

**A REMARK ON A CENTRAL LIMIT THEOREM FOR
NON-SYMMETRIC RANDOM WALKS ON CRYSTAL
LATTICES**

RYUYA NAMBA

ABSTRACT. Recently, Ishiwata, Kawabi and Kotani [4] proved two kinds of central limit theorems for non-symmetric random walks on crystal lattices from the view point of discrete geometric analysis developed by Kotani and Sunada. In the present paper, we establish yet another kind of the central limit theorem for them. Our argument is based on a measure-change technique due to Alexopoulos [1].

1. INTRODUCTION AND RESULTS

Let $X = (V, E)$ be a locally finite, connected and oriented graph. Here V is the set of all vertices and E the set of all oriented edges. For an edge $e \in E$, we denote by $o(e), t(e)$ and \bar{e} the origin, the terminus and the inverse edge of e , respectively. We denote by E_x the collection of all edges whose origin is $x \in V$. A path c in X with length n is a sequence $c = (e_1, \dots, e_n)$ of edges e_i with $t(e_i) = o(e_{i+1})$ ($i = 1, \dots, n-1$). We denote by $\Omega_{x,n}(X)$ ($x \in V, n \in \mathbb{N} \cup \{\infty\}$) the set of all paths of length n for which origin $o(c) = x$. We also denote by $o(c), t(c)$ the origin and the terminus of the path c . For simplicity, we write $\Omega_x(X) := \Omega_{x,\infty}(X)$.

Let $p : E \rightarrow (0, 1]$ be a *transition probability*, that is, a positive function on E satisfying

$$(1.1) \quad \sum_{e \in E_x} p(e) = 1 \quad (x \in V).$$

This induces the probability measure \mathbb{P}_x on $\Omega_x(X)$ and the random walk associated with the transition probability p is the time homogeneous Markov chain $(\Omega_x(X), \mathbb{P}_x, \{w_n\}_{n=0}^\infty)$ with values in X defined by $w_n(c) := o(e_{n+1})$ ($c = (e_1, e_2, \dots) \in \Omega_x(X)$). The graph X is endowed with the graph distance. For a topological space \mathcal{T} , we denote by $C_\infty(\mathcal{T})$ the space of all functions $f : \mathcal{T} \rightarrow \mathbb{R}$ vanishing at infinity with the uniform topology $\|f\|_\infty^\mathcal{T}$.

The (p) -transition operator L acting on $C_\infty(X)$ is defined by

$$Lf(x) := \sum_{e \in E_x} p(e)f(t(e)) \quad (x \in V).$$

Mathematics Subject Classification. Primary 60J10; Secondary 60F05.

Key words and phrases. crystal lattice, central limit theorem, non-symmetric random walk, (modified) harmonic realization.

The n -step transition probability $p(n, x, y)$ ($n \in \mathbb{N}$, $x, y \in V$) is given by $p(n, x, y) := L^n \delta_y(x)$, where $\delta_y(\cdot)$ stands for the Dirac delta function with pole at y . If there is a positive function $m : V \rightarrow (0, \infty)$ up to constant multiple such that

$$p(e)m(o(e)) = p(\bar{e})m(t(e)) \quad (e \in E),$$

the random walk is called (m) -symmetric or *reversible*. Otherwise, it is called (m) -non-symmetric.

As one of the most fundamental examples of infinite graphs, a *crystal lattice* has been studied by many authors from both geometric and probabilistic viewpoints. Roughly speaking, an infinite graph X is called a (Γ) -crystal lattice if X is an infinite-fold covering graph of a finite graph whose covering transformation group Γ is abelian. Typical examples we have in mind are the square lattice, the triangular lattice and the hexagonal lattice, and so on. For basic results, see [5, 6, 7, 8] and literatures therein.

In the theory of random walks on infinite graphs, to investigate the long time asymptotics, for instance, the *central limit theorem* (CLT) is a principal topic for both geometers and probabilists. Recently, Ishiwata, Kawabi and Kotani [4] proved two kinds of functional CLTs for non-symmetric random walks on crystal lattices using the theory of *discrete geometric analysis* developed by Kotani and Sunada. For more details on discrete geometric analysis, see Section 2. We also refer to [9, 7].

Before stating our results, we start with a brief review of the setting and results in [4]. Let us consider a (Γ) -crystal lattice $X = (V, E)$, where the covering transformation group Γ , acting on X freely, is a torsion free, finitely generated abelian group. Here we may assume that Γ is isomorphic to \mathbb{Z}^d , without loss of generality. We denote by $X_0 = (V_0, E_0)$ its (finite) quotient graph $\Gamma \backslash X$. Let $p : E \rightarrow (0, 1]$ be a Γ -invariant transition probability. Namely, it satisfies (1.1) and $p(\sigma e) = p(e)$ for every $\sigma \in \Gamma$, $e \in E$. Through the covering map $\pi : X \rightarrow X_0$, the transition probability p also induces a Markov chain with values in X_0 . Since the transition probability p on X is positive, the random walk on X is *irreducible*, that is, for every $x, y \in V$, there exists $n = n(x, y) \in \mathbb{N}$ such that $p(n, x, y) > 0$. And so is the random walk on X_0 . Then, thanks to the Perron–Frobenius theorem, we find a unique *invariant probability measure* on V_0 with

$$(1.2) \quad \sum_{x \in V_0} m(x) = 1 \quad \text{and} \quad m(x) = \sum_{e \in (E_0)_x} p(\bar{e})m(t(e)) \quad (x \in V_0).$$

It is also called a *stationary distribution* (see e.g., Durrett [2]). We also write $m : V \rightarrow (0, 1]$ for the Γ -invariant lift of m to X .

Let $H_1(X_0, \mathbb{R})$ and $H^1(X_0, \mathbb{R})$ be the first homology group and the first cohomology group of X_0 , respectively. We take a linear map $\rho_{\mathbb{R}}$ from $H_1(X_0, \mathbb{R})$

onto $\Gamma \otimes \mathbb{R}$ through the covering map $\pi : X \rightarrow X_0$. We define the *homological direction* of the random walk on X_0 by

$$\gamma_p := \sum_{e \in E_0} p(e)m(o(e))e \in H_1(X_0, \mathbb{R}),$$

and we call $\rho_{\mathbb{R}}(\gamma_p) \in \Gamma \otimes \mathbb{R}$ the *asymptotic direction*. We remark that the random walk is m -symmetric if and only if $\gamma_p = 0$. Moreover, $\gamma_p = 0$ implies $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$. However, the converse does not hold in general. We write g_0 for the (p -)Albanese metric on $\Gamma \otimes \mathbb{R}$. (See Section 2, for its precise definition.) We call that a periodic realization $\Phi_0 : X \rightarrow \Gamma \otimes \mathbb{R}$ is (p -)modified harmonic if

$$(1.3) \quad \sum_{e \in E_x} p(e) \left(\Phi_0(t(e)) - \Phi_0(o(e)) \right) = \rho_{\mathbb{R}}(\gamma_p) \quad (x \in V).$$

This notion was first proposed in [7] to seek the most canonical periodic realization of a topological crystal in the geometric context.

Now we are in a position to review two kinds of CLTs formulated in [4]. We first set a reference point $x_* \in V$ with $\Phi_0(x_*) = \mathbf{0}$ and put $\xi_n(c) := \Phi_0(w_n(c))$ ($n = 0, 1, 2, \dots, c \in \Omega_{x_*}(X)$). Let $C_{\mathbf{0}}([0, \infty), (\Gamma \otimes \mathbb{R}, g_0))$ be the set of all continuous paths from $[0, \infty)$ to $(\Gamma \otimes \mathbb{R}, g_0)$ starting from the origin. We equip it with the usual compact uniform topology. We define a measurable map $\mathbf{X}^{(n)} : \Omega_{x_*}(X) \rightarrow (C_{\mathbf{0}}([0, \infty), (\Gamma \otimes \mathbb{R}, g_0)), \mu)$ by

$$\mathbf{X}_t^{(n)}(c) := \frac{1}{\sqrt{n}} \left\{ \xi_{[nt]}(c) + (nt - [nt]) (\xi_{[nt]+1}(c) - \xi_{[nt]}(c)) - nt \rho_{\mathbb{R}}(\gamma_p) \right\} \quad (t \geq 0),$$

where μ is the Wiener measure on the path space $C_{\mathbf{0}}([0, \infty), (\Gamma \otimes \mathbb{R}, g_0))$. We write $\mathbf{P}^{(n)}$ ($n = 1, 2, \dots$) for the probability measure on $C_{\mathbf{0}}([0, \infty), (\Gamma \otimes \mathbb{R}, g_0))$ induced by $\mathbf{X}^{(n)}$. Then the CLT of the first kind is stated as follows:

Theorem 1.1. ([4, Theorem 2.2]) *The sequence $\{\mathbf{P}^{(n)}\}_{n=1}^{\infty}$ of probability measures converges weakly to the Wiener measure μ as $n \rightarrow \infty$. In other words, the sequence $\{\mathbf{X}^{(n)}\}_{n=1}^{\infty}$ converges to a $(\Gamma \otimes \mathbb{R}, g_0)$ -valued standard Brownian motion $(B_t)_{t \geq 0}$ starting from the origin in law.*

Next we introduce a family $\{p_{\varepsilon}\}_{0 \leq \varepsilon \leq 1}$ of transition probabilities on X by $p_{\varepsilon}(e) := p_0(e) + \varepsilon q(e)$ ($e \in E$), where

$$\begin{aligned} p_0(e) &:= \frac{1}{2} \left(p(e) + \frac{m(t(e))}{m(o(e))} p(\bar{e}) \right), \\ q(e) &:= \frac{1}{2} \left(p(e) - \frac{m(t(e))}{m(o(e))} p(\bar{e}) \right) \quad (e \in E). \end{aligned}$$

This is nothing but the interpolation of the original transition probability $p = p_1$ and the m -symmetric transition probability p_0 . We should note that, in this setting, the relation $\rho_{\mathbb{R}}(\gamma_{p_\varepsilon}) = \varepsilon \rho_{\mathbb{R}}(\gamma_p)$ plays a crucial role to obtain the CLT of the second kind. We write $g_0^{(\varepsilon)}$ for the (p_ε) -Albanese metric on $\Gamma \otimes \mathbb{R}$. Moreover, let $\Phi_0^{(\varepsilon)} : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0^{(0)})$ be the p_ε -modified harmonic realization of X .

Now set a reference point $x_* \in V$ satisfying $\Phi_0^{(\varepsilon)}(x_*) = \mathbf{0}$ for all $0 \leq \varepsilon \leq 1$ and put $\xi_n^{(\varepsilon)}(c) := \Phi_0^{(\varepsilon)}(w_n(c))$ ($n = 0, 1, 2, \dots, c \in \Omega_{x_*}(X)$). We define a measurable map $\mathbf{Y}^{(\varepsilon, n)} : \Omega_{x_*}(X) \rightarrow C_0([0, \infty), (\Gamma \otimes \mathbb{R}, g_0^{(0)}))$ by

$$\mathbf{Y}_t^{(\varepsilon, n)}(c) := \frac{1}{\sqrt{n}} \left\{ \xi_{[nt]}^{(\varepsilon)}(c) + (nt - [nt]) (\xi_{[nt]+1}^{(\varepsilon)}(c) - \xi_{[nt]}^{(\varepsilon)}(c)) \right\} \quad (t \geq 0).$$

Let ν be the probability measure on $C_0([0, \infty), (\Gamma \otimes \mathbb{R}, g_0^{(0)}))$ induced by the stochastic process $(B_t + \rho_{\mathbb{R}}(\gamma_p)t)_{t \geq 0}$, where $(B_t)_{t \geq 0}$ is a $(\Gamma \otimes \mathbb{R}, g_0^{(0)})$ -valued standard Brownian motion with $B_0 = \mathbf{0}$. Moreover let $\mathbf{Q}^{(\varepsilon, n)}$ be the probability measure on $C_0([0, \infty), (\Gamma \otimes \mathbb{R}, g_0^{(0)}))$ induced by $\mathbf{Y}^{(\varepsilon, n)}$. Then the CLT of the second kind is the following.

Theorem 1.2. ([4, Theorem 2.4]) *The sequence $\{\mathbf{Q}^{(n^{-1/2}, n)}\}_{n=1}^\infty$ of probability measures converges weakly to the probability measure ν as $n \rightarrow \infty$. In other words, the sequence $\{\mathbf{Y}^{(n^{-1/2}, n)}\}_{n=1}^\infty$ converges to a $(\Gamma \otimes \mathbb{R}, g_0^{(0)})$ -valued standard Brownian motion with drift $\rho_{\mathbb{R}}(\gamma_p)$ starting from the origin in law.*

The main purpose of the present paper is to establish yet another CLT for non-symmetric random walks on crystal lattices. Our approach is inspired by a *measure-change technique* due to Alexopoulos [1] in which several limit theorems for random walks on discrete groups of polynomial volume growth are obtained.

Now consider the (m) -non-symmetric transition probability $p : E \rightarrow (0, 1]$. In particular, we assume that $\rho_{\mathbb{R}}(\gamma_p) \neq \mathbf{0}$. Let $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0)$ be a (p) -modified harmonic realization of X . We define a function $F = F_x(\lambda) : V_0 \times \text{Hom}(\Gamma, \mathbb{R}) \rightarrow (0, \infty)$ by

$$(1.4) \quad F_x(\lambda) := \sum_{e \in (E_0)_x} p(e) \exp \left(\langle \lambda, \Phi_0(t(\tilde{e})) - \Phi_0(o(\tilde{e})) \rangle_{\Gamma \otimes \mathbb{R}} \right)$$

for $x \in V_0$, $\lambda \in \text{Hom}(\Gamma, \mathbb{R})$, where \tilde{e} stands for a lift of $e \in E_0$ to X . Then we can verify that, for every $x \in V_0$, the function $F_x(\cdot) : \text{Hom}(\Gamma, \mathbb{R}) \rightarrow (0, \infty)$ has a unique minimizer $\lambda_* = \lambda_*(x) \in \text{Hom}(\Gamma, \mathbb{R})$. (See Lemma 3.1.) We

define a positive function $\mathfrak{p} : E_0 \rightarrow (0, 1]$ by

$$(1.5) \quad \mathfrak{p}(e) := \frac{p(e) \exp \left(\langle \lambda_*(o(e)), \Phi_0(t(\tilde{e})) - \Phi_0(o(\tilde{e})) \rangle_{\Gamma \otimes \mathbb{R}} \right)}{F_{o(e)}(\lambda_*(o(e)))}$$

for $e \in E_0$. Then it is straightforward to check that the function \mathfrak{p} also gives a transition probability on X_0 . Noting the random walk $\{w_n^{(\mathfrak{p})}\}_{n=0}^\infty$ associated with \mathfrak{p} is also irreducible, the Perron–Frobenius theorem yields a unique normalized invariant measure $\mathfrak{m} : V_0 \rightarrow (0, 1]$ in the sense of (1.2). We write $\mathfrak{p} : E \rightarrow (0, 1]$ and $\mathfrak{m} : V \rightarrow (0, 1]$ for the Γ -invariant lifts of $\mathfrak{p} : E_0 \rightarrow (0, 1]$ and $\mathfrak{m} : V_0 \rightarrow (0, 1]$, respectively. We write $g_0^{(\mathfrak{p})}$ for the (\mathfrak{p} -)Albanese metric associated with the transition probability \mathfrak{p} .

Let $L_{(\mathfrak{p})}$ be the transition operator, acting on $C_\infty(X)$, associated with the transition probability \mathfrak{p} . Namely,

$$L_{(\mathfrak{p})}f(x) = \sum_{e \in E_x} \mathfrak{p}(e)f(t(e)) \quad (x \in V).$$

Recalling that the function $F = F_x(\lambda)$ has the (unique) minimizer $\lambda_* = \lambda_*(x)$ for every $x \in V_0$, it follows that

$$(1.6) \quad L_{(\mathfrak{p})}\Phi_0(x) - \Phi_0(x) = \sum_{e \in E_x} \mathfrak{p}(e) \left(\Phi_0(t(e)) - \Phi_0(o(e)) \right) = \mathbf{0} \quad (x \in V).$$

This equation means that the p -modified harmonic realization $\Phi_0 : X \rightarrow \Gamma \otimes \mathbb{R}$ is a \mathfrak{p} -harmonic realization. In particular, we obtain $\rho_{\mathbb{R}}(\gamma_{\mathfrak{p}}) = \mathbf{0}$. Here we should emphasize that the transition probability $\mathfrak{p} : E_0 \rightarrow (0, 1]$ coincides with the original one $p : E_0 \rightarrow (0, 1]$ provided that $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$.

We fix a reference point $x_* \in V$ such that $\Phi_0(x_*) = \mathbf{0}$ and put

$$\xi_n^{(\mathfrak{p})}(c) := \Phi_0(w_n^{(\mathfrak{p})}(c)) \quad (n = 0, 1, 2, \dots, c \in \Omega_{x_*}(X)).$$

We define a measurable map $\mathfrak{X}^{(n)} : \Omega_{x_*}(X) \rightarrow (C_0([0, \infty), (\Gamma \otimes \mathbb{R}, g_0^{(\mathfrak{p})})), \mu)$ by

$$(1.7) \quad \mathfrak{X}_t^{(n)}(c) := \frac{1}{\sqrt{n}} \left\{ \xi_{[nt]}^{(\mathfrak{p})}(c) + (nt - [nt]) (\xi_{[nt]+1}^{(\mathfrak{p})}(c) - \xi_{[nt]}^{(\mathfrak{p})}(c)) \right\} \quad (t \geq 0),$$

where $\mu = \mu^{(\mathfrak{p})}$ is the Wiener measure on $C_0([0, \infty), (\Gamma \otimes \mathbb{R}, g_0^{(\mathfrak{p})}))$. We write $\mathfrak{P}^{(n)}$ ($n = 1, 2, \dots$) for the probability measure on $C_0([0, \infty), (\Gamma \otimes \mathbb{R}, g_0^{(\mathfrak{p})}))$ induced by $\mathfrak{X}^{(n)}$. Then our main theorem is stated as follows:

Theorem 1.3. *The sequence $\{\mathfrak{P}^{(n)}\}_{n=1}^\infty$ of probability measures converges weakly to the Wiener measure μ as $n \rightarrow \infty$. Namely, the sequence $\{\mathfrak{X}^{(n)}\}_{n=1}^\infty$ converges to a $(\Gamma \otimes \mathbb{R}, g_0^{(\mathfrak{p})})$ -valued standard Brownian motion $(B_t^{(\mathfrak{p})})_{t \geq 0}$ starting from the origin in law.*

Finally, we give a relationship between the n -step transition probabilities $p(n, x, y)$ and $\mathbf{p}(n, x, y)$ as follows:

Theorem 1.4. *There exist some positive constants C_1 and C_2 such that*

$$C_1 p(n, x, y) \exp(nM_p) \leq \mathbf{p}(n, x, y) \leq C_2 p(n, x, y) \exp(nM_p)$$

for all $n \in \mathbb{N}$ and $x, y \in V$, where

$$M_p := \sum_{\mathbf{x} \in V_0} m(\mathbf{x}) \left(\langle \lambda_*(\mathbf{x}), \rho_{\mathbb{R}}(\gamma_p) \rangle_{\Gamma \otimes \mathbb{R}} - \log F_{\mathbf{x}}(\lambda_*(\mathbf{x})) \right).$$

For the precise long time asymptotic behavior of $p(n, x, y)$, we refer to [4, Theorem 2.5] and Trojan [10].

The rest of the present paper is organized as follows: In Section 2, we provide a brief review on the theory of discrete geometric analysis. In Section 3, we state our measure-change technique in details and give proofs of Theorems 1.3 and 1.4. We also discuss a relationship between our measure-change technique and a discrete analogue of Girsanov's theorem due to Fujita [3]. Finally, in Section 4, we give some concrete examples of non-symmetric random walks on crystal lattices.

2. A QUICK REVIEW ON DISCRETE GEOMETRIC ANALYSIS

In this section, we give basic materials of the theory of discrete geometric analysis on graphs quickly. For more details, we refer to Kotani–Sunada [7] and Sunada [9].

We consider a random walk on a finite graph $X_0 = (V_0, E_0)$ associated with a transition probability $p : E_0 \rightarrow (0, 1]$. Thanks to the Perron–Frobenius theorem, there is a unique (normalized) invariant measure $m : V_0 \rightarrow (0, 1]$ in the sense of (1.2). The random walk is called (m) -symmetric if $p(e)m(o(e)) = p(\bar{e})m(t(e))$ ($e \in E_0$).

First we define the 0-chain group and the 1-chain group of X_0 by

$$C_0(X_0, \mathbb{R}) := \left\{ \sum_{x \in V_0} a_x x \mid a_x \in \mathbb{R} \right\},$$

$$C_1(X_0, \mathbb{R}) := \left\{ \sum_{e \in E_0} a_e e \mid a_e \in \mathbb{R}, \bar{e} = -e \right\},$$

respectively. Let $\partial : C_1(X_0, \mathbb{R}) \rightarrow C_0(X_0, \mathbb{R})$ be the boundary map, given by the homomorphism satisfying $\partial(e) := t(e) - o(e)$ ($e \in E_0$). The first homology group $H_1(X_0, \mathbb{R})$ of X_0 is defined by $\text{Ker}(\partial) (\subset C_1(X_0, \mathbb{R}))$.

On the other hand, we define the 0-cochain group and the 1-cochain group of X_0 by

$$C^0(X_0, \mathbb{R}) := \{f : V_0 \rightarrow \mathbb{R}\},$$

$$C^1(X_0, \mathbb{R}) := \{\omega : E_0 \rightarrow \mathbb{R} \mid \omega(\bar{e}) = -\omega(e)\},$$

respectively. The difference operator $d : C^0(X_0, \mathbb{R}) \rightarrow C^1(X_0, \mathbb{R})$ is defined by the homomorphism with $df(e) := f(t(e)) - f(o(e))$ ($e \in E_0$). We also define $H^1(X_0, \mathbb{R}) := C^1(X_0, \mathbb{R})/\text{Im}(d)$, called the first cohomology group of X_0 .

Next we define the transition operator $L : C^0(X_0, \mathbb{R}) \rightarrow C^0(X_0, \mathbb{R})$ by

$$Lf(x) := (I - \delta_p d)f(x) = \sum_{e \in (E_0)_x} p(e)f(t(e)) \quad (x \in V_0, f \in C^1(X_0, \mathbb{R})),$$

where the operator $\delta_p : C^1(X_0, \mathbb{R}) \rightarrow C^0(X_0, \mathbb{R})$ is defined by

$$\delta_p \omega(x) := - \sum_{e \in (E_0)_x} p(e)\omega(e) \quad (x \in V_0, \omega \in C^1(X_0, \mathbb{R})).$$

We introduce the quantity γ_p , called the *homological direction* of the given random walk on X_0 , by

$$\gamma_p := \sum_{e \in E_0} \tilde{m}(e)e \in C_1(X_0, \mathbb{R}),$$

where $\tilde{m}(e) := p(e)m(o(e))$ ($e \in E_0$). It is easy to show that $\partial(\gamma_p) = 0$, that is, $\gamma_p \in H_1(X_0, \mathbb{R})$. It should be noted that the transition probability p gives an m -symmetric random walk on X_0 if and only if $\gamma_p = 0$. A 1-form $\omega \in C^1(X_0, \mathbb{R})$ is said to be *modified harmonic* if

$$\delta_p \omega(x) + C_1(X_0, \mathbb{R}) \langle \gamma_p, \omega \rangle_{C^1(X_0, \mathbb{R})} = 0 \quad (x \in V_0).$$

We denote by $\mathcal{H}^1(X_0)$ the space of modified harmonic 1-forms on X_0 , and equip it with the inner product

$$\langle\langle \omega, \eta \rangle\rangle_p := \sum_{e \in E_0} \tilde{m}(e)\omega(e)\eta(e) - \langle \gamma_p, \omega \rangle \langle \gamma_p, \eta \rangle \quad (\omega, \eta \in \mathcal{H}^1(X_0)).$$

Due to the discrete analogue of Hodge-Kodaira theorem (cf. [7, Lemma 5.2]), we may identify $(\mathcal{H}^1(X_0), \langle\langle \cdot, \cdot \rangle\rangle_p)$ with $H^1(X_0, \mathbb{R})$.

Now let $X = (V, E)$ be a Γ -crystal lattice. Namely, X is a covering graph of a finite graph X_0 with an abelian covering transformation group Γ . We write $p : E \rightarrow (0, 1]$ and $m : V \rightarrow (0, 1]$ for the Γ -invariant lifts of $p : E_0 \rightarrow (0, 1]$ and $m : V_0 \rightarrow (0, 1]$, respectively. Through the covering map $\pi : X \rightarrow X_0$, we take the surjective linear map $\rho_{\mathbb{R}} : H_1(X_0, \mathbb{R}) \rightarrow \Gamma \otimes \mathbb{R} (\cong \mathbb{R}^d)$. We consider the transpose ${}^t\rho_{\mathbb{R}} : \text{Hom}(\Gamma, \mathbb{R}) \rightarrow H^1(X_0, \mathbb{R})$, which is a injective linear map. Here $\text{Hom}(\Gamma, \mathbb{R})$ denotes the space of homomorphisms from Γ into \mathbb{R} . We induce a flat metric g_0 on the Euclidean space $\Gamma \otimes \mathbb{R}$ through the following diagram:

$$\begin{array}{ccc}
(\Gamma \otimes \mathbb{R}, g_0) & \xleftarrow{\rho_{\mathbb{R}}} & H_1(X_0, \mathbb{R}) \\
\uparrow \text{dual} & & \uparrow \text{dual} \\
\text{Hom}(\Gamma, \mathbb{R}) & \xrightarrow[t\rho_{\mathbb{R}}]{} & H^1(X_0, \mathbb{R}) \cong (\mathcal{H}^1(X_0), \langle\langle \cdot, \cdot \rangle\rangle_p).
\end{array}$$

This metric g_0 is called the *Albanese metric* on $\Gamma \otimes \mathbb{R}$.

From now on, we realize the crystal lattice X into the continuous model $(\Gamma \otimes \mathbb{R}, g_0)$ in the following manner. A *periodic realization* of X into $\Gamma \otimes \mathbb{R}$ is defined by a piecewise linear map $\Phi : X \rightarrow \Gamma \otimes \mathbb{R}$ with $\Phi(\sigma x) = \Phi(x) + \sigma \otimes 1$ ($\sigma \in \Gamma, x \in V$). We introduce a special periodic realization $\Phi_0 : X \rightarrow \Gamma \otimes \mathbb{R}$ by

$$(2.1) \quad \text{Hom}(\Gamma, \mathbb{R}) \langle \omega, \Phi_0(x) \rangle_{\Gamma \otimes \mathbb{R}} = \int_{x_*}^x \tilde{\omega} \quad (x \in V, \lambda \in \text{Hom}(\Gamma, \mathbb{R})),$$

where x_* is a fixed reference point satisfying $\Phi_0(x_*) = \mathbf{0}$ and $\tilde{\omega}$ is the lift of ω to X . Here

$$\int_{x_*}^x \tilde{\omega} = \int_c \tilde{\omega} := \sum_{i=1}^n \tilde{\omega}(e_i)$$

for a path $c = (e_1, \dots, e_n)$ with $o(e_1) = x_*$ and $t(e_n) = x$. It should be noted that this line integral does not depend on the choice of a path c . The periodic realization Φ_0 given by above enjoys the so-called *modified harmonicity* in the sense that

$$L\Phi_0(x) - \Phi_0(x) = \rho_{\mathbb{R}}(\gamma_p) \quad (x \in V).$$

We note that this equation is also written as (1.3). Further, such a realization is uniquely determined up to translation. We call the quantity $\rho_{\mathbb{R}}(\gamma_p)$ the *asymptotic direction* of the given random walk. We should emphasize that $\gamma_p = 0$ implies $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$. However, the converse does not always hold, that is, there is a case $\gamma_p \neq 0$ and $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$. (See Subsection 4.2, for an example.) If we equip $\Gamma \otimes \mathbb{R}$ with the Albanese metric, then the modified harmonic realization $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0)$ is said to be a *modified standard realization*.

3. PROOFS OF THE MAIN RESULTS

3.1. A measure–change technique. In what follows, we write $\lambda[\mathbf{x}]_{\Gamma \otimes \mathbb{R}} := \text{Hom}(\Gamma, \mathbb{R}) \langle \lambda, \mathbf{x} \rangle_{\Gamma \otimes \mathbb{R}}$ ($\lambda \in \text{Hom}(\Gamma, \mathbb{R}), \mathbf{x} \in \Gamma \otimes \mathbb{R}$) and

$$d\Phi_0(e) := \Phi_0(t(e)) - \Phi_0(o(e)) \quad (e \in E),$$

for brevity. Take an orthonormal basis

$$\{\omega_1, \dots, \omega_d\} \subset \text{Hom}(\Gamma, \mathbb{R}) \left(\subset (\mathcal{H}^1(X_0), \langle\langle \cdot, \cdot \rangle\rangle_p) \right)$$

and denote by $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ its dual basis in $\Gamma \otimes \mathbb{R}$. Namely, $\omega_i[\mathbf{v}_j]_{\Gamma \otimes \mathbb{R}} = \delta_{ij}$ ($1 \leq i, j \leq d$). Then, $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ is an orthonormal basis in $\Gamma \otimes \mathbb{R}$ with respect to the Albanese metric g_0 . We may identify $\lambda = \lambda_1 \omega_1 + \dots + \lambda_d \omega_d \in \text{Hom}(\Gamma, \mathbb{R})$ with $(\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$. Furthermore, we write $x_i := \omega_i[\mathbf{x}]_{\Gamma \otimes \mathbb{R}}$, $\Phi_0(x)_i := \omega_i[\Phi_0(x)]_{\Gamma \otimes \mathbb{R}}$ and $\partial_i := \partial/\partial \lambda_i$ ($i = 1, \dots, d, x \in V$). We denote by $O(\cdot)$ the Landau symbol.

At the beginning, consider the function $F = F_x(\lambda) : V_0 \times \text{Hom}(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R}$ defined by (1.4). We easily see that $F = F_x(\lambda)$ is a positive function on $V_0 \times \text{Hom}(\Gamma, \mathbb{R})$ with $F_x(\mathbf{0}) = 1$ ($x \in V_0$). In our setting, the following lemma plays a significant role so as to obtain Theorem 1.3.

Lemma 3.1. *For every $x \in V_0$, the function $F_x(\cdot) : \text{Hom}(\Gamma, \mathbb{R}) \longrightarrow (0, \infty)$ has a unique minimizer $\lambda_* = \lambda_*(x)$.*

Proof. Fix a fixed $x \in V_0$, we have

$$\begin{aligned} \partial_i F_x(\lambda) &= \partial_i \left(\sum_{e \in (E_0)_x} p(e) \exp \left(\lambda [d\Phi_0(\tilde{e})]_{\Gamma \otimes \mathbb{R}} \right) \right) \\ &= \partial_i \left(\sum_{e \in (E_0)_x} p(e) \exp \left(\sum_{i=1}^d \lambda_i \cdot \omega_i [d\Phi_0(\tilde{e})]_{\Gamma \otimes \mathbb{R}} \right) \right) \\ &= \sum_{e \in (E_0)_x} p(e) \exp \left(\lambda [d\Phi_0(\tilde{e})]_{\Gamma \otimes \mathbb{R}} \right) d\Phi_0(\tilde{e})_i \\ &\quad (i = 1, \dots, d, \lambda \in \text{Hom}(\Gamma, \mathbb{R})). \end{aligned}$$

In other words,

$$(3.1) \quad \begin{aligned} & \left(\partial_1 F_x(\lambda), \dots, \partial_d F_x(\lambda) \right) \\ &= \sum_{e \in (E_0)_x} p(e) \exp \left(\lambda [d\Phi_0(\tilde{e})]_{\Gamma \otimes \mathbb{R}} \right) d\Phi_0(\tilde{e}) \in \Gamma \otimes \mathbb{R} \end{aligned}$$

for $\lambda \in \text{Hom}(\Gamma, \mathbb{R})$. Repeating the above calculation, we have

$$\partial_i \partial_j F_x(\lambda) = \sum_{e \in (E_0)_x} p(e) \exp \left(\lambda [d\Phi_0(\tilde{e})]_{\Gamma \otimes \mathbb{R}} \right) d\Phi_0(\tilde{e})_i d\Phi_0(\tilde{e})_j$$

for $\lambda \in \text{Hom}(\Gamma, \mathbb{R})$ and $1 \leq i, j \leq d$. Then, we know that $(\partial_i \partial_j F_x(\cdot))_{1 \leq i, j \leq d}$, the *Hessian matrix* of the function $F_x(\cdot)$, is positive definite. Indeed, consider the quadratic form corresponding to the Hessian matrix. Since

$$\sum_{1 \leq i, j \leq d} \sum_{e \in (E_0)_x} p(e) \exp \left(\lambda [d\Phi_0(\tilde{e})]_{\Gamma \otimes \mathbb{R}} \right) d\Phi_0(\tilde{e})_i d\Phi_0(\tilde{e})_j \xi_i \xi_j$$

$$(3.2) \quad = \sum_{e \in (E_0)_x} p(e) \exp \left(\lambda [d\Phi_0(\tilde{e})]_{\Gamma \otimes \mathbb{R}} \right) \left\{ \sum_{i=1}^d d\Phi_0(\tilde{e})_i \xi_i \right\}^2 \geq 0$$

for $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ and the transition probability p is positive, we easily see that the Hessian matrix is non-negative definite. By multiplying both sides of (3.2) by $m(x)$ and taking the sum which runs over all vertices of X_0 , it readily follows that

$$\sum_{e \in E_0} \tilde{m}(e) \exp \left(\lambda [d\Phi_0(\tilde{e})]_{\Gamma \otimes \mathbb{R}} \right) \left\{ \sum_{i=1}^d d\Phi_0(\tilde{e})_i \xi_i \right\}^2 \geq 0$$

for $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. Next suppose that the left-hand side of (3.2) is zero. Then we have

$$\sum_{i=1}^d d\Phi_0(\tilde{e})_i \xi_i = 0$$

for all $e \in E_0$. This equation implies $\langle \Phi_0(x), \boldsymbol{\xi} \rangle_{\mathbb{R}^d} = \langle \Phi_0(y), \boldsymbol{\xi} \rangle_{\mathbb{R}^d}$ for all $x, y \in V$, where $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ stands for the standard inner product on \mathbb{R}^d . Let $\sigma_1, \dots, \sigma_d$ be generators of $\Gamma \cong \mathbb{Z}^d$. It follows from the periodicity of Φ_0 that $\langle \sigma_i, \boldsymbol{\xi} \rangle_{\mathbb{R}^d} = 0$ ($1 \leq i \leq d$). Hence we conclude $\boldsymbol{\xi} = \mathbf{0}$. Namely, we have proved the positive definiteness of the Hessian matrix.

This implies that the function $F_x(\cdot) : \text{Hom}(\Gamma, \mathbb{R}) \rightarrow (0, \infty)$ is strictly convex for every $x \in V_0$. Moreover, it is easily observed that

$$\lim_{|\lambda|_{\mathbb{R}^d} \rightarrow \infty} F_x(\lambda) = \infty \quad (x \in X_0),$$

due to its definition. Consequently, we know that there exists a unique minimizer $\lambda_* = \lambda_*(x) \in \text{Hom}(\Gamma, \mathbb{R})$ of $F_x(\lambda)$ for each $x \in V_0$, thereby completing the proof. \square

Now consider the positive function $\mathbf{p} : E_0 \rightarrow (0, 1]$ given by (1.5). By definition, we easily see that the function \mathbf{p} also gives a positive transition probability on X_0 . Thus the transition probability $\mathbf{p} : E_0 \rightarrow (0, 1]$ yields an irreducible random walk $(\Omega_{x_*}(X), \widehat{\mathbb{P}}_{x_*}, \{w_n^{(\mathbf{p})}\}_{n=0}^\infty)$ with values in X . Applying the Perron-Frobenius theorem again, we find a unique positive function $\mathbf{m} : V_0 \rightarrow (0, 1]$ satisfying (1.2). Put $\tilde{\mathbf{m}}(e) := \mathbf{p}(e)\mathbf{m}(o(e))$ ($e \in E_0$). We also denote by $\mathbf{p} : E \rightarrow (0, 1]$ and $\mathbf{m} : V \rightarrow (0, 1]$ the Γ -invariant lifts of $\mathbf{p} : E_0 \rightarrow (0, 1]$ and $\mathbf{m} : V_0 \rightarrow (0, 1]$, respectively. As in the previous section, we construct the (\mathbf{p} -)Albanese metric $g_0^{(\mathbf{p})}$ on $\Gamma \otimes \mathbb{R}$ associated with the transition probability \mathbf{p} . We take an orthonormal basis $\{\omega_1^{(\mathbf{p})}, \dots, \omega_d^{(\mathbf{p})}\}$ in $\text{Hom}(\Gamma, \mathbb{R}) (\subset (\mathcal{H}^1(X_0), \langle \cdot, \cdot \rangle_{\mathbf{p}}))$.

We introduce the transition operator $L_{(\mathfrak{p})} : C_\infty(X) \rightarrow C_\infty(X)$ associated with the transition probability \mathfrak{p} by

$$L_{(\mathfrak{p})}f(x) := \sum_{e \in E_x} \mathfrak{p}(e)f(t(e)) \quad (x \in V).$$

Recalling (3.1) and the definition of $\lambda_* = \lambda_*(x)$, we see that

$$\begin{aligned} & \left(\partial_1 F_x(\lambda_*(x)), \dots, \partial_d F_x(\lambda_*(x)) \right) \\ &= \sum_{e \in (E_0)_x} p(e) \exp \left(\lambda_*(x) [d\Phi_0(\tilde{e})]_{\Gamma \otimes \mathbb{R}} \right) d\Phi_0(\tilde{e}) = \mathbf{0} \end{aligned}$$

holds for every $x \in V_0$. This immediately leads to

$$(3.3) \quad L_{(\mathfrak{p})}\Phi_0(x) - \Phi_0(x) = \sum_{e \in E_x} \mathfrak{p}(e)d\Phi_0(e) = \mathbf{0} \quad (x \in V).$$

From this equation, one concludes that the given p -modified standard realization $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0)$ in the sense of (1.3) is the harmonic realization associated with the changed transition probability \mathfrak{p} .

Remark. Equation (3.3) implies $\rho_{\mathbb{R}}(\gamma_{\mathfrak{p}}) = \mathbf{0}$. We also emphasize that the transition probability $\mathfrak{p} : E_0 \rightarrow (0, 1]$ coincides with the original one $p : E_0 \rightarrow (0, 1]$ provided that $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}$.

Remark. In our setting, it is essential to assume that the given transition probability p is positive. Because, if it were not for the positivity of p , the assertion of Lemma 3.1 would not hold in general. (There is a case where the function $F_x(\cdot)$ has no minimizers.) On the other hand, to obtain Theorems 1.1 and 1.2, it is sufficient to impose that the given transition probability p is non-negative with $p(e) + p(\bar{e}) > 0$ ($e \in E$).

3.2. Proofs of Theorems 1.3 and 1.4. This subsection is devoted to proofs of Theorems 1.3 and 1.4. Following the argument as in [4, Theorem 2.2] for the random walk associated with the changed transition probability \mathfrak{p} , we can carry out the proof of Theorem 1.3. Though a minor change of the proof is required, the argument is a little bit easier due to $\rho_{\mathbb{R}}(\gamma_{\mathfrak{p}}) = \mathbf{0}$.

As the first step, we prove the following lemma.

Lemma 3.2. *For any $f \in C_0^\infty(\Gamma \otimes \mathbb{R})$, as $N \rightarrow \infty$, $\varepsilon \searrow 0$ and $N^2\varepsilon \searrow 0$, we have*

$$\left\| \frac{1}{N\varepsilon^2} (I - L_{(\mathfrak{p})}^N) P_\varepsilon f - P_\varepsilon \left(\frac{\Delta_{(\mathfrak{p})}}{2} f \right) \right\|_\infty^X \rightarrow 0.$$

Here $P_\varepsilon : C_\infty(\Gamma \otimes \mathbb{R}) \rightarrow C_\infty(X)$ ($0 \leq \varepsilon \leq 1$) is a scaling operator defined by

$$P_\varepsilon f(x) := f(\varepsilon\Phi_0(x)) \quad (x \in X)$$

and $\Delta_{(\mathfrak{p})}$ stands for the positive Laplacian $-\sum_{i=1}^d(\partial^2/\partial x_i^2)$ on $\Gamma \otimes \mathbb{R}$ associated with the \mathfrak{p} -Albanese metric $g_0^{(\mathfrak{p})}$.

Proof. We define the function $A^N(\Phi_0)_{ij} : V \rightarrow \mathbb{R}$ ($i, j = 1, \dots, d$, $N \in \mathbb{N}$) by

$$\begin{aligned} & A^N(\Phi_0)_{ij}(x) \\ & := \sum_{c \in \Omega_{x,N}(X)} \mathfrak{p}(c) \left(\Phi_0(t(c)) - \Phi_0(x) \right)_i \left(\Phi_0(t(c)) - \Phi_0(x) \right)_j \quad (x \in V), \end{aligned}$$

where $\mathfrak{p}(c) := \mathfrak{p}(e_1) \cdots \mathfrak{p}(e_N)$ for $c = (e_1, \dots, e_N) \in \Omega_{x,N}(X)$. Applying Taylor's expansion formula, we have

$$\begin{aligned} (I - L_{(\mathfrak{p})}^N)P_\varepsilon f(x) &= -\varepsilon \sum_{i=1}^d \frac{\partial f}{\partial x_i}(\varepsilon \Phi_0(x)) \sum_{c \in \Omega_{x,N}(X)} \mathfrak{p}(c) \left(\Phi_0(t(c)) - \Phi_0(x) \right)_i \\ (3.4) \quad & - \frac{\varepsilon^2}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2 f}{\partial x_i \partial x_j}(\varepsilon \Phi_0(x)) A^N(\Phi_0)_{ij}(x) + O((N\varepsilon)^3). \end{aligned}$$

We see that the first term of the right-hand side of (3.4) equals 0 due to the \mathfrak{p} -harmonicity of Φ_0 . Next we define the function $\mathcal{A}(\Phi_0)_{ij} : V_0 \rightarrow \mathbb{R}$ ($i, j = 1, \dots, d$) by

$$\mathcal{A}(\Phi_0)_{ij}(x) := \sum_{e \in (E_0)_x} \mathfrak{p}(e) d\Phi_0(\tilde{e})_i d\Phi_0(\tilde{e})_j \quad (x \in V).$$

We note that $\mathcal{A}(\Phi_0)_{ij}(\pi(x)) = A^1(\Phi_0)_{ij}(x)$ ($x \in V$, $i, j = 1, \dots, d$) because $A^N(\Phi_0)_{ij}$ is Γ -invariant. Then, using the \mathfrak{p} -harmonicity again,

$$A^N(\Phi_0)_{ij}(x) = \sum_{k=0}^{N-1} L_{(\mathfrak{p})}^k(\mathcal{A}(\Phi_0)_{ij})(\pi(x)) \quad (x \in V).$$

Applying the ergodic theorem for $L_{(\mathfrak{p})}$ (cf. [4, Theorem 3.2]), we have

$$\frac{1}{N} \sum_{k=0}^{N-1} L_{(\mathfrak{p})}^k(\mathcal{A}(\Phi_0)_{ij})(\pi(x)) = \sum_{x \in V_0} \mathfrak{m}(x) \mathcal{A}(\Phi_0)_{ij}(x) + O\left(\frac{1}{N}\right).$$

Then, (2.1) and \mathfrak{p} -harmonicity of Φ_0 imply

$$\sum_{x \in V_0} \mathfrak{m}(x) \mathcal{A}(\Phi_0)_{ij}(x) = \sum_{e \in E_0} \tilde{\mathfrak{m}}(e) \omega_i^{(\mathfrak{p})}(e) \omega_j^{(\mathfrak{p})}(e) = \langle\langle \omega_i^{(\mathfrak{p})}, \omega_j^{(\mathfrak{p})} \rangle\rangle_{\mathfrak{p}} = \delta_{ij}$$

for $1 \leq i, j \leq d$. Putting it all together, we obtain

$$\frac{1}{N\varepsilon^2} (I - L_{(\mathfrak{p})}^N)P_\varepsilon f = P_\varepsilon \left(\frac{\Delta_{(\mathfrak{p})}}{2} f \right) + O(N^2\varepsilon) + O\left(\frac{1}{N}\right).$$

Finally, letting $N \rightarrow \infty$, $\varepsilon \searrow 0$ and $N^2\varepsilon \searrow 0$, we complete the proof. \square

Lemma 3.2 immediately leads to the following lemma. (See [4, Theorem 2.1 and Lemma 4.2] for details.)

Lemma 3.3. (1) *For any $f \in C_\infty(\Gamma \otimes \mathbb{R})$, and $0 \leq s \leq t$, we have*

$$\lim_{n \rightarrow \infty} \left\| L_{(\mathfrak{p})}^{[nt]-[ns]} P_{n-1/2} f - P_{n-1/2} e^{-(t-s)\Delta_{(\mathfrak{p})}/2} f \right\|_\infty^X = 0.$$

(2) *We fix $0 \leq t_1 < \dots < t_\ell < \infty$ ($\ell \in \mathbb{N}$). Then,*

$$(\mathfrak{X}_{t_1}^{(n)}, \dots, \mathfrak{X}_{t_\ell}^{(n)}) \xrightarrow{(d)} (B_{t_1}^{(\mathfrak{p})}, \dots, B_{t_\ell}^{(\mathfrak{p})}) \quad (n \rightarrow \infty),$$

where $(B_t^{(\mathfrak{p})})_{t \geq 0}$ is a $(\Gamma \otimes \mathbb{R}, g_0^{(\mathfrak{p})})$ -valued standard Brownian motion with $B_0^{(\mathfrak{p})} = \mathbf{0}$.

Having obtained Lemma 3.3, it is sufficient to show the tightness of $\{\mathfrak{P}^{(n)}\}_{n=1}^\infty$ for completing the proof of Theorem 1.3.

Lemma 3.4. *The sequence $\{\mathfrak{P}^{(n)}\}_{n=1}^\infty$ is tight in $C_0([0, \infty), (\Gamma \otimes \mathbb{R}, g_0^{(\mathfrak{p})}))$.*

Proof. Throughout the proof, C denotes a positive constant that may change at every occurrence. We put $\|d\Phi_0\|_\infty := \max_{e \in E_0} \|d\Phi_0(\tilde{e})\|_{g_0^{(\mathfrak{p})}}$.

By virtue of the celebrated Kolmogorov's criterion, it is sufficient to show that there exists some positive constant C independent of n such that

$$(3.5) \quad \mathbb{E}^{\widehat{\mathbb{P}}_{x_*}} \left[\|\mathfrak{X}_t^{(n)} - \mathfrak{X}_s^{(n)}\|_{g_0^{(\mathfrak{p})}}^4 \right] \leq C(t-s)^2 \quad (0 \leq s \leq t, n \in \mathbb{N}).$$

We split the proof into two cases: **(I)** : $t-s < n^{-1}$, **(II)** : $t-s \geq n^{-1}$.

First we consider the case **(I)**. In both cases $ns \geq [nt]$ and $ns < [nt]$, we have

$$\begin{aligned} \|\mathfrak{X}_t^{(n)} - \mathfrak{X}_s^{(n)}\|_{g_0^{(\mathfrak{p})}} &\leq n^{1/2}(t-s) \left\{ \|\xi_{[nt]+1}^{(\mathfrak{p})} - \xi_{[nt]}^{(\mathfrak{p})}\|_{g_0^{(\mathfrak{p})}} + \|\xi_{[nt]}^{(\mathfrak{p})} - \xi_{[nt]-1}^{(\mathfrak{p})}\|_{g_0^{(\mathfrak{p})}} \right\} \\ &\leq 2\|d\Phi_0\|_\infty n^{1/2}(t-s). \end{aligned}$$

Noting $n^2(t-s)^2 < 1$, we obtain the desired estimate (3.5) for case **(I)**.

Next we consider the case **(II)**. Let \mathcal{F} be the fundamental domain in X containing $x_* \in V$ and define $\mathfrak{M}_i^\ell = \mathfrak{M}_i^\ell(\Phi_0) : V \rightarrow \mathbb{R}$ ($i = 1, \dots, d$, $\ell = 1, 2, 3, 4$) by

$$\mathfrak{M}_i^\ell(x) := \sum_{e \in E_x} \mathfrak{p}(e) d\Phi_0(e)_i^\ell \quad (x \in V).$$

We note that \mathfrak{M}_i^ℓ is Γ -invariant and $\|\mathfrak{M}_i^\ell\|_\infty^X \leq \|d\Phi_0\|_\infty^\ell$ ($i = 1, \dots, d$). Moreover, we obtain $\mathfrak{M}_i^1 \equiv 0$ ($i = 1, \dots, d$) due to the \mathfrak{p} -harmonicity of Φ_0 .

Here we give a bound on $\mathbb{E}^{\widehat{\mathbb{P}}_{x^*}} \left[\left\| \mathcal{X}_{\frac{M}{n}}^{(n)} - \mathcal{X}_{\frac{N}{n}}^{(n)} \right\|_{g_0^{(\mathfrak{p})}}^4 \right]$ ($n \in \mathbb{N}$, $M \geq N \in \mathbb{N}$). First of all, we have

$$(3.6) \quad \begin{aligned} & \mathbb{E}^{\widehat{\mathbb{P}}_{x^*}} \left[\left\| \mathcal{X}_{\frac{M}{n}}^{(n)} - \mathcal{X}_{\frac{N}{n}}^{(n)} \right\|_{g_0^{(\mathfrak{p})}}^4 \right] \\ & \leq Cn^{-2} \max_{i=1, \dots, d} \max_{x \in \mathcal{F}} \left\{ \sum_{c \in \Omega_{x, M-N}(X)} \mathfrak{p}(c) \left(\Phi_0(t(c)) - \Phi_0(x) \right)_i^4 \right\}. \end{aligned}$$

Now fix $i = 1, \dots, d$ and $x \in \mathcal{F}$. For $k = 1, \dots, M - N$, we have

$$(3.7) \quad \begin{aligned} & \sum_{c \in \Omega_{x, k}(X)} \mathfrak{p}(c) \left(\Phi_0(t(c)) - \Phi_0(x) \right)_i^4 \\ & = \sum_{c' \in \Omega_{x, k-1}(X)} \mathfrak{p}(c') \sum_{e \in E_{t(c')}} \mathfrak{p}(e) \\ & \quad \times \left\{ \left(\Phi_0(t(e)) - \Phi_0(o(e)) \right)_i + \left(\Phi_0(o(e)) - \Phi_0(x) \right)_i \right\}^4 \\ & = \sum_{c' \in \Omega_{x, k-1}(X)} \mathfrak{p}(c') \mathfrak{M}_i^4(t(c')) \\ & \quad + 4 \sum_{c' \in \Omega_{x, k-1}(X)} \mathfrak{p}(c') \left(\Phi_0(t(c')) - \Phi_0(x) \right)_i \mathfrak{M}_i^3(t(c')) \\ & \quad + 6 \sum_{c' \in \Omega_{x, k-1}(X)} \mathfrak{p}(c') \left(\Phi_0(t(c')) - \Phi_0(x) \right)_i^2 \mathfrak{M}_i^2(t(c')) \\ & \quad + \sum_{c' \in \Omega_{x, k-1}(X)} \mathfrak{p}(c') \left(\Phi_0(t(c')) - \Phi_0(x) \right)_i^4 \\ & \leq \sum_{c' \in \Omega_{x, k-1}(X)} \mathfrak{p}(c') \left(\Phi_0(t(c')) - \Phi_0(x) \right)_i^4 + \|\mathfrak{M}_i^4\|_\infty^X \\ & \quad + 4\|d\Phi_0\|_\infty \|\mathfrak{M}_i^3\|_\infty^X + 6\|\mathfrak{M}_i^2\|_\infty^X \sum_{c \in \Omega_{x, k-1}(X)} \mathfrak{p}(c) \left(\Phi_0(t(c)) - \Phi_0(x) \right)_i^2. \end{aligned}$$

Moreover, the \mathfrak{p} -harmonicity implies

$$(3.8) \quad \begin{aligned} & \sum_{c \in \Omega_{x, k-1}(X)} \mathfrak{p}(c) \left(\Phi_0(t(c)) - \Phi_0(x) \right)_i^2 \\ & = \sum_{c' \in \Omega_{x, k-2}(X)} \mathfrak{p}(c') \left\{ \left(\Phi_0(t(c')) - \Phi_0(x) \right)_i^2 + \mathfrak{M}_i^2(t(c')) \right\} \\ & \leq (k-1) \|d\Phi_0\|_\infty^2. \end{aligned}$$

It follows from (3.7) and (3.8) that

$$\begin{aligned}
 & \sum_{c \in \Omega_{x,k}(X)} \mathfrak{p}(c) \left(\Phi_0(t(c)) - \Phi_0(x) \right)_i^4 \\
 & \leq \sum_{c \in \Omega_{x,k-1}(X)} \mathfrak{p}(c) \left(\Phi_0(t(c)) - \Phi_0(x) \right)_i^4 \\
 & \quad + 5 \|d\Phi_0\|_\infty^4 + 6 \|d\Phi_0\|_\infty^4 (k-1) \\
 (3.9) \quad & \leq Ck^2.
 \end{aligned}$$

Putting $k = M - N$, $M = [nt] + i$, $N = [ns] + j$ ($i, j = 0, 1$) and combining (3.6) with (3.9), we obtain

$$\begin{aligned}
 \mathbb{E}^{\widehat{\mathbb{P}}_{x^*}} \left[\left\| \mathfrak{X}_t^{(n)} - \mathfrak{X}_s^{(n)} \right\|_{g_0^{(p)}}^4 \right] & \leq \mathbb{E}^{\widehat{\mathbb{P}}_{x^*}} \left[\max_{i,j=0,1} \left\| \mathcal{X}_{\frac{[nt]+i}{n}}^{(n)} - \mathcal{X}_{\frac{[ns]+j}{n}}^{(n)} \right\|_{g_0^{(p)}}^4 \right] \\
 & \leq Cn^{-2} ([nt] - [ns] + 1)^2 \\
 & \leq C \left(t - s + \frac{2}{n} \right)^2 \leq C \{ (t - s) + 2(t - s) \}^2 \\
 & \leq C(t - s)^2,
 \end{aligned}$$

where we used $[nt] - [ns] \leq n(t - s) + 1$ and $n^{-1} \leq t - s$. Therefore, we have shown the the desired estimate (3.5) for case **(II)**. \square

Next we prove Theorem 1.4.

Proof of Theorem 1.4. For $n \in \mathbb{N}$ and $x, y \in V$, we have

$$\begin{aligned}
 \mathfrak{p}(n, x, y) & = \sum_{\substack{(e_1, \dots, e_n) \in \Omega_{x,n}(X) \\ o(e_1) = x, t(e_n) = y}} \mathfrak{p}(e_1) \cdots \mathfrak{p}(e_n) \\
 & = \sum_{\substack{(e_1, \dots, e_n) \in \Omega_{x,n}(X) \\ o(e_1) = x, t(e_n) = y}} p(e_1) \cdots p(e_n) \cdot \exp \left(\sum_{i=1}^n \lambda_*(o(e_i)) [d\Phi_0(\tilde{e}_i)]_{\Gamma \otimes \mathbb{R}} \right) \\
 & \quad \times F_{o(e_1)}(\lambda_*(o(e_1)))^{-1} \cdots F_{o(e_n)}(\lambda_*(o(e_n)))^{-1} \\
 & = \sum_{\substack{(e_1, \dots, e_n) \in \Omega_{x,n}(X) \\ o(e_1) = x, t(e_n) = y}} p(e_1) \cdots p(e_n) \\
 & \quad \times \exp \left(\sum_{i=1}^n \lambda_*(o(e_i)) [d\Phi_0(\tilde{e}_i)]_{\Gamma \otimes \mathbb{R}} - \sum_{i=1}^n \log F_{o(e_i)}(\lambda_*(o(e_i))) \right).
 \end{aligned}$$

Using the ergodic theorem for 1-chains (cf. [7]):

$$(3.10) \quad \frac{1}{n} \sum_{i=1}^n f(e_i) = \sum_{e \in E_0} \tilde{m}(e) f(e) + O\left(\frac{1}{n}\right) \quad (f : E_0 \rightarrow \mathbb{R}),$$

we obtain

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \left(\lambda_*(o(e_i)) [d\Phi_0(\tilde{e}_i)]_{\Gamma \otimes \mathbb{R}} - \log F_{o(e_i)}(\lambda_*(o(e_i))) \right) \\ &= \sum_{e \in E_0} \tilde{m}(e) \left(\lambda_*(o(e)) [d\Phi_0(\tilde{e})]_{\Gamma \otimes \mathbb{R}} - \log F_{o(e)}(\lambda_*(o(e))) \right) + O\left(\frac{1}{n}\right) \\ &= \sum_{\mathbf{x} \in V_0} m(\mathbf{x}) \left(\lambda_*(\mathbf{x}) \left[\sum_{e \in (E_0)_{\mathbf{x}}} d\Phi_0(\tilde{e}) \right]_{\Gamma \otimes \mathbb{R}} - \log F_{\mathbf{x}}(\lambda_*(\mathbf{x})) \right) + O\left(\frac{1}{n}\right) \\ &= \sum_{\mathbf{x} \in V_0} m(\mathbf{x}) \left(\lambda_*(\mathbf{x}) [\rho_{\mathbb{R}}(\gamma_p)]_{\Gamma \otimes \mathbb{R}} - \log F_{\mathbf{x}}(\lambda_*(\mathbf{x})) \right) + O\left(\frac{1}{n}\right) \end{aligned}$$

for $x, y \in V$. Here we used the p -modified harmonicity of Φ_0 for the final line. Finally, we obtain

$$\begin{aligned} \mathfrak{p}(n, x, y) &= p(n, x, y) \exp \left(n \sum_{\mathbf{x} \in V_0} m(\mathbf{x}) \right. \\ &\quad \left. \times \left(\lambda_*(\mathbf{x}) [\rho_{\mathbb{R}}(\gamma_p)]_{\Gamma \otimes \mathbb{R}} - \log F_{\mathbf{x}}(\lambda_*(\mathbf{x})) \right) + O(1) \right) \end{aligned}$$

for $x, y \in V$. This completes the proof. \square

Remark. Let us consider a special case where the Γ -crystal lattice X is given by a covering graph of an ℓ -bouquet graph ($\ell \in \mathbb{N}$) consisting of one vertex $\mathbf{x} \in V_0$ and ℓ -loops. Without using the ergodic theorem (3.10) in the proof of Theorem 1.4, we also obtain

$$\begin{aligned} \mathfrak{p}(n, x, y) &= \sum_{\substack{(e_1, \dots, e_n) \in \Omega_{x, n}(X) \\ o(e_1) = x, t(e_n) = y}} p(e_1) \cdots p(e_n) \\ &\quad \times \exp \left(\sum_{i=1}^n \lambda_*(\mathbf{x}) [d\Phi_0(\tilde{e}_i)]_{\Gamma \otimes \mathbb{R}} \right) \cdot F_{\mathbf{x}}(\lambda_*(\mathbf{x}))^{-n} \\ (3.11) \quad &= p(n, x, y) \exp \left(\lambda_*(\mathbf{x}) [\Phi_0(y) - \Phi_0(x)]_{\Gamma \otimes \mathbb{R}} \right) \cdot F_{\mathbf{x}}(\lambda_*(\mathbf{x}))^{-n} \end{aligned}$$

for every $n \in \mathbb{N}$, $x, y \in V$.

3.3. A relationship to a discrete analogue of Girsanov’s theorem.

In this subsection, we discuss a relationship between our formula (3.11) and a discrete analogue of Girsanov’s theorem due to Fujita [3].

Let $X = (V, E)$ be a crystal lattice covered with a one-bouquet graph $X_0 = (V_0, E_0)$; $V_0 = \{\mathbf{x}\}$ and $E_0 = \{e, \bar{e}\}$, by the group action $\Gamma = \langle \sigma \rangle \cong \mathbb{Z}^1$. We consider a random walk on X_0 with the transition probability

$$p(e) = p \quad \text{and} \quad p(\bar{e}) = 1 - p \quad (0 < p < 1).$$

We introduce a bijective linear map $\rho_{\mathbb{R}} : H_1(X_0, \mathbb{R}) \rightarrow \Gamma \otimes \mathbb{R} (\cong \mathbb{R}^1)$ by $\rho_{\mathbb{R}}([e]) = \sigma$. Then we have $\gamma_p = (2p - 1)[e]$ and $\rho_{\mathbb{R}}(\gamma_p) = (2p - 1)\sigma$. Let $\{u\} \subset \text{Hom}(\Gamma, \mathbb{R}) = (H^1(X_0, \mathbb{R}), \langle \cdot, \cdot \rangle_p)$ be a dual basis of $\{\sigma \otimes 1 = \sigma\} \subset \Gamma \otimes \mathbb{R}$. We easily see that $\langle u, u \rangle_p = 4p(1 - p)$. Hence the orthogonalization $\{v\} \subset \text{Hom}(\Gamma, \mathbb{R})$ of $\{u\}$ is given by

$$v = \frac{1}{\sqrt{4p(1 - p)}}u.$$

To the end, we identify $\lambda v \in \text{Hom}(\Gamma, \mathbb{R})$ with $\lambda \in \mathbb{R}$. We denote by $\{\mathbf{v}\} \subset \Gamma \otimes \mathbb{R}$ the dual basis of $\{v\}$. Then we observe that the realization $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}; \{\mathbf{v}\})$ defined by

$$d\Phi_0(\tilde{e}) := \sigma = \frac{1}{\sqrt{4p(1 - p)}}\mathbf{v}$$

is the modified standard realization of X .

We now consider the function $F = F_{\mathbf{x}}(\lambda)$ defined by (1.4), that is,

$$F_{\mathbf{x}}(\lambda) = p \exp\left(\frac{\lambda}{\sqrt{4p(1 - p)}}\right) + (1 - p) \exp\left(-\frac{\lambda}{\sqrt{4p(1 - p)}}\right) \quad (\lambda \in \mathbb{R}).$$

Then we know that the minimizer $\lambda_* = \lambda_*(\mathbf{x})$ and $F_{\mathbf{x}}(\lambda_*)$ are given by

$$\lambda_* = \sqrt{p(1 - p)} \log \frac{p - 1}{p}, \quad F_{\mathbf{x}}(\lambda_*) = \sqrt{4p(1 - p)}.$$

We fix $x \in V$ satisfying $\Phi_0(x) = \mathbf{0}$. For $y \in V$, we write $\Phi_0(y) = k(y)\mathbf{v}$. Then the formula (3.11) implies

$$p(n, x, y) = p(n, x, y) \cdot \left(\frac{p - 1}{p}\right)^{-k(y)/2} \cdot (\sqrt{4p(1 - p)})^{-n} \quad (n \in \mathbb{N}, y \in V).$$

In [3, page 115], the above formula is called a *discrete analogue of Girsanov’s theorem* for a non-symmetric random walk $\{Z_n\}_{n=0}^{\infty}$ on \mathbb{Z}^1 given by the sum of independent random variables $\{\xi_i\}_{i=1}^{\infty}$ with $\mathbb{P}(\xi_i = 1) = p$ and $\mathbb{P}(\xi_i = -1) = 1 - p$ ($i = 1, 2, \dots$). Hence we may regard the formula (3.11) as a generalization of the above discrete Girsanov’s theorem to the case of non-symmetric random walks on the ℓ -bouquet graph.

4. EXAMPLES

In [4, Section 7], several examples of the modified standard realization of crystal lattices associated with the non-symmetric random walks are discussed. In this final section, we give two concrete examples of non-symmetric random walks on crystal lattices and calculate the changed transition probability \mathbf{p} for each example.

4.1. The hexagonal lattice. In this subsection, we consider the *hexagonal lattice*, as a typical example of crystal lattices. Let $X = (V, E)$ be a hexagonal lattice, $V = \mathbb{Z}^2 = \{\mathbf{x} = (x_1, x_2) \mid x_1, x_2 \in \mathbb{Z}\}$ and

$$E = \{(\mathbf{x}, \mathbf{y}) \in V^2 \mid \mathbf{x} - \mathbf{y} = \pm(1, 0), \mathbf{x} - \mathbf{y} = (0, (-1)^{x_1+x_2})\}.$$

(See Figure 1). We introduce a non-symmetric random walk on X in the following way. If $\mathbf{x} = (x_1, x_2) \in V$ is a vertex so that $x_1 + x_2$ is even, set

$$p(\mathbf{x}, \mathbf{x} + (1, 0)) = \frac{1}{2}, \quad p(\mathbf{x}, \mathbf{x} - (0, 1)) = \frac{1}{3}, \quad p(\mathbf{x}, \mathbf{x} - (1, 0)) = \frac{1}{6}.$$

If $x_1 + x_2$ is odd, set

$$p(\mathbf{x}, \mathbf{x} - (1, 0)) = \frac{1}{6}, \quad p(\mathbf{x}, \mathbf{x} + (0, 1)) = \frac{1}{3}, \quad p(\mathbf{x}, \mathbf{x} + (1, 0)) = \frac{1}{2}.$$

We see that X is invariant under the action $\Gamma = \langle \sigma_1, \sigma_2 \rangle (\cong \mathbb{Z}^2)$ generated by

$$\sigma_1(\mathbf{x}) = \mathbf{x} + (1, 1), \quad \sigma_2(\mathbf{x}) = \mathbf{x} + (-1, 1) \quad (\mathbf{x} \in V).$$

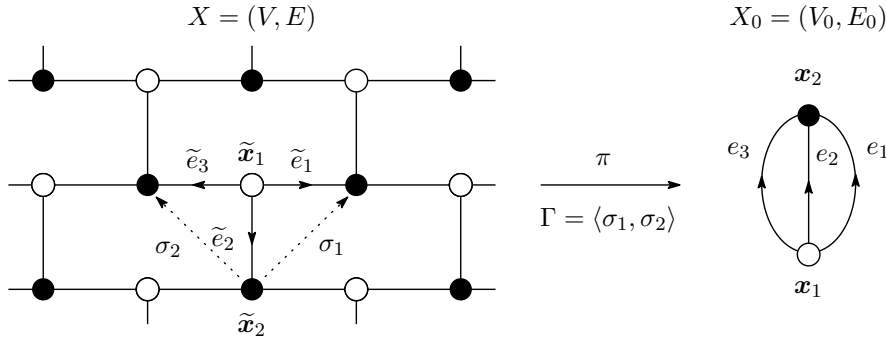


FIGURE 1. Hexagonal lattice and its quotient.

The quotient graph $X_0 = \Gamma \backslash X$ is a finite graph $X_0 = (V_0, E_0)$ consisting of two vertices $\{\mathbf{x}_1, \mathbf{x}_2\}$ with three multiple edges $E_0 = (E_0)_{\mathbf{x}_1} \cup (E_0)_{\mathbf{x}_2} = \{e_1, e_2, e_3\} \cup \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$.

We put $[c_1] := [e_1 * \bar{e}_2]$ and $[c_2] := [e_3 * \bar{e}_2]$. Then, the first homology group $H_1(X_0, \mathbb{R})$ is spanned by $\{[c_1], [c_2]\}$. Solving (1.2), we have $m(\mathbf{x}_1) = m(\mathbf{x}_2) = 1/2$. We define the surjective linear map $\rho_{\mathbb{R}} : H_1(X_0, \mathbb{R}) \rightarrow \Gamma \otimes \mathbb{R} (\cong \mathbb{R}^2)$ by $\rho_{\mathbb{R}}([c_1]) := \sigma_1$, $\rho_{\mathbb{R}}([c_2]) := \sigma_2$. Thus, the homological direction γ_p and the asymptotic direction $\rho_{\mathbb{R}}(\gamma_p)$ are given by

$$\gamma_p = \frac{1}{6}[c_1] - \frac{1}{6}[c_2], \quad \rho_{\mathbb{R}}(\gamma_p) = \frac{1}{6}\sigma_1 - \frac{1}{6}\sigma_2 (\neq \mathbf{0}),$$

respectively. We will determine the modified standard realization $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0)$. We set $\tilde{\mathbf{x}}_1 = (0, 0)$, $\tilde{\mathbf{x}}_2 = (0, 1)$ in V . Without loss of generality, we may put $\Phi_0(\tilde{\mathbf{x}}_1) = \mathbf{0} \in \Gamma \otimes \mathbb{R}$. By (1.3), we have

$$\Phi_0(\tilde{\mathbf{x}}_1) = \mathbf{0}, \quad \Phi_0(\tilde{\mathbf{x}}_2) = -\frac{1}{3}\sigma_1 - \frac{1}{3}\sigma_2.$$

Now let $\{v_1, v_2\}$ be an orthonormal basis in $\text{Hom}(\Gamma, \mathbb{R}) (\subset (\mathcal{H}^1(X_0), \langle \cdot, \cdot \rangle_p))$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ its dual basis in $(\Gamma \otimes \mathbb{R}, g_0)$. Then, we have

$$\sigma_1 = \frac{6\sqrt{7}}{7}\mathbf{v}_1 - \frac{9\sqrt{70}}{70}\mathbf{v}_2, \quad \sigma_2 = \frac{3\sqrt{70}}{10}\mathbf{v}_2$$

with the Albanese metric

$$\langle \sigma_1, \sigma_1 \rangle_{g_0} = \frac{63}{10}, \quad \langle \sigma_1, \sigma_2 \rangle_{g_0} = \frac{27}{10}, \quad \langle \sigma_2, \sigma_2 \rangle_{g_0} = \frac{63}{10},$$

by following the computations in [4, Subsection 7.3]. Hence, we find that the modified standard realization $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0)$ is given by $\Phi_0(\tilde{\mathbf{x}}_1) = \mathbf{0}$ and

$$\Phi_0(\tilde{\mathbf{x}}_2) = -\frac{2\sqrt{7}}{7}\mathbf{v}_1 - \frac{\sqrt{70}}{7}\mathbf{v}_2.$$

Now we are in a position to consider the function F introduced in (1.4). We identify $\lambda \in \text{Hom}(\Gamma, \mathbb{R})$ with $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. Then, we have

$$\begin{aligned} & F_{\mathbf{x}_1}(\lambda) \\ &= \frac{1}{2} \exp\left(\frac{4\sqrt{7}}{7}\lambda_1 - \frac{19\sqrt{70}}{70}\lambda_2\right) \\ & \quad + \frac{1}{3} \exp\left(-\frac{2\sqrt{7}}{7}\lambda_1 - \frac{\sqrt{70}}{7}\lambda_2\right) + \frac{1}{6} \exp\left(-\frac{2\sqrt{7}}{7}\lambda_1 + \frac{11\sqrt{70}}{70}\lambda_2\right), \\ (4.1) \quad & F_{\mathbf{x}_2}(\lambda) \\ &= \frac{1}{6} \exp\left(-\frac{4\sqrt{7}}{7}\lambda_1 + \frac{19\sqrt{70}}{70}\lambda_2\right) \\ & \quad + \frac{1}{3} \exp\left(\frac{2\sqrt{7}}{7}\lambda_1 + \frac{\sqrt{70}}{7}\lambda_2\right) + \frac{1}{2} \exp\left(\frac{2\sqrt{7}}{7}\lambda_1 - \frac{11\sqrt{70}}{70}\lambda_2\right). \end{aligned}$$

Differentiating both sides of (4.1) with respect to λ_1 and λ_2 , we have

$$\begin{aligned}
\partial_1 F_{\mathbf{x}_1}(\lambda) &= \frac{2\sqrt{7}}{7} \exp\left(\frac{4\sqrt{7}}{7}\lambda_1 - \frac{19\sqrt{70}}{70}\lambda_2\right) \\
&\quad - \frac{2\sqrt{7}}{21} \exp\left(-\frac{2\sqrt{7}}{7}\lambda_1 - \frac{\sqrt{70}}{7}\lambda_2\right) \\
&\quad - \frac{\sqrt{7}}{21} \exp\left(-\frac{2\sqrt{7}}{7}\lambda_1 + \frac{11\sqrt{70}}{70}\lambda_2\right), \\
\partial_2 F_{\mathbf{x}_1}(\lambda) &= -\frac{19\sqrt{70}}{140} \exp\left(\frac{4\sqrt{7}}{7}\lambda_1 - \frac{19\sqrt{70}}{70}\lambda_2\right) \\
&\quad - \frac{\sqrt{70}}{21} \exp\left(-\frac{2\sqrt{7}}{7}\lambda_1 - \frac{\sqrt{70}}{7}\lambda_2\right) \\
&\quad + \frac{11\sqrt{70}}{420} \exp\left(-\frac{2\sqrt{7}}{7}\lambda_1 + \frac{11\sqrt{70}}{70}\lambda_2\right), \\
\partial_1 F_{\mathbf{x}_2}(\lambda) &= -\frac{2\sqrt{7}}{21} \exp\left(-\frac{4\sqrt{7}}{7}\lambda_1 + \frac{19\sqrt{70}}{70}\lambda_2\right) \\
&\quad + \frac{2\sqrt{7}}{21} \exp\left(\frac{2\sqrt{7}}{7}\lambda_1 + \frac{\sqrt{70}}{7}\lambda_2\right) \\
&\quad + \frac{\sqrt{7}}{7} \exp\left(\frac{2\sqrt{7}}{7}\lambda_1 - \frac{11\sqrt{70}}{70}\lambda_2\right), \\
\partial_2 F_{\mathbf{x}_2}(\lambda) &= \frac{19\sqrt{70}}{420} \exp\left(-\frac{4\sqrt{7}}{7}\lambda_1 + \frac{19\sqrt{70}}{70}\lambda_2\right) \\
&\quad + \frac{\sqrt{70}}{21} \exp\left(\frac{2\sqrt{7}}{7}\lambda_1 + \frac{\sqrt{70}}{7}\lambda_2\right) \\
&\quad - \frac{11\sqrt{70}}{140} \exp\left(\frac{2\sqrt{7}}{7}\lambda_1 - \frac{11\sqrt{70}}{70}\lambda_2\right).
\end{aligned}$$

To find the minimizers of functions $F_{\mathbf{x}_1}(\cdot)$ and $F_{\mathbf{x}_2}(\cdot)$, it is sufficient to solve the following two algebraic equations:

$$\begin{cases} \partial_1 F_{\mathbf{x}_1}(\lambda_1, \lambda_2) = 0 \\ \partial_2 F_{\mathbf{x}_1}(\lambda_1, \lambda_2) = 0 \end{cases}, \quad \begin{cases} \partial_1 F_{\mathbf{x}_2}(\lambda_1, \lambda_2) = 0 \\ \partial_2 F_{\mathbf{x}_2}(\lambda_1, \lambda_2) = 0 \end{cases}.$$

Solving these equations, the minimizers $\lambda_*(\mathbf{x}_1)$, $\lambda_*(\mathbf{x}_2)$ of $F_{\mathbf{x}_1}(\cdot)$, $F_{\mathbf{x}_2}(\cdot)$ are given by

$$\begin{aligned}
(4.2) \quad \lambda_*(\mathbf{x}_1) &= \left(\frac{\sqrt{7}}{14} \log 26 + \frac{\sqrt{7}}{6} \log \frac{14}{3}, \frac{\sqrt{70}}{21} \log 26\right), \\
\lambda_*(\mathbf{x}_2) &= \left(\frac{\sqrt{7}}{14} \log \frac{3}{26} - \frac{\sqrt{7}}{6} \log 14, \frac{\sqrt{70}}{21} \log \frac{3}{26}\right),
\end{aligned}$$

respectively. Then, it follows from (4.1) and (4.2) that

$$F_{\mathbf{x}_1}(\lambda_*(\mathbf{x}_1)) = 7 \cdot 26^{-13/21} \cdot \left(\frac{14}{3}\right)^{-1/3},$$

$$F_{\mathbf{x}_2}(\lambda_*(\mathbf{x}_2)) = \frac{21}{26} \cdot 14^{-1/3} \cdot \left(\frac{3}{26}\right)^{-8/21}.$$

Finally, we determine the changed transition probability \mathbf{p} by

$$\begin{aligned} \mathbf{p}(e_1) &= \frac{1}{3}, & \mathbf{p}(e_2) &= \frac{1}{21}, & \mathbf{p}(e_3) &= \frac{13}{21}, \\ \mathbf{p}(\bar{e}_1) &= \frac{1}{3}, & \mathbf{p}(\bar{e}_2) &= \frac{1}{21}, & \mathbf{p}(\bar{e}_3) &= \frac{13}{21}. \end{aligned}$$

Then, the invariant measure $\mathbf{m} : \{\mathbf{x}_1, \mathbf{x}_2\} \rightarrow (0, 1]$ is also given by $\mathbf{m}(\mathbf{x}_1) = \mathbf{m}(\mathbf{x}_2) = 1/2$. Hence we know that the random walk associated with the changed transition probability \mathbf{p} is \mathbf{m} -symmetric, that is,

$$\mathbf{p}(e)\mathbf{m}(o(e)) = \mathbf{p}(\bar{e})\mathbf{m}(t(e)) \quad (e \in E_0).$$

It automatically implies $\gamma_{\mathbf{p}} = 0$ and also $\rho_{\mathbb{R}}(\gamma_{\mathbf{p}}) = \mathbf{0}$.

4.2. The dice lattice. In this subsection, we discuss a non-symmetric random walk on an infinite graph called the *dice lattice* or the *dice graph*. The dice graph $X = (V, E)$ is one of abelian covering graphs which has a free action by the lattice group $\Gamma \cong \mathbb{Z}^2$ generated by σ_1, σ_2 , and the corresponding quotient graph $X_0 = (V_0, E_0) := \Gamma \backslash X$ is a finite graph consisting of three vertices $V_0 = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$. (See Figure 2 or the description in [9].) In view of the shape of the quotient graph X_0 , we may regard the dice lattice X as something like a “*hybrid*” of a triangular lattice and a hexagonal lattice.

From now on, we consider a non-symmetric random walk on X by giving the transition probability on the quotient X_0 in the following way. We set

$$\begin{aligned} p(e_1) &= \frac{1}{4}, & p(e_2) &= \frac{1}{6}, & p(e_3) &= \frac{1}{12}, & p(e_4) &= \frac{1}{4}, & p(e_5) &= \frac{1}{6}, & p(e_6) &= \frac{1}{12}, \\ p(\bar{e}_1) &= \frac{1}{6}, & p(\bar{e}_2) &= \frac{1}{3}, & p(\bar{e}_3) &= \frac{1}{2}, & p(\bar{e}_4) &= \frac{1}{6}, & p(\bar{e}_5) &= \frac{1}{3}, & p(\bar{e}_6) &= \frac{1}{2}. \end{aligned}$$

Solving (1.2), we have $m(\mathbf{x}) = 1/2$, $m(\mathbf{y}) = m(\mathbf{z}) = 1/4$.

Next we define four 1-cycles $[c_1], [c_2], [c_3], [c_4]$ on X_0 by

$$[c_1] := [e_1 * \bar{e}_2], \quad [c_2] := [e_3 * \bar{e}_2], \quad [c_3] := [e_4 * \bar{e}_5], \quad [c_4] := [e_6 * \bar{e}_5].$$

Then $\{[c_1], [c_2], [c_3], [c_4]\}$ spans the first homology group $H_1(X_0, \mathbb{R})$. We define the linear map $\rho_{\mathbb{R}}$ from $H_1(X_0, \mathbb{R})$ onto $\Gamma \otimes \mathbb{R} \cong \mathbb{R}^2$ by

$$\rho_{\mathbb{R}}([c_1]) = \sigma_1 - \sigma_2, \quad \rho_{\mathbb{R}}([c_2]) = -\sigma_2, \quad \rho_{\mathbb{R}}([c_3]) = \sigma_2, \quad \rho_{\mathbb{R}}([c_4]) = \sigma_2 - \sigma_1.$$

Then, the homological direction γ_p and the asymptotic direction $\rho_{\mathbb{R}}(\gamma_p)$ of the random walk on X_0 are calculated as

$$\gamma_p = \frac{1}{12}([c_1] - [c_2] + [c_3] - [c_4]),$$

$$\rho_{\mathbb{R}}(\gamma_p) = \frac{1}{12}\{(\sigma_1 - \sigma_2) - (-\sigma_2) + \sigma_2 - (\sigma_2 - \sigma_1)\} = \frac{1}{6}\sigma_1 (\neq \mathbf{0}),$$

respectively.

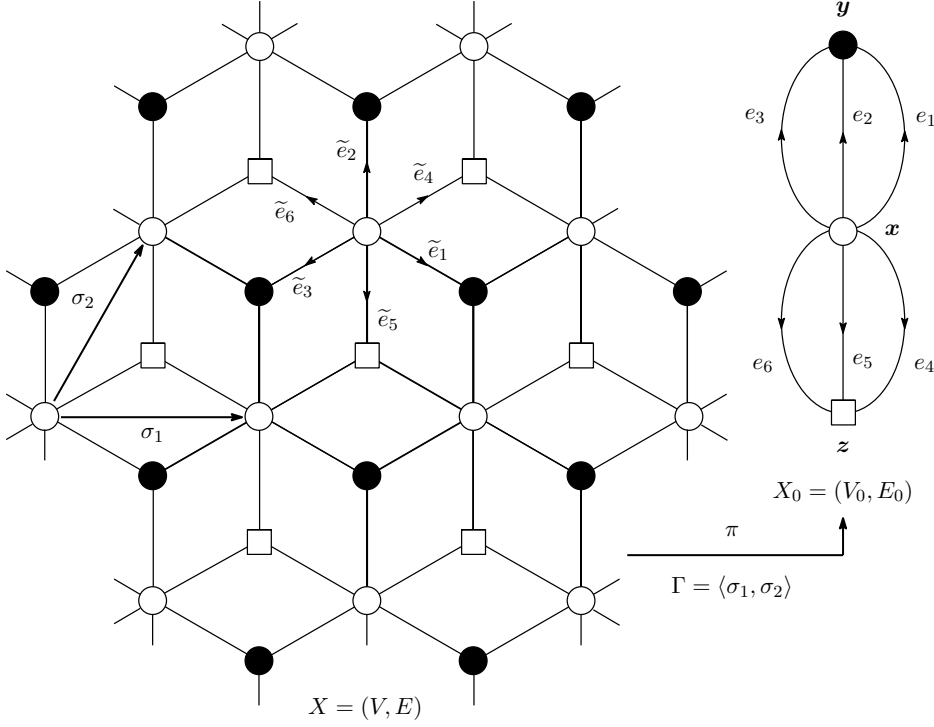


FIGURE 2. Dice lattice and its quotient.

Now we determine the modified standard realization $\Phi_0 : X \rightarrow \Gamma \otimes \mathbb{R}$. Here we may put $\Phi_0(o(\tilde{e}_i)) = \mathbf{0}$ ($i = 1, 2, 3, 4, 5, 6$), without loss of generality. Noting (1.3) and the group action Γ , we have

$$(4.3) \quad \begin{aligned} \Phi_0(t(\tilde{e}_1)) &= \frac{2}{3}\sigma_1 - \frac{1}{3}\sigma_2, & \Phi_0(t(\tilde{e}_2)) &= -\frac{1}{3}\sigma_1 + \frac{2}{3}\sigma_2, \\ \Phi_0(t(\tilde{e}_3)) &= -\frac{1}{3}\sigma_1 - \frac{1}{3}\sigma_2, & \Phi_0(t(\tilde{e}_4)) &= \frac{1}{3}\sigma_1 + \frac{1}{3}\sigma_2, \\ \Phi_0(t(\tilde{e}_5)) &= \frac{1}{3}\sigma_1 - \frac{2}{3}\sigma_2, & \Phi_0(t(\tilde{e}_6)) &= -\frac{2}{3}\sigma_1 + \frac{1}{3}\sigma_2. \end{aligned}$$

Let $\{\omega_1, \omega_2, \omega_3, \omega_4\} \subset H^1(X_0, \mathbb{R})$ be the dual basis of $\{[c_1], [c_2], [c_3], [c_4]\}$, that is, $\omega_i([c_j]) = \delta_{ij}$ ($1 \leq i, j \leq 4$). Recalling that each ω_i is a modified harmonic 1-form, we have

$$\begin{aligned} \omega_1(e_1) &= \frac{3}{4}, & \omega_1(e_2) &= \omega_1(e_3) = -\frac{1}{4}, & \omega_1(e_4) &= \omega_1(e_5) = \omega_1(e_6) = -\frac{1}{12}, \\ \omega_2(e_1) &= \omega_2(e_2) = -\frac{5}{12}, & \omega_2(e_3) &= \frac{7}{12}, & \omega_2(e_4) &= \omega_2(e_5) = \omega_2(e_6) = \frac{1}{12}, \\ \omega_3(e_1) &= \omega_3(e_2) = \omega_3(e_3) = -\frac{1}{12}, & \omega_3(e_4) &= \frac{3}{4}, & \omega_3(e_5) &= \omega_3(e_6) = -\frac{1}{4}, \\ \omega_4(e_1) &= \omega_4(e_2) = \omega_4(e_3) = \frac{1}{12}, & \omega_4(e_4) &= \omega_4(e_5) = -\frac{5}{12}, & \omega_4(e_6) &= \frac{7}{12}. \end{aligned}$$

Then, the direct computation gives us

$$\begin{aligned} \langle\langle \omega_1, \omega_1 \rangle\rangle &= \frac{1}{9}, & \langle\langle \omega_1, \omega_2 \rangle\rangle &= -\frac{1}{18}, \\ \langle\langle \omega_1, \omega_3 \rangle\rangle &= -\frac{1}{72}, & \langle\langle \omega_1, \omega_4 \rangle\rangle &= \frac{1}{72}, \\ \langle\langle \omega_2, \omega_2 \rangle\rangle &= \frac{1}{9}, & \langle\langle \omega_2, \omega_3 \rangle\rangle &= \frac{1}{72}, \\ \langle\langle \omega_2, \omega_4 \rangle\rangle &= -\frac{1}{72}, & \langle\langle \omega_3, \omega_3 \rangle\rangle &= \frac{1}{9}, \\ \langle\langle \omega_3, \omega_4 \rangle\rangle &= -\frac{1}{18}, & \langle\langle \omega_4, \omega_4 \rangle\rangle &= \frac{1}{9}. \end{aligned}$$

We introduce the basis $\{u_1, u_2\}$ in $\text{Hom}(\Gamma, \mathbb{R})$ by the dual of $\{\sigma_1 \otimes 1, \sigma_2 \otimes 1\} \subset \Gamma \otimes \mathbb{R}$. Since the dice graph X is a non-maximal abelian covering graph of X_0 with $\Gamma \cong \langle \sigma_1, \sigma_2 \rangle$, we need to find a \mathbb{Z} -basis of the lattice

$$L = \{\omega \in H^1(X_0, \mathbb{R}) \mid \omega([c]) = 0 \text{ for every cycle } \tilde{c} \text{ on } X\}.$$

It is easy to find that

$$u_1 = {}^t\rho_{\mathbb{R}}(u_1) = \omega_1 - \omega_4, \quad u_2 = {}^t\rho_{\mathbb{R}}(u_2) = -\omega_1 - \omega_2 + \omega_3 + \omega_4$$

form a \mathbb{Z} -basis of the lattice L . Carrying out the direct computation again, we have

$$(4.4) \quad \langle\langle u_1, u_1 \rangle\rangle = \frac{7}{36}, \quad \langle\langle u_1, u_2 \rangle\rangle = -\frac{1}{9}, \quad \langle\langle u_2, u_2 \rangle\rangle = \frac{2}{9}.$$

Let $\{v_1, v_2\}$ be the Gram–Schmidt orthogonalization of the basis $\{u_1, u_2\} \subset \text{Hom}(\Gamma, \mathbb{R})$. By (4.4), we have

$$v_1 = \frac{6\sqrt{7}}{7}u_1, \quad v_2 = \frac{6\sqrt{70}}{35}u_1 + \frac{3\sqrt{70}}{10}u_2.$$

We denote by $\{\mathbf{v}_1, \mathbf{v}_2\}$ the dual basis of $\{v_1, v_2\}$ in $\Gamma \otimes \mathbb{R}$. Then we obtain

$$(4.5) \quad \sigma_1 = \frac{6\sqrt{7}}{7}\mathbf{v}_1, \quad \sigma_2 = \frac{6\sqrt{70}}{35}\mathbf{v}_1 + \frac{3\sqrt{70}}{10}\mathbf{v}_2$$

Combining (4.5) with (4.3), we finally determine the modified standard realization $\Phi_0 : X \rightarrow (\Gamma \otimes \mathbb{R}, g_0) \cong (\mathbb{R}^2; \{\mathbf{v}_1, \mathbf{v}_2\})$ of X by

$$\begin{aligned} \Phi_0(t(\tilde{e}_1)) &= \frac{20\sqrt{7} - 2\sqrt{70}}{35}\mathbf{v}_1 - \frac{\sqrt{70}}{10}\mathbf{v}_2, \\ \Phi_0(t(\tilde{e}_2)) &= \frac{-10\sqrt{7} + 4\sqrt{70}}{35}\mathbf{v}_1 + \frac{\sqrt{70}}{5}\mathbf{v}_2, \\ \Phi_0(t(\tilde{e}_3)) &= \frac{-10\sqrt{7} - 2\sqrt{70}}{35}\mathbf{v}_1 - \frac{\sqrt{70}}{10}\mathbf{v}_2, \\ \Phi_0(t(\tilde{e}_4)) &= \frac{10\sqrt{7} + 2\sqrt{70}}{35}\mathbf{v}_1 + \frac{\sqrt{70}}{10}\mathbf{v}_2, \\ \Phi_0(t(\tilde{e}_5)) &= \frac{10\sqrt{7} - 4\sqrt{70}}{35}\mathbf{v}_1 - \frac{\sqrt{70}}{5}\mathbf{v}_2, \\ \Phi_0(t(\tilde{e}_6)) &= \frac{-20\sqrt{7} + 2\sqrt{70}}{35}\mathbf{v}_1 + \frac{\sqrt{70}}{10}\mathbf{v}_2. \end{aligned}$$

(See Figure 3 below.)

Now we are in a position to consider the function $F : V_0 \times \text{Hom}(\Gamma, \mathbb{R}) \rightarrow (0, \infty)$ defined by (1.4). In what follows, we identify $\lambda = \lambda_1 v_1 + \lambda_2 v_2 \in \text{Hom}(\Gamma, \mathbb{R})$ with $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. In addition, it follows from $\Phi_0(o(\tilde{e}_i)) = \mathbf{0}$ ($i = 1, 2, 3, 4, 5, 6$) that

$$(4.6) \quad \begin{aligned} d\Phi_0(\tilde{e}_1) &= \frac{20\sqrt{7} - 2\sqrt{70}}{35}\mathbf{v}_1 - \frac{\sqrt{70}}{10}\mathbf{v}_2, \\ d\Phi_0(\tilde{e}_2) &= \frac{-10\sqrt{7} + 4\sqrt{70}}{35}\mathbf{v}_1 + \frac{\sqrt{70}}{5}\mathbf{v}_2, \\ d\Phi_0(\tilde{e}_3) &= \frac{-10\sqrt{7} - 2\sqrt{70}}{35}\mathbf{v}_1 - \frac{\sqrt{70}}{10}\mathbf{v}_2, \\ d\Phi_0(\tilde{e}_4) &= \frac{10\sqrt{7} + 2\sqrt{70}}{35}\mathbf{v}_1 + \frac{\sqrt{70}}{10}\mathbf{v}_2, \\ d\Phi_0(\tilde{e}_5) &= \frac{10\sqrt{7} - 4\sqrt{70}}{35}\mathbf{v}_1 - \frac{\sqrt{70}}{5}\mathbf{v}_2, \\ d\Phi_0(\tilde{e}_6) &= \frac{-20\sqrt{7} + 2\sqrt{70}}{35}\mathbf{v}_1 + \frac{\sqrt{70}}{10}\mathbf{v}_2 \end{aligned}$$

Then, by (4.6), we have

$$\begin{aligned}
F_{\mathbf{x}}(\lambda_1, \lambda_2) &= \frac{1}{4} \exp\left(\frac{20\sqrt{7} - 2\sqrt{70}}{35} \lambda_1 - \frac{\sqrt{70}}{10} \lambda_2\right) \\
&\quad + \frac{1}{6} \exp\left(\frac{-10\sqrt{7} + 4\sqrt{70}}{35} \lambda_1 + \frac{\sqrt{70}}{5} \lambda_2\right) \\
&\quad + \frac{1}{12} \exp\left(\frac{-10\sqrt{7} - 2\sqrt{70}}{35} \lambda_1 - \frac{\sqrt{70}}{10} \lambda_2\right) \\
&\quad + \frac{1}{4} \exp\left(\frac{10\sqrt{7} + 2\sqrt{70}}{35} \lambda_1 + \frac{\sqrt{70}}{10} \lambda_2\right) \\
&\quad + \frac{1}{6} \exp\left(\frac{10\sqrt{7} - 4\sqrt{70}}{35} \lambda_1 - \frac{\sqrt{70}}{5} \lambda_2\right) \\
&\quad + \frac{1}{12} \exp\left(\frac{-20\sqrt{7} + 2\sqrt{70}}{35} \lambda_1 + \frac{\sqrt{70}}{10} \lambda_2\right), \\
F_{\mathbf{y}}(\lambda_1, \lambda_2) &= \frac{1}{6} \exp\left(\frac{-20\sqrt{7} + 2\sqrt{70}}{35} \lambda_1 + \frac{\sqrt{70}}{10} \lambda_2\right) \\
&\quad + \frac{1}{3} \exp\left(\frac{10\sqrt{7} - 4\sqrt{70}}{35} \lambda_1 - \frac{\sqrt{70}}{5} \lambda_2\right) \\
&\quad + \frac{1}{2} \exp\left(\frac{10\sqrt{7} + 2\sqrt{70}}{35} \lambda_1 + \frac{\sqrt{70}}{10} \lambda_2\right), \\
F_{\mathbf{z}}(\lambda_1, \lambda_2) &= \frac{1}{6} \exp\left(\frac{-10\sqrt{7} - 2\sqrt{70}}{35} \lambda_1 - \frac{\sqrt{70}}{10} \lambda_2\right) \\
&\quad + \frac{1}{3} \exp\left(\frac{-10\sqrt{7} + 4\sqrt{70}}{35} \lambda_1 + \frac{\sqrt{70}}{5} \lambda_2\right) \\
&\quad + \frac{1}{2} \exp\left(\frac{20\sqrt{7} - 2\sqrt{70}}{35} \lambda_1 - \frac{\sqrt{70}}{10} \lambda_2\right).
\end{aligned}$$

By solving the following equations:

$$\begin{cases} \partial_1 F_{\mathbf{x}}(\lambda_1, \lambda_2) = 0 \\ \partial_2 F_{\mathbf{x}}(\lambda_1, \lambda_2) = 0 \end{cases}, \quad \begin{cases} \partial_1 F_{\mathbf{y}}(\lambda_1, \lambda_2) = 0 \\ \partial_2 F_{\mathbf{y}}(\lambda_1, \lambda_2) = 0 \end{cases}, \quad \begin{cases} \partial_1 F_{\mathbf{z}}(\lambda_1, \lambda_2) = 0 \\ \partial_2 F_{\mathbf{z}}(\lambda_1, \lambda_2) = 0 \end{cases},$$

we find that the minimizers $\lambda_*(\mathbf{x})$, $\lambda_*(\mathbf{y})$ and $\lambda_*(\mathbf{z})$ of functions $F_{\mathbf{x}}(\cdot)$, $F_{\mathbf{y}}(\cdot)$ and $F_{\mathbf{z}}(\cdot)$ are given by

$$\begin{aligned}
\lambda_*(\mathbf{x}) &= \left(-\frac{\sqrt{7}}{6} \log 3, \frac{4\sqrt{7} - \sqrt{70}}{42} \log 3\right), \\
\lambda_*(\mathbf{y}) &= \left(-\frac{\sqrt{7}}{6} \log 3, \frac{\sqrt{70}}{21} \log 2 + \frac{2\sqrt{7} - \sqrt{70}}{21} \log 3\right), \\
\lambda_*(\mathbf{z}) &= \left(-\frac{\sqrt{7}}{6} \log 3, -\frac{\sqrt{70}}{21} \log 2 + \frac{2\sqrt{7}}{21} \log 3\right),
\end{aligned}$$

respectively, and

$$F_{\mathbf{x}}(\lambda_*(\mathbf{x})) = \frac{\sqrt{3}+1}{3}, \quad F_{\mathbf{y}}(\lambda_*(\mathbf{y})) = F_{\mathbf{z}}(\lambda_*(\mathbf{z})) = 3 \cdot 6^{-2/3}.$$

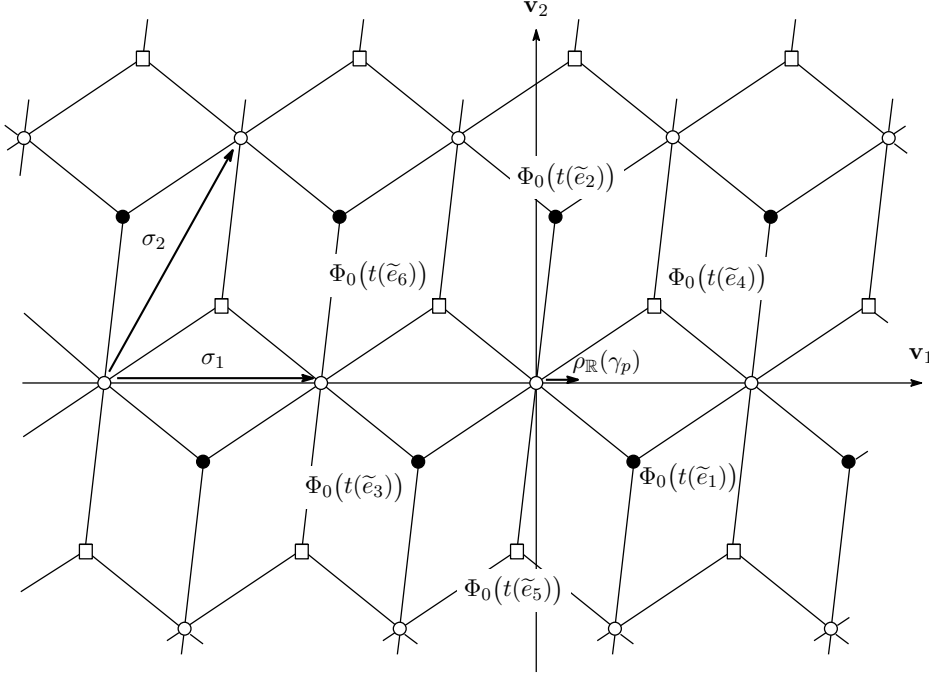


FIGURE 3. Modified standard realization of the dice lattice.

Finally, we determine the changed transition probability \mathbf{p} by

$$\begin{aligned} \mathbf{p}(e_1) &= \frac{3-\sqrt{3}}{8}, & \mathbf{p}(e_2) &= \frac{\sqrt{3}-1}{4}, & \mathbf{p}(e_3) &= \frac{3-\sqrt{3}}{8}, \\ \mathbf{p}(e_4) &= \frac{3-\sqrt{3}}{8}, & \mathbf{p}(e_5) &= \frac{\sqrt{3}-1}{4}, & \mathbf{p}(e_6) &= \frac{3-\sqrt{3}}{8}, \\ \mathbf{p}(\bar{e}_1) &= \mathbf{p}(\bar{e}_2) = \mathbf{p}(\bar{e}_3) = \mathbf{p}(\bar{e}_4) = \mathbf{p}(\bar{e}_5) = \mathbf{p}(\bar{e}_6) = \frac{1}{3}. \end{aligned}$$

Then, the invariant measure \mathbf{m} is given by $\mathbf{m}(\mathbf{x}) = 1/2$ and $\mathbf{m}(\mathbf{y}) = \mathbf{m}(\mathbf{z}) = 1/4$. Moreover, the homological direction $\gamma_{\mathbf{p}}$ and the asymptotic direction $\rho_{\mathbb{R}}(\gamma_{\mathbf{p}})$ of the random walk associated with \mathbf{p} are given by

$$\gamma_{\mathbf{p}} = \frac{5-3\sqrt{3}}{48} ([c_1] + [c_2] + [c_3] + [c_4]) (\neq 0), \quad \rho_{\mathbb{R}}(\gamma_{\mathbf{p}}) = \mathbf{0},$$

respectively. Then it follows from Theorem 1.4 that there exist some positive constants C_1 and C_2 such that

$$C_1 p(n, x, y) (\sqrt[6]{12}(\sqrt{3} - 1))^n \leq \mathfrak{p}(n, x, y) \leq C_2 p(n, x, y) (\sqrt[6]{12}(\sqrt{3} - 1))^n$$

for all $n \in \mathbb{N}$ and $x, y \in V$.

Acknowledgement. The author would like to thank his advisor Professor Hiroshi Kawabi for useful discussion and constant encouragement. He expresses his gratitude to Professors Satoshi Ishiwata, Tomoyuki Kakehi, Atsushi Katsuda and Ryokichi Tanaka for valuable advice and comments. He also would like to thank the anonymous referee for providing helpful comments and suggestions.

REFERENCES

- [1] G. Alexopoulos: *Random walks on discrete groups of polynomial growth*, Ann. Probab. **30** (2002), pp. 723–801.
- [2] R. Durrett: *Probability: Theory and Examples*, Cambridge Series in Statistical and Probabilistic Mathematics, Fourth Edition, Cambridge University Press, 2010.
- [3] T. Fujita: *Random Walks and Stochastic Analysis – From Gambling to Mathematical Finance* (in Japanese), Nihon Hyoronsha, 2008.
- [4] S. Ishiwata, H. Kawabi and M. Kotani: *Long time asymptotics of non-symmetric random walks on crystal lattices*, J. Funct. Anal. **272** (2017), pp.1553–1624.
- [5] M. Kotani: *A central limit theorem for magnetic transition operators on a crystal lattice*, J. London Math. Soc. **65** (2002), pp. 464–482.
- [6] M. Kotani and T. Sunada: *Albanese maps and off diagonal long time asymptotics for the heat kernel*, Comm. Math. Phys. **209** (2000), pp. 633–670.
- [7] M. Kotani and T. Sunada: *Large deviation and the tangent cone at infinity of a crystal lattice*, Math. Z. **254** (2006), pp. 837–870.
- [8] M. Kotani, T. Shirai and T. Sunada: *Asymptotic behavior of the transition probability of a random walk on an infinite graph*, J. Funct. Anal. **159** (1998), pp. 664–689.
- [9] T. Sunada: *Topological Crystallography: With a View Towards Discrete Geometric Analysis*, Surveys and Tutorials in the Applied Mathematical Sciences **6**, Springer Japan, 2013.
- [10] B. Trojan: *Long time behaviour of random walks on the integer lattice*, available at [arXiv:1512.09035](https://arxiv.org/abs/1512.09035).

RYUYA NAMBA
GRADUATE SCHOOL OF NATURAL SCIENCES,
OKAYAMA UNIVERSITY
OKAYAMA, 700-8530 JAPAN
e-mail address: rnamba@s.okayama-u.ac.jp

(Received January 31, 2017)

(Accepted April 11, 2017)