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STABLE SPLITTINGS OF THE COMPLEX CONNECTIVE K-THEORY OF $BSO(2n+1)$

Tsung-Hsuan Wu

Abstract. We give the stable splittings of the complex connective K-theory of the classifying space $BSO(2n+1)$, $n \geq 1$.

1. Introduction

In [6], E. Ossa has showed that

$$bu \wedge RP^\infty \wedge RP^\infty \simeq [\bigvee_{0<i,j} \Sigma^{2i+2j-2} HZ/2] \vee [\Sigma^2 bu \wedge RP^\infty].$$

In [2], B. R. Burner and J. P. C. Greenless give some studies on $bu \wedge BG$ for some finite groups G. Also, W. Stephen Wilson and D. Y. Yan [7] split $bu \wedge BO(n)$ into the suspended copies of $HZ/2$, bu, and $bu \wedge RP^\infty$. Via these splittings, we are going to split $bu \wedge BSO(2n+1)$.

First let's recall the notations we need. Let bu be the complex connective K-theory, $H\mathbb{Z}/2$ be the $\mathbb{Z}/2$ Eilenberg-Mac Lane spectrum, $RP^\infty = BO(1)$ be the infinite real projective space, $BO(n)$ be the classifying space of the n-th orthogonal group, $BSO(n)$ be the classifying space of the n-th special orthogonal group. To simplify the notations, let $H^*(X) = H^*(X, \mathbb{Z}/2)$, $\tilde{H}^*(X) = \tilde{H}^*(X, \mathbb{Z}/2)$, $H_*(X) = H_*(X, \mathbb{Z}/2)$, and $\tilde{H}_*(X) = \tilde{H}_*(X, \mathbb{Z}/2)$. We also write $\otimes$ instead of $\otimes_{\mathbb{Z}/2}$ and all the spaces, the spectra, and the homotopy equivalences are localized at prime 2.

Recall that $H^*(BO(n)) = \mathbb{Z}/2[w_1, w_2, \cdots, w_n]$, where w_i is the i-th Stiefel-Whitney class. In particular, $H^*(RP^\infty) = H^*(BO(1)) = \mathbb{Z}/2[w_1]$. Then let $b_i \in H_i(RP^\infty)$ be the dual class of $\mathrm{w}_1^i \in H^*(RP^\infty)$, $i \geq 0$, hence $H_*(BO(n))$ is the $\mathbb{Z}/2$-module generated by the monomials $b_{i_1} b_{i_2} \cdots b_{i_n}$, $\deg(b_{i_1} b_{i_2} \cdots b_{i_n}) = i_1 + i_2 + \cdots + i_n$, $b_{i_1} b_{i_2} \cdots b_{i_n} = f_*(b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes b_{i_n})$, $0 \leq i_1 \leq i_2 \leq \cdots \leq i_n$, where $f : \overset{n}{\underset{i=1}{\times}} RP^\infty \longrightarrow BO(n)$ is the classifying map. Moreover, let $h_n : BSO(n) \longrightarrow BO(n)$ be the 2-folds map, then we have $H^*(BSO(n)) = \mathbb{Z}/2[\widehat{w_2}, \widehat{w_3}, \cdots, \widehat{w_n}]$, where $\widehat{w_i} = h_n^*(w_i)$, $2 \leq i \leq n$.

Also recall that $bu_* = Z_{(2)}[v_1]$, where $\deg(v_1) = 2$, and $H^*(bu) \cong A//A(Q_0, Q_1) \cong A \otimes_E \mathbb{Z}/2$, where A is the mod 2 Steenrod algebra,

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$A(Q_0, Q_1)$ is the ideal of A generated by $Q_0 = Sq^1$ and $Q_1 = Sq^3 + Sq^2Sq^1$, and $E = \mathbb{Z}/2\langle Q_0, Q_1\rangle$, the exterior algebra on Q_0 and Q_1, is a subalgebra of A. Then by the Cartan formula $Sq^i(xy) = \sum_{j=0}^{i} Sq^j(x)Sq^{i-j}(y)$, we have $Q_k(xy) = Q_k(x)y + xQ_k(y)$, $k = 0$ or 1. Moreover, since for any space X, $\tilde{H}^*(X)$ is an E-module, we say an element x in $\tilde{H}^*(X)$ is decomposable if $x = Q_0(y) + Q_1(z)$ for some $y, z \in \tilde{H}^*(X)$, and we say an element is indecomposable if it is not decomposable.

For $n \geq 1$, let $T_{2n+1} = \{t_j \mid j \in \Lambda_{2n+1}\}$ be a largest E-linearly independent subset of $\tilde{H}^*(BSO(2n+1))$ such that each t_j is a monomial in $\tilde{H}^*(BSO(2n+1))$.

Now we state the main result of this paper.

Theorem A. *For each $n \geq 1$, $\tilde{H}^*(BSO(2n+1))$ is isomorphic to $D_{2n+1} \oplus M_{2n+1}$ as an E-module, where D_{2n+1} is an E-module with the $\mathbb{Z}/2$-generators $\widehat{w_2}^{2m_1}\widehat{w_4}^{2m_2}\cdots\widehat{w_{2n}}^{2m_n}$, $\sum_{i=1}^{n} m_i > 0$, $m_i \geq 0$, each $\widehat{w_2}^{2m_1}\widehat{w_4}^{2m_2}\cdots\widehat{w_{2n}}^{2m_n}$ has the trivial E-action, and M_{2n+1} is a free E-module with the E-basis T_{2n+1} described as above.*

Theorem B. *For each $n \geq 1$, there is a stable splitting*

$$bu \wedge BSO(2n+1) \simeq [\bigvee_{\alpha}\Sigma^{\alpha}H\mathbb{Z}/2] \vee [\bigvee_{\beta}\Sigma^{\beta}bu],$$

where $\alpha = \deg t_j$, $t_j \in T_{2n+1}$, the generators of M_{2n+1}, and the β, and their degrees, correspond to the generators of D_{2n+1}.

To prove the stable splitting of $bu \wedge BSO(2n+1)$ (Theorem B), we need to apply the stable splitting of $bu \wedge BO(n)$ [7] to decompose $\tilde{H}^*(BSO(2n+1))$ as a direct sum of an E-module $D_{2n+1}$and a free E-module M_{2n+1} (Theorem A). Then we construct the map

$$g = g_0 \vee g_1 : bu \wedge BSO(2n+1) \longrightarrow [\bigvee_{\alpha}\Sigma^{\alpha}H\mathbb{Z}/2] \vee [\bigvee_{\beta}\Sigma^{\beta}bu]$$

and prove that g induces an isomorphism on the mod 2 cohomology, hence g is a homotopy equivalence and Theorem A follows.

In fact, there is an algebraic splitting of $\tilde{H}^*(BSO(2n))$ as Theorem A, that is, $\tilde{H}^*(BSO(2n))$ is isomorphic to $D_{2n} \oplus M_{2n} \oplus B_{2n}$ as an E-module, $n \geq 1$. Unfortunately, I cannot find a suitable space or spectrum corresponding to the B_{2n} part.

The rest of paper is organized as follows : In Section 2, we will give some lemmas which link the Adams $E_2^{1,*}$ term of $\widetilde{bu}_*(X)$ to the decomposition of $\tilde{H}^*(X)$. In Section 3, we will compute the Adams $E_2^{1,*}$ term of $\widetilde{bu}_*(BO(n))$.

In Section 4, we will study the map $Bg_{2n} : BO(2n) \longrightarrow BSO(2n+1)$. In Section 5, we will prove Theorem A. In Section 6, we will prove Theorem B.

2. The E-module structure of $\tilde{H}^*(BO(n))$ and the Adams spectral sequences for $\widetilde{bu}_*(BSO(2n+1))$

In this section, we will recall the Adams spectral sequence and give some lemmas which link some useful information of the decomposition of $\tilde{H}^*(X)$ to the Adams $E_2^{1,*}$ term of $\widetilde{bu}_*(X)$ for any spaces X.

Let $A_* = \mathbb{Z}/2[\xi_1,\ \xi_2, \xi_3, \cdots]$, where ξ_k are the Milnor's generators with $\deg(\xi_k) = 2^k - 1$, be the mod 2 dual Steenrod algebra with the coproduct $\triangle(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i$. Then recall that for any space or spectrum Y, the Adams spectral sequences [1]

$$Ext_A^{*,*}(H^*(X),\ \mathbb{Z}/2) \cong Ext_{A_*}^{*,*}(\mathbb{Z}/2,\ H_*(X)) \Longrightarrow \pi_*(X_{(2)})$$

can be used to compute $\widetilde{bu}_*(Y)$ when $X = bu \wedge Y$. By a well-known change-of-rings isomorphism [3], we can replace

$$Ext_A^{*,*}(H^*(bu \wedge Y),\ \mathbb{Z}/2) \text{ with } Ext_E^{*,*}(\tilde{H}^*(Y),\ \mathbb{Z}/2),$$
$$Ext_{A_*}^{*,*}(\mathbb{Z}/2,\ H_*(bu \wedge Y)) \text{ with } Ext_{E_*}^{*,*}(\mathbb{Z}/2,\ \tilde{H}_*(Y)),$$

where $E_* = \mathbb{Z}/2\,\langle \xi_1,\ \xi_2 \rangle$ is the exterior algebra on ξ_1 and ξ_2. For simplicity of notations, let $E_2^{*,*}(Y)$ be $Ext_E^{*,*}(\tilde{H}^*(Y),\ \mathbb{Z}/2)$ and $\hat{E}_2^{*,*}(Y)$ be $Ext_{E_*}^{*,*}(\mathbb{Z}/2,\ \tilde{H}_*(Y))$. Also recall that $E_2^{*,*}(Y)$ is isomorphic to the homology of the bar complex

$$\tilde{H}^*(Y) \overset{\overline{d_1}}{\leftarrow} \overline{E} \otimes \tilde{H}^*(Y) \overset{\overline{d_2}}{\leftarrow} \overline{E} \otimes \overline{E} \otimes \tilde{H}^*(Y) \leftarrow \cdots$$

and $\hat{E}_2^{*,*}(Y)$ is isomorphic to the homology of the cobar complex

$$\tilde{H}_*(Y) \xrightarrow{\triangle_1} \overline{E}_* \otimes \tilde{H}_*(Y) \xrightarrow{\triangle_2} \overline{E}_* \otimes \overline{E}_* \otimes \tilde{H}_*(Y) \longrightarrow \cdots,$$

where $\overline{E} = E \setminus \{1\}$ and $\overline{E}_* = E_* \setminus \{1\}$.

Moreover, we have the Adams spectral sequences

$$\begin{aligned} E_2^{*,*} &\cong Ext_E^{*,*}(\mathbb{Z}/2,\ \mathbb{Z}/2) \cong \mathbb{Z}/2[\overline{v_0},\ \overline{v_1}] \cong \\ \hat{E}_2^{*,*} &\cong Ext_{E_*}^{*,*}(\mathbb{Z}/2,\ \mathbb{Z}/2) \cong \mathbb{Z}/2[\xi_1,\ \xi_2], \end{aligned}$$

where $\overline{v_0} \in E_2^{1,1}$ and $\overline{v_1} \in E_2^{1,3}$ are detected by Q_0 and Q_1 respectively, $\overline{v_0}^2$ is detected by $Q_0 \otimes Q_0$, $\overline{v_1}^2$ is detected by $Q_1 \otimes Q_1$, $\overline{v_0 v_1}$ is detected by $Q_0 \otimes Q_1 + Q_1 \otimes Q_0$, $\xi_1 \in \hat{E}_2^{1,1}$, and $\xi_2 \in \hat{E}_2^{1,3}$ (here we use the ambiguous notations, that is, we use the same symbol ξ_i in the chain level and the homology level).

Let N^* be any E-module and $E_2^{1,*}(N^*)$ be the first line of the bar complex

$$N^* \overset{\overline{d_1}}{\leftarrow} \overline{E} \otimes N^* \overset{\overline{d_2}}{\leftarrow} \overline{E} \otimes \overline{E} \otimes N^* \leftarrow \cdots .$$

Similarly, let N_* be any E_*-comodule and $\hat{E}_2^{1,*}(N_*)$ be the first line of the cobar complex

$$N_* \xrightarrow{\triangle_1} \overline{E}_* \otimes N_* \xrightarrow{\triangle_2} \overline{E}_* \otimes \overline{E}_* \otimes N_* \longrightarrow \cdots .$$

Then we have the following lemmas.

Lemma 2.1. *As E-modules, if $N^* \cong K^* \oplus L^*$, then $E_2^{1,*}(N^*) \cong E_2^{1,*}(K^*) \oplus E_2^{1,*}(L^*)$. As E_*-comodules, if $N_* \cong K_* \oplus L_*$, then $\hat{E}_2^{1,*}(N_*) \cong \hat{E}_2^{1,*}(K_*) \oplus \hat{E}_2^{1,*}(L_*)$.*

Proof. This follows immediately from the definition of the bar and cobar complexes. □

Lemma 2.2. *If $E_2^{1,*}(N^*) = 0$ and $Q_0(x) + Q_1(y) + Q_0Q_1(z) = 0$ for some $x, y, z \in N^*$, then $x = 0$ or x is decomposable, and $y = 0$ or y is decomposable.*

Proof. Since $E_2^{1,*}(N^*) = 0$ and $0 = Q_0(x) + Q_1(y) + Q_0Q_1(z) = \overline{d_1}(Q_0 \otimes x + Q_1 \otimes y + Q_0Q_1 \otimes z)$, there exists $a_1, \cdots, a_9 \in N^*$ such that

$$\begin{aligned}
& Q_0 \otimes x + Q_1 \otimes y + Q_0Q_1 \otimes z \\
= \; & \overline{d_2}(Q_0 \otimes Q_0 \otimes a_1 + Q_0 \otimes Q_1 \otimes a_2 + Q_0 \otimes Q_0Q_1 \otimes a_3 \\
& + Q_1 \otimes Q_0 \otimes a_4 + Q_1 \otimes Q_1 \otimes a_5 + Q_1 \otimes Q_0Q_1 \otimes a_6 \\
& + Q_0Q_1 \otimes Q_0 \otimes a_7 + Q_0Q_1 \otimes Q_1 \otimes a_8 + Q_0Q_1 \otimes Q_0Q_1 \otimes a_9) \\
= \; & Q_0 \otimes Q_0(a_1) + Q_0Q_1 \otimes a_2 + Q_0 \otimes Q_1(a_2) + Q_0 \otimes Q_0Q_1(a_3) \\
& + Q_1Q_0 \otimes a_4 + Q_1 \otimes Q_0(a_4) + Q_1 \otimes Q_1(a_5) + Q_1 \otimes Q_0Q_1(a_6) \\
& + Q_0Q_1 \otimes Q_0(a_7) + Q_0Q_1 \otimes Q_1(a_8) + Q_0Q_1 \otimes Q_0Q_1(a_9).
\end{aligned}$$

Then we get

$$\begin{aligned}
x &= Q_0(a_1) + Q_1(a_2) + Q_0Q_1(a_3), \\
\text{and } y &= Q_0(a_4) + Q_1(a_5) + Q_0Q_1(a_6).
\end{aligned}$$

This completes the proof. □

Lemma 2.3. *If $E_2^{1,*}(N^*) = 0$ and $Q_0Q_1(z) = 0$ for some $z \in N^*$, then $z = 0$ or z is decomposable.*

Proof. As the proof of Lemma 2.2, where $x = 0$ and $y = 0$, there exists a_1, $\cdots$, $a_9 \in N^*$ such that

$$0 = Q_0(a_1) + Q_1(a_2) + Q_0Q_1(a_3),$$

$$\begin{aligned} 0 &= Q_0(a_4) + Q_1(a_5) + Q_0Q_1(a_6), \\ z &= a_2 + a_4 + Q_0(a_7) + Q_1(a_8) + Q_0Q_1(a_9). \end{aligned}$$

Since $Q_0(a_1) + Q_1(a_2) + Q_0Q_1(a_3) = 0$ and $Q_0(a_4) + Q_1(a_5) + Q_0Q_1(a_6) = 0$, by Lemma 2.2, $a_2 = 0$ or a_2 is decomposable, and $a_4 = 0$ or a_4 is decomposable. As a result, z is also decomposable or $z = 0$. This completes the proof. □

Lemma 2.4. *Let $T = \{t_j \mid j \in \Lambda\}$ be a largest E-linearly independent subset of N^*. Then if $E_2^{1,*}(N^*) = 0$, N^* is a free E-module with the E-basis T.*

Proof. Let $M \subseteq N^*$ be the free E-submodule generated by T. We are going to show that $M = N^*$.

For any $u \in N^*$, since T is a largest E-linearly independent subset of N^*, $Q_0Q_1(u)$ can be generated by T, hence there exists a finite sum a (a could be 0) of some $t_j \in T$ such that $Q_0Q_1(u) = Q_0Q_1(a)$. Therefore, by Lemma 2.3, $Q_0Q_1(u + a) = 0$ implies $u + a = Q_0(v) + Q_1(w)$ for some v, $w \in N^*$. As above u and a, there exists finite sums b, c of some $t_j \in T$ such that $Q_0Q_1(v) = Q_0Q_1(b)$ and $Q_0Q_1(w) = Q_0Q_1(c)$. Thus we have

$$\begin{aligned} Q_1(u+a) &= Q_1Q_0(v) = Q_1Q_0(b) \\ \text{and } Q_0(u+a) &= Q_0Q_1(w) = Q_0Q_1(c), \end{aligned}$$

which means

$$\begin{aligned} Q_1(u) &= Q_1(a) + Q_1Q_0(b) \in M \\ \text{and } Q_0(u) &= Q_0(a) + Q_0Q_1(c) \in M. \end{aligned}$$

These also apply to v and w, that is, both $Q_0(v)$ and $Q_1(w)$ are in M, hence $u = a + Q_0(v) + Q_1(w)$ follows. This completes the proof. □

3. The $\hat{E}_2^{1,*}$ term of the Adams spectral sequences for $\widetilde{bu}_*(BO(n))$

To study the Adams $E_2^{1,*}$ term of $\widetilde{bu}_*(BSO(2n+1))$, we have to know the Adams $E_2^{1,*}$ term and $\hat{E}_2^{1,*}$ term of $\widetilde{bu}_*(BO(n))$. So first we recall the result in [7].

Theorem 3.1. *(Theorem 1.1 of* [7] *) As an E-module, $\tilde{H}^*(BO(n))$ is isomorphic to $D_1^* \oplus D_2^* \oplus M$, where D_1^* is a trivial E-module with E-generators*

$$w_2^{2m_1} w_4^{2m_2} \cdots w_{2k}^{2m_k} \text{ such that } \sum_{i=1}^{k} m_i > 0,\ 2k \le n,$$

D_2^* is an E-module, free over the exterior algebra on Q_0, with E-generators

$$w_1^{2j+1}w_2^{2m_1}w_4^{2m_2}\cdots w_{2t}^{2m_t} \text{ such that } \sum_{i=1}^{t} m_i \geq 0,\ j \geq 0,\ 2t \leq n-1,$$

and

$$Q_1(w_1^{2j+1}w_2^{2m_1}w_4^{2m_2}\cdots w_{2t}^{2m_t}) = Q_0(w_1^{2j+3}w_2^{2m_1}w_4^{2m_2}\cdots w_{2t}^{2m_t}),$$

and M is a free E-module.

Thus we can compute the Adams $E_2^{1,*}$ term and $\hat{E}_2^{1,*}$ term of $\widetilde{bu}_*(BO(n))$.

Lemma 3.2. *In the Adams spectral sequence*

$$Ext_{E^*}^{*,*}(\tilde{H}^*(BO(n)),\ \mathbb{Z}/2) \Longrightarrow \widetilde{bu}_*(BO(n)),$$

as a $\mathbb{Z}/2$-module, $E_2^{1,*}(BO(n))$ is generated by

$$\overline{v_0} \otimes w_2^{2m_1}w_4^{2m_2}\cdots w_{2k}^{2m_k},\ \sum_{i=1}^{k} m_i > 0,\ 2k \leq n,$$

$$\overline{v_1} \otimes w_2^{2m_1}w_4^{2m_2}\cdots w_{2k}^{2m_k},\ \sum_{i=1}^{k} m_i > 0,\ 2k \leq n,$$

$$\overline{v_0} \otimes w_1^{2j+3}w_2^{2m_1}w_4^{2m_2}\cdots w_{2t}^{2m_t} + \overline{v_1} \otimes w_1^{2j+1}w_2^{2m_1}w_4^{2m_2}\cdots w_{2t}^{2m_t},$$
$$\sum_{i=1}^{t} m_i \geq 0,\ j \geq 0,\ 2t \leq n-1.$$

Proof. Since by Theorem 3.1, $\tilde{H}^*(BO(n))$ is isomorphic to $D_1^* \oplus D_2^* \oplus M$, by Lemma 2.1, we can compute $E_2^{1,*}(D_1^*)$, $E_2^{1,*}(D_2^*)$ and $E_2^{1,*}(M)$ separately. Then since D_1^* is a trivial E-module, it is clearly that $E_2^{1,*}(D_1^*)$ has the $\mathbb{Z}/2$-generators

$$\overline{v_0} \otimes w_2^{2m_1}w_4^{2m_2}\cdots w_{2k}^{2m_k},\ \sum_{i=1}^{k} m_i > 0,\ 2k \leq n,$$

$$\overline{v_1} \otimes w_2^{2m_1}w_4^{2m_2}\cdots w_{2k}^{2m_k},\ \sum_{i=1}^{k} m_i > 0,\ 2k \leq n.$$

Moreover, $E_2^{1,*}(M) = 0$ since M is free. Therefore, it is only left $E_2^{1,*}(D_2^*)$.

Since we have

$$Q_0(w_1^{2j+1}w_2^{2m_1}w_4^{2m_2}\cdots w_{2t}^{2m_t}) = w_1^{2j+2}w_2^{2m_1}w_4^{2m_2}\cdots w_{2t}^{2m_t}$$
$$\text{and } Q_1(w_1^{2j+1}w_2^{2m_1}w_4^{2m_2}\cdots w_{2t}^{2m_t}) = w_1^{2j+4}w_2^{2m_1}w_4^{2m_2}\cdots w_{2t}^{2m_t},$$

the $\mathbb{Z}/2$-generators of $\ker \overline{d_1}$ for the bar complex of D_2^* are

$$Q_0 \otimes w_1^{2j+2} w_2^{2m_1} \cdots w_{2t}^{2m_t},\ \sum_{i=1}^{t} m_i \geq 0,\ j \geq 0,\ 2t \leq n-1,$$

$$Q_1 \otimes w_1^{2j+2} w_2^{2m_1} \cdots w_{2t}^{2m_t},\ \sum_{i=1}^{t} m_i \geq 0,\ j \geq 0,\ 2t \leq n-1,$$

$$Q_0 Q_1 \otimes w_1^{s} w_2^{2m_1} \cdots w_{2t}^{2m_t},\ \sum_{i=1}^{t} m_i \geq 0,\ s \geq 1,\ 2t \leq n-1,$$

$$\text{and } Q_0 \otimes w_1^{2j+3} w_2^{2m_1} \cdots w_{2t}^{2m_t} + Q_1 \otimes w_1^{2j+1} w_2^{2m_1} \cdots w_{2t}^{2m_t},$$
$$\sum_{i=1}^{t} m_i \geq 0,\ j \geq 0,\ 2t \leq n-1.$$

However, we also have

$$\begin{aligned}
\overline{d_2}(Q_0 \otimes Q_0 \otimes w_1^{2j+1}) &= Q_0 \otimes w_1^{2j+2},\ j \geq 0,\\
\overline{d_2}(Q_1 \otimes Q_1 \otimes w_1^{2j-1}) &= Q_1 \otimes w_1^{2j+2},\ j \geq 1,\\
\overline{d_2}(Q_0 \otimes Q_1 \otimes w_1^{2j+2}) &= Q_0 Q_1 \otimes w_1^{2j+2},\ j \geq 0,
\end{aligned}$$

$$\begin{aligned}
&\overline{d_2}(Q_1 \otimes Q_0 \otimes w_1^1 + Q_0 \otimes Q_1 \otimes w_1^1 + Q_0 \otimes Q_0 \otimes w_1^3)\\
&= Q_1 \otimes w_1^2 + Q_1 Q_0 \otimes w_1^1 + Q_0 Q_1 \otimes w_1^1 + Q_0 \otimes w_1^4 + Q_0 \otimes w_1^4 = Q_1 \otimes w_1^2,
\end{aligned}$$

$$\begin{aligned}
& \overline{d_2}(Q_0 \otimes Q_1 \otimes w_1^{2j+1} + Q_0 \otimes Q_0 \otimes w_1^{2j+3})\\
= \ & Q_0 Q_1 \otimes w_1^{2j+1} + Q_0 \otimes w_1^{2j+4} + Q_0 \otimes w_1^{2j+4} = Q_0 Q_1 \otimes w_1^{2j+1},\ j \geq 0,
\end{aligned}$$

and the fact that $Q_0 \otimes w_1^{2j+3} w_2^{2m_1} \cdots w_{2t}^{2m_t} + Q_1 \otimes w_1^{2j+1} w_2^{2m_1} \cdots w_{2t}^{2m_t}$ can not be an image of $\overline{d_2}$. This completes the proof. □

Lemma 3.3. *In the Adams spectral sequence*

$$Ext_{E_*}^{*,*}(\mathbb{Z}/2,\ \tilde{H}_*(BO(n))) \Longrightarrow \widetilde{bu}_*(BO(n)),$$

as a $\mathbb{Z}/2$-module, $\hat{E}_2^{1,*}(BO(n))$ is generated by

$$\begin{aligned}
\xi_1 \otimes b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_k}^2,\ 1 &\leq j_1 \leq j_2 \leq \cdots \leq j_k,\ 2k \leq n,\\
\xi_2 \otimes b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_k}^2,\ 1 &\leq j_1 \leq j_2 \leq \cdots \leq j_k,\ 2k \leq n,
\end{aligned}$$

$$\begin{aligned}
\xi_1 \otimes b_{2i+1} b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_t}^2,\ 0 &\leq j_1 \leq j_2 \leq \cdots \leq j_t,\ 2t \leq n-1,\ i \geq 0,\\
\xi_2 \otimes b_{2i+1} b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_t}^2,\ 0 &\leq j_1 \leq j_2 \leq \cdots \leq j_t,\ 2t \leq n-1,\ i \geq 0,
\end{aligned}$$

and subjects to the relations

$$\xi_1 \otimes b_{2i+3} b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_t}^2 = \xi_2 \otimes b_{2i+1} b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_t}^2$$

$$\text{and } \xi_1 \otimes b_1 b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_t}^2 \quad = \quad 0.$$

Proof. First recall the coaction of $\tilde{H}_*(BO(n))$ over A_* is

$$\triangle(b_i) = \sum_{j=1}^{i} (\xi^j)_{i-j} \otimes b_j,$$

where $\xi = 1 + \xi_1 + \xi_2 + \xi_3 + \cdots$ [8], and we have the coproduct $\triangle(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i$. Thus the comodule stucture of $\tilde{H}_*(BO(n))$ over $\overline{E}_* = E_* \backslash \{1\}$ is generated by

$$\begin{aligned} \triangle(b_{2i}) &= \xi_1 \otimes b_{2i-1} + \xi_2 \otimes b_{2i-3}, \\ \triangle(b_{2i-1}) &= 0, \\ \triangle(b_i^2) &= 0, \end{aligned}$$

where $i \geq 1$. Moreover, in $\overline{E}_*$, we have $\triangle(\xi_1) = 0$ and $\triangle(\xi_2) = 0$. So under the cobar complex

$$\tilde{H}_*(BO(n)) \xrightarrow{\triangle_1} \overline{E}_* \otimes \tilde{H}_*(BO(n)) \xrightarrow{\triangle_2} \overline{E}_* \otimes \overline{E}_* \otimes \tilde{H}_*(BO(n)) \longrightarrow \cdots,$$

we have

$$\begin{aligned} \triangle_2(\xi_1 \otimes b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_k}^2) &= 0, \ 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k, \ 2k \leq n, \\ \triangle_2(\xi_2 \otimes b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_k}^2) &= 0, \ 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k, \ 2k \leq n, \end{aligned}$$

$$\begin{aligned} \triangle_2(\xi_1 \otimes b_{2i+1} b_{2j_1}^2 \cdots b_{2j_t}^2) &= 0, \ 0 \leq j_1 \leq j_2 \leq \cdots \leq j_t, \ 2t \leq n-1, \ i \geq 0, \\ \triangle_2(\xi_2 \otimes b_{2i+1} b_{2j_1}^2 \cdots b_{2j_t}^2) &= 0, \ 0 \leq j_1 \leq j_2 \leq \cdots \leq j_t, \ 2t \leq n-1, \ i \geq 0, \end{aligned}$$

and under $\triangle_1$, the only methods to produce the above elements are

$$\begin{aligned} \triangle_1(b_{2i+4} b_{2j_1}^2 \cdots b_{2j_t}^2) &= \xi_1 \otimes b_{2i+3} b_{2j_1}^2 \cdots b_{2j_t}^2 + \xi_2 \otimes b_{2i+1} b_{2j_1}^2 \cdots b_{2j_t}^2, \ i \geq 0, \\ \triangle_1(b_2 b_{2j_1}^2 \cdots b_{2j_t}^2) &= \xi_1 \otimes b_1 b_{2j_1}^2 \cdots b_{2j_t}^2. \end{aligned}$$

Therefore, $\hat{E}_2^{1,*}(BO(n))$ at least contains the generators described in the statement of this lemma. Then since as $\mathbb{Z}/2$-modules,

$$\hat{E}_2^{1,*}(BO(n)) \cong E_2^{1,*}(BO(n)),$$

counting the generators of $\hat{E}_2^{1,k}(BO(n))$ we just found and the generators of $E_2^{1,k}(BO(n))$ in Lemma 3.2 for each $k \geq 1$, we can see that all the generators of $\hat{E}_2^{1,*}(BO(n))$ are found. This completes the proof. $\square$

4. The map $Bg_{2n} : BO(2n) \longrightarrow BSO(2n+1)$

In this section, first we construct the map

$$Bg_{2n} : BO(2n) \longrightarrow BSO(2n+1),$$

which is the classifying map of $g_{2n} : O(2n) \longrightarrow SO(2n+1)$ defined by $g_{2n}(\alpha) = \det \alpha \oplus \alpha$. Then we will show that $(Bg_{2n})_*$ is surjective and compute its behavior.

Lemma 4.1. *The map $(Bg_{2n})_* : \hat{E}_2^{1,*}(BO(2n)) \longrightarrow \hat{E}_2^{1,*}(BSO(2n+1))$ is surjective.*

Proof. Since the fibre of $Bg_{2n} : BO(2n) \longrightarrow BSO(2n+1)$ is

$$SO(2n+1)/O(2n) = RP^{2n}$$

and the Eular characteristic $\chi(RP^{2n}) \equiv 1 \mod 2$, there exists a Becker-Gottlieb stable transfer

$$t : BSO(2n+1) \longrightarrow BO(2n)$$

such that $Bg_{2n} \circ t \simeq id$ (localized at prime 2). Hence the composite map

$$\hat{E}_2^{1,*}(BSO(2n+1)) \underset{1:1}{\xrightarrow{t_*}} \hat{E}_2^{1,*}(BO(2n)) \underset{onto}{\xrightarrow{(Bg_{2n})_*}} \hat{E}_2^{1,*}(BSO(2n+1))$$

is an isomorphism. This completes the proof. □

Now we recall some results in [10]. We have the following commutative diagram

$$\begin{array}{ccc} BO(2n) & \xrightarrow{Bg_{2n}} & BSO(2n+1) \\ & f_{2n} \searrow & \downarrow h_{2n+1} \\ & & BO(2n+1) \end{array},$$

where h_{2n+1} is the usual 2-fold map and f_{2n} is constructed similarly as Bg_{2n}. Then we have the following lemma.

Lemma 4.2. *(Lemma 2.2 in* [10] *)In*

$$(f_{2n})_* : H_*(BO(2n)) \longrightarrow H_*(BO(2n+1)),$$

we have

$$(f_{2n})_*(b_{m_1} b_{m_2} \cdots b_{m_{2n}}) = \sum \frac{\left(\sum_{k=1}^{2n} i_k\right)!}{\prod_{k=1}^{2n} i_k!} b_{\sum_{k=1}^{2n} i_k} b_{m_1 - i_1} b_{m_2 - i_2} \cdots b_{m_{2n} - i_{2n}},$$

where the sum is taken over the sequence $(i_1, i_2, i_3, \cdots, i_{2n})$, $0 \le i_k \le m_k$, $m_k \ge 0$, $1 \le k \le 2n$.

Thus we have the following important proposition of $(Bg_{2n})_*$.

Proposition 4.3. *In*

$$(Bg_{2n})_* : H_*(BO(2n)) \longrightarrow H_*(BSO(2n+1)),$$

we have

$$(Bg_{2n})_*(b_{2i+1}b_{m_1}^2 b_{m_2}^2 \cdots b_{m_{n-1}}^2) = 0,$$

where $i \geq 0$, $m_k \geq 0$, $1 \leq k \leq n-1$.

Before we prove Proposition 4.3, we need two lemmas.

Lemma 4.4. $\frac{(2n)!}{n!n!}$ *is even for* $n \geq 1$.

Proof. It follows immediately from the following equalities

$$\frac{(2n)!}{n!n!} = \frac{2n}{n} \cdot \frac{(2n-1)!}{n!(n-1)!} = 2\binom{2n-1}{n}.$$

This completes the proof. □

Lemma 4.5. $\frac{(i_1+\sum_{k=1}^{n} 2j_k)!}{i_1!\prod_{k=1}^{n}(j_k!)^2}$ *is even for any* $i_1 \geq 0$ *and at least one* $j_k \neq 0$.

Proof. Assume $j_1 \neq 0$. Then it follows from the equality

$$\frac{(i_1 + \sum_{k=1}^{n} 2j_k)!}{i_1! \prod_{k=1}^{n} (j_k!)^2} = \frac{(i_1 + \sum_{k=1}^{n} 2j_k)!}{i_1!(2j_1)! \prod_{k=2}^{n} (j_k!)^2} \cdot \frac{(2j_1)!}{(j_1!)^2}$$

since $\frac{(i_1+\sum_{k=1}^{n} 2j_k)!}{i_1!(2j_1)!\prod_{k=2}^{n}(j_k!)^2}$ is an integer and $\frac{(2j_1)!}{(j_1!)^2}$ is even. This completes the proof. □

Proof of Proposition 4.3. By Lemma 4.2, we have the following formula

$$(f_{2n})_*(b_{2i+1}b_{m_1}^2 b_{m_2}^2 \cdots b_{m_{n-1}}^2)$$
$$= \sum \frac{\left(i_1 + \sum_{k=1}^{n-1} (j_{k,1} + j_{k,2})\right)!}{i_1! \prod_{k=1}^{n-1} (j_{k,1}! j_{k,2}!)} b_{i_1+\sum_{k=1}^{n-1}(j_{k,1}+j_{k,2})} b_{2i+1-i_1} \prod_{k=1}^{n-1} (b_{m_k - j_{k,1}} b_{m_k - j_{k,2}}).$$

Note that for a fixed sequence $(i_1, j_{1,1}, j_{1,2}, \cdots, j_{n-1,1}, j_{n-1,2})$ which contains exactly t couples $(j_{k,1},\ j_{k,2})$ with $j_{k,1} \neq j_{k,2}$, there exists 2^t corresponding sequences which are got from interchanging $j_{k,1}$ and $j_{k,2}$ in some of those t

couples, hence there are 2^t identical terms in the above sum. Then since we are using the $\mathbb{Z}/2$-coefficient, we have

$$(f_{2n})_*(b_{2i+1}b_{m_1}^2 b_{m_2}^2 \cdots b_{m_{n-1}}^2) = \sum \frac{\left(i_1 + 2\sum_{k=1}^{n-1} j_k\right)!}{i_1! \prod_{k=1}^{n-1} (j_k!)^2} b_{i_1+2\sum_{k=1}^{n-1} j_k} b_{2i+1-i_1} \prod_{k=1}^{n-1} b_{m_k-j_k}^2,$$

where $j_k = j_{k,1} = j_{k,2}$. So by Lemma 4.5,

$$\begin{aligned}
&(f_{2n})_*(b_{2i+1}b_{m_1}^2 b_{m_2}^2 \cdots b_{m_{n-1}}^2) \\
&= \sum_{i_1=0}^{2i+1} \frac{i_1!}{i_1!} b_{i_1} b_{2i+1-i_1} b_{m_1}^2 b_{m_2}^2 \cdots b_{m_{n-1}}^2 \\
&= b_{m_1}^2 b_{m_2}^2 \cdots b_{m_{n-1}}^2 \sum_{i_1=0}^{2i+1} b_{i_1} b_{2i+1-i_1} \\
&= b_{m_1}^2 b_{m_2}^2 \cdots b_{m_{n-1}}^2 (b_0 b_{2i+1} + b_1 b_{2i} + \cdots + b_{2_i} b_1 + b_{2i+1} b_0) \\
&= 0.
\end{aligned}$$

Finally since we have the commutative diagram

$$\begin{array}{ccc}
BO(2n) & \xrightarrow{Bg_{2n}} & BSO(2n+1) \\
& f_{2n} \searrow & \downarrow h_{2n+1} \\
& & BO(2n+1)
\end{array},$$

and since $(h_{2n+1})_*$ is injective, we also have $(Bg_{2n})_*(b_{2i+1}b_{m_1}^2 b_{m_2}^2 \cdots b_{m_{n-1}}^2) = 0$. This completes the proof. □

5. Proof of Theorem A

In this section, we will use Lemma 3.5, Lemma 4.1, Proposition 4.3 and the Wu formula [9] to compute the Adams $E_2^{1,*}$ term of $\widetilde{bu}_*(BSO(2n+1))$. Then we can prove Theorem B. First we recall the Wu formula.

Proposition 5.1. *(Wu formula* [9] *)* $Sq^k(w_m) = \sum_{t=0}^{k} \binom{m-k+t-1}{t} w_{k-t} w_{m+t}$, *where the binomial coefficient* $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ *is taken mod 2.*

Then let's find the Adams $E_2^{1,*}$ term of $\widetilde{bu}_*(BSO(2n+1))$.

Theorem 5.2. *As a* $\mathbb{Z}/2$*-module,* $E_2^{1,*}(BSO(2n+1))$ *is generated by* $\overline{v_0} \otimes \widehat{w_2}^{2m_1}\widehat{w_4}^{2m_2}\cdots\widehat{w_{2n}}^{2m_n}$ *and* $\overline{v_1} \otimes \widehat{w_2}^{2m_1}\widehat{w_4}^{2m_2}\cdots\widehat{w_{2n}}^{2m_n}$*, where* $\sum_{i=1}^{n} m_i > 0$*,* $m_i \geq 0$.

Proof. By the Wu formula, in $\tilde{H}^*(BSO(2n+1)) = \mathbb{Z}/2\left[\widehat{w_2}, \widehat{w_3}, \cdots, \widehat{w_{2n+1}}\right]$, we use the following diagrams

$$\begin{array}{ccc} \widehat{w_{2k+1}} & \xrightarrow{Q_0} & 0 \\ Q_1 \downarrow & & Q_1 \downarrow \\ \widehat{w_3}\widehat{w_{2k+1}} & \xrightarrow{Q_0} & 0 \end{array}, \quad 0 \leq k \leq n,$$

$$\begin{array}{ccc} \widehat{w_{2k}} & \xrightarrow{Q_0} & \widehat{w_{2k+1}} \\ Q_1 \downarrow & & Q_1 \downarrow \\ \widehat{w_3}\widehat{w_{2k}} + \widehat{w_{2k+3}} & \xrightarrow{Q_0} & \widehat{w_3}\widehat{w_{2k+1}} \end{array}, \quad 0 \leq k \leq n-1,$$

$$\begin{array}{ccc} \widehat{w_{2n}} & \xrightarrow{Q_0} & \widehat{w_{2n+1}} \\ Q_1 \downarrow & & Q_1 \downarrow \\ \widehat{w_3}\widehat{w_{2n}} & \xrightarrow{Q_0} & \widehat{w_3}\widehat{w_{2n+1}} \end{array}$$

to indicate the E-actions. It follows that the E-actions on the generators of $\tilde{H}^*(BSO(2n+1))$ must be the sum of $\widehat{w_{odd}}w$, where $w = 1$ or w is any monomial in $\tilde{H}^*(BSO(2n+1))$, hence the monomials $\widehat{w_2}^{2m_1}\widehat{w_4}^{2m_2}\cdots\widehat{w_{2n}}^{2m_n}$ are all indecomposable, $\sum_{i=1}^{n} m_i > 0$, $m_i \geq 0$. So

$$\begin{aligned} \overline{v_0} \otimes \widehat{w_2}^{2m_1}\widehat{w_4}^{2m_2}\cdots\widehat{w_{2n}}^{2m_n}, \ \sum_{i=1}^{n} m_i &> 0, m_i \geq 0, \\ \text{and } \overline{v_1} \otimes \widehat{w_2}^{2m_1}\widehat{w_4}^{2m_2}\cdots\widehat{w_{2n}}^{2m_n}, \ \sum_{i=1}^{n} m_i &> 0, m_i \geq 0, \end{aligned}$$

must be part of the $\mathbb{Z}/2$-generators of $E_2^{1,*}(BSO(2n+1))$.

Then by Lemma 3.5, Lemma 4.1 and Proposition 4.3, $\hat{E}_2^{1,*}(BSO(2n+1))$ contains at most the $\mathbb{Z}/2$-generators

$$\begin{aligned} (Bg_{2n})_*(\xi_1 \otimes b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_k}^2), \ 1 &\leq j_1 \leq j_2 \leq \cdots \leq j_k, \ 2k \leq 2n, \\ \text{and } (Bg_{2n})_*(\xi_2 \otimes b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_k}^2), \ 1 &\leq j_1 \leq j_2 \leq \cdots \leq j_k, \ 2k \leq 2n, \end{aligned}$$

Thus, counting the rank of $\hat{E}_2^{1,k}(BSO(2n+1))$ and $E_2^{1,k}(BSO(2n+1))$ as $\mathbb{Z}/2$-modules for each $k \geq 1$, we can see that all the generators of $E_2^{1,*}(BSO(2n+1))$ are found. This completes the proof. □

Proof of Theorem A. For each $n \geq 1$,recall that D_{2n+1} is the E-module with the $\mathbb{Z}/2$-generators $\widehat{w_2}^{2m_1}\widehat{w_4}^{2m_2}\cdots\widehat{w_{2n}}^{2m_n}$, $\sum_{i=1}^{n} m_i > 0$, $m_i \geq 0$, and M_{2n+1} is a free E-module with the E-basis $T_{2n+1} = \{t_j \mid j \in \Lambda_{2n+1}\}$ described in Section 1. Let N be the $\mathbb{Z}/2$-submodule of $\tilde{H}^*(BSO(2n+1))$ generated by all but this kind of monomials $\widehat{w_2}^{2m_1}\widehat{w_4}^{2m_2}\cdots\widehat{w_{2n}}^{2m_n}$, $\sum_{i=1}^{n} m_i > 0$, $m_i \geq 0$. Then since $\widehat{w_2}^{2m_1}\widehat{w_4}^{2m_2}\cdots\widehat{w_{2n}}^{2m_n}$ are indecomposable, N is an E-submodule and $\tilde{H}^*(BSO(2n+1)) \cong D_{2n+1} \oplus N$, as E-modules. Note that T_{2n+1} is contained in N since $\widehat{w_2}^{2m_1}\widehat{w_4}^{2m_2}\cdots\widehat{w_{2n}}^{2m_n}$ can not be generated by T_{2n+1}.

Then by Theorem 5.2 and Lemma 2.1, the $\mathbb{Z}/2$-generators of $E_2^{1,k}(D_{2n+1})$ are

$$\overline{v_0} \otimes \widehat{w_2}^{2m_1}\widehat{w_4}^{2m_2}\cdots\widehat{w_{2n}}^{2m_n}, \ \sum_{i=1}^{n} m_i > 0, m_i \geq 0,$$

$$\text{and } \overline{v_1} \otimes \widehat{w_2}^{2m_1}\widehat{w_4}^{2m_2}\cdots\widehat{w_{2n}}^{2m_n}, \ \sum_{i=1}^{n} m_i > 0, m_i \geq 0,$$

and $E_2^{1,k}(N) = 0$. Thus by Lemma 2.4, N is a free E-module with the E-basis T_{2n+1}, that is, $N = M_{2n+1}$. This completes the proof. □

6. Proof of Theorem B

In this section, we are going to prove Theorem A. First we recall what we need in [3]. Suppose M and N are left A-modules with the actions μ_M and μ_N, then $M \otimes N$ is also a left A-module with the action defined by the composite map

$$A \otimes M \otimes N \overset{\psi\otimes M\otimes N}{\longrightarrow} A \otimes A \otimes M \otimes N \overset{A\otimes T\otimes N}{\longrightarrow} A \otimes M \otimes A \otimes N \overset{\mu_M\otimes\mu_N}{\longrightarrow} M \otimes N,$$

where ψ is the diagonal map of A and $T(a \otimes b) = (-1)^{\dim a \dim b}(b \otimes a)$ is the twist map. We write ${}_D(M \otimes N)$ to indicate $M \otimes N$ with this left action. Similarly, ${}_L(M \otimes N)$ indicates the extended A-action over M. Then we have the following proposition.

If B is a Hopf subalgebra of A, then we know that ${}_D(M \otimes N)$ is a left B-module and $A \otimes_B N$ is a left A-module with the extended action over A. Thus we have the following proposition.

Proposition 6.1. *(Proposition 1.7 of* [3]*) If B is a Hopf subalgebra of A, M is a left A-module, and N is a left B-module, then*

$${}_D[M \otimes (A \otimes_B N)] \cong_L [A \otimes_B \ {}_D(M \otimes N)]$$

as left A-modules.

Remark 6.2. **Remark 6.2.** Let N be $\mathbb{Z}/2$ and B be E in Proposition 6.1. Since

$$_D[M \otimes (A \otimes_E \mathbb{Z}/2)] \cong_D [(A \otimes_E \mathbb{Z}/2) \otimes M] \text{ and } _D(M \otimes \mathbb{Z}/2) \cong M,$$

the isomorphism becomes

$$\theta :_L [A \otimes_E M] \cong_D [(A \otimes_E \mathbb{Z}/2) \otimes M]$$

and is given by $\theta(a \otimes x) = \sum a' \otimes 1 \otimes a''x$, with the inverse $\theta^{-1}(a \otimes 1 \otimes x) = \sum a' \otimes \chi(a'')x$, where $\psi(a) = \sum a' \otimes a''$and χ is the conjugation map. (See [1] and Proposition 1.1 of [3] for the details.)

Recall that $H^*(bu) \cong A//A(Q_0, Q_1) \cong A \otimes_E \mathbb{Z}/2$ and the Künneth theorem gives the isomorphism

$$\phi : H^*(bu \wedge X) \cong H^*(bu) \otimes \tilde{H}^*(X) \cong A \otimes_E \mathbb{Z}/2 \otimes \tilde{H}^*(X) \overset{\theta^{-1}}{\cong} A \otimes_E \tilde{H}^*(X)$$

for any space or spectrum X. Then by Theorem A, we have

$$\begin{aligned} H^*(bu \wedge BSO(2n+1)) &\overset{\phi}{\cong} A \otimes_E \tilde{H}^*(BSO(2n+1)) \\ &\cong A \otimes_E D_{2n+1} \oplus A \otimes_E M_{2n+1} \end{aligned}$$

and

$$H^*(bu \wedge BO(n)) \overset{\phi}{\cong} A \otimes_E \tilde{H}^*(BO(n)).$$

Next, to construct the homotopy equivelnce we need, we have to recall the main result in [7]. Recall that in Theorem 3.1, we have

$$\tilde{H}^*(BO(n)) \cong D_1^* \oplus D_2^* \oplus M,$$

where D_1^* is a trivial E-module with the generators $w_2^{2m_1} w_4^{2m_2} \cdots w_{2k}^{2m_k}$, $\sum_{i=1}^{k} m_i > 0,\ 2k \le n$. Note that in $\tilde{H}^*(BO(2n+1))$, $D_1^* \cong D_{2n+1}$.

Theorem 6.3. *(Theorem 1.2 of* [7]*) For each $n \ge 1$, there is a stable splitting*

$$bu \wedge BO(n) \simeq [\bigvee_{\alpha'} \Sigma^{\alpha'} H\mathbb{Z}/2] \vee [\bigvee_{\beta} \Sigma^{\beta} bu] \vee [\bigvee_{\gamma} \Sigma^{\gamma} bu \wedge RP^\infty],$$

where the α', and their degrees, correspond to the generators of M, the β, and their degrees, correspond to the generators of D_1^*, the γ, and their degrees, correspond to the generators of D_2^*.

Remark 6.4. Let f be the above homotopyequivelnce. Then

$$f^*(0 \oplus (1 \otimes \Sigma^{\beta} 1) \oplus 0) = 1 \otimes w_2^{2m_1} w_4^{2m_2} \cdots w_{2k}^{2m_k}$$

for each generator $w_2^{2m_1} w_4^{2m_2} \cdots w_{2k}^{2m_k}$ of D_1^* and the corresponding β, where $1 \otimes \Sigma^\beta 1 \in A \otimes_E \tilde{H}^*(\bigvee_\beta S^\beta) \cong H^*(\bigvee_\beta \Sigma^\beta bu)$. For the details on the map f, see the proof of Theorem 1.2 and Section 3 of [7].

Proof of Theorem B. First we construct the stable map

$$g : bu \wedge BSO(2n+1) \longrightarrow [\bigvee_\alpha \Sigma^\alpha H\mathbb{Z}/2] \vee [\bigvee_\beta \Sigma^\beta bu].$$

For each E-free generators $t_j \in \tilde{H}^\alpha(BSO(2n+1))$, $\deg t_j = \alpha$, $t_j \in T_{2n+1} = \{t_j \mid j \in \Lambda_{2n+1}\}$, let $g_{t_j} : BSO(2n+1) \longrightarrow \Sigma^\alpha H\mathbb{Z}/2$ represent t_j, which means $g_{t_j}^*(\Sigma^\alpha 1) = t_j$. Let $i : bu \longrightarrow H\mathbb{Z}/2$ be the multiplicative canonical map and μ' be the ring structure map of $H\mathbb{Z}/2$. Then we define

$$g_0 : bu \wedge BSO(2n+1) \overset{bu \wedge (\bigvee_{t_j} g_{t_j})}{\longrightarrow} bu \wedge [\bigvee_\alpha \Sigma^\alpha H\mathbb{Z}/2] \overset{\bigvee_{t_j} \nu}{\longrightarrow} [\bigvee_\alpha \Sigma^\alpha H\mathbb{Z}/2],$$

where $\alpha = \deg t_j$ and $\nu : bu \wedge H\mathbb{Z}/2 \overset{i \wedge H\mathbb{Z}/2}{\longrightarrow} H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \overset{\mu'}{\longrightarrow} H\mathbb{Z}/2$.

On the other hand, we define

$$\begin{aligned} g_1 \quad : \quad & bu \wedge BSO(2n+1) \overset{bu \wedge h_{2n+1}}{\longrightarrow} bu \wedge BO(2n+1) \overset{f}{\simeq} \\ & [\bigvee_{\alpha'} \Sigma^{\alpha'} H\mathbb{Z}/2] \vee [\bigvee_\beta \Sigma^\beta bu] \vee [\bigvee_\gamma \Sigma^\gamma bu \wedge RP^\infty] \overset{p}{\longrightarrow} [\bigvee_\beta \Sigma^\beta bu], \end{aligned}$$

where p is the projection map. Then for each generator $\widehat{w_2}^{2m_1} \widehat{w_4}^{2m_2} \cdots \widehat{w_{2n}}^{2m_n}$ of D_{2n+1}, we have

$$\begin{aligned} g_1^*(\Sigma^\beta 1) &= (bu \wedge h_{2n+1})^*(1 \otimes w_2^{2m_1} w_4^{2m_2} \cdots w_{2n}^{2m_n}) \\ &= 1 \otimes \widehat{w_2}^{2m_1} \widehat{w_4}^{2m_2} \cdots \widehat{w_{2n}}^{2m_n}. \end{aligned}$$

Therefore, we have the stable map

$$g = g_0 \vee g_1 : bu \wedge BSO(2n+1) \longrightarrow [\bigvee_\alpha \Sigma^\alpha H\mathbb{Z}/2] \vee [\bigvee_\beta \Sigma^\beta bu].$$

Now we show that g induces an isomorphism on the mod 2 cohomology. Since for $1 \in A$, we have $\psi(1) = 1 \otimes 1$ and $\chi(1) = 1$, under the map

$$\begin{aligned} \Phi_1 \quad : \quad & A \otimes_E \tilde{H}^*(\vee S^\beta) \overset{\phi^{-1}}{\cong} H^*(\vee \Sigma^\beta bu) \overset{g_1^*}{\longrightarrow} H^*\left(bu \wedge BSO(2n+1)\right) \\ \cong \quad & (A \otimes_E D_{2n+1}) \oplus (A \otimes_E M_{2n+1}) \overset{p_1}{\longrightarrow} A \otimes_E D_{2n+1}, \end{aligned}$$

where p_1 is the projection map, we have

$$\begin{aligned} \Phi_1 \quad : \quad & 1 \otimes \Sigma^\beta 1 \overset{\phi^{-1}}{\mapsto} 1 \otimes \Sigma^\beta 1 \overset{g_1^*}{\mapsto} 1 \otimes \widehat{w_2}^{2m_1} \widehat{w_4}^{2m_2} \cdots \widehat{w_{2n}}^{2m_n} \\ \mapsto \quad & (1 \otimes \widehat{w_2}^{2m_1} \widehat{w_4}^{2m_2} \cdots \widehat{w_{2n}}^{2m_n}) \oplus 0 \overset{p_1}{\mapsto} 1 \otimes \widehat{w_2}^{2m_1} \widehat{w_4}^{2m_2} \cdots \widehat{w_{2n}}^{2m_n}. \end{aligned}$$

Therefore, since the $\mathbb{Z}/2$-basis of D_{2n+1} is consisted of the monomials $\widehat{w_2}^{2m_1}\cdots\widehat{w_{2n}}^{2m_n}$, which means D_{2n+1} is isomorphic to $\tilde{H}^*(\vee S^\beta)$ as E-modules, and since the A-action on $A\otimes_E D_{2n+1}$ is just on A, and so is $A\otimes_E \tilde{H}^*(\vee S^\beta)$, Φ_1 is an isomorphism and this implies g_1^* takes $H^*(\vee\Sigma^\beta bu)$ isomorphically onto $A\otimes_E D_{2n+1}$.

Similarly, under the map

$$\begin{aligned}\Phi_2 \quad : \quad & H^*(\vee\Sigma^\alpha H\mathbb{Z}/2) \xrightarrow{g_0^*} H^*\left(bu\wedge BSO(2n+1)\right)\cong \\ & (A\otimes_E D_{2n+1})\oplus(A\otimes_E M_{2n+1}) \xrightarrow{p_2} A\otimes_E M_{2n+1},\end{aligned}$$

where p_2 is the projection map, we have

$$\Phi_2 : \Sigma^\alpha 1 \overset{g_0^*}{\mapsto} 1\otimes t_j \mapsto 0\oplus(1\otimes t_j) \overset{p_2}{\mapsto} 1\otimes t_j$$

for each E-free generators t_j and the corresponding $\Sigma^\alpha 1\in H^*(\vee\Sigma^\alpha H\mathbb{Z}/2)$. Thus Φ_2 is an isomorphism and this implies g_0^* takes the free A-module $H^*(\vee\Sigma^\alpha H\mathbb{Z}/2)$ isomorphically onto $A\otimes_E M_{2n+1}$.

As a result, we see that the composite homomorphism

$$\begin{aligned} H^*([\bigvee_\alpha \Sigma^\alpha H\mathbb{Z}/2]\vee[\bigvee_\beta \Sigma^\beta bu]) & \overset{g^*=g_0^*\oplus g_1^*}{\longrightarrow} H^*\left(bu\wedge BSO(2n+1)\right) \\ & \cong (A\otimes_E D_{2n+1})\oplus(A\otimes_E M_{2n+1})\end{aligned}$$

is an isomorphism, hence g is an equivalence at prime 2. This completes the proof. □

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Tsung-Hsuan Wu
Department of Mathematics
National Tsing Hua University
Hsinchu, Taiwan
e-mail address: thwu@math.nthu.edu.tw

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