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REVIEW ON HIGHER HOMOTOPIES IN THE THEORY OF H-SPACES

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Abstract. Higher homotopy in the theory of H-spaces started from the works by Sugawara in the 1950th. In this paper we review the development of the theory of H-spaces associated with it. Mainly there are two types of higher homotopies, homotopy associativity and homotopy commutativity. We give explanations of the polytopes used as the parameter spaces of those higher forms.

1. Introduction

This paper is prepared for the 60th special edition of Mathematical Journal of Okayama University. In the journal many important articles have been published, and among them, Sugawara's paper on H-spaces [56, 57, 58, 60, 59, 61] published in the 50th exerted a significant degree of influence on development of the homotopy theory. In particular, the concept of group-like space introduced in 1957 was rearranged to A_n-space by Stasheff [53], and developed to operad by May [45], which has played an important role not only in mathematics but also in physics. This paper includes a variety of higher homotopies, organized as follows.

In the next section, we first give the definition of H-space, the concept of which appeared after the paper by Hopf [23]. It is sometimes called Hopf space. As far as the author knows, the term "Hopf space" was first used in the paper by Moore and Smith [50] as homotopy associative H-space. Zabrodsky [69] and Kane [33] used the same term in their books. But it appeared only in their titles. Kane writes in introduction as follows:

> The use of the term "Hopf space" in the cover title was primarily designed to make the subject matter of the book as clear as possible to non topologists.

In this paper, we only use the term H-space.

We also recall the theorems by Hopf [23] and Borel [7] on the cohomology of H-space. Moreover, as specific examples, we treat spheres and the Hopf invariant problem associated with them. As an example of finite H-space, the Hilton-Roitberg manifold, the total space of a principal S^3-bundle over

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S^7, is significant. This is a counter example of the homotopy version of the Hilbert's fifth problem since it has the homotopy type of a topological group, but not the homotopy type of a Lie group.

From section 3 we review higher homotopy in the theory of H-space. In general, any higher homotopy describing a structure of H-spaces is given as a family of maps $\{h_i\colon \mathcal{L}_i \times X^i \to Y\}_{i\leq n}$. These are called as A_n-form, C_n-form and so on. Here, $\mathcal{L}_i$ is a polytope, which is homeomorphic to an interval if i is small as 2 or 3. Thus in this case the corresponding map h_2 or h_3 is just an ordinary homotopy. For greater i, the parameter space $\mathcal{L}_i$ is homeomorphic to a higher dimensional disk, whose boundary is covered by the facets. Each facet is homeomorphic to lower dimensional $\mathcal{L}_j$s $(j < i)$, and the restriction of h_i to each facet is represented by lower h_js $(j < i)$. This property expresses the compatibility of the family $\{h_i\}_{i\leq n}$.

We need a variety of parameter spaces $\mathcal{L}_i$ depending on higher homotopies such as associahedra, multiplihedra, permutohedra, resultohedra, permuto-associahedra and cyclohedra. The construction of these polytopes as subsets of the euclidean space is an important theme, and there are many attempts for it. But, in this paper, we focus on the combinatorial properties of these polytopes. Readers who are interested in the realization of them should refer to the original papers.

Section 3 recalls A_n-form, which represents higher homotopy associativity. We first deal with the A_n-form on spaces in 3.2. The parameter spaces used here are called associahedra. From the homotopy theoretic view point, topological group, topological monoid and loop space are equivalent concepts. Any space having the homotopy type of such a space admits a homotopy associative H-structure. Contrarily, the converse does not hold. The concept of group-like space by Sugawara was introduced as a property for an H-space to have the homotopy type of a topological group. A_∞-space by Stasheff is equivalent to group-like space

In 3.2, we review A_n-form on maps. A homomorphism between topological monoids induces a map between their classifying spaces. This is a homotopy invariant property. Then Sugawara [62] introduced the concept of strongly homotopy-multiplicative such that maps between topological monoids with this property induce maps between their classifying spaces. This is expressed as a form of higher homotopy, and later Stasheff [55] reorganized as A_n-form on maps between topological monoids. The parameter spaces used for the definition are cubes.

Then Stasheff extended the definition of A_n-form for the case that the sources of the maps are A_n-spaces. Extending the definition to maps between A_n-spaces is naturally required. However, the parameter spaces for it are very complicated. In fact, Stasheff says in [55, p.53] as follows:

> "It is also possible to consider maps of A_n-spaces which respect the structure up to homotopy, but the details are too complicated to mention completely here."

The parameter spaces of A_n-form for the case that only the sources of the maps are A_n-spaces are associahedra.

The definition of A_n-form on maps between A_n-spaces is due to Iwase [26]. Later it appeared in the paper [30] by Iwase and Mimura. The parameter spaces used in it are called multiplihedra.

Homotopy commutativity comes in section 4. If X is an H-space, then the loop multiplication of ΩX is homotopy commutative. On the other hand, the converse does not hold. Sugawara [62] considered on what homotopical conditions for a topological monoid X the classifying space BX of X admits an H-space structure. The property he introduced is a higher homotopy commutativity called strong homotopy-commutativity. This concept is closely related to the strong homotopy-multiplicative. In fact, if a monoid X is strong homotopy-commutative, then the multiplication X is strong homotopy-multiplicative, and so we have a multiplication of BX as $BX \times BX \simeq B(X \times X) \to BX$. We review in 4.1 strong homotopy-multiplicative and Sugawara C_n-form defined by McGibbon [46].

In 4.2, we review another higher homotopy commutativity introduced by Williams [64]. This is defined as a family of higher homotopies between maps defined by using the action of permutations. The nth parameter space is called n-permutohedron, which has vertexes corresponding to all permutations of n letters. This family of polytopes was introduced by Milgram [47] for the study of iterated loop spaces. We call this type of form as Williams C_n-form to distinguish it from Sugawara C_n-form.

Williams C_n-form seems natural since defined by considering all permutations. However, it is weaker than Sugawara C_n-form so that for any Williams C_∞-space X, the classifying space BX of X is not necessarily an H-space. Another form with a similar property to Sugawara C_n-form was defined by Hemmi and Kawamoto [21] and Kishimoto and Kono [41] by using shuffles. The polytopes used as parameter spaces of their higher form are subsets of permutohedra with vertexes corresponding to some shuffles, and are homeomorphic to polytopes called resultohedra.

The higher homotopy commutativity stated so far are all defined on topological monoids. On the other hand, in definition of Williams C_n-form, we only consider the product of at most n elements. Thus, the definition can be generalized to A_n-spaces. But, to do so, we need polytopes which is hybrid of the associahedra and the permutohedra. In 4.3, we mention the polytopes called permuto-associahedra. Incidentally, since resultohedra are realized as subsets of permutohedra, we can consider similar hybrid spaces for resultohedra. Kawamoto [38] showed that some of such spaces are homeomorphic to cyclohedra.

An interesting fact valids for finite H-spaces is the torus theorem proved by Hubbuck [24]: any connected finite H-space with homotopy commutative multiplication has the homotopy type of a torus. Prior to Hubbuck, weaker forms of the above theorem have been obtained by several authors. Araki, James and Thomas [4] first proved in the case that X is a compact connected Lie group with group multiplication. James [32] extended the result to compact connected Lie groups with any multiplications.

Lin [42] showed that the torus theorem is essentially on the mod 2 structure of the H-space. He showed that for a simply connected H-space X with finite $\mathbb{F}_2$-cohomology $H^*(X;\mathbb{F}_2)$, if X admits a homotopy commutative multiplication then $H^*(X;\mathbb{F}_2)$ is acyclic. This theorem is called the mod 2 torus theorem. The torus theorem by Hubbuck can be proved from the mod 2 torus theorem.

Then, it is natural to consider corresponding theorems for odd primes. Unfortunately, this is not so easy. In fact, Iriye and Kono [25, Theorem 1.3] showed that if p is an odd prime, then any p-localized H-space admits a homotopy commutative multiplication. In contrast with the result by Iriye-Kono, several authors have shewn theorems called mod p torus theorem for odd primes p. To get them. the H-space must have some sort of higher homotopy commutativity. We review it in 4.4.

2. H-SPACE

There are two versions of definition of H-space. The two are different in the strictness of the unit. One version is that a space X with base point $*$ is called an H-space if there is a continuous map $\mu_X\colon X \times X \to X$ such that $\mu_X(*, e) = \mu_X(*, x) = x$ for any $x \in X$. In other words, the base point $*$ is the strict unit of the multiplication μ_X. The other version is that the base point is only assumed to be a homotopy unit, that is, both maps $x \mapsto \mu_X(x, *)$ and $x \mapsto \mu_X(*, x)$ are homotopic to the identity on X.

Usually the difference of these two definitions is not so serious. In fact, the existence of the strict unit means that $\mu_X|X \vee X$ equals to the holding map $\nabla\colon X \vee X \to X$, and the existence of the homotopy unit means that $\mu_X|X \vee X$ is homotopic to ∇. Thus, if the pair $(X \times X, X \vee X)$ has the homotopy extension property for X, then any multiplication with homotopy unit can be deformed to a multiplication with strict unit. For example, if X is a CW-complex, both definitions for X are considered to be equivalent.

From now on, H-space means a pointed space X equipped with a continuous multiplication with strict unit. We usually denote the multiplication of X by μ_X, and $\mu_X(x, y)$ is abbreviated as xy.

The associativity of the multiplication and the existence of the inverse are not assumed for the definition of H-space. Even though, H-space enjoys many interesting properties. In particular, the cohomology ring with the coefficient in a field has a natural Hopf algebra structure. Here, the coassociativity of the coproduct is not assumed for Hopf algebra.

If X is a connected H-space, then for any field $\mathbb{F}$, the cohomology ring $H^*(X;\mathbb{F})$ together with the cup product $H^*(X;\mathbb{F}) \otimes H^*(X;\mathbb{F}) \to H^*(X;\mathbb{F})$ and $\mu_X^* \colon H^*(X;\mathbb{F}) \to H^*(X;\mathbb{F}) \times H^*(X;\mathbb{F})$ is a Hopf algebra. In particular, for the rational cohomology, we have the following theorem by Hopf [23].

Theorem 2.1. *Let X be a connected H-space. If the rational cohomology $H^*(X;\mathbb{Q})$ is finite type, i.e., $H^n(X;\mathbb{Q})$ is finite dimensional for any n, then as algebras we have*

$$H^*(X;\mathbb{Q}) \cong \Lambda(x_1, x_2, \dots) \otimes \mathbb{Q}[y_1, y_2, \dots],$$

for some odd dimensional generators $x_1, x_2, \dots$ and even dimensional generators $y_1, y_2, \dots$.

In particular, if X is a finite CW-complex, then

$$H^*(X;\mathbb{Q}) \cong \Lambda(x_1, x_2, \dots, x_k),$$

for some finitely many odd dimensional generators $x_1, x_2, \dots, x_k$.

An H-space with the homotopy type of a finite CW-complex is called a finite H-space. For finite H-spaces, many similar terminologies to Lie groups are used. For example, if $H^*(X;\mathbb{Q}) \cong \Lambda(x_1, x_2, \dots, x_k)$, then the number of generators k is called the rank of X, and the sequence $(\deg x_1, \dots, \deg x_k)$ is called the type of X, where we assume that $\deg x_{i-1} \leq \deg x_i$ for any i.

If we take the prime field $\mathbb{F}_p$ of characteristic p instead of $\mathbb{Q}$, where p is a prime, then we have the following theorem by Borel [7].

Theorem 2.2. *Let X be a connected H-space.*

(1) If the $\mathbb{F}_2$-cohomology $H^*(X;\mathbb{F}_2)$ is finite type, then as an algebra

$$H^*(X;\mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, \dots]/(x_1^{m_1}, x_2^{m_2}, \dots),$$

for some generators $x_1, x_2, \dots,$ where m_i is a power of 2 or ∞.

In particular, if X is a finite CW-complex, then

$$H^*(X;\mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, \dots, x_k]/(x_1^{m_1}, x_2^{m_2}, \dots, x_k^{m_k}),$$

for some finitely many generators $x_1, x_2, \dots, x_k$ with m_i a power of 2.

(2) If the $\mathbb{F}_p$-cohomology $H^*(X;\mathbb{F}_p)$ is finite type for an odd prime p, then as an algebra

$$H^*(X;\mathbb{F}_p) \cong \Lambda(x_1, x_2, \dots) \otimes \mathbb{F}_p[y_1, y_2, \dots]/(y_1^{s_1}, y_2^{s_2}, \dots),$$

for some odd dimensional generators $x_1, x_2, \dots$ and even dimensional generators $y_1, y_2, \dots$, where s_i is a power of p or ∞.

In particular, if X is a finite CW-complex, then

$$H^*(X;\mathbb{F}_p) \cong \Lambda(x_1, x_2, \dots, x_k) \otimes \mathbb{F}_p[y_1, y_2, \dots, y_l]/(y_1^{s_1}, y_2^{s_2}, \dots, y_l^{s_l}),$$

for some finitely many generators $x_1, x_2, \dots, x_k$ and finitely many even dimensional generators $y_1, y_2, \dots, y_l$ with s_i a power of p.

Typical examples of H-space are topological groups. In particular, S^1 and S^3 are H-spaces since they are Lie groups: the spaces of unit vectors in the complex numbers and in the quaternions, respectively. Moreover, S^7 is an H-space, which is the space of unit vectors of the Cayley numbers. The problem of which sphere is an H-space is one of the main problems in homotopy theory in the 50th since it is closely related to the Hopf invariant one problem.

For any map $\mu\colon X \times Y \to Z$, the Hopf construction of μ is a map $h(\mu)\colon X * Y \to \Sigma Z$. Here, $X * Y$ is the join of X and Y. If $X = Y = Z = S^n$, then we have an element in the homotopy group $\pi_{2n+1}(S^{n+1})$. Moreover, if μ is of type (m_1, m_2), that is, the degree of the maps $S^n \to S^n$ defined by $x \mapsto \mu(x, *)$ and $x \mapsto \mu(*, x)$ are m_1 and m_2, respectively, then the Hopf invariant of $h(\mu)$ is $m_1 m_2$. This shows that the existence of an H-space structure on S^n implies the existence of an element of Hopf invariant one in $\pi_{2n+1}(S^{n+1})$.

Moreover, this problem is related to many other interesting problems. Adams [1] remarked that the following statements are all equivalent:

(1) S^n admits an H-space structure.
(2) $\mathbb{R}^{n+1}$ has a structure of a division algebra over the real.
(3) S^n has a differential structure of parallelizable.
(4) There is an element of Hopf invariant one in $\pi_{2n+1}(S^{n+1})$.
(5) There is a two cell complex $Y = S^{n+1} \cup e^{2n+2}$ such that $H^*(Y;\mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^3)$ with $\deg x = n + 1$.
(6) There is a two cell complex $Y = S^m \cup e^{m+n+1}$ such that Sq^{n+1} is non trivial in $H^*(Y;\mathbb{F}_2)$

Here, Sq^{n+1} in (6) is the Steenrod squaring operation acting on $\mathbb{F}_2$-cohomology: $H^k(\ \cdot\ ;\mathbb{F}_2) \to H^{k+n+1}(\ \cdot\ ;\mathbb{F}_2)$. The operations $\{Sq^t\}$ obey well known relation called the Adem relation, and by using this relation, it can be proved that if $n+1$ is not a power of 2, then Sq^{n+1} decomposes as $Sq^{n+1} = \sum_i Sq^{a_i} Sq^{n+1-a_i}$ for some a_i. This implies that $Sq^{n+1} = 0$ in (6) if

$n+1$ is not a power of 2, and so if S^n admits an H-space structure, then $n+1$ should be a power of 2. From this observation, the first problem whether the sphere S^n admits and H-space structure or not is the case that $n = 15$. Toda [63] showed that S^{15} does not admit any H-space structures. To show this, Toda introduced a secondary composition method called Toda bracket, which has been considered as a very useful method for the calculation of homotopy groups.

For the remaining cases, Adams [1] finally showed the following

Theorem 2.3. *The sphere S^n $(n \geq 1)$ is an H-space if and only if $n = 1, 3$ or 7.*

Adams proved the above theorem by introducing some higher order cohomology operations. The method to construct the operations extended to the method to calculate the stable homotopy groups, which is now called the Adams spectral sequence.

Beside the 7-sphere S^7, finite H-spaces with non Lie type have not been found for long time. It was conjectured that any finite H-space is of the homotopy type of the product of a Lie group and finitely many seven spheres.

The first example of a finite H-space except for such spaces was found by Hilton and Roitberg [22]. In the study of cancellation problem, they considered principal S^3-bundles over S^7. Let M_λ be the total space of the principal S^3-bundle over S^7 associated with $\lambda\omega$, where ω is the generator of $\pi_7(BS^3) \cong \mathbb{Z}/12\mathbb{Z}$ such that $M_1 = Sp(2)$. They showed that $M_7 \times S^3 \simeq Sp(2) \times S^3$. Since $M_\lambda \simeq M_\delta$ if and only if $\lambda \equiv \pm\delta \bmod 12$, M_7 and $Sp(2)$ are of different homotopy types, which gives a counter example of the cancellation problem. Moreover, this result implies that $M_7 \times S^3$ is an H-space, and so M_7 is an H-space not having the homotopy type of the produce of a Lie group and finitely many 7-spheres.

This example is interesting for another reason. Stasheff [54] showed that M_7 has the homotopy type of a topological group. Hilbert's fifth problem implies that any topological manifold which is also a finite dimensional manifold is a Lie group. Since M_7 is a 10-dimensional differential manifold, his result indicates that the answer of the homotopical version of the Hilbert's fifth problem is negative.

The theorem Stasheff proved for the spaces M_λ is as follows:

Theorem 2.4 (Stasheff [54, Theorem 2])**.** *For the space M_λ, the followings hold:*

(1) *M_λ is of homotopy type of a Lie group if and only if $\lambda \equiv \pm 1 \bmod 12$.*
(2) *M_λ is of homotopy type of a topological group if and only if $\lambda \equiv \pm 1, \pm 5 \bmod 12$.*
(3) *M_λ is an H-space if and only if $\lambda \not\equiv 2 \bmod 4$.*

The construction by Hilton-Roitberg is easily generalized. In fact, after their discussion, many finite H-spaces have been found in the total spaces of principal bundles. Let $G(n,d) = SO(n), SU(n)$ or $Sp(n)$ according as $d = 1, 2$ or 4. Then we have the principal $G(n-1,d)$ bundle

$$G(n-1,d) \longrightarrow G(n,d) \longrightarrow G(n,d)/G(n-1,d) = S^{dn-1}$$

Let $M_\lambda(n,d)$ be the total space of the principal $G(n-1,d)$-bundle induced from the above principal bundle by the map on S^{dn-1} of degree λ. Then the following facts are proved by Curtis and Mislin [10], Zabrodsky [67, 65] and Hemmi [15, 16].

Theorem 2.5. (1) *If n is even, then $M_\lambda(n,1)$ is an H-space if and only if $n = 2, 4, 8$ or λ is odd. On the other hand, if n is odd, then $M_\lambda(n,1)$ is an H-space if and only if $\lambda = \pm 1$.*

(2) *$M_\lambda(n,2)$ is an H-space if and only if $n = 2, 4$ or λ is odd.*

(3) *$M_\lambda(2,4)$ is an H-space if and only if $\lambda \not\equiv 2 \bmod 4$. If $n \neq 2$, then $M_\lambda(n,4)$ is an H-space if and only if λ is odd.*

(4) *$M_\lambda(n,d)$ has the homotopy type of a topological manifold if and only if $\lambda \not\equiv 0 \bmod p$ for any prime p with $2p < dn$.*

For the homotopy types of the spaces $M_\lambda(n,d)$, Zabrodsky [68, 66] proved the following fact. Here, $k(n,d) = (dn/2-1)!$ for $d = 2, 4$. Note that the order of the cyclic group $\pi_{dn-2}(G(n-1,d))$ is $k(n,d)$ if $d = 2$ or n is odd, and $2k(d,n)$ if $d = 4$ and n is even.

Theorem 2.6. (1) *If $M_\lambda(n,d) \simeq M_\eta(n,d)$, then $\lambda \equiv \pm\eta \bmod k(n,d)$.*

(2) *Suppose that $\lambda \equiv \pm\eta \bmod k(n,d)$ if $d = 2$ or n is odd, and $\lambda \equiv \pm\eta \bmod 2k(n,d)$ if $d = 4$ and n is even. Then $M_\lambda(n,d) \simeq M_\eta(n,d)$.*

3. Homotopy associativity

Since the existence of the inverse of the multiplication is not assumed for H-space, topological monoids are also H-spaces. Moreover, the loop space ΩY is an H-space. The loop multiplication has only homotopy unit, but as is noted, there is a multiplication with strict unit which is homotopic to the loop multiplication. Moreover, as a space of the same homotopy type of the loop space, one can consider the Moore loop space, which is a space of loops with length. The loop multiplication of the Moore loop space has a strict unit and is strictly associative.

Three concepts, topological group, topological monoid and loop space, are equivalent from the homotopy theoretic view point. In fact we have the following

Theorem 3.1. *For a space X, the following three conditions are equivalent.*

(1) *X has the homotopy type of a topological group.*
(2) *X has the homotopy type of a topological monoid.*
(3) *X has the homotopy type of a loop space.*

The fact that any topological group has the homotopy type of a loop space is proved by Milnor [48]. For a topological group G, he constructed the universal principal G-bundle $p\colon EG \to BG$ with $p^{-1}(*) = G$. Then we have a homotopy fibration $\Omega(BG) \to G \to EG$. Since EG is contractible, the map $\Omega(BG) \to G$ induces isomorphisms on the homotopy groups, and so G has the homotopy type of the loop space $\Omega(BG)$.

We look at the construction of the universal bundle in a little more detail. Milnor constructed a sequences of principal G-bundles $p_i\colon E_iG \to B_iG$ $(i \geq 0)$:

$$\begin{array}{ccccccccccccc}
E_0G & \xrightarrow[\subset]{} & E_1G & \xrightarrow[\subset]{} & E_2G & \xrightarrow[\subset]{} & \cdots & \xrightarrow[\subset]{} & E_nG & \xrightarrow[\subset]{} & E_{n+1}G & \xrightarrow[\subset]{} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
B_0G & \xrightarrow[\subset]{} & B_1G & \xrightarrow[\subset]{} & B_2G & \xrightarrow[\subset]{} & \cdots & \xrightarrow[\subset]{} & B_nG & \xrightarrow[\subset]{} & B_{n+1}G & \xrightarrow[\subset]{} & \cdots
\end{array}$$

Here E_nG is the n-fold join $G * \cdots * G$ of G, $B_0G = *$, $B_1G = \Sigma G$, and $B_{n+1}G = B_nG \cup_{p_n} C(E_nG)$. Moreover, the fiber G of p_n is contractible in E_nG for any n. Then the universal G-bundle is given by $EG = \bigcup_n EG_n \to BG = \bigcup_n B_nG$.

Dold and Lashoff [11] generalized the above construction to topological monoids. Milnor used the inverse of a topological group for the construction, but Dold and Lashoff showed a similar construction can be given without the inverse.

Let M be a topological monoid. Put $E_1M = M \cup_{\mu_M} M \times CM$ and $B_1M = \Sigma M$. Then the map $E_1M \to B_1M$ induced by the projection $M \times CM \to CM$ followed by the natural map $CM \to \Sigma M$ is a quasifibration with fiber M. Moreover, they used the associativity of the multiplication of M to define an action of M on E_1M, and by using this action they showed that the quasifibration $E_1M \to B_1M$ can be extended to a quasifibration $E_2M \to B_2M$. They continued this process and constructed a sequences of principal quasifibrations $p_i\colon E_iM \to B_iM$ $(i \geq 0)$ similar to Milnor's construction:

$$\begin{array}{ccccccccccccc}
E_0M & \xrightarrow[\subset]{} & E_1M & \xrightarrow[\subset]{} & E_2M & \xrightarrow[\subset]{} & \cdots & \xrightarrow[\subset]{} & E_nM & \xrightarrow[\subset]{} & E_{n+1}M & \xrightarrow[\subset]{} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
B_0M & \xrightarrow[\subset]{} & B_1M & \xrightarrow[\subset]{} & B_2M & \xrightarrow[\subset]{} & \cdots & \xrightarrow[\subset]{} & B_nM & \xrightarrow[\subset]{} & B_{n+1}M & \xrightarrow[\subset]{} & \cdots
\end{array}$$

Then the universal quasifibration $EM = \bigcup_n EM_n \to BM = \bigcup_n B_nM$ with fiber M is constructed, and since EM is contractible, we have $M \simeq \Omega(BM)$.

Later, Milnor also showed in [49] that any loop space ΩY has the homotopy type of a topological group under some minor conditions on Y. From these results, we have Theorem 3.1.

Now for any based space Y, the path space LY is the space of all paths $w\colon I \to Y$ with $w(0)$ the base point in Y. Then the map $p\colon LY \to Y$ defined by $p(w) = w(1)$ is a fibration with the fiber ΩY. Since LY is contractible, this fact together with the results by Milnor and Dold-Lashoff shows the following

Theorem 3.2 (Spanier and Whitehead [52])**.** *A space X is of homotopy type of a loop space if and only if X is the homotopy fiber of a fibration of which the total space is contractible.*

Sugawara studied corresponding conditions for H-spaces. The first step of the Dold-Lashoff constriction does not use the associativity of the multiplication. Thus a quasifibration $E_1X \to B_1X$ for any H-space X can be constructed such that the fiber X is contractible in E_1X. Incidentally, the Hopf construction $h(\mu_X)\colon X * X \to \Sigma X$ of the multiplication μ_X is also a quasifibration such that the fiber X is contractible in the total space $X * X$ ([60, Theorem 4]). Moreover, if the multiplication of X is homotopy associative, i.e., the following diagram is homotopy commutative:

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\mu_X \times id} & X \times X \\ \downarrow{\scriptstyle id\times\mu_X} & & \downarrow{\scriptstyle \mu_X} \\ X \times X & \xrightarrow[\mu_X]{} & X \end{array}$$

then he showed that one more step of the Dold-Lashoff construction can be achieved. His results are stated as follows:

Theorem 3.3 (Sugawara [60, Theorem 1], [61])**.** *A pointed space X admits an H-space structure if and only if there is a quasifibration $p_1\colon E_1 \to B_1$ with fiber X such that X is contractible in E_1.*

Moreover, X admits a homotopy associative H-space structure if and only if p_1 can be extended to a quasifibrations $p_2\colon E_2 \to B_2$ with fiber X such that E_1 is contractible in E_2.

Here, we note that there is a difference between the admitting a homotopy associative H-structure and the having the homotopy type of a topological monoid. If X has the homotopy type of a topological monoid G. then by using the multiplication $\mu_G\colon G \times G \to G$, X admits a multiplication which is homotopy associative as follows:

$$\mu_X\colon X \times X \simeq G \times G \xrightarrow{\mu_G} G \simeq X.$$

On the other hand, even if X admits a homotopy associative multiplication, X is not necessarily having the homotopy type of a topological monoid. In fact, if p is a prime with $p \geq 5$, then p-localized odd sphere $S^{2n-1}_{(p)}$ has a homotopy associative multiplication, but it has the homotopy type of a topological group only if n divides $p-1$.

Sugawara explored homotopical conditions to construct the full step of the Dold-Lashoff construction. Then he reached the concept of higher homotopy associativity of infinite order. He called this condition group-like [59]. Later Stasheff [53] rearranged this concept as A_n-space such that group-like space is equivalent to A_∞-space. We explain the definitions of A_n-space and group like space later, and here we just state the theorem by Sugawara.

Theorem 3.4 (Sugawara [59, Theorem 1.1])**.** *A pointed space X is group-like if and only if there is a quasifibration $E \to B$ with fiber X such that E is contractible. Thus in particular, X has the homotopy type of the loop space ΩB.*

3.1. **A_n-form on spaces.** We follow the argument by Stasheff [53]. He first considered the Milnor or the Dold-Lasheff type construction.

Definition 3.5. Let n be an integer with $n \geq 2$. Then an A_n-structure of a based space X is an n-tuple of maps $p_i \colon E_i \to B_i$ $(1 \leq i \leq n)$:

$$\begin{array}{ccccccccc} X & = & E_1 & \xrightarrow[\subset]{} & E_2 & \xrightarrow[\subset]{} & \cdots & \xrightarrow[\subset]{} & E_n \\ & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & & & \downarrow{\scriptstyle p_n} \\ * & = & B_1 & \xrightarrow[\subset]{} & B_2 & \xrightarrow[\subset]{} & \cdots & \xrightarrow[\subset]{} & B_n \end{array}$$

such that the inclusion $X \to E_i$ is homotopy equivalent to the homotopy fiber of p_i, and there is a contracting homotopy $h \colon CE_{n-1} \to E_n$ with $h(CE_{i-1}) \subset E_i$ for any i.

By Theorem 3.3, a space with an A_2-structure is an H-space, and a space with an A_3-structure is a homotopy associative H-space.

Next, we define a higher homotopy which is equivalent to an A_n-structure. Stasheff called this homotopy the A_n-form on a space X.

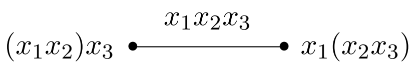
An A_n-form on a space X is a family of maps $\{M_i \colon \mathcal{K}_i \times X^i \to X\}_{2 \leq i \leq n}$. Here, the parameter space $\mathcal{K}_i$ is called i-associahedron and is homeomorphic to the $i-2$ dimensional disk. We consider the structure of $\mathcal{K}_i$ for $i = 2, 3, 4$.

An A_2-form on X, which is just a map $M_2 \colon \mathcal{K}_2 \times X^2 \to X$, should be an H-space multiplication. Thus $\mathcal{K}_2$ consists of only one point, and

$$M_2(*, x, *) = M_2(*, *, x) = x \quad \text{for any } x \in X. \tag{3.1}$$

Now, A_3-form $\{M_i \colon \mathcal{K}_i \times X^i \to X\}_{2 \leq i \leq 3}$ should represent a homotopy associativity of the multiplication of X. Since M_2 is the multiplication,

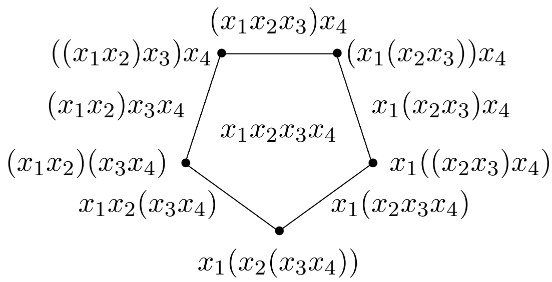
$M_3\colon \mathcal{K}_3 \times X^3 \to X$ should be an associating homotopy: the homotopy between $M_2 \circ (M_2 \times id)$ and $M_2 \circ (id \times M_2)$. Thus $\mathcal{K}_3$ is an interval, which consists of two vertexes and one edge. We represent two vertexes by $(x_1x_2)x_3$ and $x_1(x_2x_3)$, and then, the edge is represented by $x_1x_2x_3$.



Incidentally, we represent the one point space $\mathcal{K}_2$ by x_1x_2.

Next we consider the map $M_4\colon \mathcal{K}_4 \times X^4 \to X$ in the definition of the A_4-form. The space $\mathcal{K}_4$ is represented by the word $x_1x_2x_3x_4$, such that the vertexes of it are represented by inserting two meaningful pairs of parentheses into the word as follows: $((x_1x_2)x_3)x_4$, $(x_1(x_2x_3))x_4$, $x_1((x_2x_3)x_4)$, $x_1(x_2(x_3x_4))$ and $(x_1x_2)(x_3x_4)$. The edges connecting two of these vertexes are illustrated in the following figure:

(3.2)



In general, the face poset of the n-associahedron $\mathcal{K}_n$ is isomorphic to the set of all sequences $x_1x_2\ldots x_n$ inserted meaningful parentheses. For the order, $w_1 \preceq w_2$ means that w_1 is given by removing some pairs of meaningful parentheses from w_2. Thus, the maximum element is the word $x_1x_2\ldots x_n$ with no parentheses, and a k dimensional face of $\mathcal{K}_n$ corresponds to the sequence inserted $n-2-k$ pairs of parentheses. In particular, each facet corresponds to a sequence with one parenthesis as $x_1x_2\ldots x_{k-1}(x_k\ldots x_{k+s-1})x_{k+s}\ldots x_n$ for $k \geq 1$ and $2 \leq s \leq n-1$. This facet is homeomorphic to $\mathcal{K}_r \times \mathcal{K}_s$, where $r+s = n+1$, and denoted by $\mathcal{K}_k(r,s)$. Then we have a natural homeomorphism, called a face operator

$$\partial_k^{\mathcal{K}} = \partial_k^{\mathcal{K}}(r,s)\colon \mathcal{K}_r \times \mathcal{K}_s \to \mathcal{K}_k(r,s) \subset \mathcal{K}_n.$$

The family of maps $\{M_i\}$ should satisfies

$$\begin{aligned}(3.3)\quad & M_i(\partial_k^{\mathcal{K}}(r,s)(\rho,\sigma), x_1,\ldots,x_i)\\ &\qquad = M_r(\rho, x_1,\ldots,x_{k-1}, M_s(\sigma, x_k,\ldots,x_{k+s-1}), x_{k+s},\ldots,x_i)\end{aligned}$$

for $\rho \in \mathcal{K}_r$, $\sigma \in \mathcal{K}_s$ with $r+s = i+1$.

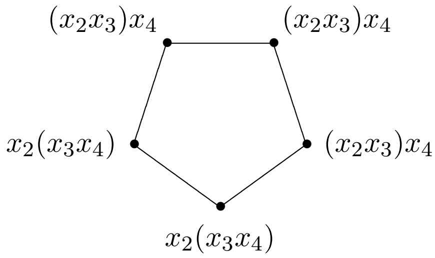
The intersection of two facets are of the forms $x_1 \dots (x_j \dots (x_{j+k-1\dots}) \dots) \dots$ or $x_1 \dots (x_k \dots) \dots (x_{j+s-1\dots}) \dots$. By using the face operators we can express the intersection as follows:

$$\partial_j^{\mathcal{K}} \circ (id \times \partial_k^{\mathcal{K}}) = \partial_{j+k-1}^{\mathcal{K}} \circ (\partial_j^{\mathcal{K}} \times id)$$
$$\partial_{j+s-1}^{\mathcal{K}} \circ (\partial_k^{\mathcal{K}} \times id) = \partial_k^{\mathcal{K}} \circ (\partial_j^{\mathcal{K}} \times id) \circ (\times T)$$

where $T\colon \mathcal{K}_s \times \mathcal{K}_t \to \mathcal{K}_t \times \mathcal{K}_s$ is the switching map.

Next we consider the effect of the unit element. The map $M_3\colon \mathcal{K}_3 \times X^3 \to X$ is a homotopy between $(x_1x_2)x_3$ and $x_1(x_2x_3)$. If one of the three element is the unit, for example, if $x_1 = *$, then we have $(x_1x_2)x_3 = x_1(x_2x_3) = x_2x_3$. Thus, the restriction of the homotopy M_3 to $\mathcal{K}_3 \times * \times X^2$ can be the constant homotopy, which means that $M_3(\rho, *, x_2, x_3) = M_2(*, x_2, x_3)$ for any $\rho \in \mathcal{K}_3$.

For the case of $\mathcal{K}_4$, the figure (3.2) with $x_1 = *$ is illustrated as follows:



Thus $M_4(\rho, *, x_2, x_3, x_4)$ $(\rho \in \mathcal{K}_4)$ reduces to $M_3(s_1^{\mathcal{K}}(\rho), x_2, x_3, x_4)$ for a map $s_1^{\mathcal{K}}\colon \mathcal{K}_4 \to \mathcal{K}_3$ with $s_1^{\mathcal{K}}((x_1x_2x_3)x_4) = s_1^{\mathcal{K}}(x_1(x_2x_3)x_4) = (x_1x_2)x_3$ and $s_1^{\mathcal{K}}(x_1x_2(x_3x_4)) = x_1(x_2x_3)$.

In general, degeneracy operations $s_j^{\mathcal{K}}\colon \mathcal{K}_i \to \mathcal{K}_{i-1}$ for $1 \le j \le i$ are defined to satisfy the following relation:

$$M_i(\tau, x_1, \dots, x_i) = M_{i-1}(s_j^{\mathcal{K}}(\tau), x_1, \dots, \hat{x}_j, \dots, x_i) \quad \text{if } x_j = *. \tag{3.4}$$

For the face of $\mathcal{K}_i$ represented by a parenthetical words w of $x_1 \dots x_{i+1}$, $s_j^{\mathcal{K}}(w)$ is given by first removing x_j from w, then renumbering x_{j+t} for $1 \le t \le i - j + 1$ to x_{j+t-1}, and finally removing unnecessary pairs of parentheses.

For the relations these operations obey see [53].

Now we give the definition of A_n-form by Stasheff [53].

Definition 3.6. Let n be an integer with $n \ge 2$. An A_n-form on a space X is a family of maps $\{M_i\colon \mathcal{K}_i \times X^i \to X\}_{2 \le i \le n}$ such that (3.1), (3.3) and (3.4) are satisfied. An A_∞-form on X is defined as a family $\{M_i\}_{2 \le i}$ with (3.1), (3.3) and (3.4). An A_n-space is a space X equipped with an A_n-form.

Any topological monoid X has the trivial A_∞-form $\{M_i\}_{2\leq i}$ defined as

$$M_i(\rho, x_1, \ldots, x_i) = x_1 \ldots, x_i.$$

Stasheff remarked that the condition (3.4) of the definition of A_n-form is technically, and is no restriction. In fact, he did not assume (3.4) for the definition of A_n-form in [55, Definition 11.2]. He claimed that if X admits an A_{n-1}-form $\{M_i\}_{i<n}$ and if $M'_n \colon \mathcal{K}_n \times X^n \to X$ is a map just satisfying the conditions (3.3) in the definition of the A_n-form then there is a map $M_n \colon \mathcal{K}_n \times X^n \to X$ such that $\{M_i\}_{2\leq i\leq n}$ satisfies all the conditions in the definition of the A_n-form ([53, Lemma 7]).

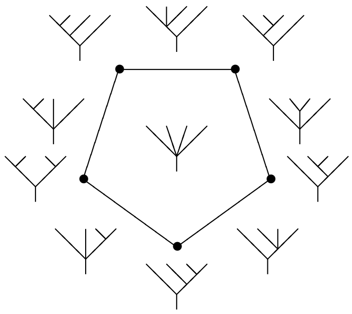
However, it is not clear that only the above argument is enough for the existence of a family of maps $\{M_i\}_{i<n}$ with all the conditions from the existence of $\{M'_i\}_{i<n}$ with (3.1) and (3.3). A detail about this problem is seen in Iwase [29]. In particular, to get the above form $\{M_i\}_{i<n}$ from $\{M'_i\}_{i<n}$, Iwase reconstructed associahedra as a convex polytopes with piecewise linearly decomposed faces. As is noted in [29], there are several attempts to realize the associahedra and multiplihedra as convex polytopes.

Now we give another description of the associahedra by using planar rooted trees introduced by Boardman and Vogt [6]. Here, a planar rooted tree is an oriented planar graph with no circuit. Moreover, the degree of any vertex is one or greater than 2. There is only one special vertex, called the root vertex. This is a vertex of degree one, and the edge connecting with it, which is called the root edge, is incoming edge. Other than the root vertex, a vertex with degree one is called a leaf vertex, and the edge connecting with it is called a leaf edge which is outgoing. There are at least two leaves in any planar rooted tree. The vertexes with degree greater that 2 are called inner nodes. Any inner node has only one outgoing edge and at least two incoming edges. Any planar rooted tree is embedded in the xy-plane with the root at the bottom and leaves at the top.

The set of planar rooted trees with n leaves corresponds to the face poset of n-associahedron $\mathcal{K}_n$. For example, the vertexes x_1x_2, $(x_1x_2)x_3$ and $x_1(x_2x_3)$ and an edge $x_1x_2x_3$ are represented as follows:

x_1x_2 $(x_1x_2)x_3$ $x_1(x_2x_3)$ $x_1x_2x_3$

The 4-associahedron $\mathcal{K}_4$ is represented as follows:



In general, the n-associahedron $\mathcal{K}_n$ is represented by a planar rooted tree with only one inner node and n leaves, which is called the n-corolla. We denote the n-corolla by $\mathcal{C}_n$. Moreover, vertexes of $\mathcal{K}_n$ are represented by binary trees with n leaves whose inner nodes have degree 3.

For any planar rooted tree with n leaves, we assign the numbers $1, 2, \ldots, n$ to the leaves from the left. Let T_1 and T_2 be planar rooted trees representing faces of $\mathcal{K}_r$ and $\mathcal{K}_s$, respectively. Then $\partial_k^{\mathcal{K}}(r, s)(T_1, T_2)$ is given by grafting the root of T_2 to the kth leaf of T_1, which means that we first remove the kth leaf vertex of T_1 and root vertex of T_2, then the kth leaf edge of T_1 and the root edge of T_2 are identified. We denote the resulting tree as $T_1 \circ_k T_2$. In particular, any facet of $\mathcal{K}_n$ is represented by $\mathcal{C}_r \circ_k \mathcal{C}_s$ for $k \geq 1$ and $2 \leq s \leq n-1$.

On the other hand, $s_j^{\mathcal{K}}(T)$ for a planar rooted tree T is given by removing the jth leaf vertex v and the jth leaf edge e. Moreover, if the inner node connecting with e has only one incoming edge except for e, then we remove this node and identify two edges connecting this node.

Now, Stasheff proved that the A_n-form is a homotopical representation of the A_n-structure. In fact, he showed the following theorem, which is the same as Theorem 3.4 if $n = \infty$.

Theorem 3.7 (Stasheff [53, Theorem 5]). *A space X admits an A_n-structure if and only if it admits an A_n-form,*

For an A_n-space X, Stasheff defined a space P_iX for $0 \leq i \leq n$ called the projective i-space of X. If X is an A_∞-space, then the family of spaces $\{P_iX\}$ is a filtration of the classifying space of X so that $\Omega(P_\infty X) \simeq X$ for $P_\infty X = \bigcup_i P_iX$. Moreover, P_iX is the base space B_{i+1} of the $i+1$th quasifibration $E_{i+1} \to B_{i+1}$ in Definition 3.5 constructed by a specific method from the given A_n-form. These spaces satisfy that $P_0X = *$, $P_1X = \Sigma X$, $P_{i-1}X \subset P_iX$ and $P_iX/P_{i-1}X \simeq \Sigma^i(X \wedge \cdots \wedge X)$ (the suspension of the i-fold smash product of X). Incidentally, if $X = S^0, S^1$ or S^3, then P_iX

is the real, the complex or the quaternionic projective i-space. If $X = S^7$, then P_2X is the Cayley projective plane.

Now, as examples of A_n-space, we recall the sphere extensions described in section 2: $M_\lambda(n,d)$ is the total space of the principal $G(n-1,d)$-bundle induced from the principal bundle $G(n-1,d) \to G(n,d) \to S^{dn-1}$ by the map on S^{dn-1} of degree λ. In the following theorem, we consider two cases $G(n,2) = SU(n)$ and $G(n,4) = Sp(n)$.

Theorem 3.8 (Iwase and Mimura [30, Theorem, 6.5, Corollary 6.6])**.** *Let k be a positive integer with $k \geq 2$. If λ is prime to $k!$, then $M_\lambda(n,d)$ admits an A_k-form. Moreover, for the case of $k = 3$, the converse is also true provided that n does not divide $2 \cdot 3^*$.*

At the end of this section, we give brief comments on two topics related to the higher homotopy associativity of H-spaces. One is the A_n-algebra. As is described in section 2, the cohomology algebra of an H-space with coefficients in a filed has natural Hopf algebra structure. Stasheff [53] showed that the singular chain complex of an A_n-space has an extra structure called A_n-algebra. This structure is useful for the study of the cohomology of the projective spaces of given A_n-space. Now a days, the theory of A_n-algebra has become an important subject on its own. We don't give the explicit definition of the A_n-algebra here. For readers who are interested in it, Keller's paper [40] is a good reference.

The second one is the higher Hopf invariant introduced by Iwase [28], which is a further generalization of the generalized Hopf invariant defined by Berstein and Hilton [5]. This invariant relates to the higher homotopy associativity of H-spaces. In fact, Iwase [28, Example 2.7] showed that if X is an A_n-space, then there is a map $f\colon E_{n+1}X \to P_nX$ with higher Hopf invariant one, where $\{p_{i+1}\colon E_{i+1}X \to P_iX\}$ is the A_n-structure given from the A_n-form on X. It is also shown that the converse is true if X is a sphere, and he conjectured that the converse is true in general ([28, Conjecture 2.8]). Moreover, this invariant relates to the theory of LS category. Here, the LS category of a space X, $\mathrm{cat}(X)$, is the least number m such that there is a covering of X by $m+1$ closed subsets of X, each of which is contractible in X. Indeed, Iwase [27] used the concept of higher Hopf invariant to construct counter examples of the Ganea conjecture, which asserts that $\mathrm{cat}(X \times S^n) = \mathrm{cat}(X) + 1$ for any space X and $n \geq 1$.

3.2. A_n-form on maps.

For the case of maps between topological monoids: Next problem we have to consider is to find homotopical conditions on which maps between topological monoids induce maps between classifying spaces.

If $f\colon X \to Y$ is a homomorphism between topological monoids. Then it induces a map $Bf\colon BX \to BY$ such that $\Omega(Bf)$ is homotopic f by identifying $\Omega(BX)$ with X and $\Omega(BY)$ with Y. If $g\colon X \to Y$ is homotopic to the homomorphism f, $\Omega(Bf)$ is homotopic g, and so we can say g induces a map between classifying spaces.

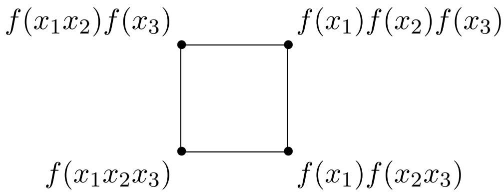
Sugawara [62] described the condition as a higher homotopy, called strongly homotopy multiplicative. Later Stasheff [53] defined A_n-map between topological monoids, which is the same as strongly homotopy multiplicative map if $n = \infty$.

As in the case of spaces, we need suitable parameter spaces to define A_n-form on maps. The parameter spaces for this case are the cubes: An A_n-form of a map $f\colon X \to Y$ is a family $\{F_i\colon I^{i-1} \times X^i \to Y\}_{1\le i\le n}$. The map F_1 is identified with f, and F_2 is a homotopy between $f \circ \mu_X$ and $\mu_Y \circ (f \times f)$:

$$F_2(0, x_1, x_2) = f(x_1x_2), \quad F_2(1, x_1, x_2) = f(x_1)f(x_2)$$

The map F_3 represents the higher homotopy between two homotopies $f(x_1x_2x_3) \sim f(x_1)f(x_2x_3) \sim f(x_1)f(x_2)f(x_3)$ and $f(x_1x_2x_3) \sim f(x_1x_2)f(x_3) \sim f(x_1)f(x_2)f(x_3)$. Thus, this map is illustrated as follows:

(3.5)



In this way, the idea is very easy to understand. In particular, F_i should satisfy the following equations for $1 \le k \le i-1$:

(3.6)

$$\begin{aligned} &F_i(t_1, \dots, t_{i-1}, x_1, \dots, x_i) \\ &\quad = F_{i-1}(t_1, \dots, \hat{t}_k, \dots, t_{i-1}, x_1, \dots, x_kx_{k+1}, \dots, x_i) && \text{for } t_k = 0 \\ &\quad = F_k(t_1, \dots, t_{k-1}, x_1, \dots, x_k)F_{i-k}(t_{k+1}, \dots, t_{i-1}, x_{k+1}, \dots, x_i) && \text{for } t_k = 1 \end{aligned}$$

Moreover, if $x_j = *$ for $1 \le j \le i$, then we need

(3.7)

$$\begin{aligned} &F_i(t_1, \dots, t_{i-1}, x_1, \dots, x_i) \\ &\quad = F_{i-1}(t_2, \dots, t_{i-1}, x_2, \dots, x_i) && \text{for } j = 1 \\ &\quad = F_{i-1}(t_1, \dots, \max\{t_{j-1}, t_j\}, \dots, t_{i-1}, x_1, \dots, \hat{x}_j, \dots, x_i) && \text{for } 1 < j < i \\ &\quad = F_{i-1}(t_1, \dots, t_{i-2}, x_1, \dots, x_{i-1}) && \text{for } j = i \end{aligned}$$

Definition 3.9 (Sugawara [62], Stasheff [53])**.** Let n be a positive integer. An A_n-form on a map $f\colon X \to Y$ of topological monoids X and Y is a family of maps $\{F_i \colon I^{i-1} \times X^i \to Y\}_{1 \leq i \leq n}$ such that $F_1 = f$ by identifying $I^0 \times X$ with X, and (3.6) and (3.7) are satisfied:. An A_∞-form on f is a family $\{F_i\}_{1 \leq i}$ satisfying the above conditions.

An A_n-map is a map $f \colon X \to Y$ equipped with an A_n-form. An A_∞-map is also called strong homotopy-multiplicative.

We note that the condition (3.7) was not assumed in the original definitions of A_n-form and strong homotopy-multiplicative.

Stasheff showed that a map between topological monoid admits an A_n-form if and only if it induces a map between A_n-structures constructed by a specific method. This result is also proved by Sugawara for the case of $n = \infty$. More strongly we have the following

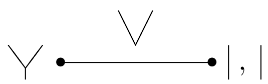
Theorem 3.10 (Stasheff [55, Theorem 8.4])**.** *A map $f \colon X \to Y$ between topological monoids is an A_n-map if and only if the map $\Sigma f \colon \Sigma X \to \Sigma Y$ extends to a map of projective n-spaces $P_n f \colon P_n X \to P_n Y$.*

For the case of maps from A_n-spaces to topological monoids: Next we consider the case of maps from A_n-spaces to topological monoids. Let X be an A_n-space with A_n-form $\{M_i\}$, and Y a topological monoid. We investigate polytopes $\mathcal{L}_i$ needed to define an A_n-form $\{F_i \colon \mathcal{L}_i \times X^i \to Y\}_{i \leq n}$ on f.

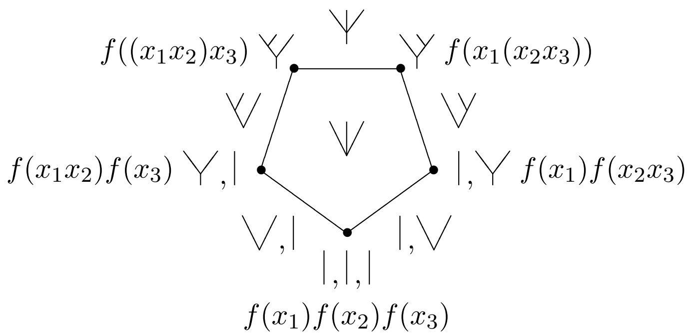
To describe the combinatorial structure of the polytopes $\mathcal{L}_i$ we use planar rooted trees again. But, in this case we allow the 1-corolla which is a planar rooted tree with only one edge. Moreover, we consider the one point union $T = T_1 \vee T_2 \vee \cdots \vee T_k$ of planar rooted trees T_i, which is the union of T_i with identifying all the root vertexes of them. Thus T has one root vertex and k root edges. The root edges are arranged such that the root edge of T_i is the ith root edge of T from the left. Thus the degree of the root vertex of a planar rooted tree in this case can be greater than 1.

To avoid any unnecessarily confusion, we distinguish two types of trees as follows. We call any tree representing a face of the associahedra as a tree with one root edge. On the other hand, for the new type of trees we call trees with multiple root edges. A tree with one root edge is also considered as a tree with multiple root edges. Trees with multiple root edges with leaves less than 4 is listed as follows:

Now $F_1 \colon \mathcal{L}_1 \times X \to Y$ is identified with f, and so we have $\mathcal{L}_1 = *$. Since A_2-maps are H-maps, $\mathcal{L}_2$ must be an interval such that F_2 is a homotopy between $f \circ \mu_X$ and $\mu_Y \circ (f \times f)$ as well as the case of maps between topological monoids. The multiplication μ_X is represented by the 2-corolla  in the definition of A_n-form on spaces, and so we also use the 2-corolla to represent the vertex of $\mathcal{L}_2$ corresponding to $f \circ \mu_X$. In this notation, we consider that the root edge represents the operation of the map f. Thus the 1-corolla, which is a line $|$, represents the map f, and the vertex of $\mathcal{L}_2$ corresponding to $\mu_Y \circ (f \times f)$ is represented by the sequence of two 1-corollas $\mathcal{C}_1, \mathcal{C}_1 = |\,,|$. We remark that we don't use any symbol representing the multiplication of Y. This makes sense because Y is a topological monoid and so the multiplication of any i letters is unique. Then we represent the edge of $\mathcal{L}_2$ by $\mathcal{C}_1 \vee \mathcal{C}_1 = \bigvee$.



The polytope $\mathcal{L}_3$ must be a pentagon, which is illustrated as follows:



In general, the wedge of i-copies of the 1-corolla $\mathcal{C}_1 \vee \cdots \vee \mathcal{C}_1$, which we denote by $\mathcal{F}_i$, represents the maximum cell $\mathcal{L}_i$, and any face of it is represented by a sequence of trees with multiple root edges such that the total number of the leaves of the trees is i.

We consider the dimension of the face. Let $l(T)$ be the number of leaves and $v(T)$ the number of inner vertexes of T. Then for $V = T_1 \vee \cdots \vee T_k$ with trees with one root edge T_i,

$$\begin{aligned}\dim V &= \sum_i \dim T_i + k - 1 \\ &= \sum_i (l(T_i) - v(T_i) - 1) + k - 1 \\ &= l(V) - v(V) - 1.\end{aligned}$$

Moreover, for a sequence of trees with multiple root edges $V_1, \ldots, V_k$, we have

$$\dim(V_1, \ldots, V_k) = \sum_i \dim V_i - k = \sum_i l(V_i) - \sum_i v(V_i) - k.$$

In particular, $\dim \mathcal{L}_i = i - 1$, and so the dimension of a facet is $i - 2$. Thus any facet is of the form $\mathcal{C}_i$, $\mathcal{F}_k \vee \mathcal{C}_s \vee \mathcal{F}_{i-k-s}$ or $\mathcal{F}_r, \mathcal{F}_{i-r}$.

Now we show that the polytope $\mathcal{L}_i$ is isomorphic to the associahedron $\mathcal{K}_{i+1}$. To do so we give an isomorphism between the two face posets given by Kishimoto and Kono [41]. For any tree with one root edge T representing a face of $\mathcal{K}_{i+1}$, we remove the $i+1$th (the right most) leaf vertex and the root vertex together with the edges connecting these two vertexes. Then the resulting sequence of trees with multiple root edges is the corresponding one representing a face of $\mathcal{L}_i$. For example, the corresponding sequence of the following left tree is the right one.

By using the above correspondence, we can also express the facets of $\mathcal{L}_i$ from the facets of $\mathcal{K}_{i+1}$. A facet of $\mathcal{K}_{i+1}$ is represented by a tree of the form $\mathcal{C}_r \circ_k \mathcal{C}_s$ $(r+s = i+2)$ which has only inner node. Then the corresponding sequence of trees representing a facet of $\mathcal{L}_i$ is given as follows:

$$\begin{aligned} &\mathcal{F}_{k-1} \vee \mathcal{C}_s \vee \mathcal{F}_{r-k-1} && \text{if } k < r \\ &\mathcal{F}_{r-1}, \mathcal{F}_{s-1} && \text{if } k = r \end{aligned}$$

Note that if $r = 2$ and $k = 1$, then the first one is the corolla $\mathcal{C}_s$.

From the above argument, we have the following boundary conditions:
(3.8)

$$\begin{aligned} &F_i(\partial_k^{\mathcal{K}}(\rho, \sigma), x_1, \ldots, x_i) \\ &\quad = F_{r-1}(\rho, x_1, \ldots, x_{k-1}, M_s(\sigma, x_k, \ldots, x_{k+s-1}), x_{k+s}, \ldots, x_i) \quad \text{if } k < r \\ &\quad = F_{r-1}(\rho, x_1, \ldots, x_{r-1}) F_{s-1}(\sigma, x_r, \ldots, x_i) \quad \text{if } k = r \end{aligned}$$

The degeneracy maps of the associahedra already defined can be used for A_n-form on maps, since they represent the case that one of the elements of X is the unit. We remark that the degeneracy maps $s_j^{\mathcal{K}}$ on $\mathcal{K}_i$ is defined for $1 \leq j \leq i$, but for this case we need for $1 \leq j \leq i-1$. Thus we have the following

$$F_i(\tau, x_1, \ldots, x_i) = F_{i-1}(s_j^{\mathcal{K}}(\tau), x_1, \ldots, \hat{x}_j, \ldots, x_i) \quad \text{if } x_j = *. \tag{3.9}$$

Now we give the definition of A_n-form on maps from A_n-spaces to topological monoids.

Definition 3.11. Let n be a positive integer. An A_n-form on a map $f \colon X \to Y$ from an A_n-space X with A_n-form $\{M_i\}$ to a topological monoid Y is a family of maps $\{F_i \colon \mathcal{K}_{i+1} \times X^i \to Y\}_{1 \leq i \leq n}$ such that $F_1 = f$ by identifying $\mathcal{K}_2 \times X$ with X, and (3.8) and (3.9) are satisfied. An A_∞-form on f is a family $\{F_i\}_{1 \leq i}$ satisfying the above conditions.

An A_n-map is a map $f \colon X \to Y$ equipped with an A_n-form.

If X is a topological monoid, then for the trivial A_∞-form on X, two definitions of A_n-form on a map $f\colon X \to Y$ are equivalent. In fact, there is a projection $\psi_i\colon \mathcal{K}_{i+1} \to I^{i-1}$ such that for any A_n-form $\{F_i\colon I^{i-2} \times X^i \to Y\}_{1\leq i\leq n}$ in Definition 3.10, $\{F_i \circ (\psi_i \times id)\colon \mathcal{K}_{i+1} \times X^i \to Y\}_{1\leq i\leq n}$ is an A_n-form in Definition 3.11.

We note that Theorem 3.10 also holds for A_n-maps from A_n-spaces to topological monoids. Moreover the following facts holds.

Theorem 3.12 (Stasheff [55, Theorem 11.10])**.** *If X is an A_n-space, then the map $X \to \Omega P_n X$, the adjoint of the inclusion $\Sigma X \to P_n X$, is an A_n-map.*

For the case of maps between A_n-spaces: Finally we consider maps between A_n-spaces. The parameter spaces are constructed by Iwase [26], which are called multiplihedra. Later the multiplihedra appeared in the paper [30] by Iwase and Mimura. We denote the n-multiplihedron by $\mathcal{J}_n$. As is noted before, the construction of the multiplihedra is very complicated, but the combinatorial structure can be easily imagine from the above discussion.

For a map from an A_n-space to a topological monoid, the order of the multiplication of elements of the topological monoid can be ignored. To define the A_n-form $\{F_i\colon \mathcal{J}_i \times X^i \to Y\}_{i\leq n}$ on a map $f\colon X \to Y$ between A_n-spaces, we have to take account the A_n-structure of the target space. Thus, a face of the i-multiplihedron $\mathcal{J}_i$ is of the form $T; V_1, \ldots, V_t$ or V. The first one consists of trees with multiple root edges V_j $(1 \leq j \leq t)$ and a tree with one root edge T. The sequence $V_1, \ldots, V_t$ represents a sequence of maps given by the A_i-form of X and lower F_js, and T represents the A_i-form of Y such that the number of leaves of T is t. The second one V is a tree with multiple root edges which represents a map given by the A_i-form of X followed by f. Iwase gave a different expression of the multiplihedra in [29]. He used bearded trees, which are essentially equivalent to our expression. In fact, from a sequence $T; V_1, \ldots, V_t$, we can get a bearded tree by grafting the root vertex of tree V_j to the jth leaf vertex of T, and put a beard at each grafting point. For V, we put a beard at the root edge.

Now the dimension of the face of $\mathcal{J}_i$ corresponding to $T; V_1, \ldots, V_t$ is given as

$$\dim T; V_1, \ldots, V_t = \dim T + \dim V_1, \ldots, V_t = i - \left(v(T) + \sum_j v(V_j) \right) - 1$$

Thus, if $T; V_1, \ldots, V_t$ represents a facet, then $v(T) + \sum_j v(V_j) = 1$, which means that $V_j = \mathcal{F}_{r_j}$ for some $i_j \geq 1$ and $T = \mathcal{C}_t$. We denote this facet by $\mathcal{J}(t; r_1, \ldots, r_t)$, and we have a homeomorphism

$$\partial^{\mathcal{J}}(t; r_1, \ldots, r_t)\colon \mathcal{K}_t \times \mathcal{J}_{r_1} \times \cdots \times \mathcal{J}_{r_t} \to \mathcal{J}(t; r_1, \ldots, r_t) \subset \mathcal{J}_i$$

On the other hand, if V represent a facet, then $V = \mathcal{F}_{k-1} \vee \mathcal{C}_s \vee \mathcal{F}_{r-k}$, where $r+s = i+1$ with $1 \leq k \leq r$ since $\dim V = i - v(V) - 1$. We not that if $r = k = 1$, then $V = \mathcal{C}_i$. We denote this facet as $\mathcal{J}_k(r,s)$, and we have a homeomorphism

$$\partial_k^{\mathcal{J}}(r,s)\colon \mathcal{J}_r \times \mathcal{K}_s \to \mathcal{J}_k(r,s) \subset \mathcal{J}_i.$$

These face operators are defined to satisfy the following

$$\begin{aligned} &F_i(\partial_k^{\mathcal{J}}(r,s)(\rho,\sigma), x_1, \ldots, x_i) \\ &\quad = F_{r-1}(\rho, x_1, \ldots, x_{k-1}, M_s^X(\sigma, x_k, \ldots, x_{k+s-1}), x_{k+s}, \ldots, x_i) \\ &F_i(\partial^{\mathcal{J}}(t; r_1, \ldots, r_t)(\tau, \rho_1, \ldots, \rho_t), x_1, \ldots, x_i) \\ &\quad = M_t^Y(\tau, F_{r_1}(\rho_1, x_1, \ldots, x_{r_1}), \ldots, F_{r_t}(\rho_t, x_{r_1+\cdots+r_{t-1}+1}, \ldots, x_i) \end{aligned} \tag{3.10}$$

for $r+s = i+2$, where $\{M_i^X\}$ and $\{M_i^Y\}$ are A_n-forms of X and Y, respectively.

The degeneracy map $s_j^{\mathcal{J}}\colon \mathcal{J}_i \to \mathcal{J}_{i-1}$ $(1 \leq j \leq i)$ is essentially given by removing the jth leaf so that the following conditions are satisfied.

$$F_i(\tau, x_1, \ldots, x_i) = F_{i-1}(s_j^{\mathcal{J}}(\tau), x_1, \ldots, \hat{x}_j, \ldots, x_i) \quad \text{if } x_j = *. \tag{3.11}$$

For the relations these operations obey see [30, 29].

Now we give the definition of A_n-form on maps between A_n-spaces.

Definition 3.13. Let n be a positive integer. An A_n-form on a map $f\colon X \to Y$ between A_n-spaces X with A_n-form $\{M_i^X\}$ and Y with A_n-form $\{M_i^Y\}$ is a family of maps $\{F_i\colon \mathcal{J}_i \times X^i \to Y\}_{1 \leq i \leq n}$ such that $F_1 = f$ by identifying $\mathcal{J}_1 \times X$ with X and (3.10) and (3.11) are satisfied. An A_∞-form on f is a family $\{F_i\}_{1 \leq i}$ satisfying the above conditions.

An A_n-map is a map $f\colon X \to Y$ equipped with an A_n-form.

Iwase and Mimura defined A_n-structure of maps between A_n-spaces, which is analogous to the one for spaces by Stasheff (Definition 3.5). To give an explicit definition, we fix a special A_n-structure $p_i^X\colon E_iX \to P_iX$ $(1 \leq i \leq n)$ for an A_n-space X derived from given A_n-form on X. Note that P_iX is the projective i-space of X. Then, Stasheff showed that there are spaces D_iX with $E_iX \subset D_iX \subset E_{i+1}X$ and maps $\sigma_i^X\colon D_iX \to P_{i+1}X$ with $p_{i+1}^X|D_iX = \sigma_i^X$ such that (D_iX, E_iX) has the homotopy type of (CE_iX, E_iX) and $(\sigma_i^X, p_i^X)\colon (D_iX, E_iX) \to (P_{i+1}X, P_iX)$ is a relative homeomorphism ([53, Proposition 24]).

Definition 3.14. Let n be an integer with $n \geq 2$. Then an A_n-structure of a map $f\colon X \to Y$ between A_n-spaces X and Y is a pair of n-tuples of maps $(D_if, E_if)\colon (D_iX, E_iX) \to (D_iY, E_iY)$ and $(P_if, P_{i-1}f)\colon (P_iX, P_{i-1}X) \to$

$(P_iY, P_{i-1}Y)$ $(1 \leq i \leq n)$ such that the following diagram is commutative:

$$\begin{array}{ccc} (D_iX, E_iX) & \xrightarrow{(D_if, E_if)} & (D_iY, E_iY) \\ {\scriptstyle (\sigma_i^X, p_i^X)}\downarrow & & \downarrow{\scriptstyle (\sigma_i^Y, p_i^Y)} \\ (P_{i+1}X, P_iX) & \xrightarrow[(P_{i+1}f, P_if)]{} & (P_{i+1}Y, P_iY) \end{array}$$

Then Iwase and Mimura showed the following

Theorem 3.15 ([30, Theorem 3.1])**.** *A map $f\colon X \to Y$ between A_n-spaces X and Y admits an A_n-structure if and only if f admits an A_n-form.*

For the corresponding fact to Theorem 3.10 for maps between A_n-spaces, we need to be careful. In fact, we can only prove the following

Theorem 3.16 (Iwase and Mimura [30, Theorem 3.2], Hemmi [19, Theorem 7.2])**.** *If a map $f\colon X \to Y$ between A_n-spaces is an A_n-map, then the map $\Sigma f\colon \Sigma X \to \Sigma Y$ extends to a map of projective n-spaces $P_nf\colon P_nX \to P_nY$. The converse holds provided that the A_n-form of Y can extend to an A_{n+1}-form.*

The extra assumption is needed to show the converse in the above theorem. In fact, we have the following fact for the retraction $r\colon \Omega\Sigma X \to X$, which exists for any H-space X.

Theorem 3.17 (Iwase and Mimura [30, P10)], Hemmi [18, Theorem 1.3, Theorem 3.1])**.** *If X is an A_n-space, then $\Sigma r\colon \Sigma\Omega\Sigma X \to \Sigma X$ extends to a map $P_n\Omega\Sigma X \to P_nX$. Moreover, if r is an A_n-map, then the A_n-form of X can extend to an A_{n+1}-form,*

As is the case noted in the last paragraph, if Y is a topological monoid, then for the trivial A_∞-form on Y, two definitions Definition 3.13 and Definition 3.11 are equivalent. In fact, there is a projection $\phi_i\colon \mathcal{J}_i \to \mathcal{K}_{i+1}$ such that for any A_n-form $\{F_i\colon \mathcal{K}_{i+1} \times X^i \to Y\}_{1 \leq i \leq n}$ in Definition 3.11, $\{F_i \circ (\phi_i \times id)\colon \mathcal{J}_i \times X^i \to Y\}_{1 \leq i \leq n}$ is an A_n-form in Definition 3.13.

4. Homotopy commutativity

4.1. **Strong homotopy-commutativity.** If the classifying space BX is an H-space, then the multiplication of a topological monoid X is homotopy commutative. Contrarily, the converse is not true in general. Sugawara [62] showed the following

Theorem 4.1. *Let X be a topological monoid. The multiplication $\mu\colon X \times X \to X$ is an A_∞-map if and only if X is an H-space.*

If the multiplication μ_X is an A_∞-map, then we have a map $B\mu_X\colon B(X\times X)\to BX$ with $\Omega(B\mu_X)\simeq\mu_X$. Since $B(X\times X)=BX\times BX$, $B\mu_X$ gives an H-space multiplication on BX.

Incidentally, Hemmi and Kawamoto [21, Proposition 4.2] showed that the multiplication of X is an A_n-map if and only if X is an $H(n)$-space. Here, $H(n)$-space is introduced by Félix and Tanré [12]. The definition is given by using the Ganea fibrations on X. From the definition, an $H(1)$-space is just a space, and an $H(\infty)$-space is an H-space. They introduced this concept to find conditions for the mapping space $\mathrm{Map}_*(Y,Z)$ to be an H-space. They showed that if the Lusternik-Schnirelman category of a space Y is less than or equal to n and Z is an $H(n)$-space, then $\mathrm{Map}_*(Y,Z)$ is an H-space ([12, Proposition 1])

Sugawara showed that the condition that the multiplication μ_X of a topological monoid X is an A_∞-map is represented by a higher homotopy commutativity of μ_X. He called the property strong homotopy-commutativity. Associating his idea McGibbon [46] defined a concept of Sugawara C_n-form. Sugawara C_∞-form is just the strongly homotopy-commutativity.

Definition 4.2. Let n be a positive integer. A Sugawara C_n-form on a topological monoid X is a family of maps $\{C_i\colon I^i\times X^{2i}\to X\}_{1\le i<n}$ such that the following conditions are satisfied:

(1) $C_1(0,x,y)=xy$ and $C_1(1,x,y)=yx$.

(2) If $t_k=0$, then

$$\begin{aligned}
&C_i(t_1,\dots,t_i,x_1,\dots,x_i,y_1,\dots,y_i)\\
&\quad= x_1C_{i-1}(t_2,\dots,t_i,x_2,\dots,x_i,y_1y_2,\dots,y_i)\\
&\qquad\text{for } k=1\\
&\quad= C_{i-1}(t_1,\dots,\hat{t}_k,\dots,t_i,x_1,\dots,x_{k-1}x_k,\dots,x_i,y_1,\dots,y_ky_{k+1},\dots,y_i)\\
&\qquad\text{for } 1<k<i\\
&\quad= C_{i-1}(t_1,\dots,t_{i-1},x_1,\dots,x_{i-1}x_i,y_1,\dots,\dots,y_{i-1})y_i\\
&\qquad\text{for } k=i
\end{aligned}$$

(3) If $t_k=1$, then

$$\begin{aligned}
&C_i(t_1,\dots,t_i,x_1,\dots,x_i,y_1,\dots,y_i)\\
&\quad= C_{k-1}(t_1,\dots,t_{k-1},x_1,\dots,x_{k-1},y_1,\dots,\dots,y_{k-1})y_kx_k\\
&\qquad\qquad C_{i-k}(t_{k+1},\dots,t_i,x_{k+1},\dots,x_k,y_{k+1},\dots,\dots,y_i)
\end{aligned}$$

A Sugawara A_∞-form on X is a family $\{C_i\}_{1\le i}$ satisfying the above conditions.

A Sugawara C_n-space is a topological monoid X equipped with a Sugawara C_n-form.

The map C_1 is a commuting homotopy of the multiplication of X.

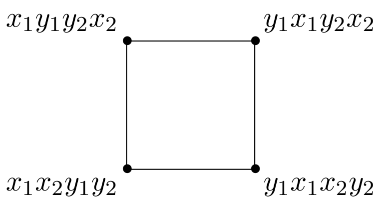
If a topological monoid X is strong homotopy-commutative, then the family of maps $\{M_i\colon I^{i-1} \times (X \times X)^i \to X\}$ defined as follows is an A_∞-form on the multiplication of X:

$$M_1(*, (x, y)) = xy$$
$$M_i(t_1, \dots, t_{i-1}, (x_1, y_1), \dots, (x_i, y_i))$$
$$= x_1 C_{i-1}(t_1, \dots, t_{i-1}, x_2, \dots, x_i, y_1, \dots, y_{i-1}) y_i$$

Thus by Theorem 4.1 we have the following

Theorem 4.3 (Sugawara [62, Theorem 4.2]). *A topological monoid X is Sugawara C_∞-space if and only if the multiplication $\mu_X\colon X \times X \to X$ is an A_∞-map in the sense of Definition 3.9. Thus, these properties are equivalent to that the classifying space BX of X is an H-space.*

The definition of Sugawara C_n-form is not so easy to understand. For example, the map C_2, which is illustrated as follows, seems not so natural in the sense of homotopy commutativity.



4.2. **Homotopy commutativity by Permutohedra and Resultohedra.** Williams [64] considered another type of higher homotopy commutativity by considering all permuted multiplications. The ith parameter space $\mathcal{P}_i$ of his higher homotopy has vertexes corresponding to the all permutations $\mathcal{S}_i$ on the i-letters $\{1, 2, \dots, i\}$. Thus $\mathcal{P}_2$ is an interval, and $\mathcal{P}_3$ is a hexagon.

He defined a C_n-form on a topological monoid X as a family of maps $\{C_i\colon \mathcal{P}_i \times X^{2i} \to X\}_{1 \le i \le n}$ such that

$$C_i(v, x_1, \dots, x_i) = x_{\sigma^{-1}(1)} \dots x_{\sigma^{-1}(i)}.$$

for the vertex v of $\mathcal{P}_i$ corresponding to a permutation σ.

The polyhedron $\mathcal{P}_n$ is called the n-permutohedron, and first constructed by Milgram [47] for the study of iterated loop spaces. This polytope can be easily realized as follows.

Let n be a positive integer, and consider the point $q_n = (1, 2, \dots, n)$ in $\mathbb{R}^n$. The n-permutohedron $\mathcal{P}_n$ is the convexhull of the set $\{\sigma q_n = (\sigma(1), \dots, \sigma(n)) \mid \sigma \in \mathcal{S}_n\}$. By definition, $\mathcal{P}_n$ is homeomorphic to the $n-1$ dimensional disk.

The faces of $\mathcal{P}_n$ are easily described by the ordered partitions of the set $x[n] = \{x_1, x_2, \ldots, x_n\}$, which means a sequence $\alpha_1 \ldots \alpha_k$ of nonempty disjoint subsets of $x[n]$ such that $\alpha_1 \cup \cdots \cup \alpha_k = x[n]$. If α_i consists of n_i elements, then the partition $\alpha_1 \ldots \alpha_k$ is called of type $(n_1, \ldots, n_k)$ with length k.

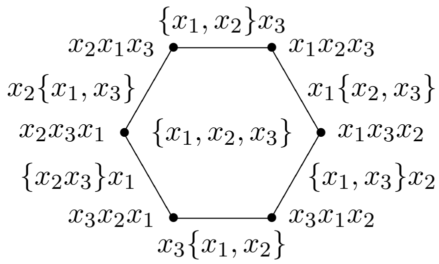
For any ordered partition $\alpha_1 \ldots \alpha_k$ of type $(n_1, \ldots, n_k)$, we consider the subset $\mathcal{S}(\alpha_1 \ldots \alpha_k)$ of $\mathcal{S}_n$ consisting of elements $\sigma \in \mathcal{S}_n$ such that

$$\sigma^{-1}(j) \in \alpha_t \quad \text{for } n_1 + \cdots + n_{t-1} < j \leq n_1 + \cdots + n_t. \tag{4.1}$$

For example, if $\alpha_1\alpha_2 = \{2, 4\}\{1, 3\}$, then $x_{\sigma^{-1}(1)}x_{\sigma^{-1}(2)}x_{\sigma^{-1}(3)}x_{\sigma^{-1}(4)}$ for all $\sigma \in \mathcal{S}(\alpha_1\alpha_2)$ are as follows:

$$x_2x_4x_1x_3, \quad x_2x_4x_3x_1, \quad x_4x_2x_1x_3, \quad x_4x_2x_3x_1.$$

Then the convexhull of $\{\sigma q_n \mid \sigma \in \mathcal{S}(\alpha_1 \ldots \alpha_k)\}$ is the face of $\mathcal{P}_n$ corresponding to $\alpha_1 \ldots \alpha_k$. This face, which we denote by $\mathcal{P}_n(\alpha_1 \ldots \alpha_k)$, is homeomorphic to $\mathcal{P}_{n_1} \times \cdots \times \mathcal{P}_{n_k}$, and so the dimension is $\sum_i n_i - k = n - k$. In particular, a facet corresponds to an ordered partition with length 2. Moreover, a vertex corresponds to the one of type $(1, \ldots, 1)$ with length n, and so is denoted by $\{x_{j_1}\}\{x_{j_2}\} \ldots \{x_{j_n}\}$. From now on if α_t consists of one letter, say $\alpha_t = \{x_j\}$, then we identify α_t with x_j. Thus, any vertex is represented by a sequence $x_{j_1}x_{j_2} \ldots x_{j_n}$ with $\{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, n\}$. Then $\mathcal{P}_3$ is illustrated as follows:



We describe the facets of $\mathcal{P}_n$ more explicitly. Let $[n] = \{1, 2, \ldots, n\}$. Then any increasing map $f\colon [i] \to [n]$ for $i \leq n$ gives a subset $\{x_{f(1)}, x_{f(2)}, \ldots, x_{f(i)}\}$ of $x[n]$. This is one to one correspondence between the increasing maps $[i] \to [n]$ and the subsets of $x[n]$ with i elements, Thus for any subset $\alpha \subset x[n]$ we use the same letter for the corresponding increasing map as $\alpha\colon [i] \to [n]$.

Let $\alpha_1\alpha_2$ be an ordered partition of $x[n]$ of type (r, s) with $r + s = n$. Then a homeomorphism $\partial^{\mathcal{P}}_{\alpha_1\alpha_2}\colon \mathcal{P}_r \times \mathcal{P}_s \to \mathcal{P}_n(\alpha_1\alpha_2) \subset \mathcal{P}_n$ is defined. In particular, for a vertex of $\mathcal{P}_r \times \mathcal{P}_s$ we have

$$\begin{aligned}&\partial^{\mathcal{P}}_{\alpha_1\alpha_2}(x_{i_1}x_{i_2} \ldots x_{i_r}, x_{j_1}x_{j_2} \ldots x_{j_s}) \\ &\qquad = x_{\alpha_1^{-1}(i_1)}x_{\alpha_1^{-1}(i_2)} \cdots x_{\alpha_1^{-1}(i_r)}x_{\alpha_2^{-1}(j_1)}x_{\alpha_2^{-1}(j_2)} \cdots x_{\alpha_2^{-1}(j_s)}.\end{aligned}$$

Then the family of maps $\{C_i\}_{i\le n}$ should satisfy the following

$$(4.2)\qquad \begin{aligned} &C_i(\partial^{\mathcal{P}}_{\alpha\,\beta}(\rho,\sigma),x_1,\dots,x_i)\\ &\quad = C_r(\rho,x_{\alpha^{-1}(1)},\dots,x_{\alpha^{-1}(r)})C_s(\sigma,x_{\beta^{-1}(1)},\dots,x_{\beta^{-1}(s)}) \end{aligned}$$

for an ordered partition $\alpha\beta$ of type (r,s) with $t+s=n$.

The degeneracy map $s^{\mathcal{P}}_j\colon \mathcal{P}_i \to \mathcal{P}_{i-1}$ is defined as follows. From any ordered partition $\alpha_1\dots\alpha_k$ of $x[i]$, we remove x_j and then renumber x_{j+t} for $1\le t\le i-j+1$ to x_{j+t-1}. Then the resulting partition is $s^{\mathcal{P}}_j(\alpha_1\dots\alpha_k)$. We have

$$(4.3)\qquad C_i(\tau,x_1,\dots,x_i) = C_{i-1}(s^{\mathcal{P}}_j(\tau),x_1,\dots,\hat{x}_j,\dots,x_i)\quad \text{if } x_j=*.$$

This C_n-form introduced by Williams [64] is referred to as Williams C_n-form in this paper.

Definition 4.4. Let n be a positive integer. A Williams C_n-form on a topological monoid X is a family of maps $\{C_i\colon \mathcal{P}_i\times X^{2i}\to X\}_{1\le i\le n}$ such that $C_1 = id$ by identifying $\mathcal{P}_1\times X$ with X, and (4.2) and (4.3) are satisfied. A Williams C_∞-form on X is a family $\{C_i\}_{1\le i}$ satisfying the above conditions.

A Williams C_n-space is a topological monoid X equipped with a C_n-form.

If the classifying space BX of a topological monoid X is an H-space, then X admits a Williams C_∞-form. On the other hand, for any Williams C_∞-space X, BX is not necessarily an H-space. Thus, Williams C_∞-space is exactly weaker than Sugawara C_∞-space. In fact, if E is the two stage Postnikov space with k-invariant $\iota^{p+1}\in H^{2np+2n}(K(\mathbb{Z},2n);\mathbb{Z}/p\mathbb{Z})$ for a prime p, then E is not an H-space but ΩE admits a Williams C_∞-form (see [46, Example 5]).

To characterize Williams C_n-spaces, we recall the n-fold reduced product $J_n(X)$ of a space X defined by James [31]. $J_n(X)$ is an identification space of n-fold product X^n, and $J_\infty(X)$ has the homotopy type of $\Omega\Sigma X$ if X is a connected CW-complex.

Theorem 4.5 (Williams [64, Main Theorem 14])**.** *For a topological monoid X, the following conditions are equivalent.*

1. *X admits a Williams C_n-form.*
2. *The Hopf construction for X extends to a principal quasifibration $p\colon E\to B$ such that B is the homotopy type of $J_n(\Sigma X)$.*
3. *There is an A_n-map $d\colon \Omega J_n(\Sigma X)\to X$ such that $d\circ j\simeq id$, where $j\colon X\to \Omega J_n(\Sigma X)$ is the adjoint of the inclusion $\Sigma X\subset J_n(\Sigma X)$.*

Consider the case of $n=\infty$ in the above theorem. Then in the third property, the map $j\colon X\to \Omega J_\infty(\Sigma X)\simeq \Omega^2\Sigma^2 X$ is homotopic to the double suspension map. Thus we have the following

Theorem 4.6 (Williams [64, Corollary 18]). *A topological monoid X is a Williams C_∞-space if and only if the double suspension map $X \to \Omega^2\Sigma^2 X$ has the homotopy right inverse $\Omega^2\Sigma^2 X \to X$ which is an A_∞-map.*

Hemmi and Kawamoto [21] showed that the Williams' approach can be modified to get a criterion for the classifying space of a topological monoid to be an H-space. Their approach is to use polytopes given by shuffles.

Let m and n be positive integers. An (m,n)-shuffle is an element $\sigma \in \mathcal{S}_{m+n}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(m) \quad \text{and} \quad \sigma(m+1) < \sigma(m+2) < \cdots < \sigma(m+n).$$

If σ is an (m,n)-shuffle, then $x_{\sigma^{-1}(1)}x_{\sigma^{-1}(2)}\dots x_{\sigma^{-1}(m+n)}$ represents a product such that x_i appears before x_{i+1} if $i \neq m$. To see it more easily we replace x_{m+j} by y_j for $1 \le j \le n$. Then the above product is a product of $x_1,\dots,x_m,y_1,\dots,y_n$ such that x_i appears before x_{i+1} for $1 \le i < m$ and y_j appears before y_{j+1} for $1 \le j < n$. We call such a product as an (m,n)-shuffle product of $x_1,\dots,x_m,y_1,\dots,y_n$ and denote the set of all (m,n)-shuffle products by of $\{[x_1,\dots,x_m],[y_1,\dots,y_n]\}$. For example, we have the following

$$\begin{aligned}&\{[x_1,x_2],[y_1,y_2]\}\\ &\qquad = \{x_1x_2y_1y_2, x_1y_1x_2y_2, x_1y_1y_2x_2, y_1x_1x_2y_2, y_1x_1y_2x_2, y_1y_2x_1x_2\}.\end{aligned}$$

We omit the bracket if there in only one letter in the bracket. For example, we just write as $\{[x_1x_2,x_3],y_1\}$ instead for $\{[x_1x_2,x_3],[y_1]\}$.

Let $\mathcal{S}_{m,n}$ be the subset of $\mathcal{S}_{m+n}$ consisting of all (m,n)-shuffles. Gelfand Kapranov and Zelevinsky [13] constructed a polytope in $\mathbb{R}^{m+n+2}$ whose vertexes correspond to $\mathcal{S}_{m,n}$. This polytope is called a resultohedron and denoted by $\mathcal{N}_{m,n}$. We can realize $\mathcal{N}_{m,n}$ as a subset of $m+n$-permutohedron $\mathcal{P}_{m+n}$ by considering the convexhull of $\{\sigma q_{m+n} \mid \sigma \in \mathcal{S}_{m,n}\}$. We put $\mathcal{N}_{m.0} = \mathcal{N}_{0,n} = *$.

Now we consider the faces of $\mathcal{N}_{m,n}$. For the vertexes we only consider (m,n)-shuffle products $\{[x_1,\dots,x_m],[y_1,\dots,y_n]\}$. But, for higher dimensional faces we need to consider more complicated types.

Here, we just give some examples. $\mathcal{N}_{3,1}$ is a tetrahedron with vertexes: $v_1 = x_1x_2x_3y_1$, $v_2 = x_1x_2y_1x_3$, $v_3 = x_1y_1x_2x_3$ and $v_4 = y_1x_1x_2x_3$. The edges v_1v_2, v_2v_3 and v_3v_4 are denoted by $x_1x_2\{x_3,y_1\}$, $x_1\{x_2,y_1\}x_3$ and $\{x_1,y_1\}x_2x_3$, respectively. On the other hand, the edges v_1v_3, v_1v_4 and v_2v_4 are denoted by $x_1\{x_2x_3,y_1\}$, $\{x_1x_2x_3,y_1\}$ and $\{x_1x_2,y_1\}x_3$, respectively. Moreover, the faces $v_1v_2v_3$, $v_1v_2v_4$, $v_1v_3v_4$ and $v_2v_3v_4$ are denoted by $x_1\{[x_2,x_3],y_1\}$, $\{[x_1x_2,x_3],y_1\}$, $\{[x_1,x_2x_3],y_1\}$ and $\{[x_1,x_2],y_1\}x_3$, respectively. Then the whole space $\mathcal{N}_{3,1}$ is denoted by $\{[x_1,x_2,x_3],y\}$.

For $\mathcal{N}_{2,2}$ we have six vertexes $v_1 = x_1x_2y_1y_2$, $v_2 = x_1y_1x_2y_2$, $v_3 = y_1x_1x_2y_2$, $v_4 = x_1y_1y_2x_2$, $v_5 = y_1x_1y_2x_2$ and $v_6 = y_1y_2x_1x_2$. There are eleven edges v_1v_2, v_1v_3, v_1v_4, v_1v_6, v_2v_3, v_2v_4, v_3v_5, v_3v_6, v_4v_5, v_4v_6 and v_5v_6. Moreover, we have seven faces. The face $v_1v_2v_3$ is denoted by $\{[x_1, x_2].y_1\}y_2$. On the other hand, the face with vertexes v_2, v_3, v_4, v_5 is denoted by a sequence $\{x_1, y_1\}\{x_2, y_2\}$.

For more complicated case, the following sequence represents a 9 dimensional face of $\mathcal{N}_{8,8}$

$$\{x_1, y_1y_2\}\{[x_2x_3, x_4, x_5x_6], [y_3, y_4y_5]\}x_7\{x_8, [y_5, y_7y_8]\}.$$

In general, any face of $\mathcal{N}_{m,n}$ is denoted by a sequence $\alpha_1 \dots \alpha_t$ such that each α_t is of the form $\{[\xi_{i(t-1)+1}, \dots, \xi_{i(t)}], [\zeta_{j(t-1)+1}, \dots, \zeta_{j(t)}]\}$, where ξ_a is a product of some x_ls, and ζ_b is a product of some y_ks.

There are basically two types of facets. The first type is represented as follows:

$$\begin{aligned}
\mathcal{N}_{m,n}(0,*): &\quad x_1\{[x_2, \dots, x_m], [y_1, \dots, y_n]\} \\
\mathcal{N}_{m,n}(i,*): &\quad \{[x_1, \dots, x_ix_{i+1}, \dots, x_m], [y_1, \dots, y_n]\} \quad (1 \le i \le m-1) \\
\mathcal{N}_{m,n}(m,*): &\quad \{[x_1, \dots, x_{m-1}], [y_1, \dots, y_n]\}x_m \\
\mathcal{N}_{m,n}(*,0): &\quad y_1\{[x_1, \dots, x_m], [y_2, \dots, y_n]\} \\
\mathcal{N}_{m,n}(*,j): &\quad \{[x_1, \dots, x_m], [y_1, \dots, y_jy_{j+1}, \dots, y_n]\} \quad (1 \le j \le n-1) \\
\mathcal{N}_{m,n}(*,n): &\quad \{[x_1, \dots, x_m], [y_1, \dots, y_{n-1}]\}y_n
\end{aligned}$$

By definition we have homeomorphisms:

$$\partial^{\mathcal{N}}_{i,*} \colon \mathcal{N}_{m-1,n} \to \mathcal{N}_{m,n}(i,*)$$
$$\partial^{\mathcal{N}}_{*,j} \colon \mathcal{N}_{m,n-1} \to \mathcal{N}_{m,n}(*,j)$$

The other type is represented by the product of two shuffle products as

$$\{[x_1, \dots, x_i], [y_1, \dots, y_j]\}\{[x_{i+1}, \dots, x_m], [y_{j+1}, \dots, y_n]\}$$

for $0 < i < m$ and $0 < j < n$. We denote this facet by $\mathcal{N}_{m,n}(i,j)$. By definition we have a homeomorphism

$$\partial^{\mathcal{N}}_{i,j} \colon \mathcal{N}_{i,j} \times \mathcal{N}_{m-i,n-j} \to \mathcal{N}_{m,n}(i,j)$$

Moreover, degeneracy operations $s^{\mathcal{N}}_{k,*} \colon \mathcal{N}_{m,n} \to \mathcal{N}_{m-1,n}$ for $1 \le k \le m$ and $s^{\mathcal{N}}_{*,l} \colon \mathcal{N}_{m,n} \to \mathcal{N}_{m,n-1}$ for $1 \le l \le n$ are defined corresponding to the cases of $x_i = *$ and $y_j = *$. For the relations these operations obey see [21, Lemma 2.2].

There are two definitions of higher homotopy commutativity by the resultohedra: $C_k(n)$-form by Hemmi and Kawamoto [21], and $C(k,l)$-form by Kishimoto and Kono [41].

Definition 4.7. Let X be a topological monoid. A family of maps $\{Q_{r,s}\colon \mathcal{N}_{r,s}\times X^{r+s}\to X\}$ satisfying the following properties is called a $C_k(n)$-form on X if the maps are defined for $r,s\geq 0$, $1\leq r+s\leq n$ and $s\leq k$, and called a $C(k,l)$-form on X if the maps are defined for $0\leq r\leq k$ and $0\leq s\leq l$.

(1) $Q_{r,0}(*,x_1,\dots,x_r)=x_1\dots x_r$ and $Q_{0,s}(*,y_1,\dots,y_s)=y_1\dots y_s$

(2) For $0\leq i\leq m$ and $0\leq j\leq n$, we have

$$\begin{aligned}
&Q_{r,s}(\partial^{\mathcal{N}}_{i,*}(a),x_1,\dots,x_r,y_1,\dots,y_s)\\
&\qquad= x_1Q_{r-1,s}(a,x_2,\dots,x_r,y_1,\dots,y_s) && \text{if } i=0\\
&\qquad= Q_{r-1,s}(a,x_1,\dots,x_ix_{i+1},\dots,x_r,y_1,\dots,y_s) && \text{if } 0<i<r\\
&\qquad= Q_{r-1,s}(a,x_1,\dots,x_{r-1},y_1,\dots,y_s)x_r && \text{if } i=m\\
&Q_{r,s}(\partial^{\mathcal{N}}_{*,j}(a),x_1,\dots,x_r,y_1,\dots,y_s)\\
&\qquad= y_1Q_{r,s-1}(a,x_1,\dots,x_r,y_2,\dots,y_s) && \text{if } j=0\\
&\qquad= Q_{r,s-1}(a,x_1,\dots,x_r,y_1,\dots,y_jy_{j+1},\cdots,y_s) && \text{if } 0<j<s\\
&\qquad= Q_{r,s-1}(a,x_1,\dots,x_r,y_1,\dots,y_{s-1})y_s && \text{if } j=n
\end{aligned}$$

(3) For $0<i<m$ and $0<j<n$, we have

$$\begin{aligned}
&Q_{r,s}(\partial^{\mathcal{N}}_{i,j}(a,b),x_1,\dots,x_r,y_1,\dots,y_s)\\
&\quad= Q_{i,j}(a,x_1,\dots,x_i,y_1,\dots,y_j)Q_{r-i,s-j}(b,x_{i+1},\dots,x_r,y_{j+1},\dots,y_s)
\end{aligned}$$

(4) If $x_i=*$ for $1\leq i\leq r$, then

$$\begin{aligned}
&Q_{r,s}(a,x_1,\dots,x_r,y_1,\dots,y_s)\\
&\qquad= Q_{r-1,s}(s^{\mathcal{N}}_{i,*}(a),x_1,\dots,\hat{x}_i,\dots,x_r,y_1,\dots,y_s)
\end{aligned}$$

and if $y_j=*$ for $1\leq j\leq s$, then

$$\begin{aligned}
&Q_{r,s}(a,x_1,\dots,x_r,y_1,\dots,y_s)\\
&\qquad= Q_{r,s-1}(s^{\mathcal{N}}_{*,j}(a),x_1,\dots,x_r,y_1,\dots,\hat{y}_j,\dots,y_s)
\end{aligned}$$

A $C_k(n)$-space is a topological monoid X equipped with a $C_k(n)$-form, and a $C(k,l)$-space is X equipped with a $C(k,l)$-form.

A homotopy commutative topological monoid is a $C_k(2)$-space with $k=1,2$ and a $C(1,1)$-space. Moreover, any abelian monoid admits a $C_\infty(\infty)$-form. The relation with the Williams C_n-space is given as follows

Proposition 4.8 (Hemmi-Kawamoto [21, Proposition 4.5])**.** *Any $C_k(n)$-space is a C_n-space.*

The above fact is proved by decomposing permutohedra by resultohedra.

Now we describe properties that the classifying space of a $C_k(n)$-space and a $C(k,l)$-space enjoy. A space X is called an $H(k,l)$-space if there is an axial map

$$\mu\colon P_k(\Omega X) \times P_l(\Omega X) \to X$$

so that $\mu(x,*) = \iota_k(x)$ and $\mu(*,y) = \iota_s(y)$ for the inclusion $\iota_i\colon P_i(\Omega X) \to P_\infty(\Omega X) \simeq X$. If there is an axial map

$$\mu\colon \bigcup_{0\le s\le k} P_{n-s}(\Omega X) \times P_s(\Omega X) \to X,$$

then X is called an $H_k(n)$-space. Both $H(k,\infty)$-space and $H_k(\infty)$-space are equivalent to T_k-space by Aguadé [2]. In particular, H-space, $H(\infty,\infty)$-space and $H_\infty(\infty)$-space are all equivalent.

Then we have the following

Theorem 4.9 (Hemmi-Kawamoto [21, Theorem A, Corollary 1.1], Kishimoto-Kono [41, Theorem 1.6])**.** *Let X be a connected topological monoid.*

(1) *X admits an $C_k(n)$-form if and only if BX is an $H_k(n)$-space.*
(2) *X admits an $C_n(n)$-form if and only if BX is an $H(n)$-space.*
(3) *X admits an $C(k,l)$-form if and only if BX is an $H(k,l)$-space.*

Hasui, Kishimoto and Tsutaya [14] generalized the above concepts and defined $C(k_1,\dots,k_r)$-space, which is a topological monoid X admitting an axial map $P_{k_1}X \times \cdots \times P_{k_r}X \to BX$.

4.3. **Hybrid of associativity and commutativity.** In the definition of higher homotopy commutativity, we need to multiply only n-elements of the space. Thus it is natural to think of extending the definition to A_n-spaces. To do so we need polytopes which are hybrid of associahedra and permutohedra (or resultohedra).

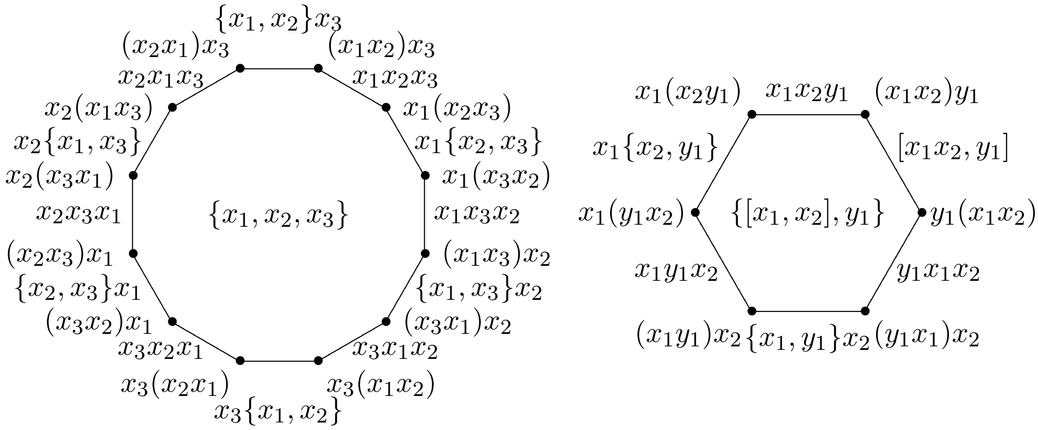
The extension of Williams C_n-form to A_n-spaces is given by Hemmi and Kawamoto [20]. The parameter spaces used there are permuto-associahedra, which are introduced by Kapranov [34]. On the other hand, the extension of $C_1(n)$-form to A_n-spaces is given by Kawamoto [38], which is called B_n-form. He showed that the parameter spaces for B_n-form are homeomorphic to cyclohedra, which are introduced by Bott and Taubes [8] to study topological descriptions of self-linking invariants of knots. For $C_k(n)$-form with $k > 1$, no papers have been published treating this theme.

In any cases, the poset structures for the polytopes are easy to imagine. Any face of $\mathcal{P}_n$ and $\mathcal{N}_{m,n}$ are represented by a sequence $\alpha_1\alpha_2\dots\alpha_k$ for some α_i suitably defined. Thus, to consider the case for A_n-spaces, we just insert some pairs of meaningful parentheses to $\alpha_1\alpha_2\dots\alpha_k$. For example, from $\{x_1, x_5\}x_2\{x_3, x_4\}$ in $\mathcal{P}_5$, which is homeomorphic to $\mathcal{P}_2 \times \mathcal{P}_2$, we have following sequences:

$$\{x_1, x_5\}x_2\{x_3, x_4\}, \quad (\{x_1, x_5\}x_2)\{x_3, x_4\}, \quad \{x_1, x_5\}(x_2\{x_3, x_4\})$$

The first sequence represent a face homeomorphic to $\mathcal{K}_3 \times \mathcal{P}_2 \times \mathcal{P}_2$, and the other sequences represent two faces of the first one which are homeomorphic to $\mathcal{K}_2 \times \mathcal{P}_2 \times \mathcal{P}_2$.

The following are the hybrid polytopes given from $\mathcal{P}_3$ and $\mathcal{N}_{2,1}$.



From the above argument, all faces of $\mathcal{P}_n$ represent all facets of the n-permuto-associahedron Γ_n, and any lower dimensional face of Γ_n is given by inserting some pairs of parentheses in the sequence representing a face of $\mathcal{P}_n$. Let $\alpha_1\alpha_2 \dots \alpha_k$ be an ordered partition of $x[n]$ of type $(n_1, \dots, n_k)$ representing a face of $\mathcal{P}_n$. Then the facet of Γ_n corresponding to this sequence is homeomorphic to $\mathcal{K}_k \times \mathcal{P}_{n_1} \times \cdots \times \mathcal{P}_{n_k}$. We denote this facet as $\Gamma_n(\alpha_1 \dots \alpha_k)$, and we have a homeomorphism

$$\partial^{\Gamma}_{\alpha_1 \dots \alpha_k} \colon \mathcal{K}_k \times \Gamma_{n_1} \times \cdots \times \Gamma_{n_k} \to \Gamma_n(\alpha_1 \dots \alpha_k).$$

The degeneracy map $s^{\Gamma}_i \colon \Gamma_n \to \Gamma_{n-1}$ for $1 \leq i \leq n$ is defined by removing x_i from $x[n]$ and renumber x_{j+1} for $j > i$ to x_j. For example, we consider a face $(\{x_2\}\{x_1, x_4\})\{x_3, x_5\}$. Then we have the following

$$\begin{aligned} s^{\Gamma}_i((\{x_2\}\{x_1, x_4\})\{x_3, x_5\}) &= (\{x_1\}\{x_3\})\{x_2, x_4\} && \text{if } i = 1 \\ &= \{x_1, x_3\}\{x_2, x_4\} && \text{if } i = 2 \\ &= (\{x_2\}\{x_1, x_3\})\{x_4\} && \text{if } i = 3 \\ &= (\{x_2\}\{x_1\})\{x_3, x_4\} && \text{if } i = 4 \\ &= (\{x_2\}\{x_1, x_4\})\{x_3\} && \text{if } i = 5 \end{aligned}$$

For the relations these operations obey see [20, Propositions 2.1 and 2.2]. The extension of the Williams C_n-form to A_n-spaces is called AC_n-form by Hemmi-Kawamoto [20, Definition 3.1]. We skip the definition here.

On the other hand, in the case of $C_1(n)$-form, the correspondence of the parameter spaces to the cyclohedra is not so easy to imagine. We do not give detail here, so readers who are interested in it should refer to the original paper [38].

4.4. **Homotopy commutative finite H-space.** At the end of this paper, we review the mod p torus theorems proved by several authors.

As is noted in the first section, Hubbuck's torus theorem can be deduced from the mod 2 torus theorem by Lin ([42]): for a simply connected H-space X with finite $\mathbb{F}_2$-cohomology $H^*(X;\mathbb{F}_2)$, if X admits a homotopy commutative multiplication then $H^*(X;\mathbb{F}_2)$ is acyclic. For an odd prime p, Aguadé and Smith [3, Corollary] showed a similar result under some strong assumption: if X is a topological monoid with finite exterior $\mathbb{F}_p$-cohomology algebra for a prime p such that the classifying space BX is an H-space, then X is mod p equivalent to a torus. It is clear that the assumption of this theorem seems too strong. Therefore, many authors have assumed higher homotopy commutativity of order p in some sense, instead.

Assertion 4.10. *Let X be a simply connected H-space with finite $\mathbb{F}_p$-cohomology $H^*(X;\mathbb{F}_p)$ for an odd prime p. If X admits some kind of higher homotopy commutativity of order p, then $H^*(\tilde{X};\mathbb{F}_p)$ is acyclic,*

The first one showed the above theorem is McGibbon [46, Theorem 3]. In his theorem, the assumption of the higher homotopy commutativity of order p was the Sugawara C_p-form. Then, Hemmi [17, Theorem 1.1] used quasi C_p-form as the assumption. Here, a quasi C_n-form on an A_n-space X is a family of maps $\{\varphi_i\colon J_i(\Sigma) \to P_i(X)\}_{1\le i\le n}$ such that $\varphi_1 = id$, $\varphi_i|J_{i-1}(\Sigma X) = \varepsilon_{i-1}\circ\varphi_{i-1}$ and $\rho_i\circ\varphi_i \simeq \sum_{\tau\in\mathfrak{S}_i}\tau\rho_i$, where the symmetric group $\mathfrak{S}_i$ acts on the i-fold smash product $X^{[i]}$ as the permutation of the factors, and the summation $\sum_{\tau\in\mathfrak{S}_i}\tau\rho_i$ is defined by using the comultiplication of $(\Sigma X)^{[i]} \simeq \Sigma^i(X^{[i]})$. Moreover, many authors have extended the mod p torus theorem for H-spaces with $\mathbb{F}_p$-cohomology not necessarily finite, such as Slack [51], Lin and Williams [44], Lin [43], Broto and Crespo [9], Kawamoto [35], [36], [37] and Kawamoto and Lin [39].

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