Math. J. Okayama Univ. **60** (2018), 155–164

ARITHMETIC OF POSITIVE INTEGERS HAVING PRIME SUMS OF COMPLEMENTARY DIVISORS

Kenichi Shimizu

Abstract. We study a class of integers called SP numbers (Sum Prime numbers). An SP number is by definition a positive integer d that gives rise to a prime number $(a+b)/\gcd(4, 1+d)$ from every factorization $d = ab$. We also discuss properties of SP numbers in relations with arithmetic of imaginary quadratic fields (least split primes, exponents of ideal class groups). Further we point out that special cases of SP numbers provide the problems of distribution of prime numbers (twin primes, Sophi-Germain primes, quadratic progressions). Finally, we consider the problem whether there exist infinitely many SP numbers.

1. Introduction

The aim of this paper is to investigate a class of integers considered in Shimizu and Goto [6]. Our motivation was to generalize relationship among the well known phenomena that $x^2 + x + 41$ =primes for $x = 0, 1, \cdots, 39$, that the class number of $\mathbb{Q}(\sqrt{-163})$ is one and that 163 is an SP number.

Let d be a positive integer with $4 \nmid d$. Then, it is easy to see that, for every factorization $d = ab$ $(a, b \geq 1)$, the sum $a + b$ is divisible by the greatest common divisor $(4, 1+d)$, that is, $(4, 1+d) = 1, 2$ or 4 according as $d \equiv 2, 1$ or 3 (mod 4). Taking this into consideration, we shall say that d is an *SP number* (Sum Prime number) if $(a+b)/(4, 1+d)$ is a prime number for every factorization $d = ab$.

For example, 30 ($\equiv 2 \bmod 4$) is an SP number: for the factorizations $30 = 1 \cdot 30 = 2 \cdot 15 = 3 \cdot 10 = 5 \cdot 6$, each sum of two divisors is always a prime number, that is, $1 + 30 = 31, 2 + 15 = 17, 3 + 10 = 13$ and $5 + 6 = 11$.

In this paper, we give a sufficient condition for a square-free integer d to be an SP number. Let q_D be the least prime number that splits completely in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ with the discriminant $-D$, where $D = 4d$ or $D = d$ if $d \equiv 1, 2 \pmod 4$ or $d \equiv 3 \pmod 4$, respectively. We show the following theorem.

Theorem 3.1. *Suppose $d \neq 1, 3$. If $q_D > \sqrt{(1+d)/(4, 1+d)}$, then d is an SP number.*

We also study in Section 4 special cases of SP numbers related to prime distribution. For example, an SP number $2p$ with p an odd prime produces

---

Mathematics Subject Classification. Primary 11A41;Secondary 11R11,11R29.
Key words and phrases. SP number, prime number, imaginary quadratic field.

a triple of primes $\{p, p+2, 2p+1\}$ (twin and Sophi-Germain primes), and an SP number p^2 is related to a quadratic progression for an odd prime number p.

Finally in Section 5, we investigate the following conjecture.

Conjecture 5.1. *There exist infinitely many SP numbers.*

For a prime number p, denote by $S(p)$ the set of SP numbers d taking the prime value $p = (a+b)/(4, 1+d)$ for a factorization $d = ab$. Then the above Conjecture 5.1 will be shown to be equivalent to the following conjecture.

Conjecture 5.3. *There exist infinitely many prime numbers p such that $S(p) \neq \emptyset$.*

Recently, topics related to SP numbers are discussed in very interesting papers [1] and [3].

2. Preliminary results

First, let us list the SP numbers less than 500.

The twenty-one SP numbers $\equiv 2 \pmod 4$ are:

2, 6, 10, 22, 30, 42, 58, 70, 78, 82, 102, 130, 190, 210, 310, 330, 358, 382, 442, 462, 478.

The thirty-six SP numbers $\equiv 1 \pmod 4$ are:

5, 9, 13, 21, 25, 33, 37, 57, 61, 73, 85, 93, 105, 121, 133, 145, 157, 165, 177, 193, 205, 213, 217, 253, 273, 277, 313, 345, 357, 361, 385, 393, 397, 421, 445, 457.

The twenty-four SP numbers $\equiv 3 \pmod 4$ are:

7, 11, 19, 27, 43, 51, 67, 75, 91, 115, 123, 147, 163, 187, 211, 235, 267, 283, 331, 355, 403, 427, 435, 451.

Next, we remark some properties of SP numbers with square factors.

Proposition 2.1. *Suppose that d is a positive integer satisfying $d = k^2 m$ for integers $k > 1$ and $m \geq 1$.*

(1) *If $d \equiv 2 \pmod 4$, then d is not an SP number.*

(2) *If $d \equiv 1 \pmod 4$ is an SP number, then $m = 1$ and k is an odd prime number.*

(3) *If $d \equiv 3 \pmod 4$ is an SP number, then $m = 3$ and k is an odd prime number.*

Proof. (1) An integer $k + km = (1+m)k$ is not a prime number since $k > 1$. Thus $d = k^2 m$ is not an SP number.

(2) Since d is an SP number, an integer $(k + km)/2 = k \cdot (1 + m)/2$ is a prime number. By $k > 1$, it follows that k is an odd prime number and $(1 + m)/2 = 1$, namely, $m = 1$.

(3) Since d is an SP number, an integer $(k+km)/4 = k\cdot(1+m)/4$ is a prime number. Thus k is an odd prime number and $(1+m)/4 = 1$, namely, $m = 3$. □

As for square-free SP numbers, we note the following

Proposition 2.2. *Suppose $d > 3$. If $d \equiv 3 \pmod 4$ is a square-free SP number, then $d = 7$ or $d \equiv 3 \pmod 8$.*

Proof. Let $d = ab \equiv 3 \pmod 4$. Then we may assume that $a \equiv 1 \pmod 4$ and $b \equiv 3 \pmod 4$. Put $a = 4k+1$ and $b = 4l+3$, where k and l are non-negative integers satisfying $(k,\ l) \neq (0,\ 0)$.

Since d is an SP number, $(a+b)/4 = k+l+1$ is a prime number. If $k+l+1 = 2$, then $(k,\ l) = (1,\ 0)$ or $(k,\ l) = (0,\ 1)$. If $(k,\ l) = (1,\ 0)$, then $d = 15$, which is not an SP number. If $(k,\ l) = (0,\ 1)$, then $d = 7$, which is an SP number. If $k+l+1$ is an odd prime number, then $k+l$ is even. Since $ab = (4k+1)(4l+3) = 16kl + 4\{2k+(k+l)\} + 3$, we get $d = ab \equiv 3 \pmod 8$. □

3. SP numbers and imaginary quadratic fields

In this section we shall discuss SP numbers in relation with arithmetic of the imaginary quadratic fields. Assume that d is a square-free positive integer throughout this section.

Denote by $\mathbb{Q}(\sqrt{-d})$ the imaginary quadratic field with the discriminant $-D$, where

$$D = \begin{cases} 4d & \text{if } d \equiv 1,\ 2 \pmod 4; \\ d & \text{if } d \equiv 3 \pmod 4. \end{cases} \tag{3.1}$$

Let $\omega = \sqrt{-d}$ or $\omega = (1+\sqrt{-d})/2$ according as $d \equiv 1,\ 2$ or $d \equiv 3 \pmod 4$. Define $f_D(x)$ by

$$f_D(x) := N(x+\omega) = \begin{cases} x^2 + d & \text{if } d \equiv 1,\ 2 \pmod 4; \\ x^2 + x + (1+d)/4 & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

where $N(x+\omega)$ is the norm of $x+\omega$. A prime number p is called a split prime of $\mathbb{Q}(\sqrt{-d})$ if $\left(\frac{-D}{p}\right) = 1$, a ramified prime if $\left(\frac{-D}{p}\right) = 0$ and an inert prime if $\left(\frac{-D}{p}\right) = -1$, where $\left(\frac{-D}{p}\right)$ is the Kronecker symbol. Let q_D be the least split prime of $\mathbb{Q}(\sqrt{-d})$. We give a sufficient condition for d to be an SP number.

Theorem 3.1. *Suppose* $d \neq 1, 3$. *If* $q_D > \sqrt{(1+d)/(4, 1+d)}$, *then* d *is an SP number.*

For the proof of Theorem 3.1 we show the following lemmas.

Lemma 3.2. *The prime divisors of* $f_D(x)$ *are only split primes or ramified primes of* $\mathbb{Q}(\sqrt{-d})$.

Proof. Assume that p is an inert prime and that $p \mid f_D(x)$. Then we have $p \mid (x+\omega)(x+\omega')$, where $x+\omega'$ is the conjugate of $x+\omega$. Since (p) is a prime in $\mathbb{Q}(\sqrt{-d})$, p divides $x+\omega$ or $x+\omega'$, which is a contradiction. □

Lemma 3.3. *Suppose* $d \neq 1, 3$. *If* $d = ab$ $(a \geq 1,\ b \geq 1)$, *then* $(a+b)/(4, 1+d)$ *is divisible only by split primes in* $\mathbb{Q}(\sqrt{-d})$.

Proof. Since $d \neq 1, 3$, it holds that $(a+b)/(4, 1+d) > 1$.

We claim that $(a+b)/(4, 1+d)$ divides $f_D(x)$ for some integer x, because $f_D(a) = a(a+b)$, $f_D(a) = 2a \cdot (a+b)/2$ or $f_D((a-1)/2) = a \cdot (a+b)/4$ according as $d \equiv 2, 1$ or $3 \pmod 4$. Hence the prime divisors of $(a+b)/(4, 1+d)$ are split primes or ramified primes by Lemma 3.2.

Further we show that $(a+b)/(4, 1+d)$ is not divisible by ramified primes. Assume that $(a+b)/(4, 1+d)$ is divisible by a ramified prime p. When $d \equiv 1 \pmod 4$, the ramified prime $p = 2$ does not divide $(a+b)/(4, 1+d) = (a+b)/2$. Thus we can assume that $p \mid d$. If $p \mid a$, then $p \mid b$ by $p \mid \{(a+b)/(4, 1+d)\}$, which is a contradiction since d is a square-free integer by assumption in this section. By the same way $p \mid b$ also derives a contradiction. Thus $(a+b)/(4, 1+d)$ is divisible only by split primes. □

Proof of Theorem 3.1.

Assume that d is not an SP number. Then $(a+b)/(4, 1+d)$ is not a prime number for some factorization $d = ab$. By Lemma 3.3, the prime divisors of $(a+b)/(4, 1+d)$ are only split primes in $\mathbb{Q}(\sqrt{-d})$. Hence we get $q_D^2 \leq (a+b)/(4, 1+d) \leq (1+d)/(4, 1+d)$, and consequently $q_D \leq \sqrt{(1+d)/(4, 1+d)}$. Therefore $q_D > \sqrt{(1+d)/(4, 1+d)}$ implies that d is an SP number. □

Let e_D be the exponent of the class group of $\mathbb{Q}(\sqrt{-d})$, which is the least positive integer n such that $\mathfrak{a}^n$ is principal for any ideal $\mathfrak{a}$ in $\mathbb{Q}(\sqrt{-d})$.

Shimizu and Goto [6] gave the following sufficient condition for a square-free positive integer d to be an SP number.

Theorem 3.4. *Suppose* $d \neq 1, 3$.

(1) *If* $d \equiv 1,\ 2 \pmod 4$, *then* $e_D \leq 2$ *implies that* d *is an SP number.*

(2) *If* $d \equiv 3 \pmod 4$ *and* $(1+d)/4$ *is not a square, then* $e_D \leq 2$ *implies that* d *is an SP number.*

In [5], H. Möller stated the same result as a property of an imaginary quadratic field with $e_D \leq 2$ without the notion of SP numbers. Some conditions for $e_D \leq 2$ are considered in [7] and [8].

Note that only sixty-five imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ with $e_D \leq 2$ are known and that at most one possible such unknown could exist in the range of $|D| > 5460$ (cf.[2] Table 5, [9] Theorem 1).

Example. When $|D| = 5460 = 4 \cdot 1365$, $d = 1365$ is the SP number and gives rise to eight prime numbers by $1365 = 3 \cdot 5 \cdot 7 \cdot 13 \equiv 1 \pmod 4$: $(1+1365)/2 = 683$, $(3+5\cdot 7\cdot 13)/2 = 229$, $(5+3\cdot 7\cdot 13)/2 = 139$, $(7+3\cdot 5\cdot 13)/2 = 101$, $(13+3\cdot 5\cdot 7)/2 = 59$, $(3\cdot 5+7\cdot 13)/2 = 53$, $(3\cdot 7+5\cdot 13)/2 = 43$, $(3\cdot 13+5\cdot 7)/2 = 37$.

We compare the condition $q_D > \sqrt{(1+d)/(4, 1+d)}$ with the condition $e_D \leq 2$.

Theorem 3.5. (cf.[5], [8]) (1) *If* $d \equiv 2 \pmod 4$*, then* $e_D \leq 2$ *if and only if* $q_D > \sqrt{1+d}$.

(2) *If* $d \equiv 1 \pmod 4$*, then* $e_D \leq 2$ *implies* $q_D > \sqrt{(1+d)/2}$.

(3) *If* $d \equiv 3 \pmod 4$*, then* $e_D \leq 2$ *implies* $q_D \geq \sqrt{(1+d)/4}$. *The equality holds if and only if* $(1+d)/4$ *is a square.*

Proof. See [5] (p.184, Lemma 4) and [8] (p.264, Corollary 4.3). □

Theorem 3.5 (1) derives that there exist only finitely many SP numbers $d \equiv 2 \pmod 4$ satisfying $q_D > \sqrt{1+d}$. If $d \equiv 1 \pmod 4$, then the converse of Theorem 3.5 (2) does not hold for $d =$73, 193, 205, 217, 553 and 697, and there exist no other such integers d less than 10^8. Hence, for $d > 697$, we predict that $e_D \leq 2$ is equivalent to $q_D > \sqrt{(1+d)/2}$. If $d \equiv 3 \pmod 4$, then the converse of Theorem 3.5 (3) is unknown at the time of writing this paper, but appears to be true.

Thus it seems that there exist only finitely many imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ satisfying $q_D > \sqrt{(1+d)/(4, 1+d)}$.

In SP numbers listed in Section 2 up to 500, the following square-free SP numbers d do not satisfy $q_D > \sqrt{(1+d)/(4, 1+d)}$:

$d =$82, 310, 358, 382, 442 and 478 $\equiv 2 \pmod 4$;

$d =$61, 145, 157, 213, 277, 313, 393, 397, 421, 445 and 457 $\equiv 1 \pmod 4$;

$d =$211, 283, 331, 355 and 451 $\equiv 3 \pmod 4$.

There are also many SP numbers more than 500 that do not satisfy $q_D > \sqrt{(1+d)/(4, 1+d)}$. In Section 5, we will discuss the conjecture that infinitely many SP numbers should exist.

Finally, we pose the following conjectures about necessary conditions for SP numbers. Let t_D denote the number of distinct prime divisors of D.

Conjecture 3.6. *If d is an SP number, then $t_D \leq q_D$.*

Conjecture 3.6 holds for all SP numbers less than 10^6.

We expect q_D in the above may be replaced with a constant as in the following

Conjecture 3.7. *If d is an SP number with D as in* (3.1)*, then*

$$\begin{cases} t_D \leq 5 & (d \equiv 2,\ 3 \bmod 4); \\ t_D \leq 6 & (d \equiv 1 \bmod 4). \end{cases}$$

Conjecture 3.7 holds for all SP numbers less than 10^9.

4. Special cases of SP numbers

In this section we study special cases of SP numbers, which are related to the problem of distribution of prime numbers.

(I) For an odd prime number p, if $2p \equiv 2 \pmod 4$ is an SP number, then $\{p,\ 2+p,\ 1+2p\}$ is a triple of prime numbers. In particular, p and $2+p$ are twin primes, and p is a Sophi-Germain prime since $\{p,\ 1+2p\}$ is a prime pair.

(II) For odd prime numbers p_1 and p_2 ($p_1 < p_2$), if $p_1p_2 \equiv 1 \pmod 4$ is an SP number, then we have four prime numbers of the form $\{p_1, (p_1 + p_2)/2,\ p_2,\ (1 + p_1p_2)/2\}$. In particular, $\{p_1,\ (p_1 + p_2)/2,\ p_2\}$ is a triple of prime numbers in an arithmetical progression.

As stated in Proposition 2.1, if an SP number d is divisible by a square, then $d = p^2$ or $3p^2$ for an odd prime number p.

(III) If p^2 is an SP number for an odd prime number p, then $\{p,\ (1+p^2)/2\}$ is a prime pair. Putting $p = 2n+1$, we get a prime pair $\{2n+1,\ 2n^2+2n+1\}$.

(IV) If $3p^2$ is an SP number for an odd prime number p, then $\{p,\ (3 + p^2)/4,\ (1+3p^2)/4\}$ is a triple of prime numbers. Putting $p = 2n+1$, we get a triple of prime numbers $\{2n + 1,\ n^2 + n + 1,\ 3n^2 + 3n + 1\}$.

5. The number of SP numbers

Finally we discuss the number of SP numbers.

The following table gives the number of SP numbers less than 10^n ($2 \leq n \leq 7$).

n	SP numbers
100	31
1000	123
10000	532
100000	2728
1000000	15402
10000000	98294

We pose the following:

Conjecture 5.1. *There exist infinitely many SP numbers.*

We have good grounds for believing that Conjecture 5.1 holds. First, as stated in Section 4, if $2p$ is an SP number for an odd prime number p, then $2 + p$ and $1 + 2p$ are both prime numbers. Conversely, if $2 + p$ and $1 + 2p$ are both prime numbers, then $2p$ is an SP number. It is conjectured that there exist infinitely many triples of prime numbers $\{p,\ 2+p,\ 1+2p\}$ (see, e.g., [1]). Thus we predict that there exist infinitely many SP numbers $2p$. Second, we believe Conjecture 5.1 for the reason that Theorem 5.4 holds below.

For a prime number p, denote by $S(p)$ the set of SP numbers d taking the prime value $p = (a + b)/(4, 1 + d)$ for a factorization $d = ab$. For example, $S(7) = \{6, 10, 13, 27, 33, 75, 115, 147, 187\}$ by the following table.

SP numbers	$p = a + b$
$6 = 1 \cdot 6$	$7 = 1 + 6$
$10 = 2 \cdot 5$	$7 = 2 + 5$

SP numbers	$p = (a + b)/2$
$13 = 1 \cdot 13$	$7 = (1 + 13)/2$
$33 = 3 \cdot 11$	$7 = (3 + 11)/2$

SP numbers	$p = (a + b)/4$
$27 = 1 \cdot 27$	$7 = (1 + 27)/4$
$75 = 3 \cdot 25$	$7 = (3 + 25)/4$
$115 = 5 \cdot 23$	$7 = (5 + 23)/4$
$147 = 7 \cdot 21$	$7 = (7 + 21)/4$
$187 = 11 \cdot 17$	$7 = (11 + 17)/4$

It is easy to see that $S(p)$ is a finite set for every prime number p. We pose the following conjecture.

Conjecture 5.2. *For every prime number p we have $S(p) \neq \emptyset$.*

Conjecture 5.2 holds for all prime numbers less than 3×10^6. Even if Conjecture 5.2 does not hold, it appears that there exist very few prime numbers p such that $S(p) = \emptyset$. Thus we expect the following

Conjecture 5.3. *There exist infinitely many prime numbers p such that $S(p) \neq \emptyset$.*

We have the following theorem.

Theorem 5.4. *Conjecture* 5.1 *is equivalent to Conjecture* 5.3*.*

For the proof of Theorem 5.4, let us prepare two lemmas.

Let $m(p)$ be the least SP number in $S(p)$ if $S(p) \neq \emptyset$. For example, $m(7) = 6$.

Lemma 5.5. *The inequality $m(p) \geq p - 1$ holds for any prime number p with $S(p) \neq \emptyset$.*

Proof. If $d \in S(p)$, then there exists a factorization $d = ab$ such that $p = (a + b)/(4, 1 + d)$. Since $p = (a + b)/(4, 1 + d) \leq (1 + d)/(4, 1 + d)$, it holds $p \leq p(4, 1+d) \leq 1+d$, and consequently $p \leq 1+d$, that is, $d \geq p-1$. Hence $m(p) \geq p - 1$ holds. □

Let P_k be the set of prime numbers p satisfying $S(p) \neq \emptyset$ and $m(p) = k$ for a positive integer k.

Lemma 5.6. *The number of elements in P_k is finite for every positive integer k.*

Proof. By Lemma 5.5 we get $k = m(p) \geq p - 1$. Thus $p \leq k + 1$ holds, that is, P_k is a finite set. □

Proof of Theorem 5.4.

We show that Conjecture 5.1 implies Conjecture 5.3. Suppose that there exist infinitely many SP numbers. Since $(1 + d)/(4, 1 + d)$ is the prime number for every SP number d, we obtain $d \in S((1 + d)/(4, 1 + d))$. Hence $S((1 + d)/(4, 1 + d)) \neq \emptyset$ for every SP number d. Therefore there exist infinitely many prime numbers $p = (1 + d)/(4, 1 + d)$ such that $S(p) \neq \emptyset$.

Conversely, we show that Conjecture 5.3 implies Conjecture 5.1. Suppose that there exist infinitely many prime numbers $\{p_1,\ p_2,\ p_3,\ \cdots\}$ such that $S(p_i) \neq \emptyset$ $(i = 1,\ 2,\ 3,\ \cdots)$. Then we can take the sequence of SP numbers $\{m(p_1),\ m(p_2),\ m(p_3),\ \cdots\}$. If there exist only finitely many SP numbers, then

$$m(p_{i_1}) = m(p_{i_2}) = m(p_{i_3}) = \cdots$$

holds for infinitely many prime numbers $\{p_{i_1},\ p_{i_2},\ p_{i_3} \cdots\}$ that is a subsequence of $\{p_1,\ p_2,\ p_3,\ \cdots\}$. This contradicts Lemma 5.6. Hence there exist infinitely many SP numbers. □

Consider factorizations of a positive integer d except for the trivial factorization $d = 1 \cdot d$. We claim that there exist infinitely many positive integers d such that $(a+b)/(4, 1+d)$ is a prime number for every non-trivial factorization $d = ab$ $(a > 1,\ b > 1)$. It follows from the result of Green and Tao. They give the following theorem.

Theorem (Green and Tao [4]). *The prime numbers contain infinitely many arithmetic progressions of length n for all n.*

Thus for every positive integer n there exists an n-term arithmetic progression of prime numbers

$$p,\ p+r,\ p+2r,\ p+3r,\ \cdots,\ p+(n-1)r, \tag{5.1}$$

where p is an odd prime number and r is an even integer. Suppose $n \geq 5$ and let $p_i = p + 2ir$ and $p_j = p + 2jr$ for integers i and j satisfying $1 \leq i < j \leq (n-1)/2$. Then $(p_i + p_j)/2 = p + (i+j)r$ belongs to the arithmetical progression (5.1) since $2 < i + j < n - 1$. Thus $(p_i + p_j)/2$ is a prime number and $p_i p_j = (p+2ir)(p+2jr) = p^2 + 2(i+j)r + 4ijr^2 \equiv 1 \pmod 4$. Hence $d = p_i p_j \equiv 1 \pmod 4$ gives rise to the prime number $(p_i + p_j)/2$ for the non-trivial factorization. We can take the distinct $d = p_i p_j$ for the distinct positive integer n. Therefore there exist infinitely many such positive integers $d = p_i p_j \equiv 1 \pmod 4$.

References

[1] M. Becheanu, F. Luca and I. E. Shparlinski: On the sums of complementary divisors, Int. J. Number Theory **3**, 635-648 (2007).

[2] A. I. Borevich and I. R. Shafarevich: Number Theory, Academic Press (1966).

[3] M. T. Damir, B. Faye, F. Luca and A. Tall: Fibonacci numbers with prime sums of complementary divisors, Integers **14**, 1-9 (2014)

[4] B. Green and T. Tao: The primes contain arbitrarily long arithmetic progressions, Ann. of Math., **167**(2), 481-547 (2008)

[5] H. Möller: Imaginär-quadratische Zahlkörper mit einklassigen Geschlechtern, Acta Arith. **30**, 179-186 (1976).

[6] K. Shimizu and K. Goto: A Property of Integers Related to Quadratic Fields. The Journal of the Faculty of Education Tottori Univ. Vol. **47**, No.1, 5-12 (1998).

[7] K. Shimizu: Imaginary quadratic fields whose exponents are less than or equal to two, Math. J. Okayama Univ. Vol.**50**, 85-99 (2008).

[8] K. Shimizu: Imaginary quadratic fields whose exponents are less than or equal to two, II, Algebraic Number Theory and Related Topics 2010, RIMS Kôkyûroku Bessatsu **B32**, 255-269 (2012).

[9] P. J. Weinberger: Exponents of the class groups of complex quadratic fields, Acta Arith. **22**, 117–124 (1973).

Kenichi Shimizu
10-5, Mitachinishi 3-chome, Himeji-shi, Hyogo, 670-0074, Japan
e-mail address: s-2357@gaia.eonet.ne.jp

(*Received June 16, 2016*)
(*Accepted May 18, 2017*)