$ introduced in Section 3 to $\mathbb{R}^{2n}$ by means of the map $j : (p,q) \to q - \lambda i p$, that is, we consider the action of G_0 on $\mathbb{R}^{2n}$ defined by

$$g_0 \cdot (p,q) := j^{-1}(g_0 \cdot (p,q)) = (p + x_0, q + \lambda y_0)$$

where $g_0 = ((z_0,\bar{z}_0),c_0)$ and $z_0 = x_0 + iy_0$ with $x_0, y_0 \in \mathbb{R}^n$. Then we have the following result.

Proposition 4.1. [19] *Let $\Psi_\lambda : \mathbb{R}^{2n} \to \mathfrak{g}_0^*$ be the map defined by*

$$\langle \Psi_\lambda(p,q), X\rangle := \mathrm{Re}((q - \lambda i p)\bar{a}) + \lambda c$$

for each $X = ((a,\bar{a}),c) \in \mathfrak{g}_0$. Then

(1) *For each $X \in \mathfrak{g}_0$ and each $(p,q) \in \mathbb{R}^{2n}$, we have*

$$W_0^{-1}(d\sigma_0(X))(p,q) = i\langle \Psi_\lambda(p,q), X\rangle.$$

(2) *For each $g_0 \in G_0$ and each $(p,q) \in \mathbb{R}^{2n}$, we have $\Psi_\lambda(g_0 \cdot (p,q)) = \mathrm{Ad}^*(g_0)\,\Psi_\lambda(p,q)$.*

(3) *The map Ψ_λ is a diffeomorphism from $\mathbb{R}^{2n}$ onto $\mathcal{O}_\lambda$.*

Let $\mathcal{L}_2(L^2(\mathbb{R}^n))$ be the Hilbert space of all Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. Then W_0 induces a unitary operator from $L^2(\mathbb{R}^{2n})$ onto $\mathcal{L}_2(L^2(\mathbb{R}^n))$ [27].

For each $g_0 \in G_0$ and each function f on $\mathbb{R}^{2n}$, we denote by $\tilde{L}^0_{g_0} f$ the function on $\mathbb{R}^{2n}$ defined by $\tilde{L}^0_{g_0} f(p,q) = f(g_0^{-1}\cdot(p,q))$. Then we have the following result.

Proposition 4.2. [27], [36], [19] *W_0 is G_0-equivariant with respect to σ_0, that is, for each $g_0 \in G_0$ and each $f \in L^2(\mathbb{R}^{2n})$, we have*

$$W_0(\tilde{L}^0_{g_0} f) = \sigma_0(g_0) W_0(f) \sigma_0(g_0)^{-1}.$$

Equivalently, W_0^{-1} is G_0-equivariant with respect to σ_0, that is, for each $g_0\in G_0$ and each $A_0\in\mathcal{L}_2(L^2(\mathbb{R}^n))$, we have

$$W_0^{-1}(\sigma_0(g_0)A_0\sigma_0(g_0)^{-1})=\tilde{L}^0_{g_0}W_0^{-1}(A_0).$$

We can then obtain an adapted Weyl correspondence for σ_0 as follows. Let $\mathcal{A}$ be the space of all functions f on $\mathcal{O}_\lambda$ such that $(f\circ\Psi_\lambda)(p,q)$ is a smooth function which is polynomial in the variable q. Let $\mathcal{B}$ be the space of all differential operators on $\mathbb{R}^n$ with coefficients in $C^\infty(\mathbb{R}^n)$. Then from Proposition 4.1 and Proposition 4.2 we can deduce the following result.

Proposition 4.3. [19] *The map $\mathcal{W}_0:\mathcal{A}\to\mathcal{B}$ defined by $\mathcal{W}_0(f):=W_0(f\circ\Psi_\lambda)$ is an adapted Weyl correspondence which is G_0-equivariant with respect to σ_0 .*

Finally, note that W_0 (hence $\mathcal{W}_0$) can be related to U^0 (see Section 3) as follows. Let I_{B_0} be the unitary map from $\mathcal{L}_2(L^2(\mathbb{R}^n))$ onto $\mathcal{L}_2(\mathcal{F}_0)$ defined by $I_{B_0}(A)=B_0AB_0^{-1}$ and let J be the map from $L^2(\mathbb{C}^n,\mu_\lambda)$ onto $L^2(\mathbb{R}^{2n})$ defined by $J(F)=F\circ j$. Then we have the following proposition.

Proposition 4.4. [36], [40] *We have $U^0I_{B_0}=(W_0J)^{-1}$.*

5. Heisenberg motion groups

Let K be a closed subgroup of $U(n)$. Recall K acts on G_0 by $k\cdot((z,\bar z),c)=((kz,\overline{kz}),c)$, see Section 2. Then we can form the semidirect product $G:=G_0\rtimes K$ with respect to this action. The group G is called a Heisenberg motion group. The elements of G can be written as $((z,\bar z),c,k)$ where $z\in\mathbb{C}^n$, $c\in\mathbb{R}$, $k\in K$ and the multiplication of G is given by

$$\begin{aligned}((z,\bar z),c,k)&\cdot((z',\bar z'),c',k')\\ &=((z,\bar z)+(kz',\overline{kz'}),c+c'+\tfrac12\omega((z,\bar z),(kz',\overline{kz'})),kk').\end{aligned}$$

Let $\mathfrak{k}$ and $\mathfrak{g}$ be the Lie algebras of K and G. The Lie brackets of $\mathfrak{g}$ are given by

$$\begin{aligned}[((w,\bar w),c,A),&((w',\bar w'),c',A')]\\ &=((Aw'-A'w,\overline{Aw'}-\overline{A'w}),\omega((w,\bar w),(w',\bar w')),[A,A']).\end{aligned}$$

Now, we give the formulas for the adjoint and coadjoint actions of G.

Let $g=((z_0,\bar z_0),c_0,k_0)\in G$ where $z_0\in\mathbb{C}^n$, $c_0\in\mathbb{R}$, $k_0\in K$ and $X=((w,\bar w),c,A)\in\mathfrak{g}$ where $w\in\mathbb{C}^n$, $c\in\mathbb{R}$ and $A\in\mathfrak{k}$. We can easily verify that

$$\mathrm{Ad}(g)X=\frac{d}{dt}(g\exp(tX)g^{-1})|_{t=0}=((w',\bar w'),c',\mathrm{Ad}(k_0)A)$$

where $w':=k_0w-(\mathrm{Ad}(k_0)A)z_0$ and

$$c':=c+\omega\big((z_0,\bar z_0),(k_0w,\overline{k_0w})\big)-\tfrac12\omega\big((z_0,\bar z_0),(\mathrm{Ad}(k_0)A)z_0,\overline{\mathrm{Ad}(k_0)A)z_0}\big).$$

Now, let us denote by $\xi=((u,\bar{u}),d,\phi)$, where $u\in\mathbb{C}^n$, $d\in\mathbb{R}$ and $\phi\in\mathfrak{k}^*$, the element of $\mathfrak{g}^*$ defined by

$$\langle\xi,((w,\bar{w}),c,A)\rangle=\omega((u,\bar{u}),(w,\bar{w}))+dc+\langle\phi,A\rangle.$$

Also, for $u,v\in\mathbb{C}^n$, we denote by $(v,\bar{v})\times(u,\bar{u})$ the element of $\mathfrak{k}^*$ defined by

$$\langle(v,\bar{v})\times(u,\bar{u}),A\rangle:=\omega((u,\bar{u}),(Av,\bar{A}\bar{v}))$$

for $A\in\mathfrak{k}$. Then, from the formula for the adjoint action of G, we deduce that, for each $\xi=((u,\bar{u}),d,\phi)\in\mathfrak{g}^*$ and $g=((z_0,\bar{z}_0),c_0,k_0)\in G$, we have

$$\begin{aligned}&\mathrm{Ad}^*(g)\xi\\&=\big((k_0u-dz_0,\overline{k_0u-dz_0}),d,\mathrm{Ad}^*(k_0)\phi+(z_0,\bar{z}_0)\times(k_0u-\tfrac{d}{2}z_0,\overline{k_0u-\tfrac{d}{2}z_0})\big).\end{aligned}$$

From this, we see that if a coadjoint orbit of G contains a point $((u,\bar{u}),d,\phi)$ with $d\neq 0$ then it also contains a point of the form $((0,0),d,\phi_0)$. Such an orbit is called *generic.*

We consider the unitary irreducible representations of G associated with the integral generic orbits. These representations are called *generic* and we can realize them in Fock spaces as holomorphic induced representations by using the general method of [39], Chapter XII.

More precisely, let us consider a unitary irreducible representation ρ of K on a (finite-dimensional) complex vector space V and let us fix an element $\xi_0=((0,0),d,\phi_0)$ of $\mathfrak{g}^*$. We assume that $d\neq 0$ and that the orbit $o(\phi_0)$ of ϕ_0 for the coadjoint action of K is associated with ρ as in [21] and [46].

Let $\tilde{K}$ be the subgroup of G defined by $\tilde{K}:=\{((0,0),c,k)\,:\,c\in\mathbb{R},\,k\in K\}$ and let $\tilde{\rho}$ be the representation of $\tilde{K}$ on V defined by $\tilde{\rho}((0,0),c,k)=e^{i\lambda c}\rho(k)$ for each $c\in\mathbb{R}$ and $k\in K$. Then we can easily verify that the representation π of G which is holomorphically induced from $\tilde{\rho}$ can be realized in the Hilbert space $\mathcal{F}$ of all holomorphic functions $f:\mathbb{C}^n\to V$ such that

$$\Vert f\Vert^2_{\mathcal{F}}:=\int_{\mathbb{C}^n}\Vert f(z)\Vert^2_V\,e^{-|z|^2/2\lambda}\,d\mu_\lambda(z)<+\infty$$

as

$$(\pi(g)f)(z)=\exp\left(i\lambda c_0+\tfrac{1}{2}i\bar{z}_0z-\tfrac{\lambda}{4}|z_0|^2\right)\,\rho(k)\,f(k^{-1}(z+i\lambda z_0))$$

where $g=((z_0,\bar{z}_0),c_0,k)\in G$ and $z\in\mathbb{C}^n$.

Note that we have $\mathcal{F}=\mathcal{F}_0\otimes V$. For $f_0\in\mathcal{F}_0$ and $v\in V$, we denote by $f_0\otimes v$ the function $z\to f_0(z)v$. It is clear that

$$\langle f_0\otimes v,f_1\otimes w\rangle_{\mathcal{F}}=\langle f_0,f_1\rangle_{\mathcal{F}_0}\langle v,w\rangle_V$$

for each $f_0,f_1\in\mathcal{F}_0$ and each $v,w\in V$. Moreover, if A_0 is an operator of $\mathcal{F}_0$ and A_1 is an operator of V then we denote by $A_0\otimes A_1$ the operator of

$\mathcal{F}$ defined by $(A_0 \otimes A_1)(f_0 \otimes v) = A_0 f_0 \otimes A_1 v$ for each $f_0 \in \mathcal{F}_0$ and each $v \in V$. Then we have the decomposition formula

$$\pi((z_0, \bar{z}_0), c_0, k) = \pi_0((z_0, \bar{z}_0), c_0)\tau(k) \otimes \rho(k) \tag{5.1}$$

for each $z_0 \in \mathbb{C}^n$, $c_0 \in \mathbb{R}$ and $k \in K$. This is precisely Formula (3.18) in [7].

Now, we introduce the Schrödinger representations of G by extending B_0 to V-valued functions. More precisely, we consider the map B from $L^2(\mathbb{R}^n, V) \cong L^2(\mathbb{R}^n) \otimes V$ to $\mathcal{F} \cong \mathcal{F}_0 \otimes V$ defined by $B := B_0 \otimes I_V$. Then we have the integral formula

$$B(f)(z) = (\lambda/\pi)^{n/4} \int_{\mathbb{R}^n} e^{(1/4\lambda)z^2 + ixz - (\lambda/2)x^2} f(x)\, dx$$

for each $f \in L^2(\mathbb{R}^n, V)$.

This allows us to imitate the case of the Heisenberg groups and to define the Schrödinger representation σ of G on $L^2(\mathbb{R}^n, V)$ by $\sigma(g) := B^{-1}\pi(g)B$.

Similarly, for each $k \in K$ we define the operator $\tilde{\tau}$ of $L^2(\mathbb{R}^n)$ by $\tilde{\tau}(k) := B_0^{-1}\tau(k)B_0$. Then, from Equation 5.1 we immediately obtain the decomposition formula

$$\sigma(g) = \sigma_0(g_0)\tilde{\tau}(k) \otimes \rho(k) \tag{5.2}$$

for each $g_0 \in G_0$, $k \in K$ and $g = (g_0, k) \in G$.

6. Berezin correspondence for Heisenberg motion groups

In this section, we introduced the Berezin correspondence S associated with π and show that S is G-equivariant.

Recall that the Berezin calculus on $o(\phi_0)$ associates with each operator A_1 on V a complex-valued function $s(A_1)$ on the orbit $o(\phi_0)$ which is called the symbol of the operator A_1 (see [9]). We denote by $Sy(o(\phi_0))$ the space of all such symbols. Moreover, for each $k \in K$ and each function u on $o(\phi_0)$, we denote by $\tilde{l}_k u$ the function on $o(\phi_0)$ defined by $\tilde{l}_k u(\phi) = u(\mathrm{Ad}^*(k)^{-1}\phi)$.

The following properties of the Berezin calculus are well-known, see [4], [12], [24] and [46].

Proposition 6.1. (1) *The map $A_1 \to s(A_1)$ is injective.*

(2) *For each operator A_1 on V, we have $s(A_1^*) = \overline{s(A_1)}$.*

(3) *For each operator A_1 on V, $k \in K$ and $\phi \in o(\phi_0)$, we have*

$$s(A_1)(\mathrm{Ad}^*(k)\phi) = s(\rho(k)^{-1}A_1\rho(k))(\phi)$$

and, equivalently, for each operator A_1 on V and each $k \in K$, we have

$$s(\rho(k)A_1\rho(k)^{-1}) = \tilde{l}_k s(A_1).$$

(4) *For* $X \in \mathfrak{k}$ *and* $\phi \in o(\phi_0)$*, we have* $s(d\rho(X))(\phi) = i\langle \phi, X\rangle$*.*

In particular, we see that s is an adapted Weyl transform on $o(\phi_0)$ in the sense of Definition 1.

Now, S is defined as follows. For each operator A_0 on $\mathcal{F}_0$ and each operator A_1 on V, we set $S(A_0 \otimes A_1) := S^0(A_0) \otimes s(A_1)$ and then we extend S by linearity to operators on $\mathcal{F}$.

Consider the action of G on $\mathbb{C}^n \times o(\phi_0)$ defined by $g \cdot (z, \phi) = (g \cdot z, \mathrm{Ad}^*(k)^{-1}\phi)$ where $g = (g_0, k) \in G$. Then, for each $g \in G$ and each function F on $\mathbb{C}^n \times o(\phi_0)$, we denote by $L_g F$ the function on $\mathbb{C}^n \times o(\phi_0)$ defined by $L_g F(z, \phi) := F(g^{-1} \cdot (z, \phi))$.

Proposition 6.2. *The map* S *is* G*-equivariant with respect to* π*, that is, for each operator* A *on* $\mathcal{F}$ *and each* $g \in G$*, we have* $S(\pi(g)^{-1} A \pi(g)) = L_{g^{-1}} S(A)$*.*

Proof. It is sufficient to consider the case where $A = A_0 \otimes A_1$ for A_0 operator on $\mathcal{F}_0$ and A_1 operator on V.

Let $g = (g_0, k) \in G$. By Equation 5.1, we have

$$\pi(g)^{-1} A \pi(g) = \tau(k)^{-1} \pi_0(g_0)^{-1} A_0 \pi_0(g_0) \tau(k) \otimes \rho(k)^{-1} A_1 \rho(k).$$

Then, by using Proposition 3.1, Proposition 3.3 and Proposition 6.1, we get

$$\begin{aligned} S(\pi(g)^{-1} A \pi(g)) &= S^0(\tau(k)^{-1} \pi_0(g_0)^{-1} A_0 \pi_0(g_0) \tau(k)) \otimes s(\rho(k)^{-1} A_1 \rho(k)) \\ &= l_{k^{-1}} S^0(\pi_0(g_0)^{-1} A_0 \pi_0(g_0)) \otimes \tilde{l}_{k^{-1}} s(A_1) \\ &= l_{k^{-1}} L^0_{g_0^{-1}}(S^0(A_0)) \otimes \tilde{l}_{k^{-1}} s(A_1). \end{aligned}$$

This implies that

$$\begin{aligned} S(\pi(g)^{-1} A \pi(g))(z, \phi) &= S^0(A_0)(kz - i\lambda z_0) s(A_1)(\mathrm{Ad}^*(k)\phi) \\ &= (S^0(A_0) \otimes s(A_1))(g \cdot z, \mathrm{Ad}^*(k)\phi) \end{aligned}$$

for each $(z, \phi) \in \mathbb{C}^n \times o(\phi_0)$. This gives the desired result. $\square$

7. Weyl correspondence for Heisenberg motion groups

In this section, we first introduce the Berezin-Weyl correspondence, in the spirit of [15].

Recall that the Berezin calculus s is an isomorphism from $\mathrm{End}(V)$ onto $Sy(o(\varphi_0))$, see Section 6. We say that a complex-valued smooth function $f : (p, q, \phi) \to f(p, q, \phi)$ is a symbol on $\mathbb{R}^{2n} \times o(\phi_0)$ if for each $(p, q) \in \mathbb{R}^{2n}$ the function $f(p, q, \cdot) : \phi \to f(p, q, \phi)$ is in $Sy(o(\phi_0))$. In this case, we denote $\hat{f}(p, q) := s^{-1}(f(p, q, \cdot))$. A symbol f on $\mathbb{R}^{2n} \times o(\phi_0)$ is called an S-symbol if the function $\hat{f}$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^{2n}, \mathrm{End}(V))$

of rapidly decreasing smooth functions on $\mathbb{R}^{2n}$ with values in $\mathrm{End}(V)$. We define similarly the notion of L^2-symbol. For each S-symbol on $\mathbb{R}^{2n}\times o(\varphi_0)$, we define the operator $W(f)$ on the Hilbert space $L^2(\mathbb{R}^n,V)=L^2(\mathbb{R}^n)\otimes V$ by

$$W(f)\varphi(p)=(2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{itq}\,\hat{f}(p+(1/2)t,q)\,\varphi(p+t)\,dt\,dq.$$

Note that W can be extended to much larger classes of symbols in the same way as W_0, see Section 4. It is also clear that we have

$$W(f_0\otimes f_1)=W_0(f_0)\otimes s^{-1}(f_1)$$

for each $f_0\in\mathcal{S}(\mathbb{R}^n)$ and $f_1\in Sy(o(\phi_0))$.

Now, we consider the action of G on $\mathbb{R}^{2n}\times o(\phi_0)$ defined by

$$g\cdot(p,q,\phi):=(g_0\cdot(p,q),\mathrm{Ad}^*(k)\phi)$$

for each $g=(g_0,k)\in G$ and each $(p,q,\phi)\in\mathbb{R}^{2n}\times o(\phi_0)$. Then, for each function f on $\mathbb{R}^{2n}\times o(\phi_0)$, we denote by $\tilde{L}_g f$ the function defined on $\mathbb{R}^{2n}\times o(\phi_0)$ by $\tilde{L}_g f(p,q,\phi):=f(g^{-1}\cdot(p,q,\phi))$.

Proposition 7.1. *The map W^{-1} is G-equivariant with respect to σ, that is, we have $W^{-1}(\sigma(g)^{-1}A\sigma(g))=\tilde{L}_{g^{-1}}(W^{-1}(A))$ for each $g\in G$ and each Hilbert-Schmidt operator A on $L^2(\mathbb{R}^n,V)$.*

Equivalently, W is G-equivariant with respect to σ, that is, for each $g\in G$ and each L^2-symbol f, we have $\sigma(g)^{-1}W(f)\sigma(g)=W(L_{g^{-1}}f)$.

Proof. The proof is based on the equivariance of s and U^0. As usual, we can assume, without loss of generality, that $A=A_0\otimes A_1$ with A_0 operator on $L^2(\mathbb{R}^n)$ and A_1 operator on V.

Let $g=(g_0,k)\in G$. Then, by Equation 5.2, we have

$$\begin{aligned}&W^{-1}(\sigma(g)^{-1}A\sigma(g))\\&\quad=(W_0^{-1}\otimes s)\big(\tilde{\tau}(k)^{-1}\sigma_0(g_0)^{-1}A_0\sigma_0(g_0)\tilde{\tau}(k))\otimes\rho(k)^{-1}A_1\rho(k)\big)\\&\quad=W_0^{-1}\big(\tilde{\tau}(k)^{-1}\sigma_0(g_0)^{-1}A_0\sigma_0(g_0)\tilde{\tau}(k)\big)\otimes s(\rho(k)^{-1}A_1\rho(k)).\end{aligned}$$

But, by using successively Proposition 4.4, the second assertion of Proposition 3.3 and Proposition 3.2, we can write

$$\begin{aligned}&W_0^{-1}\big(\tilde{\tau}(k)^{-1}\sigma_0(g_0)^{-1}A_0\sigma_0(g_0)\tilde{\tau}(k)\big)\\&\qquad=(JU^0I_{B_0})\big(\tilde{\tau}(k)^{-1}\sigma_0(g_0)^{-1}A_0\sigma_0(g_0)\tilde{\tau}(k)\big)\\&\qquad=JU^0\big(\tau(k)^{-1}B_0\sigma_0(g_0)^{-1}A_0\sigma_0(g_0)B_0^{-1}\tau(k)\big)\\&\qquad=(Jl_{k^{-1}}U^0)\big(\pi_0(g_0)^{-1}B_0A_0B_0^{-1}\pi_0(g_0)\big)\\&\qquad=(Jl_{k^{-1}}L^0_{g_0^{-1}}U^0I_{B_0})(A_0)\end{aligned}$$

which implies that

$$\begin{aligned}
W_0^{-1}\big(\tilde{\tau}(k)^{-1}\sigma_0(g_0)^{-1}A_0\sigma_0(g_0)\tilde{\tau}(k)\big)(p,q)& \\
&= (l_{k^{-1}}L^0_{g_0^{-1}}U^0 I_{B_0})(A_0)(j(p,q)) \\
&= U^0 I_{B_0}(A_0)(g_0\cdot(k\cdot j(p,q))) \\
&= JU^0 I_{B_0}(A_0)(j^{-1}(g_0\cdot(k\cdot j(p,q))) \\
&= W_0^{-1}(A_0)(g\cdot(p,q)).
\end{aligned}$$

On the other hand, by (3) of Proposition 6.1, we have

$$s(\rho(k)^{-1}A_1\rho(k))(\phi) = s(A_1)(\mathrm{Ad}^*(k)\phi).$$

Then we can conclude that

$$\begin{aligned}
W^{-1}(\sigma(g)^{-1}A\sigma(g)) &= \tilde{L}_g(W_0^{-1}(A_0)\otimes s(A_1)) \\
&= \tilde{L}_g(W_0^{-1}\otimes s)(A_0\otimes A_1) = \tilde{L}_g W^{-1}(A).
\end{aligned}$$

Thus we have proved the first assertion of the proposition. The second assertion immediately follows. □

8. Adapted Weyl correspondences

In this section, we first compute $W^{-1}(d\sigma(X))$ for $X \in \mathfrak{g}$. We have the following result.

Proposition 8.1. [23]

(1) *For each* $X = (X_0, A)$ *with* $X_0 \in \mathfrak{g}_0$ *and* $A \in \mathfrak{k}$, *we have*

$$d\sigma(X) = (d\sigma_0(X_0) + d\tilde{\tau}(A))\otimes I_V + I_{\mathcal{F}_0}\otimes d\rho(A).$$

(2) *For each* $A = (a_{kl}) \in \mathfrak{k}$, *we have*

$$d\tilde{\tau}(A) = \frac{1}{2\lambda}\sum_{k,l} a_{kl}\frac{\partial^2}{\partial p_k\partial p_l} + \frac{1}{2}\sum_{k,l} a_{kl}\left(p_k\frac{\partial}{\partial p_l} - p_l\frac{\partial}{\partial p_k}\right) - \frac{\lambda}{2}p(Ap) + \frac{1}{2}\,\mathrm{Tr}(A).$$

Note that (1) is a simple consequence of Equation 5.2. From this proposition, we can deduce the following result.

Proposition 8.2. *For each* $X = ((a,\bar{a}), c, A) \in \mathfrak{g}$ *and* $(p,q,\phi) \in \mathbb{R}^{2n}\times o(\phi_0)$, *we have*

$$\begin{aligned}
W^{-1}(d\sigma(X))(p,q,\varphi) &= i\lambda c + \frac{1}{2}\,\mathrm{Tr}(A) + \frac{i}{2}\left(\bar{a}j(p,q) + a\overline{j(p,q)}\right) \\
&\quad - \frac{1}{2\lambda}\overline{j(p,q)}(Aj(p,q)) + s(d\rho(A))(\phi).
\end{aligned}$$

Proof. Let $X=((a,\bar{a}),c,A)\in\mathfrak{g}$. Consider the following symbols:

$$\begin{aligned} f_1(p,q,\phi) &:= \frac{i}{2}\left(\bar{a}j(p,q)+a\overline{j(p,q)}\right)\\ f_2(p,q,\phi) &:= -\frac{1}{2\lambda}\overline{j(p,q)}(Aj(p,q))\\ f_3(p,q,\phi) &:= s(d\rho(A))(\phi). \end{aligned}$$

Then we have

$$f_1(p,q,\phi)=\frac{i}{2}(a+\bar{a})q-\frac{\lambda}{2}(a-\bar{a})p$$

and, by using Equation 4.1, we get

$$W(f_1)=-\frac{\lambda}{2}(a-\bar{a})p-\frac{1}{2}\sum_{k=1}^{n}(a_k+\bar{a}_k)\frac{\partial}{\partial p_k}.$$

Similarly, writing

$$f_2(p,q,\phi)=-\frac{1}{2\lambda}\left(q(Aq)+\lambda^2p(Ap)+\lambda ip(Aq)-\lambda iq(Ap)\right),$$

we get

$$W(f_2)=\frac{1}{2\lambda}\sum_{k,l}a_{kl}\frac{\partial^2}{\partial p_k\partial p_l}-\frac{1}{2}\sum_{k,l}a_{kl}\left(p_l\frac{\partial}{\partial p_k}-p_k\frac{\partial}{\partial p_l}\right)-\frac{\lambda}{2}p(Ap).$$

On the other hand, by (4) of Proposition 6.1, we have $(W(f_3)\varphi)(p)=(d\rho(A)\varphi)(p)$ for each $\varphi\in C_0^\infty(\mathbb{R}^n,V)$. The result then follows by Proposition 8.1. $\square$

Note that, since $\mathfrak{k}\subset u(n)$, for each $A\in\mathfrak{k}$ and each $(p,q)\in\mathbb{R}^{2n}$, we have $\overline{j(p,q)}Aj(p,q)\in i\mathbb{R}$ and $\mathrm{Tr}(A)\in i\mathbb{R}$. Then, for $(p,q,\phi)\in\mathbb{R}^n\times o(\phi_0)$, the map $X\to -iW^{-1}(d\sigma(X))(p,q,\phi)$ is a real-valued linear map on $\mathfrak{g}$. We denote this map by $\Psi(p,q,\phi)$.

Proposition 8.3. (1) *For each $X\in\mathfrak{k}$ and each $(p,q,\phi)\in\mathbb{R}^{2n}\times o(\phi_0)$, we have*

$$W^{-1}(d\sigma(X))(p,q,\phi)=i\langle\Psi(p,q,\phi),X\rangle.$$

Also, for each $(p,q,\phi)\in\mathbb{R}^{2n}\times o(\phi_0)$, we have

$$\Psi(p,q,\phi)$$
$$=\left((-ij(p,q),i\overline{j(p,q)}),\lambda,-\frac{i}{2}\,\mathrm{Tr}+(j(p,q),\overline{j(p,q)})\times(j(p,q),\overline{j(p,q)})+\phi\right).$$

(2) *For each $g\in G$ and each $(p,q,\phi)\in\mathbb{R}^{2n}\times o(\phi_0)$, we have $\Psi(g\cdot(p,q,\phi))=\mathrm{Ad}^*(g)\Psi(p,q,\phi)$.*

(3) *Assume that $K\subset SU(n)$. Then Ψ is a diffeomorphism from $\mathbb{R}^{2n}\times o(\phi_0)$ onto the coadjoint orbit $\mathcal{O}(\xi_0)\subset\mathfrak{g}^*$ of ξ_0.*

Proof. (1) immediately follows from the definition of Ψ and (2) from the G-equivariance of W^{-1}. To prove (3), first note that we have $\Psi(0,0,\phi_0)=\xi_0$ since the hypothesis $K\subset SU(n)$ implies that $\mathrm{Tr}(A)=0$ for each $A\in\mathfrak{k}$. Then, by (2), we see that Ψ is a surjective map from $\mathbb{R}^n\times o(\phi_0)$ onto $\mathcal{O}(\xi_0)$. On the other hand, by (1), Ψ is injective, hence bijective.

It remains to show that Ψ is regular. By (2) again, it is sufficient to verify that Ψ is regular at $(0,0,\phi_0)$. But we have

$$(d\Psi)_{(0,0,\phi_0)}(u,v,\mathrm{ad}^*(A)\phi_0)=((-i(v-i\lambda u),i(v+i\lambda u)),0,\mathrm{ad}^*(A)\phi_0)$$

for each $(u,v)\in\mathbb{R}^{2n}$ and $A\in\mathfrak{k}$, hence the result. □

Finally, we obtain an adapted Weyl correspondence for σ by transferring W to $O(\xi_0)$. We say that a smooth function f on $O(\xi_0)$ is a symbol on $O(\xi_0)$ (respectively a P-symbol, an S-symbol) if $f\circ\Psi$ is a symbol (respectively a P-symbol, an S-symbol) for W. From the properties of W, we obtain the following proposition.

Proposition 8.4. *Let $\mathcal{A}$ be the space of P-symbols on $O(\xi_0)$ and let $\mathcal{B}$ be the space of differential operators on $\mathbb{R}^n$ with coefficients in $C^\infty(\mathbb{R}^n,V)$. Then the map $\mathcal{W}:\mathcal{A}\to\mathcal{B}$ that assigns to each $f\in\mathcal{A}$ the operator $W(f\circ\Psi)$ on $L^2(\mathbb{R}^n,V)$ is an adapted Weyl correspondence in the sense of Definition 1. Moreover, $\mathcal{W}$ is G-equivariant with respect to σ.*

References

[1] S. T. Ali and M. Englis, *Quantization methods: a guide for physicists and analysts*, Rev. Math. Phys., **17**, no. 4 (2005), 391–490.

[2] J. Arazy and H. Upmeier, *Weyl Calculus for Complex and Real Symmetric Domains*, Harmonic analysis on complex homogeneous domains and Lie groups (Rome, 2001). Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **13**, no. 3-4 (2002), 165–181.

[3] J. Arazy and H. Upmeier, *Invariant symbolic calculi and eigenvalues of invariant operators on symmeric domains*, Function spaces, interpolation theory and related topics (Lund, 2000), 151–211, de Gruyter, Berlin, 2002.

[4] D. Arnal , M. Cahen and S. Gutt, *Representations of compact Lie groups and quantization by deformation*, Acad. R. Belg. Bull. Cl. Sc. 3e série **LXXIV**, no. 45 (1988), 123–141.

[5] D. Arnal, J.-C. Cortet, *Nilpotent Fourier Transform and Applications*, Lett. Math. Phys., **9** (1985), 25–34.

[6] L. Auslander and B. Kostant, *Polarization and Unitary Representations of Solvable lie Groups*, Invent. Math., **14** (1971), 255–354.

[7] C. Benson, J. Jenkins, R. L. Lipsmann and G. Ratcliff, *A geometric criterion for Gelfand pairs associated with the Heisenberg group*, Pacific J. Math., **178**, no. 1 (1997), 1–36.

[8] C. Benson, J. Jenkins and G. Ratcliff, *The orbit method and Gelfand pairs associated with nilpotent Lie groups*, J. Geom. Anal., **9** (1999), 569–582.

[9] F. A. Berezin, *Quantization*, Math. USSR Izv., **8**, no. 5 (1974), 1109–1165.

[10] F. A. Berezin, *Quantization in complex symmetric domains*, Math. USSR Izv., **9**, no. 2 (1975), 341–379.

[11] P. Bernat, N. Conze, M. Duflo, M. Levy-Nahas, M. Rais, P. Renouard and M. Vergne, Representations des groupes de Lie résolubles, Dunod, Paris 1972.

[12] B. Cahen, *Deformation Program for Principal Series Representations*, Lett. Math. Phys, **36** (1996), 65–75.

[13] B. Cahen, *Quantification d'une orbite massive d'un groupe de Poincaré généralisé*, C. R. Acad. Sci. Paris, série I, **325** (1997), 803–806.

[14] B. Cahen, *Contractions of $SU(1,n)$ and $SU(n+1)$ via Berezin quantization*, J. Anal. Math., **97** (2005), 83–102.

[15] B. Cahen, *Weyl quantization for semidirect products*, Differential Geom. Appl., **25** (2007), 177–190.

[16] B. Cahen, *Berezin quantization on generalized flag manifolds*, Math. Scand., **105** (2009), 66–84.

[17] B. Cahen, *Weyl quantization for Cartan motion groups*, Comment. Math. Univ. Carolin., **52**, no. 1 (2011), 127–137.

[18] B. Cahen, *Berezin Quantization and Holomorphic Representations*, Rend. Sem. Mat. Univ. Padova, **129** (2013), 277–297.

[19] B. Cahen, *Stratonovich-Weyl correspondence for the diamond group*, Riv. Mat. Univ. Parma, **4** (2013), 197–213.

[20] B. Cahen, *A contraction of the principal series by Berezin-Weyl quantization*, Rocky Mountain J. Math., **43**, no. 2 (2013), 417–441.

[21] B. Cahen, *Stratonovich-Weyl correspondence via Berezin Quantization*, Rend. Istit. Mat. Univ. Trieste, **46** (2014), 157–180.

[22] B. Cahen, *Berezin transform and Stratonovich-Weyl correspondence for the multi-dimensional Jacobi group*, Rend. Sem. Mat. Univ. Padova, **136** (2016), 69–93.

[23] B. Cahen, *Schrödinger model and Stratonovich-Weyl correspondence for Heisenberg motion groups*, Rend. Istit. Mat. Univ. Trieste, **48** (2016), 509–530.

[24] M. Cahen, S. Gutt and J. Rawnsley, *Quantization on Kähler manifolds I, Geometric interpretation of Berezin quantization*, J. Geom. Phys., **7** (1990), 45–62.

[25] M. Combescure and D. Robert, Coherent States and Applications in Mathematical Physics, Springer, 2012.

[26] P. Cotton and A.H. Dooley, *Contraction of an Adapted Functional Calculus*, J. Lie Theory, **7** (1997), 147–164.

[27] B. Folland, Harmonic Analysis in Phase Space, Princeton Univ. Press, 1989.

[28] J. M. Gracia-Bondìa, *Generalized Moyal quantization on homogeneous symplectic spaces*, Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990), 93–114, Contemp. Math., 134, Amer. Math. Soc., Providence, RI, 1992.

[29] J. M. Gracia-Bondìa and J. C. Vàrilly, *The Moyal Representation for Spin*, Ann. Physics, **190** (1989), 107–148.

[30] M. Gotay, *Obstructions to Quantization*, in: Mechanics: From Theory to Computation (Essays in Honor of Juan-Carlos Simo), J. Nonlinear Science Editors, Springer New-York, 2000, pp. 271-316.

[31] B. C. Hall, *Holomorphic methods in analysis and mathematical physics*, In: First Summer School in Analysis and Mathematical Physics (S. Pérez and C. Villegas-Blas, Eds.), 1–59, Contemp. Math. 260, Amer. Math. Soc., Providence, RI, 2000.

[32] B. C. Hall and J. J. Mitchell, *The Segal-Bargmann transform for noncompact symmetric spaces of the complex type*, J. Funct. Anal., **227** (2005), 338–371.

[33] L. Hörmander, The analysis of linear partial differential operators, Vol. 3, Section 18.5, Springer-Verlag, Berlin, Heidelberg, New-York, 1985.

[34] A. A. Kirillov, *Lectures on the Orbit Method, Graduate Studies in Mathematics*, Vol. 64, American Mathematical Society, Providence, Rhode Island, 2004.

[35] B. Kostant, *Quantization and unitary representations*, in: Modern Analysis and Applications, Lecture Notes in Mathematics 170, Springer-Verlag, Berlin, Heidelberg, New-York, 1970, pp. 87–207.

[36] S. Luo, *Polar decomposition and isometric integral transforms*, Int. Transf. Spec. Funct., **9**, no. 4 (2000), 313–324.

[37] S. Luo, *On the Bargmann transform and the Wigner transform*, Bull. London Math. Soc., **30** (1998), 413–418.

[38] J. Möllers, *A geometric quantization of the Kostant-Sekiguchi correspondence for scalar type unitary highest weight representations*, Documenta Math. **18** (2013), 785–855.

[39] K-H. Neeb, Holomorphy and Convexity in Lie Theory, de Gruyter Expositions in Mathematics, Vol. 28, Walter de Gruyter, Berlin, New-York 2000.

[40] B. Ørsted and G. Zhang, *Weyl Quantization and Tensor Products of Fock and Bergman Spaces*, Indiana Univ. Math. J., **43**, no. 2 (1994), 551–583.

[41] R. L. Stratonovich, *On distributions in representation space*, Soviet Physics. JETP, **4** (1957), 891–898.

[42] M. E. Taylor, Noncommutative Harmonic Analyis, Mathematical Surveys and Monographs, Vol. 22, American Mathematical Society, Providence, Rhode Island 1986.

[43] A. Unterberger and H. Upmeier, *Berezin transform and invariant differential operators*, Commun. Math. Phys., **164**, no. 3 (1994), 563–597.

[44] A. Voros, *An Algebra of Pseudo differential operators and the Asymptotics of Quantum Mechanics*, J. Funct. Anal., **29** (1978), 104–132.

[45] N. J. Wildberger, *Convexity and unitary representations of a nilpotent Lie group*, Invent. Math., **89** (1989), 281–292.

[46] N. J. Wildberger, *On the Fourier transform of a compact semisimple Lie group*, J. Austral. Math. Soc. A, **56** (1994), 64–116.

Benjamin Cahen
Département de mathématiques
Université de Lorraine, Site de Metz, UFR-MIM
3 rue Augustin Fresnel
BP 45112, 57073 Metz CEDEX 03
France
e-mail address: benjamin.cahen@univ-lorraine.fr

(*Received May 19, 2016*)
(*Accepted September 8, 2017*)