

ON THE EXISTENCE OF NON-FINITE COVERINGS OF  
STABLE CURVES OVER COMPLETE DISCRETE  
VALUATION RINGS

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ABSTRACT. Let  $R$  be a complete discrete valuation ring with algebraically residue field of characteristic  $p > 0$  and  $X$  a stable curve over  $R$ . In the present paper, we study the geometry of coverings of  $X$ . Under certain assumptions, we prove that, by replacing  $R$  by a finite extension of  $R$ , there exists a morphism of stable curves  $f : Y \rightarrow X$  over  $R$  such that the morphism  $f_\eta : Y_\eta \rightarrow X_\eta$  induced by  $f$  on generic fibers is finite étale and the morphism  $f_s : Y_s \rightarrow X_s$  induced by  $f$  on special fibers is non-finite.

INTRODUCTION

Let  $R$  be a complete discrete valuation ring with algebraically closed residue field  $k$ ,  $K$  the quotient field of  $R$ , and  $\overline{K}$  an algebraic closure of  $K$ . We use the notation  $S$  to denote the spectrum of  $R$ . Write  $\eta, \overline{\eta}$  and  $s$  for the generic point of  $S$ , the geometric generic point of  $S$ , and the closed point of  $S$  corresponding to the natural morphisms  $\text{Spec } K \rightarrow S$ ,  $\text{Spec } \overline{K} \rightarrow S$ , and  $\text{Spec } k \rightarrow S$ , respectively. Let  $X$  be a stable curve over  $S$  of genus  $g_X$ . Write  $X_\eta, X_{\overline{\eta}}$ , and  $X_s$  for the generic fiber of  $X$ , the geometric generic fiber of  $X$ , and the special fiber of  $X$ , respectively. Moreover, we suppose that  $X_\eta$  is smooth over  $\eta$ .

In the present paper, we study the geometry of coverings of  $X$ . Let  $f_\eta : Y_\eta \rightarrow X_\eta$  be an arbitrary Galois étale covering over  $\eta$ . By replacing  $S$  by a finite extension of  $S$ ,  $f_\eta$  extends uniquely to a stable covering  $f : Y \rightarrow X$  over  $S$  (cf. Definition 1.3 and Remark 1.3.1). Note that  $f_s$  is not a finite morphism in general. A closed point  $x$  of  $X$  is called a vertical point of  $X$  if, by replacing  $S$  by a finite extension of  $S$ , there exists a stable covering such that the fiber of  $x$  is not a finite set (cf. Definition 1.5). Write  $X^{\text{ver}}$  for the set of the vertical points of  $X$ . We may post a question as follows:

**Question 0.1.** *What is  $X^{\text{ver}}$ ?*

If  $\text{char}(K) = \text{char}(k) = 0$ , up to composition with an inner automorphism, there is a natural isomorphism between the étale fundamental group of  $X_{\overline{\eta}}$  and the admissible fundamental group of  $X_s$  (cf. Definition 1 and [12,

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Théorème 2.2]). Thus, for any finite étale covering of  $X_\eta$ , by replacing  $S$  by a finite extension of  $S$ , the morphism of special fibers induced by the étale covering of  $X_\eta$  is an admissible covering. Then we obtain  $X^{\text{ver}}$  is empty.

If  $\text{char}(k) = p > 0$ , we may consider whether or not the set  $X^{\text{ver}}$  is empty. The motivation of Question 0.1 partly comes from anabelian geometry. Question 0.1 was first considered by S. Mochizuki (cf. [2, the proof of Theorem 9.2]). If  $\text{char}(K) = 0$ , Mochizuki proved that  $X^{\text{ver}}$  is not empty. Then he reduced the Grothendieck conjecture for proper, hyperbolic curves over number fields to the Grothendieck conjecture for proper, singular, stable curves over finite fields, which is then reduced to the Grothendieck conjecture for affine curves over finite fields which had been proven by A. Tamagawa.

In the present paper, we focus on Question 0.1 in the case where  $\text{char}(K) = \text{char}(k) = p > 0$ . The main theorem of the present paper is as follows.

**Theorem 0.2.** *Suppose that  $\text{char}(K) = \text{char}(k) = p > 0$ .*

(i) *Suppose that  $X$  is smooth over  $S$ ,  $X_{\bar{\eta}}$  cannot be defined over an algebraic closure of  $\mathbb{F}_p$ , and  $X_s$  can be defined over an algebraic closure of  $\mathbb{F}_p$ . Then  $X^{\text{ver}}$  is not empty.*

(ii) *Suppose that  $X_s$  is an irreducible singular stable curve over  $s$ . Then  $X^{\text{ver}}$  is not empty.*

**Remark 0.2.1.** Suppose that  $R = \mathbb{F}_q[[t]]$ , where  $q = p^r$  such that  $r \neq 0$ . Let  $X_1$  and  $X_2$  be stable curves over  $S$ . Moreover, we suppose that the generic fibers of  $X_{1,\eta}$  and  $X_{2,\eta}$  are smooth over  $S$ . The Grothendieck conjecture for curves over local fields of positive characteristic asks whether or not the natural morphism

$$\text{Isom}_{\text{schemes}}(X_{1,\eta}, X_{2,\eta}) \rightarrow \text{Isom}(\pi_1(X_{1,\eta}), \pi_1(X_{2,\eta})) / \text{Inn}(\pi_1(X_{2,\eta}))$$

is a bijection. Here, the left-hand side is the set of  $\mathbb{F}_p$ -isomorphisms of curves and the right-hand side is the set of continuous isomorphisms of profinite groups. The difficult part for proving this conjecture is the surjectivity. Suppose that the fundamental groups of the generic fibers of  $X_1$  and  $X_2$  are isomorphic. By applying Theorem 0.2, similar arguments to the arguments given in the proofs of [2] imply that the special fibers of  $X_1$  and  $X_2$  are isomorphic. However, we do not know how to recover the  $X_1$  and  $X_2$  from the special fibers and the fundamental groups. At the present, no results are known about the Grothendieck conjecture for curves over local fields of positive characteristic.

On the other hand, Tamagawa approached the Grothendieck conjecture for curves in positive characteristic by changing the point of view. He showed that, the isomorphism class of a hyperbolic curve in positive characteristic may possibly be determined by the isomorphism class of its **geometric** tame

fundamental group (i.e., without Galois action). More precisely, Tamagawa proved that (cf. [9]),

if  $U$  is a hyperbolic curve of type  $(0, n)$  over an algebraically closed field of characteristic  $p > 0$ , then we can detect completely whether  $U$  can be defined over  $\overline{\mathbb{F}}_p$  or not, group-theoretically from the tame fundamental group of  $U$ ; moreover, if  $U$  is defined over  $\overline{\mathbb{F}}_p$ , then the isomorphism class of the tame fundamental group of  $U$  determines completely the isomorphism class of  $U$  as scheme.

Furthermore, for the case of higher genus, the following finiteness theorem proved by M. Raynaud, F. Pop, and M. Saïdi under certain conditions and by Tamagawa in full generality (cf. [6], [3], [10]):

Let  $U$  be a hyperbolic curve over  $\overline{\mathbb{F}}_p$ , then there are only finitely many isomorphism classes of hyperbolic curves over  $\overline{\mathbb{F}}_p$  whose tame fundamental groups are isomorphic to the tame fundamental group of  $U$ .

Recently, the author extended the Tamagawa's results above to the case of (possibly singular) pointed stable curves (cf. [14]). Furthermore, the author proved that, the set of open morphisms between the tame fundamental groups of two curves of type  $(0, n)$  over  $\overline{\mathbb{F}}_p$  is not empty if and only if the curves are isomorphic as schemes (cf. [15]); this result can be regarded as a weak Hom-version Grothendieck conjecture for curves of type  $(0, n)$  over  $\overline{\mathbb{F}}_p$ .

**Remark 0.2.2.** If  $\text{char}(K) = 0$  and  $\text{char}(k) = p > 0$ , Tamagawa proved the following result: suppose that  $R$  is a finite extension of the Witt ring  $W(\overline{\mathbb{F}}_p)$ ; then  $X^{\text{ver}}$  is equal to the set of closed points of  $X$ . Moreover, by applying Tamagawa's idea, we extend Tamagawa's result to the case where  $k$  is an arbitrary algebraically closed field of characteristic  $p > 0$  (cf. Appendix of the present paper).

Moreover, for a given stable covering, we have the following theorem.

**Theorem 0.3.** *Let  $G$  be a finite group,  $f : Y \rightarrow X$  a  $G$ -stable covering over  $S$ , and  $f_s : Y_s \rightarrow X_s$  the morphism induced by  $f$  on the special fibers.*

*(i) Suppose that  $X_s$  is smooth over  $s$ . Then  $f_s$  is a finite morphism if and only if  $f_s$  is an étale covering.*

*(ii) Suppose that  $X_s$  is irreducible, and  $G$  is a solvable group. Then  $f_s$  is a finite morphism if and only if  $f_s$  is an admissible covering.*

Finally, let  $G$  be a finite  $p$ -group,  $f : Y \rightarrow X$  a non-finite  $G$ -stable covering over  $S$ , and  $x \in X^{\text{ver}}$ . We mention that an explicit formula for  $p$ -rank (cf.

Definition 1.2) of  $f^{-1}(x)$  is given by Raynaud, Saïdi, and the author (cf. [4], [7], [13]).

## 1. PRELIMINARIES

In this section, we recall some definitions and results which will be used in the present paper.

**Definition 1.1.** Let  $C_1$  and  $C_2$  be stable curves over an algebraically closed field  $l$  and  $\phi : C_2 \rightarrow C_1$  a morphism of stable curves over  $\text{Spec} l$ .

We shall call  $\phi$  a **Galois admissible covering** over  $\text{Spec} l$  (or Galois admissible covering for short) if the following conditions hold: (i) there exists a finite group  $G \subseteq \text{Aut}_k(C_2)$  such that  $C_2/G = C_1$ , and  $\phi$  is equal to the quotient morphism  $C_2 \rightarrow C_2/G$ ; (ii) for each  $c_2 \in C_2^{\text{sm}}$ ,  $\phi$  is étale at  $c_2$ , where  $(-)^{\text{sm}}$  denotes the smooth locus of  $(-)$ ; (iii) for any  $c_2 \in C_2^{\text{sing}}$ , the image  $\phi(c_2)$  is contained in  $C_1^{\text{sing}}$ , where  $(-)^{\text{sing}}$  denotes the singular locus of  $(-)$ ; (iv) for each  $c_2 \in C_2^{\text{sing}}$ , the local morphism between two nodes (cf. (iii)) induced by  $\phi$  may be described as follows:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{C_1, \phi(c_2)} \cong l[[u, v]]/uv & \rightarrow & \hat{\mathcal{O}}_{C_2, c_2} \cong l[[s, t]]/st \\ u & \mapsto & s^n \\ v & \mapsto & t^n, \end{array}$$

where  $(n, \text{char}(l)) = 1$  if  $\text{char}(l) = p > 0$ ; moreover, write  $D_{c_2} \subseteq G$  for the decomposition group of  $c_2$ ; then  $\tau(s) = \zeta_{\#D_{c_2}} s$  and  $\tau(t) = \zeta_{\#D_{c_2}}^{-1} t$  for each  $\tau \in D_{c_2}$ , where  $\zeta_{\#D_{c_2}}$  is a primitive  $\#D_{c_2}$ -th root of unit.

We shall call  $\phi$  an **admissible covering** if there exists a morphism of stable curves  $\phi' : C'_2 \rightarrow C_2$  over  $\text{Spec} l$  such that the composite morphism  $\phi \circ \phi' : C'_2 \rightarrow C_1$  is a Galois admissible covering over  $\text{Spec} l$ .

Let  $Z$  be the disjoint union of finitely many stable curves over  $\text{Spec} l$ . We shall call a morphism  $Z \rightarrow C_1$  over  $\text{Spec} l$  **multi-admissible** if the restriction of  $Z \rightarrow C_1$  to each connected component of  $Z$  is admissible. We define a category  $\text{Cov}^{\text{adm}}(C_1)$  as follows: (i) the objects of  $\text{Cov}^{\text{adm}}(C_1)$  are the empty object and the multi-admissible coverings of  $C_1$ ; (ii) for any  $A, B \in \text{Cov}^{\text{adm}}(C_1)$ ,  $\text{Hom}(A, B)$  consists of all the morphisms whose restriction to each connected component of  $B$  is a multi-admissible covering. It is well-known that  $\text{Cov}^{\text{adm}}(C_1)$  is a Galois category. Thus, by choosing a base point  $c_1 \in C_1$ , we obtain a fundamental group  $\pi_1^{\text{adm}}(C_1, c_1)$  which is called the **admissible fundamental group** of  $C_1$ . For simplicity, we omit the base point and denote the admissible fundamental group by  $\pi_1^{\text{adm}}(C_1)$ .

For more details on admissible coverings and the fundamental groups for (pointed) stable curves, see [1], [2].

**Remark 1.1.1.** If  $C_1$  is a smooth projective curve over  $\text{Spec } l$ , then every admissible covering of  $C_1$  is étale. Thus, we have  $\pi_1^{\text{adm}}(C_1) = \pi_1(C_1)$ , where  $\pi_1(-)$  denotes the étale fundamental group of  $(-)$ .

**Remark 1.1.2.** Let  $\overline{\mathcal{M}}_{g,n}$  be the moduli stack of pointed stable curves of type  $(g, n)$  over  $\text{Spec } \mathbb{Z}$  and  $\mathcal{M}_{g,n}$  the open substack of  $\overline{\mathcal{M}}_{g,n}$  parametrizing pointed smooth curves. Write  $\overline{\mathcal{M}}_{g,n}^{\text{log}}$  for the log stack obtained by equipping  $\overline{\mathcal{M}}_{g,n}$  with the natural log structure associated to the divisor with normal crossings  $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$  relative to  $\text{Spec } \mathbb{Z}$ .

Let  $C_1$  be a stable curve over an algebraically closed field  $l$  of genus  $g$ . Then  $C_1 \rightarrow \text{Spec } l$  induces a morphism  $\text{Spec } l \rightarrow \overline{\mathcal{M}}_{g,0}$ . Write  $s_{C_1}^{\text{log}}$  for the log scheme whose underlying scheme is  $\text{Spec } l$ , and whose log structure is the pulling-back log structure induced by the morphism  $\text{Spec } l \rightarrow \overline{\mathcal{M}}_{g,0}$ . We obtain a natural morphism  $s_{C_1}^{\text{log}} \rightarrow \overline{\mathcal{M}}_{g,0}^{\text{log}}$  induced by the morphism  $\text{Spec } l \rightarrow \overline{\mathcal{M}}_{g,0}$  and a stable log curve  $C_1^{\text{log}} := s_{C_1}^{\text{log}} \times_{\overline{\mathcal{M}}_{g,0}^{\text{log}}} \overline{\mathcal{M}}_{g,1}^{\text{log}}$  over  $s_{C_1}^{\text{log}}$  whose underlying scheme is  $C_1$ . Then the admissible fundamental group  $\pi_1^{\text{adm}}(C_1)$  of  $C_1$  is naturally isomorphic to the geometric log étale fundamental group of  $C_1^{\text{log}}$  (i.e.,  $\text{Ker}(\pi_1(C_1^{\text{log}}) \rightarrow \pi_1(s_{C_1}^{\text{log}}))$ ).

**Definition 1.2.** Let  $C$  be a stable curve over an algebraically closed field of characteristic  $p > 0$ . We define the  **$p$ -rank**  $\sigma(C)$  of  $C$  to be

$$\sigma(C) := \dim_{\mathbb{F}_p} H_{\text{ét}}^1(C, \mathbb{F}_p).$$

From now on, we fix some notations. Let  $R$  be a complete discrete valuation ring with algebraically closed residue field  $k$  of characteristic  $p > 0$ ,  $K$  the quotient field of  $R$ , and  $\overline{K}$  an algebraic closure of  $K$ . We use the notation  $S$  to denote the spectrum of  $R$ . Write  $\eta, \overline{\eta}$  and  $s$  for the generic point of  $S$ , the geometric generic point of  $S$ , and the closed point of  $S$  corresponding to the natural morphisms  $\text{Spec } K \rightarrow S$ ,  $\text{Spec } \overline{K} \rightarrow S$ , and  $\text{Spec } k \rightarrow S$ , respectively. Let  $X$  be a stable curve over  $S$  of genus  $g_X$ . Write  $X_\eta := X \times_S \eta$  for the generic fiber of  $X$ ,  $X_{\overline{\eta}} := X \times_S \overline{\eta}$  for the geometric generic fiber of  $X$ , and  $X_s := X \times_S s$  for the special fiber of  $X$ , respectively. Moreover, we suppose that  $X_\eta$  is smooth over  $\eta$ . Write  $\Gamma_{X_s}$  for the dual graph of  $X_s$  and  $v(\Gamma_{X_s})$  for the set of vertices of  $\Gamma_{X_s}$ .

**Definition 1.3.** Let  $f : Y \rightarrow X$  a morphism of stable curves over  $S$ , and  $G$  a finite group. We shall call  $f$  a  **$G$ -stable covering** over  $S$  if the morphism  $f_\eta : Y_\eta \rightarrow X_\eta$  over  $\eta$  induced by  $f$  on generic fibers is a Galois étale covering whose Galois group is isomorphic to  $G$ .

**Remark 1.3.1.** Let  $W_\eta \rightarrow X_\eta$  be any geometrically connected Galois étale covering over  $\eta$  whose Galois group is isomorphic to  $G$ . By replacing  $S$  by

a finite extension of  $S$ ,  $W_\eta$  admits a stable model  $W$  over  $S$ . Moreover, the action of  $G$  on  $W_\eta$  induces an action of  $G$  on  $W$ . Then [4, Appendice Corollaire] implies that  $W/G$  is a semi-stable curve over  $S$ . Since  $X$  is a stable curve over  $S$ , we have a natural morphism  $W/G \rightarrow X$  over  $S$ . Thus, the morphism  $W_\eta \rightarrow X_\eta$  may extend to a  $G$ -stable covering over  $S$ .

On the other hand, we would like to mention that Definition 1.3 is a special case of [13, Definition 3.3]. For more details on semi-stable coverings and the existence of semi-stable coverings, see [13, Definition 3.3 and Proposition 3.4].

**Remark 1.3.2.** Let  $f : Y \rightarrow X$  be a  $G$ -stable covering over  $S$ , and  $y$  any closed point of  $Y$ . Then  $f$  induces a morphism  $f_y : \text{Spec } \widehat{\mathcal{O}}_{Y,y} \rightarrow \text{Spec } \widehat{\mathcal{O}}_{X,f(y)}$  over  $S$ . Suppose that  $f_s : Y_s \rightarrow X_s$  over  $s$  on special fibers induced by  $f$  is generically étale.

If  $y$  is a smooth point, then  $f(y) \in X$  is a smooth point too. Then Zariski-Nagata purity implies that the morphism  $f_y$  is étale.

If  $y$  is a singular point, then  $f(y) \in X$  is a singular point too. Then Zariski-Nagata purity and [11, Lemma 2.1 (iii)] imply that the morphism of local rings  $\widehat{\mathcal{O}}_{X_s,f(y)} \rightarrow \widehat{\mathcal{O}}_{Y_s,y}$  induced by  $f_y$  satisfies the condition (iv) of Definition 1.1.

Thus, we have that  $f_s$  is an admissible covering over  $s$  if and only if  $f_s$  is generically étale.

The following highly non-trivial result was proved by Tamagawa (cf. [10, Theorem 8.1], [10, Corollary 3.11]).

**Proposition 1.4.** *Let*

$$sp : \pi_1(X_{\bar{\eta}}) \rightarrow \pi_1^{\text{adm}}(X_s)$$

*be the specialization map.*

(i) *Suppose that  $X$  is smooth over  $S$ ,  $X_{\bar{\eta}}$  cannot be defined over an algebraic closure field of  $\mathbb{F}_p$ , and  $X_s$  can be defined over an algebraically closed field of  $\mathbb{F}_p$ . Then there exists a finite étale covering  $f : Y \rightarrow X$  over  $S$  such that  $Y_{\bar{\eta}} := Y \times_{\eta} \bar{\eta}$  and  $Y_s$  are connected, and  $\sigma(Y_{\bar{\eta}}) - \sigma(Y_s) > 0$ . In particular,  $sp$  is not an isomorphism.*

(ii) *Suppose that  $X_s$  is a singular curve. Then there exists a finite group  $G$ , and, by replacing  $S$  by a finite extension of  $S$ , there exists a  $G$ -stable covering  $f : Y \rightarrow X$  over  $S$  such that  $f_s : Y_s \rightarrow X_s$  is an admissible covering and  $\sigma(Y_{\bar{\eta}}) - \sigma(Y_s) > 0$ . In particular,  $sp$  is not an isomorphism.*

**Remark 1.4.1.** Let  $f : Y \rightarrow X$  be a  $G$ -stable covering over  $S$ . Proposition 1.4 implies that the morphism  $f_s : Y_s \rightarrow X_s$  on special fibers induced by  $f$  is not an admissible covering in general.

**Definition 1.5.** Let  $x$  be a closed point of  $X$ . We shall call  $x$  a **vertical point** of  $X$  if there exists a finite group  $G$ , and, after replacing  $S$  by a finite extension of  $S$ , there exists a  $G$ -stable covering  $f : Y \rightarrow X$  over  $S$  such that  $\dim(f^{-1}(x)) = 1$ . We use the notation  $X^{\text{ver}}$  to denote the set of vertical points of  $X$ .

There is a criterion for the existence of vertical points of  $G$ -stable coverings proved by Tamagawa (cf. [11, Proposition 4.3 (ii) (WRamS2)  $\Rightarrow$  (Vert) $_x$ ]).

**Proposition 1.6.** *Let  $x \in X$  be a closed point and  $f : Y \rightarrow X$  a  $G$ -stable covering over  $S$ . Suppose that for each irreducible component  $Z := \overline{\{z\}}$  of  $\text{Spec } \widehat{\mathcal{O}}_{X_s, x}$ , and for each point  $w$  of the fiber  $Y \times_X z$ , the natural morphism from the integral closure  $W^s$  of  $Z$  in  $k(w)^s$  to  $Z$  is wildly ramified, where  $k(w)^s$  denotes the maximal separable subextension of  $k(w)$  in  $k(z)$ . Then  $x$  is a vertical point of  $X$ .*

## 2. GEOMETRY OF COVERINGS OF CURVES

We maintain the notations introduced in Section 1.

**Lemma 2.1.** *Let  $f : Y \rightarrow X$  be a finite  $G$ -stable covering over  $S$  and  $y$  a closed point of  $Y$ . If  $y$  is a node (resp. smooth point) of the special fiber  $Y_s$  of  $Y$ , then  $f(y)$  is a node (resp. smooth point) of  $X_s$ .*

*Proof.* Write  $I_y \subseteq G$  for the inertia group of  $y$ . Consider the quotient morphism  $q_{I_y} : Y \rightarrow Y/I_y$ . Note that [4, Appendice Corollaire] implies that  $Y/I_y$  is a semi-stable curve over  $S$ . The morphism  $f$  induces a natural morphism  $d_{I_y} : Y/I_y \rightarrow X$  such that  $f = d_{I_y} \circ q_{I_y}$ . Then  $d_{I_y}$  is étale at  $q_{I_y}(y)$ . Thus, to verify the lemma, we may assume that  $I_y = G$ .

Suppose that  $y$  is a smooth point of  $Y_s$ . Then  $x$  is a smooth point (cf. [4, Proposition 5]).

Suppose that  $y$  is a node of  $Y_s$ . Let  $Y_1$  and  $Y_2$  be the irreducible components (which may be equal) of  $Y_s$  which contain  $y$ . Write  $D_{Y_1} \subseteq G$  and  $D_{Y_2} \subseteq G$  for the decomposition groups of  $Y_1$  and  $Y_2$ , respectively. The proof of [4, Proposition 5] implies that one of the following statements holds: (i) if  $D_{Y_1}$  and  $D_{Y_2}$  are not equal to  $G$ , then  $f(y)$  is a smooth point; (ii) if  $D_{Y_1} = D_{Y_2} = G$ , then  $f(y)$  is a node.

Next, we prove that the case (i) does not happen. If  $D_{Y_1}$  and  $D_{Y_2}$  are not equal to  $G$ , then, for each  $\tau \in G \setminus D_{Y_1}$  (resp.  $\tau \in G \setminus D_{Y_2}$ ),  $\tau(Y_1) = Y_2$  and  $\tau(Y_2) = Y_1$ . Thus, we obtain that  $D := D_1 = D_2$ . Moreover,  $D$  is a normal subgroup of  $G$ . By replacing  $I_y$  by  $I_y/D$  and  $Y$  by  $Y/D$  and applying the case (ii), we may assume that  $D$  is trivial. Then  $f_s$  is étale at the generic points of  $Y_1$  and  $Y_2$ . Consider the local morphism  $f_y : \text{Spec } \mathcal{O}_{Y, y} \rightarrow \text{Spec } \mathcal{O}_{X, f(y)}$  induced by  $f$ . Since  $f_y$  is étale at all the points of  $\text{Spec } \mathcal{O}_{Y, y}$  corresponding

to the prime ideals of  $\mathcal{O}_{Y,y}$  of height 1, the Zariski-Nagata purity theorem implies that  $f_y$  is étale. This means that if  $f(y)$  is a smooth point,  $y$  is a smooth point too. This is a contradiction. We complete the proof of the lemma.  $\square$

**Remark 2.1.1.** The lemma also has been treated by Raynaud (cf. [5, Lemme 6.3.5]).

**Definition 2.2.** Let  $f : Y \rightarrow X$  be a  $\mathbb{Z}/p\mathbb{Z}$ -stable covering over  $S$ ,  $v$  an element of  $v(\Gamma_{X_s})$ , and  $X_v$  the irreducible component of  $X_s$  corresponding to  $v$ . Write  $\beta_{X_v}$  for the generic point of  $X_v$  and  $k(\beta_{X_v})$  for the residue field of  $\beta_{X_v}$ . We shall call  $f$  a  **$v$ -wildly ramified covering** if there exists a point  $\beta_{Y_v} \in f_s^{-1}(\beta_{X_v})$  such that the extension of residue fields  $k(\beta_{Y_v})/k(\beta_{X_v})$  induced by  $f$  is purely inseparable. We shall call  $f$  a **wildly ramified covering** if there exists an element  $v \in v(\Gamma_{X_s})$  such that  $f$  is a  $v$ -wildly ramified covering.

Next, we prove our main theorem.

**Theorem 2.3.** *Suppose that  $\text{char}(K) = \text{char}(k) = p > 0$ .*

(i) *Suppose that  $X$  is smooth over  $S$ ,  $X_{\bar{\eta}}$  cannot be defined over an algebraic closure of  $\mathbb{F}_p$ , and  $X_s$  can be defined over an algebraic closure of  $\mathbb{F}_p$ . Then  $X^{\text{ver}}$  is not empty.*

(ii) *Suppose that  $X_s$  is an irreducible singular stable curve over  $s$ . Then  $X^{\text{ver}}$  is not empty.*

*Proof.* First, let us prove (i). Proposition 1.4 (i) implies that, by replacing  $X$  by a finite étale covering of  $X$ , we may assume that  $\sigma(X_{\bar{\eta}}) - \sigma(X_s) > 0$ .

For any geometrically connected Galois étale covering  $V_{\eta} \rightarrow X_{\eta}$  whose Galois group is isomorphic to  $\mathbb{Z}/p\mathbb{Z}$ , by replacing  $S$  by a finite extension of  $S$ , we may assume that  $V_{\eta}$  admits a stable model  $V$  over  $S$ . Suppose that  $X^{\text{ver}}$  is empty. Then the  $\mathbb{Z}/p\mathbb{Z}$ -stable covering  $V \rightarrow X$  over  $S$  induced by  $V_{\eta} \rightarrow X_{\eta}$  is a finite morphism, and Lemma 2.1 implies that the special fiber  $V_s$  of  $V$  is a smooth curve over  $s$ . Write  $\beta_{V_s}$  for the generic point of  $V_s$  and  $I_{V_s} \subseteq \mathbb{Z}/p\mathbb{Z}$  for the inertia group of  $\beta_{V_s}$ . By applying [4, Proposition 5], we obtain that the quotient  $V/I_{V_s}$  is a smooth curve over  $S$ . Since  $I_{V_s}$  is the inertia group of  $\beta_{V_s}$ , the quotient morphism  $V_s \rightarrow V_s/I_{V_s}$  is a radical morphism. Moreover, since the natural morphism  $V_s/I_{V_s} \rightarrow (V/I_{V_s})_s := V/I_{V_s} \times_S s$  is a radical morphism, the natural morphism of special fibers  $V_s \rightarrow (V/I_{V_s})_s$  induced by the quotient  $V \rightarrow V/I_{V_s}$  is a radical morphism. Then we have  $g_{V_s} = g_{V_{\bar{\eta}}} > g_{V_{\bar{\eta}}/I_{V_s}} = g_{V_s/I_{V_s}} = g_{V_s}$ , where  $g_{(-)}$  denotes the genus of  $(-)$ . We obtain that  $I_{V_s}$  is trivial. Thus,  $V_s$  is an étale covering over  $X_s$ . This means that  $\sigma(X_{\bar{\eta}}) - \sigma(X_s) = 0$ . This is a contradiction. We complete the proof of (i).

Next, we start to prove (ii). By replacing  $S$  by a finite extension of  $S$ , there exist a finite group  $G$  and a  $G$ -stable covering  $d : X' \rightarrow X$  over  $S$  such that (i) the morphism  $d_s : X'_s \rightarrow X_s$  induced by  $d$  on the special fibers is an admissible covering; (ii) each irreducible component of  $X'_s$  is smooth over  $s$ ; (iii)  $\sigma(X'_{\bar{\eta}}) - \sigma(X'_s) > 0$  (cf. Proposition 1.4 (ii)), where  $X'_{\bar{\eta}} := X' \times_S \bar{\eta}$ . In order to prove (ii), from now on, we suppose that  $X^{\text{ver}} = \emptyset$ .

By replacing  $S$  by a finite extension of  $S$  again, we may assume that each Galois étale covering of  $X'_\eta := X' \times_S \eta$  over  $\eta$ , whose Galois group is isomorphic to  $\mathbb{Z}/p\mathbb{Z}$ , admits a stable model over  $S$ .

**Claim 1:** There exists a  $\mathbb{Z}/p\mathbb{Z}$ -stable covering of  $f : Y \rightarrow X'$  such that  $f$  is a wildly ramified covering.

Let us prove Claim 1. If Claim 1 is not true, by Definition 6, we have that, for any  $\mathbb{Z}/p\mathbb{Z}$ -stable covering  $Z \rightarrow X'$ , the morphism of the special fibers  $Z_s \rightarrow X'_s$  induced by  $Z \rightarrow X'$  is generically étale. Then Remark 1.3.2 implies that  $Z_s \rightarrow X'_s$  is an admissible covering. This means that  $\sigma(X'_{\bar{\eta}}) = \sigma(X'_s)$ . Thus, we obtain a contradiction. This completes the proof of Claim 1.

Since  $g_{Y_s} > g_{X'_s}$ , we obtain that  $f_s : Y_s \rightarrow X'_s$  is not a radicial morphism. Write  $\Gamma_{X'_s}$  for the dual graph of  $X'_s$ ,  $v(\Gamma_{X'_s})$  for the set of the vertices of  $\Gamma_{X'_s}$ , and  $e(\Gamma_{X'_s})$  for the set of the edges of  $\Gamma_{X'_s}$ . Thus, there exist two vertices  $w_1$  and  $w_2$  of  $v(\Gamma_{X'_s})$  which are linked by an edge  $e \in e(\Gamma_{X'_s})$  such that  $f$  is a  $w_1$ -wildly ramified covering, and  $f$  is not a  $w_2$ -wildly ramified covering. Moreover, we have the following claim.

**Claim 2:** There exists a  $\mathbb{Z}/p\mathbb{Z}$ -stable covering  $g : Z \rightarrow X'$  over  $S$  such that  $g$  is a  $w_2$ -wildly ramified covering, and  $g$  is not a  $w_1$ -wildly ramified covering.

Let us prove Claim 2. Let  $v, v' \in v(\Gamma_{X'_s})$  be two vertices which are linked by an edge of  $\Gamma_{X'_s}$ . Set  $v \rightsquigarrow v'$  if each  $v$ -wildly ramified coverings is a  $v'$ -wildly ramified coverings. We define a subgraph  $\Gamma_v$  of  $\Gamma_{X'_s}$  associated to  $v \in v(\Gamma_{X'_s})$  to be the maximal subgraph of  $\Gamma_{X'_s}$  such that the following conditions hold: (i) A vertex  $v'' \in v(\Gamma_{X'_s})$  is a vertex of  $\Gamma_v$  if there is a path  $v_0 e_{0,1} v_1 e_{1,2} \dots v_{n-1} e_{n-1,n} v_n$  in  $\Gamma_{X'_s}$ , where  $v_i \in v(\Gamma_{X'_s})$  and  $e_{i,i+1} \in e(\Gamma_{X'_s})$  for  $i = 0, \dots, n-1$ , such that (a)  $v_0 = v$  and  $v_n = v''$ ; (b)  $v_i$  and  $v_{i+1}$  are linked by  $e_{i,i+1}$ ; (c)  $v_0 \rightsquigarrow v_1, \dots, v_{n-1} \rightsquigarrow v_n$ ; (ii) for any edge  $e \in e(\Gamma_{X'_s})$ , write  $v_1^e$  and  $v_2^e$  for the vertices of  $\Gamma_{X'_s}$  which are abutted by  $e$ ; an edge  $e \in e(\Gamma_{X'_s})$  is an edge of  $\Gamma_v$  if  $v_1^e$  and  $v_2^e$  are vertices of  $\Gamma_v$ . If Claim 2 is not true, by the definition of  $\Gamma_{w_2}$ , we have  $w_1 \in \Gamma_{w_2}$ . Then we obtain  $\Gamma_{w_1} \subseteq \Gamma_{w_2}$ . On the other

hand, since the  $v(\Gamma_{X'_s})$  is transitive under the action of  $G$ , we have  $\Gamma_{w_1} = \Gamma_{w_2}$ . In particular, we have  $v_1 \rightsquigarrow v_2$ . This contradicts Claim 1, so Claim 2 follows.

Write  $Y_\eta$  (resp.  $Y_s, Z_\eta, Z_s$ ) for the generic fiber of  $Y$  (resp. the special fiber  $Y$ , the generic fiber of  $Z$ , the special fiber of  $Z$ ). By replacing  $S$  by a finite extension of  $S$ , we may assume that  $W_\eta := Y_\eta \times_{X'_\eta} Z_\eta$  admits a stable model  $W$  over  $S$ . Write  $W_s$  for the special fiber  $W$ . Then the natural projections  $W_\eta \rightarrow Y_\eta$  and  $W_\eta \rightarrow Z_\eta$  over  $X'_\eta$  induce two morphisms of stable curves  $W \rightarrow Y$  and  $W \rightarrow Z$  over  $X'$ , respectively. Write  $T$  for the fiber product  $Y \times_{X'} Z$ ,  $T_\eta$  (resp.  $T_s$ ) for the generic fiber of  $T$  (resp. the special fiber of  $T$ ). Then we obtain a natural morphism  $n : W \rightarrow T$  over  $X'$  induced by the morphisms  $W \rightarrow Y$  and  $W \rightarrow Z$ . Write  $h$  for the morphism  $W \rightarrow X'$  induced by the natural morphism  $W_\eta \rightarrow X'_\eta$  and  $h'$  for the natural morphism  $T \rightarrow X'$ . Note that we have  $h = h' \circ n$ . Since we assume that  $X^{\text{ver}} = \emptyset$ ,  $h$  is a finite morphism. Thus,  $n$  is a finite morphism too. This means that  $W$  is equal to the normalization of  $T$ .

Write  $X'_{w_1}$  (resp.  $X'_{w_2}$ ) for the irreducible component of  $X'_s$  corresponding to  $w_1$  (resp.  $w_2$ ),  $Y_1$  (resp.  $Y_2$ ) for the closed subscheme  $Y \times_{X'} X'_{w_1} \subset Y_s$  (resp.  $Y \times_{X'} X'_{w_2} \subset Y_s$ ),  $Z_1$  (resp.  $Z_2$ ) for the closed subscheme  $Z \times_{X'} X'_{w_1} \subset Z_s$  (resp.  $Z \times_{X'} X'_{w_2} \subset Z_s$ ),  $T_1$  (resp.  $T_2$ ) for the closed subscheme  $T \times_{X'} X'_{w_1} \subset T_s$  (resp.  $T \times_{X'} X'_{w_2} \subset T_s$ ), and  $W_1$  (resp.  $W_2$ ) for the closed subscheme  $W \times_{X'} X'_{w_1} \subset W$  (resp.  $W \times_{X'} X'_{w_2} \subset W_s$ ). By the construction of  $Y$  and  $Z$ , we have  $T \rightarrow Y$  is étale at the generic point of  $Y_1$ , and  $T \rightarrow Z$  is étale at the generic point of  $Z_2$ . Thus,  $\mathcal{O}_{T, \beta_{T_1}}$  and  $\mathcal{O}_{T, \beta_{T_2}}$  are normal, where  $\beta_{T_1}$  and  $\beta_{T_2}$  denote the respective generic points of  $T_1$  and  $T_2$ . Then  $n|_{W_1} : W_1 \rightarrow T_1$  and  $n|_{W_2} : W_2 \rightarrow T_2$  are birational. Moreover, since  $T_1$  and  $T_2$  are smooth over  $s$ ,  $W_1$  and  $W_2$  are smooth over  $s$  too. Then  $n|_{W_1}$  and  $n|_{W_2}$  are isomorphisms.

Write  $x_e \in X'_{w_1} \cap X'_{w_2}$  for the node corresponding to  $e$  and  $u_e$  for the unique closed point of  $W$  such that  $h(u_e) = x_e$ . Write  $\widehat{X}'_{w_1} := \overline{\{\widehat{\beta}_{X'_{w_1}}\}}$  and  $\widehat{X}'_{w_2} := \overline{\{\widehat{\beta}_{X'_{w_2}}\}}$  for the irreducible components of  $\text{Spec } \widehat{\mathcal{O}}_{X'_s, x_e}$ , respectively, where  $\widehat{\beta}_{X'_{w_1}}$  and  $\widehat{\beta}_{X'_{w_2}}$  denote the generic points of  $\text{Spec } \widehat{\mathcal{O}}_{X'_s, x_e}$ . Note that since  $X'_{w_1}$  and  $X'_{w_2}$  are smooth over  $s$ ,  $\mathcal{O}_{\widehat{X}'_{w_1}}$  and  $\mathcal{O}_{\widehat{X}'_{w_2}}$  are discrete valuation rings. Write  $\widehat{h}_s$  for the morphism  $\text{Spec } \widehat{\mathcal{O}}_{W_s, u_e} \rightarrow \text{Spec } \widehat{\mathcal{O}}_{X'_s, x_e}$  induced by  $h$ ,  $\widehat{\beta}_{W_1} := (\widehat{h}_s)^{-1}(\widehat{\beta}_{X'_{w_1}})$  and  $\widehat{\beta}_{W_2} := (\widehat{h}_s)^{-1}(\widehat{\beta}_{X'_{w_2}})$  for the generic points of the irreducible components of  $\text{Spec } \widehat{\mathcal{O}}_{W_s, u_e}$ ,  $k(\widehat{\beta}_{W_1})$  and  $k(\widehat{\beta}_{W_2})$  for the residue fields of  $\widehat{\beta}_{W_1}$  and  $\widehat{\beta}_{W_2}$ , respectively. Write  $\widehat{W}_1^s$  and  $\widehat{W}_2^s$  for the respective integral closures of  $\widehat{X}'_{w_1}$  and  $\widehat{X}'_{w_2}$  in  $k(\widehat{\beta}_{W_1})^s$  and  $k(\widehat{\beta}_{W_2})^s$ , where  $k(\widehat{\beta}_{W_1})^s$  and

$k(\widehat{\beta}_{W_2})^s$  denote the respective maximal separable subextension of  $k(\widehat{\beta}_{X'_{w_1}})$  and  $k(\widehat{\beta}_{X'_{w_2}})$  in  $k(\widehat{\beta}_{W_1})$  and  $k(\widehat{\beta}_{W_2})$ . Note that  $\mathcal{O}_{\widehat{W}_1^s}$  and  $\mathcal{O}_{\widehat{W}_2^s}$  are discrete valuation rings.

**Claim 3:** The morphism  $\mathcal{O}_{\widehat{X}'_{w_1}} \rightarrow \mathcal{O}_{\widehat{W}_1^s}$  (resp.  $\mathcal{O}_{\widehat{X}'_{w_2}} \rightarrow \mathcal{O}_{\widehat{W}_2^s}$ ) induced by the natural morphism  $\widehat{W}_1^s \rightarrow \widehat{X}'_{w_1}$  (resp.  $\widehat{W}_2^s \rightarrow \widehat{X}'_{w_2}$ ) is a wildly ramified extension.

Let us prove Claim 3. Write  $\beta_{X'_{w_1}}$  (resp.  $\beta_{X'_{w_2}}$ ) for the generic point of the irreducible component  $X'_{w_1}$  (resp.  $X'_{w_2}$ ). Write  $t_e \in T_1 \cap T_2$  (resp.  $\beta_{T_1}, \beta_{T_2}$ ) for  $n(u_e)$  (resp. the generic point of  $T_1$ , the generic point of  $T_2$ ). We have

$$T_1 \rightarrow T_1^s \rightarrow X'_{w_1}$$

and

$$T_2 \rightarrow T_2^s \rightarrow X'_{w_2},$$

where  $T_1^s$  and  $T_2^s$  denote the smooth projective curves over  $s$  whose function fields are the maximal separable subextensions of  $k(\beta_{T_1})/k(\beta_{X'_{w_1}})$  and  $k(\beta_{T_2})/k(\beta_{X'_{w_2}})$ , respectively. Then by the construction of  $T$ , we have  $T_1^s$  and  $T_2^s$  are isomorphic to  $Z_1$  and  $Y_2$ , respectively. Thus,  $T_1^s \rightarrow X_{w_1}$  and  $T_2^s \rightarrow X_{w_2}$  are wildly ramified at the closed points  $(T_1 \rightarrow T_1^s)(t_e)$  and  $(T_2 \rightarrow T_2^s)(t_e)$ , respectively. Since  $n|_{W_1}$  and  $n|_{W_2}$  are isomorphisms, we have  $\widehat{W}_1^s \cong \text{Spec } \widehat{\mathcal{O}}_{T_1^s, t_e}$  and  $\widehat{W}_2^s \cong \text{Spec } \widehat{\mathcal{O}}_{T_2^s, t_e}$ . Then Claim 3 follows.

Then Proposition 1.6 and Claim 3 imply that  $x_e$  is a vertical point. Thus,  $X_s^{\text{ver}}$  is not empty. This is a contradiction. Then we complete the proof of the theorem.  $\square$

**Remark 2.3.1.** Suppose that  $\text{char}(K) = 0$ . If  $k$  is an algebraic closure of  $\mathbb{F}_p$ , A. Tamagawa proved that  $X^{\text{ver}}$  is equal to the set of closed points of  $X$ . If  $k$  is an arbitrary algebraically closed field of characteristic  $p$ , we may prove that  $X^{\text{ver}}$  is dense in  $X_s$  and contains all the nodes of  $X_s$  (see Appendix of the present paper).

**Theorem 2.4.** *Let  $G$  be a finite group,  $f : Y \rightarrow X$  a  $G$ -stable covering over  $S$ , and  $f_s : Y_s \rightarrow X_s$  the morphism induced by  $f$  on the special fibers.*

(i) *Suppose that  $X_s$  is smooth over  $s$ . Then  $f_s$  is a finite morphism if and only if  $f_s$  is an étale covering.*

(ii) *Suppose that  $X_s$  is irreducible, and that  $G$  is a solvable group. Then  $f_s$  is a finite morphism if and only if  $f_s$  is an admissible covering.*

*Proof.* First, let us prove (i). We only need to prove that, if  $f_s$  is a finite morphism, then  $f_s$  is an étale covering. By Lemma 2.1, we obtain that  $Y$  is smooth over  $S$ . Write  $I_{Y_s}$  for the inertia group of the special fiber  $Y_s$ . Thus, the finite morphism  $Y/I_{Y_s} \rightarrow X$  is étale. Since  $Y = Y_s/I_{Y_s} \rightarrow (Y/I_{Y_s})_s := Y/I_{Y_s} \times_S s$  is a radicial morphism, thus the genus of  $Y_s$  and  $(Y/I_{Y_s})_s$  are equal. On the other hand, the genus of  $(Y/I_{Y_s}) \times_S \eta$  is equal to the genus of  $(Y/I_{Y_s})_s$ , and the genus of  $Y_\eta$  is larger than  $(Y/I_{Y_s}) \times_S \eta$ . Thus, we have  $I_{Y_s} = \{1\}$ . Then  $f_s$  is étale.

Next, let us prove (ii). We only need to prove that, if  $f_s$  is a finite morphism, then  $f_s$  is an admissible covering.

If  $(\#G, p) = 1$ , where  $\#(-)$  denotes the cardinality of  $(-)$ , then the specialization theorem of prime to  $p$  admissible fundamental groups (cf. [12, Théorème 2.2]) implies that  $f_s$  is an admissible covering. Thus, we may assume that  $p \mid \#G$ .

Suppose that  $\#G = p$ . For each irreducible component  $Y_v$  of special fiber  $Y_s$ , write  $I_{Y_v} \subseteq G$  for the inertia group of  $Y_v$ . Since  $G$  is abelian, and  $X_s$  is irreducible, we obtain that  $I_{Y_v}$  does not depend on the choices of  $Y_v$ . Then we use the notation  $I$  to denote  $I_{Y_v}$ . Similar arguments to the arguments given in the proof of (i) imply that  $g_{Y_s} = g_{Y_s/I} = g_{Y_\eta/I} = g_{Y_\eta}$ . We have  $I = \{1\}$ . Then Remark 1.3.2 implies that  $f_s$  is an admissible covering.

Suppose that  $\#G > p$ . Let

$$\{1\} \subset A := G^{(m)} \subset G^{(m-1)} \subset \dots \subset G$$

be the derived series of  $G$ . Write  $A_p$  for the Sylow  $p$ -subgroup of  $A$  and  $G'$  for  $G/A_p$ . Then we obtain a  $A_p$ -stable covering  $f^{A_p} : Y \rightarrow Y/A_p$  over  $S$  and a  $G'$ -stable covering  $f^{G'} : Y/A_p \rightarrow X$  over  $S$  such that  $f = f^{G'} \circ f^{A_p}$ . By induction, we obtain that  $f_s^{G'}$  is an admissible covering. Let us prove that  $f_s^{A_p}$  is an admissible covering.

Let  $Y_v \subseteq Y_s$  be an irreducible component and  $I_{Y_v} \subseteq A_p$  the inertia group of  $Y_v$ . For each irreducible component  $Y_{v'} \subseteq Y_s$ , write  $I_{Y_{v'}} \subseteq A_p$  for the inertia group of  $Y_{v'}$ . Since  $X_s$  is irreducible, we have  $\#I_{Y_{v'}} = \#I_{Y_v}$ . If  $I_{Y_v}$  is trivial, then Remark 1.3.2 implies that  $f_s^{A_p}$  is an admissible covering. Thus, we may assume that  $I_{Y_v}$  is not trivial. Moreover, we have the following Claim.

**Claim:** For each irreducible component  $Y_{v'} \subseteq Y_s$ , we have  $I_{Y_{v'}} = I_{Y_v}$ .

Let us prove the claim. Write  $A'_p$  for the quotient group  $A_p/I_{Y_v}$ . Consider the quotient of  $Y$  by  $I_{Y_v}$ , we obtain two morphisms of stable curves  $f^{I_{Y_v}} : Y \rightarrow Y/I_{Y_v}$  and  $f^{A'_p} : Y/I_{Y_v} \rightarrow Y/A_p$  such that  $f^{A_p} = f^{A'_p} \circ f^{I_{Y_v}}$ .

If  $Y_s$  is irreducible, then Claim is trivial. Thus, we may assume that  $Y_s$  is not irreducible. Let  $Y_w (\neq Y_v)$  be an irreducible component such that  $Y_v \cap Y_w \neq \emptyset$ ,  $I_{Y_w} \subseteq A_p$  the inertia subgroup of  $Y_w$ , and  $y \in Y_v \cap Y_w$  a node of  $Y_s$ . Write  $Y_v^{I_{Y_v}}$  (resp.  $Y_w^{I_{Y_v}}$ ,  $y_{I_{Y_v}}$ ,  $Y_v^{A_p}$ ,  $Y_w^{A_p}$ ,  $y_{A_p}$ ) for  $f^{I_{Y_v}}(Y_v)$  (resp.  $f^{I_{Y_v}}(Y_w)$ ,  $f^{I_{Y_v}}(y)$ ,  $f^{A_p}(Y_v)$ ,  $f^{A_p}(Y_w)$ ,  $f^{A_p}(y)$ ). By the construction, we obtain that  $f_s^{I_{Y_v}}|_{Y_v}$  is radicial, and  $f_s^{A_p'}|_{Y_v^{I_{Y_v}}}$  is generically étale.

If  $I_{Y_v} \neq I_{Y_w}$ , then  $f_s^{A_p'}|_{Y_w^{I_{Y_v}}}$  is not generically étale. Thus, the local morphism  $\mathcal{O}_{Y_v^{A_p}, y_{A_p}} \rightarrow \mathcal{O}_{Y_w^{I_{Y_v}}, y_{I_{Y_v}}}$  is a wildly ramified extension. Write  $K(Y_{v_1})^{\text{sep}}$  for the separable closure of function field  $K(Y_v^{A_p})$  in the function field  $K(Y_v)$ . Note that  $K(Y_v)^{\text{sep}}$  is the function field of  $Y_v^{I_{Y_v}}$ . Thus, the morphism of  $\mathcal{O}_{Y_v^{A_p}, y_{A_p}}$  to the integral closure of  $\mathcal{O}_{Y_w^{A_p}, y_{A_p}}$  in  $K(Y_v)^{\text{sep}}$  is a wildly ramified extension.

On the other hand, write  $K(Y_w)^{\text{sep}}$  for the separable closure of function field  $K(Y_w^{A_p})$  in the function field  $K(Y_w)$ . Similar arguments to the arguments given above imply that the morphism of  $\mathcal{O}_{Y_w^{A_p}, y_{A_p}}$  to the integral closure of  $\mathcal{O}_{Y_w^{A_p}, y_{A_p}}$  in  $K(Y_w)^{\text{sep}}$  is a wildly ramified extension. By Proposition 1.6, we have that  $y_{A_p}$  is a vertical point. This contradict to the assumption that  $f_s$  is finite. Thus, we have  $I_{Y_v} = I_{Y_w}$ . Moreover, since  $Y_s$  is connected, similar arguments to the arguments given above imply that, for any irreducible component  $Y_{v'}$ , the inertia subgroup  $I_{Y_{v'}}$  is equal to  $I_{Y_v}$ .

We use the notation  $I$  to denote  $I_{Y_v}$ . Similar arguments to the arguments given in the proof of (i) imply that  $g_{Y_s} = g_{Y_s/I} = g_{Y_{\bar{\eta}}/I} < g_{Y_{\bar{\eta}}}$ . This is a contradiction. Then  $I$  is trivial. Then Remark 1.3.2 implies that  $f_s$  is an admissible covering. □

### 3. APPENDIX

In this subsection, we assume that  $\text{char}(K) = 0$  and  $\text{char}(k) = p > 0$ . Let  $M_{g,r,k}$  be the coarse moduli space of  $\mathcal{M}_{g,r} \times_{\text{Spec } \mathbb{Z}} \text{Spec } k$ . For any point  $x \in M_{g,r,k}$ , write  $\bar{x}$  for a geometric point above  $x$  and  $(C_{\bar{x}}, D_{\bar{x}})$  for a pointed smooth curve corresponding to the point  $\bar{x}$  (well-defined up to isomorphism). Then the isomorphism type of the geometric tame fundamental group  $\pi_1^{\text{tame}}((C_{\bar{x}}, D_{\bar{x}}))$  is independent of the choice of  $\bar{x}$  and  $(C_{\bar{x}}, D_{\bar{x}})$  (and

the implicit base point on  $(C_{\bar{x}}, D_{\bar{x}})$  used to define  $\pi_1^{\text{tame}}((C_{\bar{x}}, D_{\bar{x}}))$ . We have the following result proved by Saïdi and Tamagawa.

**Proposition 3.1.** *Let  $U \subseteq M_{g,r,k}$  a subvariety of positive dimension. Then the geometric tame fundamental group  $\pi_1^{\text{tame}}$  is not constant on  $U$ .*

*Proof.* See [8, Theorem 3.12]. □

By using the idea of the proof of Tamagawa's result (cf. [11, Theorem 0.2 (v)]), we have the following theorem.

**Theorem 3.2.** *The set of vertical points  $X^{\text{ver}}$  contains all the nodes of  $X_s$ , and the closure of  $X^{\text{ver}}$  in  $X_s$  is equal to  $X_s$ .*

*Proof.* By applying [2, Lemma 2.9], we may assume that each irreducible component of  $X_s$  is smooth over  $s$ , and the genus is  $\geq 2$ .

Let  $S^{\text{log}}$  be a log regular scheme whose underlying scheme is  $S$ , and whose log structure is determined by the closed point of  $S$ . Then there is a natural morphism  $S^{\text{log}} \rightarrow \overline{\mathcal{M}}_{g_X,0}^{\text{log}}$  determined by  $X \rightarrow S$ . Thus, we obtain a stable log curve  $X^{\text{log}} := \overline{\mathcal{M}}_{g_X,1}^{\text{log}} \times_{\overline{\mathcal{M}}_{g_X,0}^{\text{log}}} S^{\text{log}}$  whose underlying scheme is  $X$ .

Let  $x$  be a closed point of  $X_s$ . Write  $X_v$  for an irreducible component which contains  $x$ . We define a pointed smooth curve  $U_v^\bullet$  of type  $(g_v, r_v)$  to be  $(X_v, X_v \cap X_s^{\text{sing}})$ , where  $X_s^{\text{sing}}$  denotes the set of nodes of  $X_s$ . Then we obtain a morphism  $s \rightarrow \overline{\mathcal{M}}_{g_v, r_v}^{\text{log}}$  determined by  $U_v^\bullet \rightarrow s$  and a stable log curve  $X_v^{\text{log}} := \overline{\mathcal{M}}_{g_v, r_v+1}^{\text{log}} \times_{\overline{\mathcal{M}}_{g_v, r_v}^{\text{log}}} s$  over  $s$  whose underlying scheme is  $X_v$ .

Moreover, we consider the 2<sup>nd</sup> log configuration space  $Y^{\text{log}} := \overline{\mathcal{M}}_{g_X,2}^{\text{log}} \times_{\overline{\mathcal{M}}_{g_X,0}^{\text{log}}} S^{\text{log}}$  (resp.  $W_v^{\text{log}} := \overline{\mathcal{M}}_{g_v, r_v+2}^{\text{log}} \times_{\overline{\mathcal{M}}_{g_v, r_v}^{\text{log}}} s$ ) associated to  $X \rightarrow S$  (resp.  $U_v^\bullet \rightarrow s$ ). Note that there is a natural morphism  $Y^{\text{log}} \rightarrow X^{\text{log}}$  (resp.  $W_v^{\text{log}} \rightarrow X_v^{\text{log}}$ ) induced by the natural morphism of log stacks  $\overline{\mathcal{M}}_{g_X,2}^{\text{log}} \rightarrow \overline{\mathcal{M}}_{g_X,1}^{\text{log}}$  (resp.  $\overline{\mathcal{M}}_{g_v, r_v+2}^{\text{log}} \rightarrow \overline{\mathcal{M}}_{g_v, r_v+1}^{\text{log}}$ ) given by forgetting the last marked point. Write  $Y$  (resp.  $W_v$ ) for the underlying scheme of  $Y^{\text{log}}$  (resp.  $W_v^{\text{log}}$ ). Write  $\beta_X$  (resp.  $\beta_v, \widehat{\beta}_X, \widehat{\beta}_v$ ) for the generic point of  $X$  (resp. the generic point of  $X_v$ , the generic point of  $\text{Spec } \widehat{\mathcal{O}}_{X, \beta_v}$ , the generic point of  $\text{Spec } \widehat{\mathcal{O}}_{X_v, x}$ ).

Write  $\beta_v^{\text{log}}$  for the log scheme whose underlying scheme is  $\beta_v$ , and whose log structure is the pulling-back log structure of  $X^{\text{log}}$  via the natural morphism  $\beta_v \rightarrow X$ . Since the restricting log structure of  $X_v^{\text{log}}$  on the generic point  $\beta_v$  is trivial, we have a natural morphism  $\beta_v \rightarrow X_v^{\text{log}}$ . Write  $(Z'_v)^{\text{log}}$  for  $W_v^{\text{log}} \times_{X_v^{\text{log}}} \beta_v$ . Write  $(Z''_v)^{\text{log}}$  for the stable log curve whose underlying

scheme is  $W_v \times_{X_v} \beta_v$ , and whose log structure is the pulling-back log structure of  $Y^{\log} \times_{X^{\log}} \beta_v^{\log}$  via the natural morphism  $W_v \times_{X_v} \beta_v \rightarrow Y^{\log} \times_{X^{\log}} \beta_v^{\log}$ .

Let  $(-)$  be a point (resp. a log point). We use the notation  $\widetilde{(-)}$  (resp.  $\overline{(-)}$ ) to denote a geometric point (resp. log geometric point) associated to  $(-)$ . By the specialization theorem of log étale fundamental groups (cf. [12, Théorème 2.2]), we obtain the following commutative diagram of fundamental groups. Moreover, all seven rows in the commutative diagram are exact.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(Y \times_X \overline{\beta}_X) & \longrightarrow & \pi_1(Y \times_X X_\eta) & \longrightarrow & \pi_1(X_\eta) & \longrightarrow & 1 \\
& & a_1 \uparrow & & b_1 \uparrow & & c_1 \uparrow & & \\
1 & \longrightarrow & \pi_1(Y \times_X \overline{\beta}_X) & \longrightarrow & \pi_1(Y \times_X \beta_X) & \longrightarrow & \pi_1(\beta_X) & \longrightarrow & 1 \\
& & a_2 \uparrow & & b_2 \uparrow & & c_2 \uparrow & & \\
1 & \longrightarrow & \pi_1(Y \times_X \widehat{\overline{\beta}}_X) & \longrightarrow & \pi_1(Y \times_X \widehat{\beta}_X) & \longrightarrow & \pi_1(\widehat{\beta}_X) & \longrightarrow & 1 \\
& & a_3 \downarrow & & b_3 \downarrow & & c_3 \downarrow & & \\
1 & \longrightarrow & \pi_1(Y^{\log} \times_{X^{\log}} \widetilde{\beta}_v^{\log}) & \longrightarrow & \pi_1(Y^{\log} \times_{X^{\log}} \beta_v^{\log}) & \longrightarrow & \pi_1(\beta_v^{\log}) & \longrightarrow & 1 \\
& & a_4 \uparrow & & b_4 \uparrow & & \parallel & & \\
1 & \longrightarrow & \pi_1(W_v^{\log} \times_{(Z'_v)^{\log}} \widetilde{\beta}_v^{\log}) & \longrightarrow & \pi_1(W_v^{\log} \times_{(Z'_v)^{\log}} \beta_v^{\log}) & \longrightarrow & \pi_1(\beta_v^{\log}) & \longrightarrow & 1 \\
& & a_5 \downarrow & & b_5 \downarrow & & c_5 \downarrow & & \\
1 & \longrightarrow & \pi_1((Z'_v)^{\log} \times_{\beta_v} \overline{\beta}_v) & \longrightarrow & \pi_1((Z'_v)^{\log}) & \longrightarrow & \pi_1(\beta_v) & \longrightarrow & 1 \\
& & a_6 \uparrow & & b_6 \uparrow & & c_6 \uparrow & & \\
1 & \longrightarrow & \pi_1((Z'_v)^{\log} \times_{\beta_v} \widehat{\overline{\beta}}_v) & \longrightarrow & \pi_1((Z'_v)^{\log} \times_{\beta_v} \widehat{\beta}_v) & \longrightarrow & \pi_1(\widehat{\beta}_v) & \longrightarrow & 1.
\end{array}$$

Note that the definition of geometric (log) étale fundamental groups implies that  $a_1, a_2, a_5$ , and  $a_6$  are isomorphisms (resp.  $b_1, b_2, b_5, c_1, c_2, c_3$ , and  $c_5$  are surjections,  $b_2, b_6, c_2$ , and  $c_6$  are injections). Since there is a natural morphism  $\text{Spec } \widehat{\mathcal{O}}_{X, \beta_v} \rightarrow X$ , we obtain a log scheme  $T^{\log}$  whose underlying scheme is  $\text{Spec } \widehat{\mathcal{O}}_{X, \beta_v}$ , and whose log structure is the pulling-back log structure of  $X^{\log}$ . Then by considering  $Y^{\log} \times_{X^{\log}} T^{\log}$  and the specialization theorem of log étale fundamental groups, we obtain that  $a_3, b_3$  are surjections. Moreover, [11, Proposition 2.5] and [11, Example 2.10] imply that  $a_4$  is an injection. Then we obtain  $b_4$  is an injection too.

For each  $i = 1, \dots, 7$ , write  $1 \rightarrow \Delta_i \rightarrow \Pi_i \rightarrow G_i \rightarrow 1$  for the  $i^{\text{th}}$  row of the above commutative diagram,  $\rho_i : G_i \rightarrow \text{Out}(\Delta_i)$  for the outer representation, and  $\text{Im}_i$  for the image of  $\rho_i$ . Then by [11, Remark 2.3], we obtain

$$\text{Im}_1 = \text{Im}_2 \leftrightarrow \text{Im}_3 \rightarrow \text{Im}_4 \twoheadrightarrow \text{Im}_5 = \text{Im}_6 \leftrightarrow \text{Im}_7.$$

Moreover, [11, Lemma 5.2] implies that  $\text{Im}_3 \rightarrow \text{Im}_4$  is a surjection. Write  $D_{\widehat{\beta}_X}$  for the image of  $c_1 \circ c_2$ ,  $I_{\beta_v}$  for the kernel of  $c_5 \circ c_3$ , and  $I_x$  for  $\pi_1(\widehat{\beta}_v)$ . Then we obtain a morphism  $\pi_1(\widehat{\beta}_X)/I_{\beta_v} \rightarrow \text{Im}_6$ . On the other hand, since  $\text{Im}_1 \leftrightarrow \text{Im}_3$ , we have that  $\rho_3(\pi_1(\widehat{\beta}_X)) = \rho_1(D_{\widehat{\beta}_X})$ . Thus, we obtain a morphism  $D_{\widehat{\beta}_X}/c_1 \circ c_2(I_{\beta_v}) \rightarrow \text{Im}_6$ . Furthermore, we have a commutative diagram as follows:

$$\begin{array}{ccccc} \pi_1(\widehat{\beta}_X) & \longrightarrow & D_{\widehat{\beta}_X}/c_1 \circ c_2(I_{\beta_v}) & \longrightarrow & \text{Im}_6 \\ \downarrow & & \uparrow & & \parallel \\ \pi_1(\beta_v) & \xlongequal{\quad} & \pi_1(\beta_v) & \longrightarrow & \text{Im}_6 \\ & & \uparrow & & \uparrow \\ & & I_x & \longrightarrow & \text{Im}_7. \end{array}$$

Write  $E_x^{\log}$  for the log scheme whose underlying scheme is  $\text{Spec } \widehat{\mathcal{O}}_{X_v, x}$ , and whose log structure is the pulling-back log structure of  $X_v^{\log}$  induced by the natural morphism  $\text{Spec } \widehat{\mathcal{O}}_{X_v, x} \rightarrow X_v$ . Write  $x^{\log}$  for the log scheme obtained by restricting  $E_x^{\log}$  to the closed point  $x \in \text{Spec } \widehat{\mathcal{O}}_{X_v, x}$ , and  $\widetilde{x}^{\log}$  for a geometric log point associated to  $x^{\log}$ . Consider a stable log curve  $W_v^{\log} \times_{X_v^{\log}} E_x^{\log} \rightarrow E_x^{\log}$  over  $E_x^{\log}$ . Then we obtain the specialization map

$$sp_x : \pi_1(W_v^{\log} \times_{X_v^{\log}} \widetilde{\beta}_v^{\log}) \rightarrow \pi_1(W_v^{\log} \times_{X_v^{\log}} \widetilde{x}^{\log}).$$

Suppose that  $sp_x$  is not an isomorphism. Thus, by applying [11, Proposition 4.1 (ii)], we have the image of wild inertia subgroup  $I_x^w$  in  $D_{\widehat{\beta}_X}/c_1 \circ c_2(I_{\beta_v})$  is infinite. Then by Proposition 1.6, we have  $x \in X^{\text{ver}}$ .

If  $x$  is a node of  $X_s$ , then the underlying scheme of  $W_v^{\log} \times_{X_v^{\log}} \widetilde{x}^{\log}$  is a singular curve. Then Proposition 1.4 implies that  $sp_x$  is not an isomorphism. This means that  $X^{\text{ver}}$  contains all the nodes of  $X_s$ . If  $x$  is a smooth closed point of  $X_s$ , then the underlying scheme of  $W_v^{\log} \times_{X_v^{\log}} \widetilde{x}^{\log}$  is a smooth curve over  $x$ . By applying Proposition 3.1, we obtain that the closure of  $X^{\text{ver}}$  in  $X_s$  is equal to  $X_s$ . This completes the proof of the theorem.  $\square$

**Remark 3.2.1.** Note that since the geometric étale fundamental group of  $X_\eta$  is topologically finitely generated, there are only countably many étale

coverings that each may contribute at most finitely many points to  $X^{\text{ver}}$ . Thus,  $X^{\text{ver}}$  is countable.

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