A State-Space Modeling via the Galerkin Approximation for a Boundary Control System

Jun Imai * **Katsuyuki Noso** * **Akiko Takahashi** *
Shigeyuki Funabiki *

* *Okayama University, Tsushima-naka, Kita-ku, Okayama 700-8530 JAPAN (e-mail: imai-j@okayama-u.ac.jp).*

Abstract: For linear distributed parameter systems with a finite number of boundary inputs, we propose a framework to implement the method of weighted residuals using candidate trial functions without boundary homogenization. Proposed scheme utilizes inner product matrix, or Grammian, of the trial functions to separate appropriate homogenized basis functions and the other trial functions matching inhomogeneous boundary conditions. The finite-dimensional approximate model by using the proposed scheme is represented in descriptor form and it is proved to be straightforwardly transformed into state space form. Feasibility of the method is illustrated by a brief controller design example using the approximate model of a heat conduction rod with Dirichlet boundary input.

Keywords: Distributed parameter systems, Method of weighted residuals, Boundary conditions, Reduced-order models

1. INTRODUCTION

A distributed parameter system is, in general, of infinite dimension, and some finite-dimensional approximation is practically needed for finite-dimensional controller design (Curtain (2003)). The method of weighted residuals (MWR) is recognized as a typical and effective finite-dimensional approximation framework for a wide class of distributed parameter systems (Finlayson (1972)). An appropriate choice of trial function can be crucial to yield useful reduced-order models for controller design. In a few decades, the proper orthogonal decomposition (POD) has been attracting much attention, which systematically generates trial functions from data or snapshots (Holmes et al. (1996); Kunisch and Volkwein (2001)). Since many systems have inputs on their boundary, boundary conditions often become inhomogeneous and homogenization of trial functions can be performed to apply MWR to such a boundary control system (Graham et al. (1999)). But, in fact, the analysis of such a system with inhomogeneous boundary conditions is often much involved, and resultant state space representation of the approximate model is easy to get complicated.

In this paper, we generalize the idea of homogenization of trial functions and propose an approximate modelling scheme for a class of boundary control systems with a finite number of inputs. This is to provide semi-explicit descriptor systems which consist of ordinary differential equations and algebraic constraint conditions, preserving the structure of the original system made of partial differential equations and boundary conditions. Here descriptor systems are employed because correspondence between the system and the model is relatively straightforward and easy to understand. Conventional MWR or Galerkin method, in most cases, requires trial functions that match homogeneous and inhomogeneous boundary conditions, respectively, but the proposed method requires only some mild conditions on candidate trial functions. Trial functions such as frequency response profiles and POD basis, for instance, can be directly adopted without preconditioning as candidate trial functions for the proposed method.

In the proposed method, some matrices of integration concerning trial functions are formed, and then the trial functions are internally and automatically homogenized to yield basis functions just by matrix computation. The original boundary conditions are converted to finite-dimensional algebraic constraint conditions. They are combined and we have an approximate descriptor system model. Our main contribution is that we show a state-space model is always obtained from the descriptor system under certain conditions given.

In the rest of the paper, we formulate a boundary control system in section 2, and in section 3 an implementation scheme of the Galerkin approximation is presented and the relationship of the proposed and conventional schemes is discussed. Section 4 gives examples of approximation and H_∞ controller design using eigenfunctions and frequency response profiles, for heat conduction rod with Dirichlet input. Finally, in section 5 we give concluding remarks.

2. A BOUNDARY CONTROL SYSTEM

Let H and U be Hilbert spaces, and an operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$, and a linear bounded operator $\Gamma : D(A) \to U$ which is onto, are given. We assume that $\mathcal{A}_0$, a restriction of $\mathcal{A}$ on $\operatorname{Ker}\Gamma$, generates a strongly continuous semigroup on H. Then the linear time-invariant system

$$\begin{aligned}\frac{dv(t,\cdot)}{dt} &= \mathcal{A}v(t,\cdot),\ \text{on } H,\ (t>t_0)\\ 0 &= \Gamma v(t,\cdot)+\bar{u}(t,\cdot),\ \text{on } U,\ (t>t_0)\\ v(t_0,\cdot) &= v_0(\cdot),\ \text{on } H\end{aligned} \tag{1}$$

is called boundary control system(Reis (2006)). We further assume that the control $\bar{u}$ is of finite-dimension; that is, $\bar{u}(t,\cdot)=\mathcal{F}u_2(t)$ for a linear bounded operator $\mathcal{F}:\mathbb{R}^{m_2}\to U$. Hereafter, the system we consider is

$$\begin{aligned}\frac{dv(t,\cdot)}{dt} &= \mathcal{A}v(t,\cdot)+\mathcal{B}u_1(t),\ \text{on } H,\ (t>t_0)\\ 0 &= \Gamma v(t,\cdot)+\mathcal{F}u_2(t),\ \text{on } U,\ (t>t_0)\\ v(t_0,\cdot) &= v_0,\ \text{on } H\end{aligned} \tag{2}$$

where $\mathcal{B}:\mathbb{R}^{m_1}\to H$ is a linear bounded operator. We identify $v\in H$ as a function $v(\xi)$ defined on Ω here $\Omega\subset\mathbb{R}^p,(p=1,2,3)$ is simply connected spatial domain with smooth boundary $\partial\Omega$. $u_1(t)\in\mathbb{R}^{m_1}$ is for distributed input and $u_2(t)\in\mathbb{R}^{m_2}$ for boundary input.

A typical example of boundary control system described by a partial differential equation is as follows:

$$\begin{aligned}&\frac{\partial v(t,\xi)}{\partial t}=\Delta v(t,\xi)+b(\xi)u_1(t),(\xi\in\Omega,t>t_0)\\ &0=\alpha(\xi)v(t,\xi)+(1-\alpha(\xi))\frac{\partial v}{\partial\nu}+f(\xi)u_2(t),(\xi\in\partial\Omega,t>t_0)\\ &v(t_0,\xi)=v_0(\xi),(\xi\in\Omega)\end{aligned} \tag{3}$$

where Δ is Laplacian and $0\le\alpha(\xi)\le 1$. $\partial/\partial\nu$ is outside normal derivative on $\partial\Omega$.

It can be written as $\mathcal{B}=[b_1(\xi)\ \cdots\ b_{m_1}(\xi)],(\xi\in\Omega)$, and $b_i(\cdot)\in H,(i=1,\ldots,m_1)$ are referred to as input effect functions. For the boundary input operator $\mathcal{F}$, we can also write as $\mathcal{F}=[f_1(\xi)\ \cdots\ f_{m_2}(\xi)],(\xi\in\partial\Omega)$ where $f_i(\cdot)\in U$ for $i=1,\ldots,m_2$.

3. AN APPROXIMATION SCHEME

First, we choose candidate trial functions $\chi_1(\xi),\ldots,\chi_N(\xi)\in D(\mathcal{A})$ aiming at representing the solution properly by their linear combination. But no explicit boundary conditions are placed on these trial functions at this stage. We denote $\text{span}\{\chi_1,\ldots,\chi_N\}$ by H_N.

We assume that an approximate solution $\hat{v}$ to (2) be of the form

$$\hat{v}(t,\xi)=c_1(t)\chi_1(\xi)+\cdots+c_N(t)\chi_N(\xi). \tag{4}$$

On the candidate trial functions $X(\xi)=[\chi_1(\xi)\cdots\chi_N(\xi)]^T$, we pose the following conditions:

(i) $X(\xi)$ is linearly independent[1], that is, using the notation $\langle\chi,\zeta\rangle:=\int_\Omega\chi(\xi)\zeta(\xi)d\xi$,

$$\det\left\langle X,X^T\right\rangle\neq 0. \tag{5}$$

(ii) There exists $\boldsymbol{c}=[c_1\ c_2\ \cdots c_N]^T$ such that

$$0=\Gamma X^T\boldsymbol{c}+\mathcal{F}u_2 \tag{6}$$

for each $u_2\in\mathbb{R}^{m_2}$. That is,

$$\text{Im}\,(\Gamma X^T)\supset\text{Im}\,\mathcal{F}. \tag{7}$$

[1] This condition can be removed if X is replaced by $X'=S_1X$ where $S_1^T=(\text{Ker}\left\langle X,X^T\right\rangle)^\perp$, but here we omit the detail.

$$\text{Ker}\,(\Gamma X^T)\neq\{0\}. \tag{8}$$

Since $\Gamma X^T:\mathbb{R}^N\to U$, $\text{Ker}\,(\Gamma X^T)$ consists of linearly independent $\boldsymbol{c}_i\in\mathbb{R}^N$ such that $\Gamma X^T\boldsymbol{c}_i=0$ for $i=1,2\ldots$. So $n=\dim[\text{Ker}\,(\Gamma X^T)]>0$ is essentially the number of trial functions matching the homogeneous boundary conditions, and this n becomes the order of the approximate model.

For the candidate trial functions $\chi_i(\cdot)$ $(i=1,\ldots,N)$, the number of the conditions out of homogeneous boundary conditions is $r=N-n$. $\rho=r-m_2$ is the number of the candidate trial functions not matching any boundary conditions. (They are useless and can be discarded; corresponding c_i is made identically 0.) In summary, inhomogeneous boundary conditions turn into r constraints and using a set $\Upsilon(\xi)=[\upsilon_1(\xi)\cdots\upsilon_r(\xi)]^T$ of non-zero functions on U, the boundary conditions of (2) are equivalent to algebraic constraint conditions

$$\left\langle\Upsilon,\Gamma X^T\right\rangle_U\boldsymbol{c}+\langle\Upsilon,\mathcal{F}\rangle_U u_2=0 \tag{9}$$

where $\langle f,g\rangle_U:=\int_{\partial\Omega}f(\xi)g(\xi)d\xi$.

The equation residual is

$$R=X^T\frac{d\boldsymbol{c}}{dt}-\mathcal{A}X^T\boldsymbol{c}-\mathcal{B}u \tag{10}$$

where $\mathcal{A}X^T=[\mathcal{A}\chi_1\cdots\mathcal{A}\chi_N]$ and the number of conditions of weighted residual

$$\langle W,R\rangle=0 \tag{11}$$

should be n where

$$W=[w_1(\xi)\ \cdots\ w_n(\xi)]^T$$

is linearly independent. That is, the number of weight functions should be n for the existence and the uniqueness of the approximate solution because if the number of W is greater than n then no solutions exist due to over-conditioning and if smaller than n then solution is not unique due to under-conditioning.

3.1 Approximate model in descriptor form

The condition on weighted residuals for the partial differential equations turns to be

$$\left\langle W,X^T\right\rangle\frac{d\boldsymbol{c}}{dt}=\left\langle W,\mathcal{A}X^T\right\rangle\boldsymbol{c}+\langle W,\mathcal{B}\rangle u_1 \tag{12}$$

and the equivalent constraint condition for the boundary condition is

$$\left\langle\Upsilon,\Gamma X^T\right\rangle_U\boldsymbol{c}+\langle\Upsilon,\mathcal{F}\rangle_U u_2=0. \tag{13}$$

Equations (12) and (13) are united as in the following semi-explicit descriptor system:

$$E\frac{d\boldsymbol{c}}{dt}=A\boldsymbol{c}+Bu,\quad E\boldsymbol{c}(t_0)=\langle W,v_0\rangle, \tag{14}$$

$$\hat{v}(t,\xi)=C(\xi)\boldsymbol{c}(t), \tag{15}$$

where $\boldsymbol{c}(t)\in\mathbb{R}^N$ and

$$\begin{aligned}E&=\begin{bmatrix}\left\langle W,X^T\right\rangle\\ 0\end{bmatrix},\quad A=\begin{bmatrix}\left\langle W,\mathcal{A}X^T\right\rangle\\ \left\langle\Upsilon,\Gamma X^T\right\rangle_U\end{bmatrix},\\ B&=\begin{bmatrix}\langle W,\mathcal{B}\rangle & 0\\ 0 & \langle\Upsilon,\mathcal{F}\rangle_U\end{bmatrix},\\ C(\xi)&=X^T(\xi)\end{aligned} \tag{16}$$

with $u = [u_1^T\ u_2^T]^T$. The unique solution to (14)-(16) exists only if the pencil

$$sE - A = \begin{bmatrix} \langle W, sX^T - \mathcal{A}X^T \rangle \\ \langle \Upsilon, \Gamma X^T \rangle_U \end{bmatrix}$$

is regular, or nonsingular for some $s \in \mathbb{C}$. Even if the condition holds, the system (14)-(16) does not necessarily have any state space representations. In case of $sE-A$ regular, singular value decomposition of E yields orthogonal matrices U, V, and nonsingular Σ_n such that

$$E = U \begin{bmatrix} \Sigma_n & 0 \\ 0 & 0 \end{bmatrix} V^T, \tag{17}$$

then matrices of a canonical form

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = U^T A V,\ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = U^T B \tag{18}$$

$$[C_1(\xi)\ C_2(\xi)] = C(\xi)V,\ \begin{bmatrix} \boldsymbol{x} \\ \bar{x} \end{bmatrix} = \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \boldsymbol{c} = V^T \boldsymbol{c} \tag{19}$$

are determined where $V = [V_1\ V_2]$.

If A_4 is nonsingular, the descriptor system is said to be of index-1, and a corresponding state space model is shown in the following:

$$\frac{d\boldsymbol{x}(t)}{dt} = A_n \boldsymbol{x}(t) + B_n u(t), \quad \boldsymbol{x}(t_0) = V_1^T \langle W, v_0 \rangle, \tag{20}$$

$$\hat{v}(t,\xi) = C_n(\xi)\boldsymbol{x}(t) + D_n(\xi)u(t) \tag{21}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ and

$$\begin{aligned} A_n &= \Sigma_n^{-1}(A_1 - A_2 A_4^{-1} A_3),\ (n \times n) \\ B_n &= \Sigma_n^{-1}(B_1 - A_2 A_4^{-1} B_2),\ (n \times (m_1 + m_2)) \\ C_n(\xi) &= C_1(\xi) - C_2(\xi) A_4^{-1} A_3,\ (H_N \times n) \\ D_n(\xi) &= -C_2(\xi) A_4^{-1} B_2. \quad (N_N \times (m_1 + m_2)) \end{aligned} \tag{22}$$

Lemma 1. (Shiotsuki (2011); Dai (1989)) The transfer function of the descriptor system (14) is identical to that of a state space model (20)–(22) if A_4 is nonsingular.

3.2 The Galerkin model

In the Galerkin method, weight functions are taken to be identical to basis functions. We choose W to span a subspace $Y = H_N \cap \operatorname{Ker}\Gamma$. That is, $\operatorname{span} W = Y$, and from elements in $\operatorname{span} X$ we pick up ones that match the homogeneous boundary conditions. We consider a $n \times N$ matrix T_0 which consists of basis vectors of the subspace X. That is, $T_0 = [\boldsymbol{c_1}\ \cdots\ \boldsymbol{c_n}]^T$ where linearly independent $\boldsymbol{c_1}, \ldots, \boldsymbol{c_n}$ are fundamental solutions of homogeneous linear equation $\Gamma X^T \boldsymbol{c} = 0$. Note that $\operatorname{span} T_0^T = \operatorname{Ker}\Gamma X^T = \operatorname{Ker}\left\langle (\Gamma X^T)^T, \Gamma X^T \right\rangle$, or we write as in

$$T_0 = \operatorname{Null}\left(\left\langle (\Gamma X^T)^T, \Gamma X^T \right\rangle\right)^T.$$

After all, in the Galerkin method, we propose to choose the weight functions as

$$W = T_0 X. \tag{23}$$

On the other hand, let $X^\perp$ be the orthogonal complement of X for H_N. Determine linearly independent $\boldsymbol{c}_{n+1}, \ldots, \boldsymbol{c}_{n+r}$ such that

$$\operatorname{Ker} T_0 = \operatorname{span}[\boldsymbol{c}_{n+1}, \ldots, \boldsymbol{c}_{n+r}] \tag{24}$$

then setting $T_1 = [\boldsymbol{c_{n+1}}\ \cdots\ \boldsymbol{c_{n+r}}]^T$, then $\operatorname{span} T_1^T = \operatorname{Ker} T_0$, or

$$T_1 = \operatorname{Null}(T_0)^T$$

and basis vectors of $X^\perp$ is proved to be obtained by $T_1 X$.

If we choose $\Upsilon = (\Gamma(T_1 X)^T)^T = ((\Gamma X^T T_1^T)^T = T_1(\Gamma X^T)^T$ then

$$E\frac{d\boldsymbol{c}}{dt} = A\boldsymbol{c} + Bu, \quad E\boldsymbol{c}(t_0) = T_0 \langle X, v_0 \rangle, \tag{25}$$

$$\hat{v}(t,\xi) = C(\xi)\boldsymbol{c}(t) \tag{26}$$

where $\boldsymbol{c}(t) \in \mathbb{R}^N$ and

$$\begin{aligned} E &= \begin{bmatrix} T_0 \langle X, X^T \rangle \\ 0 \end{bmatrix}, \\ A &= \begin{bmatrix} T_0 \langle X, \mathcal{A}X^T \rangle \\ T_1 \left\langle (\Gamma X^T)^T, \Gamma X^T \right\rangle_U \end{bmatrix}, \\ B &= \begin{bmatrix} T_0 \langle X, \mathcal{B} \rangle & 0 \\ 0 & T_1 \left\langle (\Gamma X^T)^T, \mathcal{F} \right\rangle_U \end{bmatrix}, \\ C(\xi) &= X^T(\xi). \end{aligned} \tag{27}$$

Claim 2. The descriptor system given by (25)-(27) is equivalent to a state space model (20)-(22) in the sense of a common transfer function.

Proof. See the next subsection. □

The above algorithm is summarized as follows:

Proposed calculating procedure for approximate model

(1) Choose a set of candidate trial functions $X(\xi) := [\chi_1(\xi) \cdots \chi_\nu(\xi)]^T$, which is linearly independent, that is,
$$M := \langle X, X^T \rangle\ : (\nu \times \nu)$$
is nonsingular.

(2) Integrate
$$\begin{aligned} K &:= \langle X, \mathcal{A}X^T \rangle\ : (\nu \times \nu), \\ L &:= \langle X, \mathcal{B} \rangle\ : (\nu \times m_1), \\ N &:= \left\langle (\Gamma X^T)^T, \Gamma X^T \right\rangle_U\ : (\nu \times \nu). \end{aligned}$$

(3) Compute $T_0 := \operatorname{Null}(N)^T$, $T_1 := \operatorname{Null}(T_0)^T$.

(4) Approximate model in descriptor form($\boldsymbol{c} \in \mathbb{R}^{n+m_2}$):
$$E\frac{d\boldsymbol{c}}{dt} = A\boldsymbol{c} + Bu, \quad E\boldsymbol{c}(t_0) = T_0 \langle X, v_0 \rangle, \tag{28}$$
$$\hat{v}(t,\xi) = C(\xi)\boldsymbol{c}(t), \tag{29}$$
$$\begin{aligned} E &= \begin{bmatrix} T_0 M T_0^T & 0 \\ 0 & 0 \end{bmatrix} =: \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}, \\ A &= \begin{bmatrix} T_0 K T_0^T & T_0 K T_1^T \\ 0 & T_1 N T_1^T \end{bmatrix} =: \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}, \\ B &= \begin{bmatrix} T_0 L & 0 \\ 0 & T_1 F \end{bmatrix} =: \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\ C(\xi) &= X(\xi)[T_0^T\ T_1^T] =: [C_1(\xi)\ C_2(\xi)]. \end{aligned} \tag{30}$$

(5) Approximate model in state-space form($\boldsymbol{x} \in \mathbb{R}^n$):
$$\frac{d\boldsymbol{x}}{dt} = A_n \boldsymbol{x} + B_n u(t), \quad \boldsymbol{x}(t_0) = T_0 \langle X, v_0 \rangle, \tag{31}$$
$$\hat{v}(t,\xi) = C_n(\xi)\boldsymbol{x}(t) + D_n(\xi)u(t), \tag{32}$$

$$
\begin{aligned}
A_n &= E_1^{-1}A_1,\\
B_n &= E_1^{-1}(B_1 - A_2A_4^{-1}B_2),\\
C_n(\xi) &= C_1(\xi),\\
D_n(\xi) &= -C_2(\xi)A_4^{-1}B_2.
\end{aligned} \tag{33}
$$

3.3 Equivalence with the conventional scheme

This subsection is to illustrate that the proposed scheme is equivalent to the conventional one in a sense of state space systems, so the proposed model (25)-(27) prove to be of index-1 and it is transformed into a state space model.

For given candidate trial functions $X = [\chi_1(\xi)\cdots\chi_N(\xi)]^T$, we choose new trial functions

$$
\begin{bmatrix}\Phi\\ \Psi\end{bmatrix} = \begin{bmatrix}T_0\\ T_1\end{bmatrix} X = TX. \tag{34}
$$

where $T = [T_0^T\ T_1^T]^T$. In fact,

$$
\hat{v} = X^T\boldsymbol{c} = \begin{bmatrix}\Phi\\ \Psi\end{bmatrix}^T \begin{bmatrix}T_0\\ T_1\end{bmatrix}^{-T} \boldsymbol{c} = \begin{bmatrix}\Phi\\ \Psi\end{bmatrix}^T \boldsymbol{c}',\ \boldsymbol{c}' = \begin{bmatrix}T_0\\ T_1\end{bmatrix}^{-T} \boldsymbol{c}
$$

Here $\boldsymbol{c}$ is transformed into $\boldsymbol{c}'$ by $\boldsymbol{c} = T^T\boldsymbol{c}'$. We choose Φ so that $\Gamma\Phi^T = 0$. And $c_1' = x_1, \dots\ c_n' = x_n$. Further we can take $W = \Phi,\ \Upsilon = (\Gamma\Psi^T)^T$. Then using $\langle\Phi, \Psi^T\rangle = 0$ we have

$$
\begin{aligned}
E &= \begin{bmatrix}\langle\Phi,\Phi^T\rangle & 0\\ 0 & 0\end{bmatrix},\\
A &= \begin{bmatrix}\langle\Phi,\mathcal{A}\Phi^T\rangle & \langle\Phi,\mathcal{A}\Psi^T\rangle\\ 0 & \langle(\Gamma\Psi^T)^T,\Gamma\Psi^T\rangle_U\end{bmatrix},\\
B &= \begin{bmatrix}\langle\Phi,\mathcal{B}\rangle & 0\\ 0 & \langle(\Gamma\Psi^T)^T,\mathcal{F}\rangle_U\end{bmatrix}.
\end{aligned} \tag{35}
$$

Lemma 3. $\langle(\Gamma\Psi^T)^T,\Gamma\Psi^T\rangle_U$ is nonsingular.

Proof. $\langle(\Gamma\Psi^T)^T,\Gamma\Psi^T\rangle_U$ is a Gramian of

$$
(\Gamma\Psi^T)(\xi) = [(\Gamma\psi_1)(\xi)\cdots(\Gamma\psi_{m_2})(\xi)],\ \xi\in\partial\Omega,
$$

and this is nonsingular if and only if $\Gamma\Psi^T$ is linearly independent. $\Gamma\Psi^T$ is linearly independent if $\sum_k x_k\Gamma\psi_k = 0$ implies $x_1 = \cdots = x_{m_2} = 0$. In fact, if $\sum_k x_k\Gamma\psi_k = \Gamma(\sum_k x_k\psi_k) = 0$ then $\psi_k \notin \operatorname{Ker}\Gamma\backslash\{0\}$ and $\sum_k x_k\psi_k \notin \operatorname{Ker}\Gamma\backslash\{0\}$ hold, so $\sum_k x_k\psi_k = 0$. But, $\{\psi_k\}_{k=1}^{m_2}$ is linearly independent because T_0 is row full rank, so if $\sum_k x_k\psi_k = 0$ then $x_1 = \cdots = x_{m_2} = 0$. So we proved that $\Gamma\Psi^T$ is linearly independent. □

So we see that, the system is of index-1, and it is reduced to a state space model according to (20)-(22):

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \langle\Phi,\Phi^T\rangle^{-1}\langle\Phi,\mathcal{A}\Phi^T\rangle\boldsymbol{x} + \langle\Phi,\Phi^T\rangle^{-1}\langle\Phi,\mathcal{B}\rangle u_1 +\\
&\quad -\langle\Phi,\Phi^T\rangle^{-1}\langle\Phi,\mathcal{A}\Psi^T\rangle\langle(\Gamma\Psi^T)^T,\Gamma\Psi^T\rangle_U^{-1}\\
&\qquad \times\langle(\Gamma\Psi^T)^T,\mathcal{F}\rangle_U u_2,\\
\hat{v}(t,\xi) &= \Phi^T(\xi)\boldsymbol{x} - \Psi^T(\xi)\\
&\qquad \times\langle(\Gamma\Psi^T)^T,\Gamma\Psi^T\rangle_U^{-1}\langle(\Gamma\Psi^T)^T,\mathcal{F}\rangle_U u_2.
\end{aligned} \tag{36}
$$

This result coincides with the conventional one for boundary control systems (Sagara and Imai (1991)).

In the above, we saw that conditions from the Galerkin projection of a partial differential equation and the others from boundary conditions are united in separate form to yield a relatively simple approximate model in the form of a descriptor system. We employ an interior method (Finlayson (1972)) where the approximate solution fully satisfies boundary conditions, and trial functions as well as an approximate solution should completely match the boundary conditions.

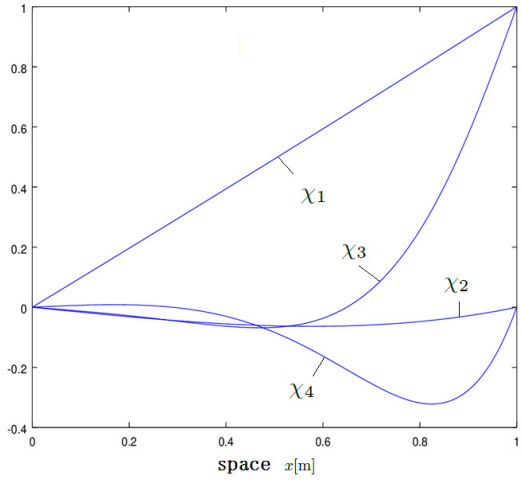

Fig. 1. The candidate trial functions: frequency response profile (χ_1 and χ_2 for $\omega_1 = 1$rad/s, and χ_3 and χ_4 for $\omega_1 = 40$rad/s)

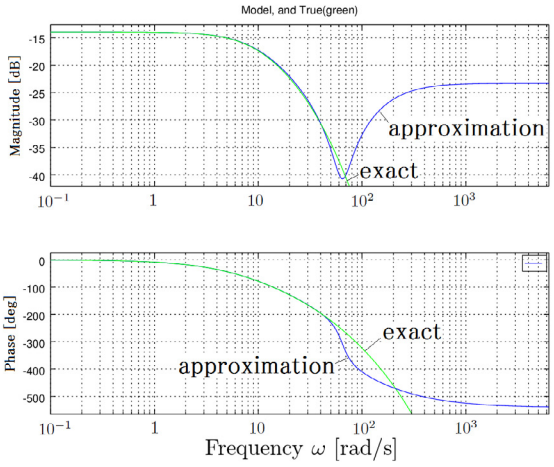

Fig. 2. Bode plots for approximate model $G_3(s, 4/5)$ via frequency response profile($\omega_1 = 1$rad/s,$\omega_2 = 40$rad/s)(blue line) and for the exact system $G(s, 4/5)$(green line)

4. APPROXIMATION AND CONTROLLER DESIGN EXAMPLE

In this section, we are going to provide examples of approximation and H_∞ controller design using eigenfunctions and frequency response profiles, for heat conduction rod with Dirichlet input.

Example. (A heat conduction rod with Dirichlet input)

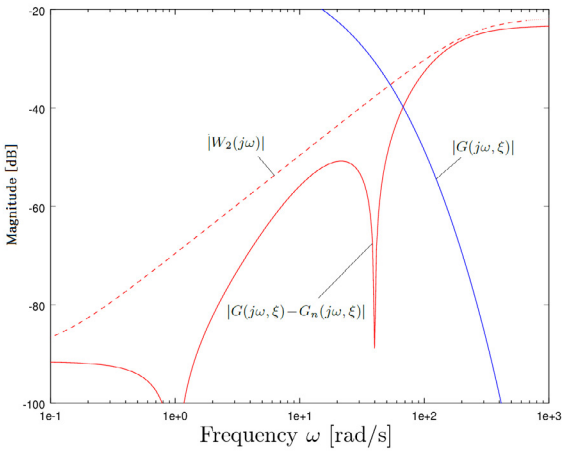

Fig. 3. Error plot $|G(j\omega,\xi) - G_n(j\omega,\xi)|$ of the approximate model via frequency response profile ($\omega_1 = 1$rad/s, $\omega_2 = 40$rad/s, $n = 3$)(solid red line), magnitude of the system $|G(j\omega,\xi)|$ (blue line), and a bode magnitude plot of a real rational function $|W_2(j\omega)|$ that covers the error(dashed line)($\xi = 4/5$)

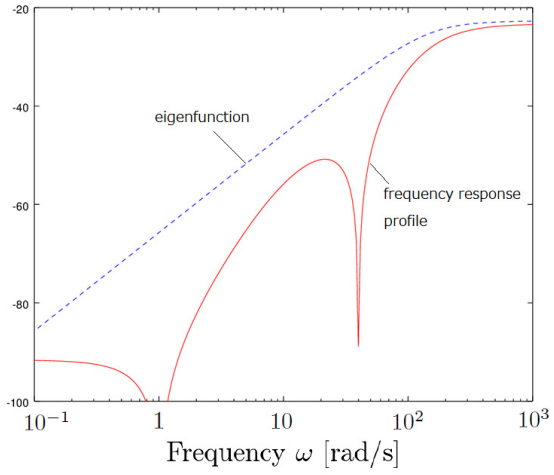

Fig. 4. Error plot $|G(j\omega,4/5) - G_n(j\omega,4/5)|$ of the approximate model via frequency response profile ($\omega_1 = 1$rad/s, $\omega_2 = 40$rad/s, $n = 3$)(solid red line), and via eigenfunctions ($n = 3$)(dashed blue line)

$$\begin{aligned} \frac{\partial v}{\partial t}(t,\xi) &= \frac{\partial^2 v}{\partial \xi^2}(t,\xi), \ (0<\xi<1), \\ v(t,0) &= 0, \ v(t,1) = u(t), \\ v(t_0,\xi) &= v_0(\xi), (0 \le \xi \le 1). \end{aligned} \tag{37}$$

The system has unknown variable $v(t,\xi)$ of the temperature on spatial domain $\Omega = [0,1]$, and the boundary input $u(t)$ is heat per unit time. For the system of a heat conduction rod, we apply the Galerkin approximation using frequency response profiles as trial functions. A transfer function from u to $v(\cdot,\xi)$ is

$$G(s,\xi) = \frac{\sinh(\sqrt{s}\xi)}{\sinh\sqrt{s}}. \tag{38}$$

The candidate trial functions $\chi_i(\xi)$ is chosen to be

$$\chi_{2i-1}(\xi) = \mathrm{Im}\,[G(j\omega_i,\xi)], \quad \chi_{2i}(\xi) = \mathrm{Re}\,[G(j\omega_i,\xi)]$$

for $i = 1, 2$. Suppose a sinusoidal function $u(t) = \sin\omega_i t$ is applied to the system as input, then we have in steady state, $v(t,\xi) = \chi_{2i-1}(\xi)$ for $t = (2\pi k + \pi/2)/\omega_i (k = 0,1,2,\ldots)$ and $v(t,\xi) = \chi_{2i}(\xi)$ for $t = (2\pi k)/\omega_i (k = 0,1,2,\ldots)$. Choosing such trial functions is, in a sense, equivalent to considering the interpolatory projection method (Gugercin et al. (2013)). That is, the transfer function of the approximate model $G_n(s,\xi)$ shall coincide with $G(s,\xi)$ at $s = \jmath\omega_i (i = 1.2)$.

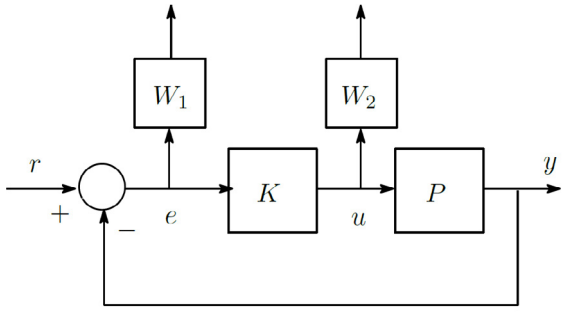

Fig. 5. Block diagram of the control system

Here $\omega_1 = 1$ rad/s and $\omega_2 = 40$ rad/s. The number of candidate trial functions is $\nu = 4$, and they are plotted in Fig. 1.

For $X = [\chi_1\ \chi_2\ \chi_3\ \chi_4]^T$, a Gramian is computed using numerical integration as in

$$\left\langle X, X^T \right\rangle = \left[\int_0^1 \chi_i(\xi)\chi_j(\xi)d\xi\right]_{ij} =$$

$$\begin{bmatrix} 2.07\times10^{-3} & -2.22\times10^{-2} & 5.49\times10^{-3} & -2.16\times10^{-3} \\ -2.22\times10^{-4} & 0.329 & -8.63\times10^{-2} & 0.112 \\ 5.49\times10^{-3} & -8.63\times10^{-2} & 2.78\times10^{-2} & -2.79\times10^{-2} \\ -2.16\times10^{-3} & 0.112 & -2.79\times10^{-2} & 8.39\times10^{-2} \end{bmatrix}.$$

Since rank $\left\langle X, X^T \right\rangle = 4$, we see X is linearly independent. Then we have

$$\begin{aligned} \Gamma X^T &= \begin{bmatrix} \chi_1(0) & \chi_2(0) & \chi_3(0) & \chi_4(0) \\ \chi_1(1) & \chi_2(1) & \chi_3(1) & \chi_4(1) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \end{aligned}$$

so basis of the null space of $\left\langle (\Gamma X^T)^T, \Gamma X^T \right\rangle_U = (\Gamma X^T)^T$. ΓX^T is readily obtained by numerical analysis(e.g., `null` function in MATLAB), and we have

$$T_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}.$$

We see the order of the approximate model is $n = \dim[\mathrm{Ker}\,(\Gamma X^T)] = 3$. To obtain a basis of the orthogonal compliment $Y^\perp$ of $Y = H_\nu \cap \mathrm{Ker}\,\Gamma$ on $H_\nu = \mathrm{span}\,X$, a basis of the null space of T_0 is computed as in

$$T_1 = \begin{bmatrix} 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}.$$

Using

$$\left\langle (\Gamma X^T)^T, \mathcal{F} \right\rangle_U = (\Gamma X^T)^T \mathcal{F} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix},$$

$\bar{E}$, $\bar{A}$, and $\bar{B}$ in (27) are obtained and they are substituted into (17)-(19), and (22) to yield a state-space model.

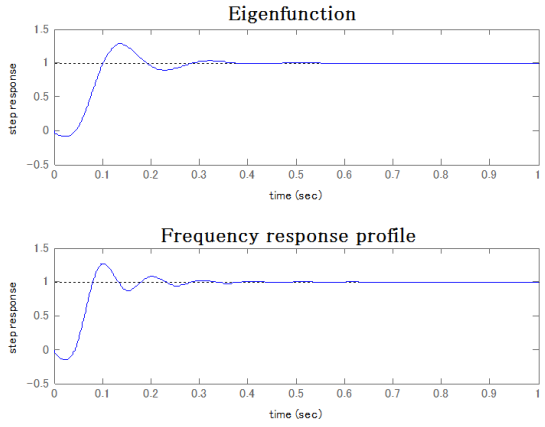

Fig. 6. Step response of the control system designed based on the approximate model using frequency response profiles ($n = 3$)

A bode plot of the approximate model is depicted in Fig. 2 where $G_3(s, \xi) = D(\xi) + C(\xi)(sI - A)^{-1}B$ and $\xi = 4/5$. The additive error is plotted in Fig. 3. We see errors in high frequencies are rather large. It is the known issue in the model reduction of descriptor systems by interpolatory projection methods(Gugercin et al. (2013)).

We can determine a real rational function

$$W_2(s) = 3.3 \times 10^{-5} \cdot \frac{1 + s/0.1}{1 + s/250} \tag{39}$$

that covers the additive error as in

$$|G(j\omega, 4/5) - G_n(j\omega, 4/5)| \leq |W_2(j\omega)|. \tag{40}$$

We denote by $P(s)$, the approximate transfer function $G_3(s, 4/5)$ from boundary input u to temperature $y = v(\cdot, \xi)$ at $\xi = 4/5$. We have a performance specification $\|W_1 S\|_\infty < 1$ for the sensitivity function

$$S(s) = \frac{1}{1 + P(s)K(s)} \tag{41}$$

where a frequency weight $W_1(s)$ is chosen as

$$W_1(s) = \frac{1/b + s/10a}{1 + s/a}. \tag{42}$$

Here a is an frequency upper bound of sensitivity reduction, b the upper bound of the sensitivity magnitude in low frequencies ($b = 0.005$), and sensitivity in high frequency is bounded by 10. Frequency range where sensitivity reduction is guaranteed (i.e., $|W_1(j\omega)| \geq 1$) is $\omega \leq a/b$. The maximum a is 0.085 such that stabilizing H_∞-controller K exists so that the H_∞ norm of the 1-input 2-output control system in Fig. 5 is less than 1. Step response in this case is plotted in Fig. 6.

For comparison, the result is presented for the case where eigenfunctions are employed as candidate trial functions. Choosing $\chi_1(\xi) = \xi$, $\chi_2(\xi) = \sin \pi\xi$, $\chi_3(\xi) = \sin 2\pi\xi$, and $\chi_4(\xi) = \sin 3\pi\xi$, then

$$\Gamma X^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$T_0 = \begin{bmatrix} 0 & -0.27 & 0.535 & -0.802 \\ 0 & 0.534 & 0.775 & 0.338 \\ 0 & -0.802 & 0.338 & 0.498 \end{bmatrix}, \quad T_1 = [1\ 0\ 0\ 0]$$

so an approximate model of order $n = 3$ is obtained. Additive error of the model is plotted in Fig. 4. In this case, a real rational function that covers the additive error is

$$W_2(s) = 5.5 \times 10^{-5} \cdot \frac{1 + s/0.1}{1 + s/130}. \tag{43}$$

And we only have $a = 0.076$ in similar design as in frequency response profiles, and the performance is a bit inferior to the previous design, as we can compare the step responses shown in Fig. 5.

5. CONCLUSION

In this paper, we proposed an implementation scheme of the Galerkin method for a class of boundary control systems. In conventional schemes, previous requirement for trial functions can be somewhat severe and computation of approximate modeling tends to be inconvenient. Feasibility and convenience of the method were demonstrated by a brief example of a heat conduction rod with Dirichlet boundary input.

REFERENCES

Curtain, R. (2003). Model reduction for control design for distributed parameter systems. In R. Smith and M. Demetriou (eds.), *Research Directions in Distributed Parameter Systems*, volume 21, 95–121. SIAM.

Dai, L. (1989). *Singular Control Systems*, volume 118 of *Lecture notes in control and information sciences*. Springer.

Finlayson, B. (1972). *The Method of Weighted Residuals and Variational Principles*. Academic Press.

Graham, W., Peraire, J., and Tang, K. (1999). Optimal control of vortex shedding using low-order models: Part 1 - open loop model development. *International Journal for Numerical Methods in Engineering*, 44, 945–972.

Gugercin, S., Stykel, T., and Wyatt, S. (2013). Model reduction of descriptor systems by interpoolatory projection methods. *SIAM Journal on Scientific Computing*, 35(5), B1010–B1033.

Holmes, P., Lumley, J., and Berkooz, G. (1996). *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*. Cambridge University Press.

Kunisch, K. and Volkwein, S. (2001). Galerkin proper orthogonal decomposition methods for parabolic problems. *Numerische Mathematik*, 90, 117–148.

Reis, T. (2006). *Systems Theoretic Aspects of PDAEs and Applications to Electrical Circuits*. Shaker, Aachen, Germany.

Sagara, S. and Imai, J. (1991). On an approximation model via galerkin method for a parabolic distributed parameter system with boundary inputs. *Transactions of the Society of Instrument and Control Engineers*, 27(12), 1366–1373. In Japanese.

Shiotsuki, T. (2011). *Analysis of Linear Systems*. Corona publishing. In Japanese.