Canonical and n-canonical modules of a Noetherian algebra

MITSUYASU HASHIMOTO*

Department of Mathematics, Okayama University
Okayama 700–8530, JAPAN
mh@okayama-u.ac.jp

Dedicated to Professor Shiro Goto on the occasion of his 70th birthday

Abstract

We define canonical and n-canonical modules on a module-finite algebra over a Noether commutative ring and study their basic properties. Using n-canonical modules, we generalize a theorem on (n, C)-syzygy by Araya and Iima which generalize a well-known theorem on syzygies by Evans and Griffith. Among others, we prove a non-commutative version of Aoyama's theorem which states that a canonical module descends with respect to a flat local homomorphism.

Contents

*This work was supported by JSPS KAKENHI Grant Number 26400045.

2010 *Mathematics Subject Classification.* Primary 16E05; Secondary 16E65. Key Words and Phrases. canonical module; n-canonical module; (n, C)-syzygy; n-C-spherical approximation.

1. Introduction

(1.1) In [EvG], Evans and Griffith proved a criterion of a finite module over a Noetherian commutative ring R to be an nth syzygy. This was generalized to a theorem on (n, C)-syzygy for a semidualizing module C over R by Araya and Iima [ArI]. The main purpose of this paper is to prove a generalization of these results in the following settings: the ring R is now a finite R-algebra Λ, which need not be commutative; and C is an n-canonical module.

(1.2) The notion of n-canonical module was introduced in [Has] in an algebro-geometric situation for commutative rings. The criterion for a module to be an nth syzygy for $n = 1, 2$ by Evans–Griffith was generalized using n-canonical modules there, and the standard 'codimension-two argument' (see e.g., [Hart2, (1.12)]) was also generalized to a theorem on schemes with 2-canonical modules [Has, (7.34)].

(1.3) Let $(R, \mathfrak{m})$ be a complete semilocal Noetherian ring, and $\Lambda \neq 0$ a module-finite R-algebra. Let $\mathbb{I}$ be a dualizing complex of R. Then $\mathbf{R}\mathrm{Hom}_R(\Lambda, \mathbb{I})$ is a dualizing complex of Λ. Its lowest non-vanishing cohomology is denoted by K_Λ, and is called the canonical module of Λ. If $(R, \mathfrak{m})$ is semilocal but not complete, then a Λ-bimodule is called a canonical module if it is the canonical module after completion. An n-canonical module is defined using the canonical module. A finite right (resp. left, bi-)module C of Λ is said to be n-canonical over R if (1) C satisfies Serre's (S'_n) condition as an R-module, that is, for any $P \in \operatorname{Spec} R$, $\operatorname{depth}_{R_P} C_P \geq \min(n, \dim R_P)$. (2) If $P \in \operatorname{Supp}_R C$ with $\dim R_P < n$, then $\widehat{C_P}$ is isomorphic to $K_{\widehat{\Lambda_P}}$ as a right (left, bi-) module of $\widehat{\Lambda_P}$, where $\widehat{\Lambda_P}$ is the PR_P-adic completion of Λ_P.

(1.4) In order to study non-commutative n-canonical modules, we study a non-commutative analogue of the theory of canonical modules developed by Aoyama [Aoy], Aoyama–Goto [AoyG], and Ogoma [Ogo] in commutative algebra. Among them, we prove an analogue of Aoyama's theorem [Aoy] which states that the canonical module descends with respect to flat homomorphisms (Theorem 7.5).

(1.5) Our main theorem is the following.

Theorem 1.6 (Theorem 8.4, cf. [EvG, (3.8)], [ArI, (3.1)])**.** *Let R be a Noetherian commutative ring, and Λ a module-finite R-algebra, which need not be commutative. Let $n \geq 1$, and C be a right n-canonical Λ-module. Set $\Gamma = \operatorname{End}_{\Lambda^{\mathrm{op}}} C$. Let $M \in \operatorname{mod} C$. Then the following are equivalent.*

1 *M is (n, C)-TF.*

2 *M has an (n, C)-universal pushforward.*

3 *M is an (n, C)-syzygy.*

4 *M satisfies the (S'_n) condition as an R-module, and $\operatorname{Supp}_R M \subset \operatorname{Supp}_R C$.*

Here we say that M has an (n, C)-universal pushforward if there is an exact sequence

$$0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1}$$

such that C^i lies in $\operatorname{add} C$ and the sequence is still exact after applying $(?)^\dagger = \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(?, C)$.

(1.7) The (n, C)-TF condition is a modified version of Takahashi's n-C-torsion freeness [Tak]. Under the assumptions of the theorem, let $(?)^\dagger = \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(?, C)$, $\Gamma = \operatorname{End}_{\Lambda^{\mathrm{op}}} C$, and $(?)^\ddagger = \operatorname{Hom}_\Gamma(?, C)$. We say that M is $(1, C)$-TF (resp. M is $(2, C)$-TF) if the canonical map $\lambda_M : M \to M^{\dagger\ddagger}$ is injective (resp. bijective). If $n \geq 3$, we say that M is (n, C)-TF if M is $(2, C)$-TF and $\operatorname{Ext}^i_\Gamma(M^\dagger, C) = 0$ for $1 \leq i \leq n - 2$, see Definition 4.5. Even if Λ is a commutative ring, a non-commutative ring Γ appears in a natural way, so even in this case, the definition is slightly different from Takahashi's original one. We prove that $\operatorname{TF}(n, C)$, the class of modules which satisfy (n, C)-TF property, and $\operatorname{UP}(n, C)$, the class of modules having (n, C)-universal pushforwards are equal in general (Lemma 4.7). This is a modified version of Takahashi's result [Tak, (3.2)].

(1.8) As an application of the main theorem, we formulate and prove a different form of the existence of n-C-spherical approximations by Takahashi [Tak], using n-canonical modules, see Corollary 8.5 and Corollary 8.6. Our results are not strong enough to deduce [Tak, Corollary 5.8] in commutative case. For related categorical results, see below.

(1.9) Section 2 is preliminaries on the depth and Serre's conditions on modules. In Section 3, we discuss $\mathcal{X}_{n,m}$-approximation, which is a categorical abstraction of approximations of modules appeared in [Tak]. Everything is done categorically here, and Theorem 3.16 is an abstraction of [Tak, (3.5)], in view of the fact that $\mathrm{TF}(n, C) = \mathrm{UP}(n, C)$ in general (Lemma 4.7). In Section 4, we discuss (n, C)-TF property, and prove Lemma 4.7 and related lemmas. In Section 5, we define the canonical module of a module-finite algebra Λ over a Noetherian commutative ring R, and prove some basic properties. In Section 6, we define the n-canonical module of Λ, and prove some basic properties, generalizing some constructions and results in [Has, Section 7]. In Section 7, we prove a non-commutative version of Aoyama's theorem which says that the canonical module descends with respect to flat local homomorphisms (Theorem 7.5). As a corollary, as in the commutative case, we immediately have that a localization of a canonical module is again a canonical module. This is important in Section 8. In Section 8, we prove Theorem 8.4, and the related results on n-C-spherical approximations (Corollary 8.5, Corollary 8.6) as its corollaries. Before these, we prove non-commutative analogues of the theorems of Schenzel and Aoyama–Goto [AoyG, (2.2), (2.3)] on the Cohen–Macaulayness of the canonical module (Proposition 8.2 and Corollary 8.3).

(1.10) Acknowledgments: Special thanks are due to Professor Osamu Iyama for valuable advice and discussion. Special thanks are also due to Professor Tokuji Araya. This work was motivated by his advice, and Proposition 8.2 is an outcome of discussion with him.

The author is also grateful to Professor Kei-ichiro Iima, Professor Takesi Kawasaki, Professor Shunsuke Takagi, Professor Ryo Takahashi, Professor Kohji Yanagawa, and Professor Yuji Yoshino for valuable advice.

2. Preliminaries

(2.1) Unless otherwise specified, a module means a left module. Let B be a ring. Hom_B or Ext_B mean the Hom or Ext for left B-modules. B^{op} denotes the opposite ring of B, so a B^{op}-module is nothing but a right B-module. Let

$B\,\mathrm{Mod}$ denote the category of B-modules. $B^{\mathrm{op}}\,\mathrm{Mod}$ is also denoted by $\mathrm{Mod}\,B$. For a left (resp. right) Noetherian ring B, $B\,\mathrm{mod}$ (resp. $\mathrm{mod}\,B$) denotes the full subcategory of $B\,\mathrm{Mod}$ (resp. $\mathrm{Mod}\,B$) consisting of finitely generated left (resp. right) B-modules.

(2.2) For derived categories, we employ standard notation found in [Hart].

For an abelian category $\mathcal{A}$, $D(\mathcal{A})$ denotes the unbounded derived category of $\mathcal{A}$. For a plump subcategory (that is, a full subcategory which is closed under kernels, cokernels, and extensions) $\mathcal{B}$ of $\mathcal{A}$, $D_{\mathcal{B}}(\mathcal{A})$ denotes the triangulated subcategory of $D(\mathcal{A})$ consisting of objects $\mathbb{F}$ such that $H^i(\mathbb{F}) \in \mathcal{B}$ for any i. For a ring B, We denote $D(B\,\mathrm{Mod})$ by $D(B)$, and $D_{B\,\mathrm{mod}}(B\,\mathrm{Mod})$ by $D_{\mathrm{fg}}(B)$ (if B is left Noetherian).

(2.3) Throughout the paper, let R denote a commutative Noetherian ring. If R is semilocal (resp. local) and $\mathfrak{m}$ its Jacobson radical, then we say that $(R,\mathfrak{m})$ is semilocal (resp. local). We say that $(R,\mathfrak{m},k)$ is semilocal (resp. local) if $(R,\mathfrak{m})$ is semilocal (resp. local) and $k = R/\mathfrak{m}$.

(2.4) We set $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$ and consider that $-\infty < \mathbb{R} < \infty$. As a convention, for a subset Γ of $\hat{\mathbb{R}}$, $\inf \Gamma$ means $\inf(\Gamma \cup \{\infty\})$, which exists uniquely as an element of $\hat{\mathbb{R}}$. Similarly for sup.

(2.5) For an ideal I of R and $M \in \mathrm{mod}\,R$, we define

$$\mathrm{depth}_R(I, M) := \inf\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_R(R/I, M) \neq 0\},$$

and call it the I-depth of M [Mat, section 16]. It is also called the M-grade of I [BS, (6.2.4)]. When $(R,\mathfrak{m})$ is semilocal, we denote $\mathrm{depth}(\mathfrak{m}, M)$ by $\mathrm{depth}_R M$ or $\mathrm{depth}\, M$, and call it the depth of M.

(2.6) For a subset F of $X = \mathrm{Spec}\, R$, we define $\mathrm{codim}\, F = \mathrm{codim}_X F$, the *codimension* of F in X, by $\inf\{\mathrm{ht}\, P \mid P \in F\}$. So $\mathrm{ht}\, I = \mathrm{codim}\, V(I)$ for an ideal I of R. For $M \in \mathrm{mod}\,R$, we define $\mathrm{codim}\, M := \mathrm{codim}\,\mathrm{Supp}_R M = \mathrm{ht}\,\mathrm{ann}\, M$, where ann denotes the annihilator. For $n \geq 0$, we denote the set $\mathrm{ht}^{-1}(n) = \{P \in \mathrm{Spec}\, R \mid \mathrm{ht}\, P = n\}$ by $R^{\langle n\rangle}$. For a subset Γ of $\mathbb{Z}$, $R^{\langle\Gamma\rangle}$ means $\mathrm{ht}^{-1}(\Gamma) = \bigcup_{n\in\Gamma} R^{\langle n\rangle}$. Moreover, we use notation such as $R^{\langle\leq 3\rangle}$, which stands for $R^{\langle\{n\in\mathbb{Z}\mid n\leq 3\}\rangle}$, the set of primes of height at most 3. For $M \in \mathrm{mod}\,R$, the set of minimal primes of M is denoted by $\mathrm{Min}\, M$.

We define $M^{[n]} := \{P \in \mathrm{Spec}\, R \mid \mathrm{depth}\, M_P = n\}$. Similarly, we use notation such as $M^{[<n]}(= \{P \in \mathrm{Spec}\, R \mid \mathrm{depth}\, M_P < n\})$.

(2.7) Let $M, N \in \operatorname{mod} R$. We say that M satisfies the $(S_n^N)^R$-condition or (S_n^N)-condition if for any $P \in \operatorname{Spec} R$, $\operatorname{depth}_{R_P} M_P \geq \min(n, \dim_{R_P} N_P)$. The $(S_n^R)^R$-condition or (S_n^R)-condition is simply denoted by $(S_n')^R$ or (S_n'). We say that M satisfies the $(S_n)^R$-condition or (S_n)-condition if M satisfies the (S_n^M)-condition. (S_n) (resp. (S_n')) is equivalent to say that for any $P \in M^{[<n]}$, M_P is a Cohen–Macaulay (resp. maximal Cohen–Macaulay) R_P-module. That is, $\operatorname{depth} M_P = \dim M_P$ (resp. $\operatorname{depth} M_P = \dim R_P$). We consider that $(S_n^N)^R$ is a class of modules, and also write $M \in (S_n^N)^R$ (or $M \in (S_n^N)$).

Lemma 2.8. *Let $0 \to L \to M \to N \to 0$ be an exact sequence in* $\operatorname{mod} R$*, and $n \geq 1$.*

1 *If L and N satisfy (S_n'), then M satisfies (S_n').*

2 *If N satisfies (S_{n-1}') and M satisfies (S_n'), then L satisfies (S_n').*

Proof. **1** follows from the depth lemma:

$$\forall P \quad \operatorname{depth}_{R_P} M_P \geq \min(\operatorname{depth}_{R_P} L_P, \operatorname{depth}_{R_P} N_P),$$

and the fact that maximal Cohen–Macaulay modules are closed under extensions. **2** is similar. □

Corollary 2.9. *Let*

$$0 \to M \to L_n \to \cdots \to L_1$$

be an exact sequence in $\operatorname{mod} R$*, and assume that L_i satisfies (S_i') for $1 \leq i \leq n$. Then M satisfies (S_n').*

Proof. This is proved using Lemma 2.8.**2** repeatedly. □

Lemma 2.10 (cf. [IW, (3.4)])**.** *Let*

$$\mathbb{L} : 0 \to L_s \xrightarrow{\partial_s} L_{s-1} \xrightarrow{\partial_{s-1}} \cdots \to L_1 \xrightarrow{\partial_1} L_0$$

be a complex in $\operatorname{mod} R$ *such that*

1 *For each $i \in \mathbb{Z}$ with $1 \leq i \leq s$, $L_i \in (S_i')$.*

2 *For each $i \in \mathbb{Z}$ with $1 \leq i \leq s$,* $\operatorname{codim} H_i(\mathbb{L}) \geq s - i + 1$.

Then $\mathbb{L}$ is acyclic.

Proof. Using induction on s, we may assume that $H_i(\mathbb{L}) = 0$ for $i > 1$. Assume that $\mathbb{L}$ is not acyclic. Then $H_1(\mathbb{L}) \neq 0$, and we can take $P \in \operatorname{Ass}_R H_1(\mathbb{L})$. By assumption, $\operatorname{ht} P \geq s$. Now localize at P and considering the complex $\mathbb{L}_P$ over R_P, we get a contradiction by Acyclicity Lemma [PS, (1.8)]. □

Example 2.11. Let $f : M \to N$ be a map in $\operatorname{mod} R$.

1 If $M \in (S'_1)$ and f_P is injective for $P \in \operatorname{Min} R$, then f is injective. To prove this, consider the complex

$$0 \to M \xrightarrow{f} N = L_0$$

and apply Lemma 2.10.

2 ([LeW, (5.11)]) If $M \in (S'_2)$, $N \in (S'_1)$, and f_P is bijective for primes P of R of height at most one, then f is bijective. Consider the complex

$$0 \to M \xrightarrow{f} N \to 0 = L_0$$

this time.

Lemma 2.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and $N \in \operatorname{mod} R$. Assume that N satisfies the (S_n) condition. If $P \in \operatorname{Min} N$ with $\dim R/P < n$, then we have*

$$\dim R/P = \operatorname{depth} N = \dim N < n.$$

If, moreover, N satisfies (S'_n), then $\operatorname{depth} N = \dim R$.

Proof. Ischebeck proved that if $M, N \in \operatorname{mod} R$ and $i < \operatorname{depth} N - \dim M$, then $\operatorname{Ext}^i_R(M, N) = 0$ [Mat, (17.1)]. As $\operatorname{Ext}^0_R(R/P, N) \neq 0$, we have that $\operatorname{depth}_R N \leq \dim R/P < n$. The rest is easy. □

Corollary 2.13. *Let $n \geq 1$, and M and N be finite R-modules. Assume that M satisfies the (S_n) condition and N satisfies the (S'_n) condition. If $\operatorname{Min} M \subset \operatorname{Min} N$, then M satisfies the (S'_n) condition.*

Proof. Let $P \in M^{[<n]}$. As M satisfies (S_n), $\operatorname{depth} M_P = \dim M_P$. Take $Q \in \operatorname{Min} M$ such that $Q \subset P$ and $\dim R_P/QR_P = \dim M_P < n$. As $\operatorname{Min} M \subset \operatorname{Min} N$, we have that $QR_P \in \operatorname{Min} N_P$. By Lemma 2.12, $\dim R_P = \dim R_P/QR_P = \operatorname{depth} M_P$, and hence M satisfies (S'_n). □

Corollary 2.14. *Let $n \geq 1$, and assume that R satisfies the (S_n) condition. Then for any finite R-module M, M satisfies the (S'_n) condition if and only if M satisfies both (S_n) and (S'_1).*

Proof. Obviously, if M satisfies the (S'_n) condition, then it satisfies both (S_n) and (S'_1). For the converse, apply Corollary 2.13 for $N = R$. □

(2.15) Let $M, N \in \operatorname{mod} R$. We say that M satisfies the $(S'_n)_N$-condition, or $M \in (S'_n)_N = (S'_n)^R_N$, if $M \in (S'_n)$ and $\operatorname{Supp}_R M \subset \operatorname{Supp}_R N$.

Lemma 2.16. *Let $n \geq 1$, and M and N be finite R-modules. Assume that N satisfies (S'_n). Then the following are equivalent.*

1 *M satisfies $(S'_n)_N$.*

2 *M satisfies (S_n) and $\operatorname{Min} M \subset \operatorname{Min} N$.*

Proof. **1⇒2**. As (S'_n) implies (S_n), we have that M satisfies (S_n). As M satisfies (S'_n) with $n \geq 1$, we have that $\operatorname{Min} M \subset \operatorname{Min} R$. By assumption, $\operatorname{Min} M \subset \operatorname{Supp} N$. So $\operatorname{Min} M \subset \operatorname{Min} R \cap \operatorname{Supp} N \subset \operatorname{Min} N$.

2⇒1. M satisfies (S'_n) by Corollary 2.13. $\operatorname{Supp} M \subset \operatorname{Supp} N$ follows from $\operatorname{Min} M \subset \operatorname{Min} N$. □

(2.17) There is another case that (S_n) implies (S'_n). An R-module N is said to be *full* if $\operatorname{Supp}_R N = \operatorname{Spec} R$. A finitely generated faithful R-module is full.

Lemma 2.18. *Let $M, N \in \operatorname{mod} R$. If N is a full R-module, then M satisfies the (S'_n) condition if and only if M satisfies the (S^N_n) condition. If $\operatorname{ann}_R N \subset \operatorname{ann}_R M$, then M satisfies the $(S^N_n)^R$ condition if and only if M satisfies the $(S'_n)^{R/\operatorname{ann}_R N}$ condition.*

Proof. Left to the reader. □

Lemma 2.19. *Let I be an ideal of R, and S a module-finite commutative R-algebra. For a finite S-module M, we have that $\operatorname{depth}_R(I, M) = \operatorname{depth}_S(IS, M)$. In particular, if R is semilocal, then $\operatorname{depth}_R M = \operatorname{depth}_S M$.*

Proof. Note that $H^i_I(M) \cong H^i_{IS}(M)$ by [BS, (4.2.1)]. By [BS, (6.2.7)], we get the lemma immediately. □

Lemma 2.20. *Let $\varphi : R \to S$ be a finite homomorphism of rings, M a finite S-module, and $n \geq 0$.*

1 *If M satisfies (S'_n) as an R-module, then it satisfies (S'_n) as an S-module.*

2 *Assume that for any $Q \in \operatorname{Min} S$, $\varphi^{-1}(Q) \in \operatorname{Min} R$ (e.g., S satisfies (S'_1) as an R-module). If M satisfies (S'_n) as an S-module and R_P is quasi-unmixed for any prime P of R with $\operatorname{depth} R_P < n$, then M satisfies (S'_n) as an R-module.*

Proof. We only prove **2**. Let $P \in \operatorname{Spec} R$, and $\operatorname{depth}_{R_P} M_P < n$. Then by Lemma 2.19 and [BS, (6.2.7)], there exists some $Q \in \operatorname{Spec} S$ such that $\varphi^{-1}(Q) = P$ and

$$\operatorname{depth}_{S_Q} M_Q = \inf_{\varphi^{-1}(Q')=P} \operatorname{depth}_{S_{Q'}} M_{Q'} = \operatorname{depth}_{S_P} M_P = \operatorname{depth}_{R_P} M_P < n.$$

Then $\operatorname{ht} Q = \operatorname{depth}_{R_P} M_P$. So it suffices to show $\operatorname{ht} P = \operatorname{ht} Q$. By assumption, R_P is quasi-unmixed. So R_P is equi-dimensional and universally catenary [Mat, (31.6)]. By [Gro, (13.3.6)], $\operatorname{ht} P = \operatorname{ht} Q$, as desired. □

(2.21) We say that R satisfies (R_n) (resp. (T_n)) if R_P is regular (resp. Gorenstein) for any prime P of R with $\operatorname{ht} P \leq n$.

3. $\mathcal{X}_{n,m}$-approximation

(3.1) Let $\mathcal{A}$ be an abelian category, and $\mathcal{C}$ its additive subcategory closed under direct summands. Let $n \geq 0$. We define

$${}^{\perp_n}\mathcal{C} := \{a \in \mathcal{A} \mid \operatorname{Ext}^i_{\mathcal{A}}(a, c) = 0 \quad 1 \leq i \leq n\}.$$

Let $a \in \mathcal{A}$. A sequence

$$\mathbb{C} : 0 \to a \to c^0 \to c^1 \to \cdots \to c^{n-1} \tag{1}$$

is said to be an $(n, \mathcal{C})$-pushforward if it is exact with $c^i \in \mathcal{C}$. If in addition,

$$\mathbb{C}^\dagger : 0 \leftarrow a^\dagger \leftarrow (c^0)^\dagger \leftarrow (c^1)^\dagger \leftarrow \cdots \leftarrow (c^{n-1})^\dagger$$

is exact for any $c \in \mathcal{C}$, where $(?)^\dagger = \operatorname{Hom}_{\mathcal{A}}(?, c)$, we say that $\mathbb{C}$ is a universal $(n, \mathcal{C})$-pushforward.

If $a \in \mathcal{A}$ has an $(n, \mathcal{C})$-pushforward, we say that a is an $(n, \mathcal{C})$-syzygy, and we write $a \in \operatorname{Syz}(n, \mathcal{C})$. If $a \in \mathcal{A}$ has a universal $(n, \mathcal{C})$-pushforward, we say that $a \in \operatorname{UP}_{\mathcal{A}}(n, \mathcal{C}) = \operatorname{UP}(n, \mathcal{C})$. Obviously, $\operatorname{UP}_{\mathcal{A}}(n, \mathcal{C}) \subset \operatorname{Syz}_{\mathcal{A}}(n, \mathcal{C})$.

(3.2) We write $\mathcal{X}_{n,m}(\mathcal{C}) = \mathcal{X}_{n,m} := {}^{\perp_n}\mathcal{C} \cap \operatorname{UP}(m, \mathcal{C})$ for $n, m \geq 0$. Also, for $a \neq 0$, we define

$$\mathcal{C}\dim a = \inf\{m \in \mathbb{Z}_{\geq 0} \mid \text{there is a resolution } 0 \to c_m \to c_{m-1} \to \cdots \to c_0 \to a \to 0\}.$$

We define $\mathcal{C}\dim 0 = -\infty$. We define $\mathcal{Y}_n(\mathcal{C}) = \mathcal{Y}_n := \{a \in \mathcal{A} \mid \mathcal{C}\dim a < n\}$. A sequence $\mathbb{E}$ is said to be $\mathcal{C}$-exact if it is exact, and $\mathcal{A}(\mathbb{E}, c)$ is also exact for each $c \in \mathcal{C}$. Letting a $\mathcal{C}$-exact sequence an exact sequence, $\mathcal{A}$ is an exact category, which we denote by $\mathcal{A}_{\mathcal{C}}$ in order to distinguish it from the abelian category $\mathcal{A}$ (with the usual exact sequences).

(3.3) Let $\mathcal{C}_0 \subset \mathcal{A}$ be a subset. Then ${}^{\perp_n}\mathcal{C}_0$, $\mathrm{UP}(n, \mathcal{C}_0)$, $\mathcal{X}_{n,m}(\mathcal{C}_0)$, $\mathcal{C}_0\mathrm{dim}$, and $\mathcal{Y}_n(\mathcal{C}_0) = \mathcal{Y}_n$ mean ${}^{\perp_n}\mathcal{C}$, $\mathrm{UP}(n, \mathcal{C})$, $\mathcal{X}_{n,m}(\mathcal{C})$, $\mathcal{C}\mathrm{dim}$, and $\mathcal{Y}_n(\mathcal{C})$, respectively, where $\mathcal{C} = \mathrm{add}\, \mathcal{C}_0$, the smallest additive subcategory containing $\mathcal{C}_0$ and closed under direct summands. A $\mathcal{C}_0$-exact sequence means a $\mathcal{C}$-exact sequence. A sequence $\mathbb{E}$ in $\mathcal{A}$ is $\mathcal{C}_0$-exact if and only if it is exact, and for any $c \in \mathcal{C}_0$, $\mathcal{A}(\mathbb{E}, c)$ is exact. If $c \in \mathcal{A}$, ${}^{\perp_n}c$, $\mathrm{UP}(n, c)$ and so on mean ${}^{\perp_n}\mathrm{add}\, c$, $\mathrm{UP}(n, \mathrm{add}\, c)$ and so on.

(3.4) By definition, any object of $\mathcal{C}$ is an injective object in $\mathcal{A}_{\mathcal{C}}$.

(3.5) Let $\mathcal{E}$ be an exact category, and $\mathcal{I}$ an additive subcategory of $\mathcal{E}$. Then for $e \in \mathcal{E}$, we define

$$\mathrm{Push}_{\mathcal{E}}(n, \mathcal{I}) := \{e \in \mathcal{E} \mid \text{There exists an exact sequence } 0 \to e \to c^0 \to c^1 \to \cdots \to c^{n-1} \text{ with } c^i \in \mathcal{I}\}.$$

Note that $\mathrm{Push}_{\mathcal{E}}(0, \mathcal{I})$ is the whole $\mathcal{E}$. Thus $\mathrm{Push}_{\mathcal{A}_{\mathcal{C}}}(n, \mathcal{C}) = \mathrm{UP}_{\mathcal{A}}(n, \mathcal{C})$.

If $a \in \mathcal{E}$ is a direct summand of an object of $\mathcal{I}$, then a admits an exact sequence

$$0 \to a \to c^0 \to c^1 \to \cdots$$

with $c^i \in \mathcal{I}$, and hence $a \in \bigcap_{n \geq 0} \mathrm{Push}_{\mathcal{E}}(n, \mathcal{I})$.

Lemma 3.6. *Let $\mathcal{E}$ be an exact category. Let $\mathcal{I}$ be an additive subcategory of $\mathcal{E}$ consisting of injective objects. Let*

$$0 \to a \xrightarrow{f} a' \xrightarrow{g} a'' \to 0$$

be an exact sequence in $\mathcal{E}$ and $m \geq 0$. Then

1 *If $a \in \mathrm{Push}(m, \mathcal{I})$ and $a'' \in \mathrm{Push}(m, \mathcal{I})$, then $a' \in \mathrm{Push}(m, \mathcal{I})$.*

2 *If $a' \in \mathrm{Push}(m+1, \mathcal{I})$ and $a'' \in \mathrm{Push}(m, \mathcal{I})$, then $a \in \mathrm{Push}(m+1, \mathcal{I})$.*

3 *If $a \in \mathrm{Push}(m+1, \mathcal{I})$, $a' \in \mathrm{Push}(m, \mathcal{I})$, then $a'' \in \mathrm{Push}(m, \mathcal{I})$.*

Proof. Let $i : \mathcal{E} \hookrightarrow \mathcal{A}$ be the Gabriel–Quillen embedding [TT]. We consider that $\mathcal{E}$ is a full subcategory of $\mathcal{A}$ closed under extensions, and a sequence in $\mathcal{E}$ is exact if and only if it is so in $\mathcal{A}$.

We prove **1**. We use induction on m. The case that $m = 0$ is trivial, and so we assume that $m > 0$. Let

$$0 \to a \to c \to b \to 0$$

be an exact sequence such that $c \in \mathcal{I}$ and $b \in \mathrm{Push}(m-1, \mathcal{I})$. Let

$$0 \to a'' \to c'' \to b'' \to 0$$

be an exact sequence such that $c'' \in \mathcal{I}$ and $b'' \in \mathrm{Push}(m-1, \mathcal{I})$. As $\mathcal{C}(a', c) \to \mathcal{C}(a, c)$ is surjective, we can form a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & a & \xrightarrow{f} & a' & \xrightarrow{g} & a'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & c & \xrightarrow{\binom{1}{0}} & c \oplus c'' & \xrightarrow{(1\ 0)} & c'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & b & \longrightarrow & b' & \longrightarrow & b'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in $\mathcal{A}$. As $\mathcal{E}$ is closed under extensions in $\mathcal{A}$, this diagram is a diagram in $\mathcal{E}$. By induction hypothesis, $b' \in \mathrm{Push}(m-1, \mathcal{I})$. Hence $a' \in \mathrm{Push}(m, \mathcal{I})$.

We prove **2**. Let $0 \to a' \to c \to b' \to 0$ be an exact sequence in $\mathcal{E}$ such that $c \in \mathcal{I}$ and $b' \in \mathrm{Push}(m, \mathcal{I})$. Then we have a commutative diagram in $\mathcal{E}$ with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & a & \xrightarrow{f} & a' & \xrightarrow{g} & a'' \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & c & \xrightarrow{1_c} & c & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & a'' & \longrightarrow & b & \longrightarrow & b' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}.$$

Applying **1**, which we have already proved, $b \in \mathrm{Push}(m, \mathcal{I})$, since a'' and b' lie in $\mathrm{Push}(m, \mathcal{I})$. So $a \in \mathrm{Push}(m+1, \mathcal{I})$, as desired.

We prove **3**. Let $0 \to a \to c \to b \to 0$ be an exact sequence in $\mathcal{E}$ such that $c \in \mathcal{I}$ and $b \in \mathrm{Push}(m, \mathcal{I})$. Take the push-out diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & a & \xrightarrow{f} & a' & \xrightarrow{g} & a'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1_{a''}} & & \\
0 & \longrightarrow & c & \longrightarrow & u & \longrightarrow & a'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & b & \xrightarrow{1_b} & b & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
.
$$

Then $u \in \mathrm{Push}(m, \mathcal{I})$ by **1**, which we have already proved. Since $c \in \mathcal{I}$, the middle row splits. Then by the exact sequence $0 \to a'' \to u \to c \to 0$ and **2**, we have that $a'' \in \mathrm{Push}(m, \mathcal{I})$, as desired. □

Corollary 3.7. *Let $\mathcal{E}$ and $\mathcal{I}$ be as in Lemma 3.6. Let $m \geq 0$, and $a, a' \in \mathcal{E}$. Then $a \oplus a' \in \mathrm{Push}(m, \mathcal{I})$ if and only if $a, a' \in \mathrm{Push}(m, \mathcal{I})$.*

Proof. The 'if' part is obvious by Lemma 3.6.**1**, considering the exact sequence

$$0 \to a \to a \oplus a' \to a' \to 0. \tag{2}$$

We prove the 'only if' part by induction on m. If $m = 0$, then there is nothing to prove. Let $m > 0$. Then by induction hypothesis, $a' \in \mathrm{Push}(m - 1, \mathcal{I})$. Then applying Lemma 3.6.**2** to the exact sequence (2), we have that $a \in \mathrm{Push}(m, \mathcal{I})$. $a' \in \mathrm{Push}(m, \mathcal{I})$ is proved similarly. □

Corollary 3.8. *Let*

$$0 \to a \xrightarrow{f} a' \xrightarrow{g} a'' \to 0$$

be a $\mathcal{C}$-exact sequence in $\mathcal{A}$ and $m \geq 0$. Then

1 *If $a \in \mathrm{UP}(m, \mathcal{C})$ and $a'' \in \mathrm{UP}(m, \mathcal{C})$, then $a' \in \mathrm{UP}(m, \mathcal{C})$.*

2 *If $a' \in \mathrm{UP}(m + 1, \mathcal{C})$ and $a'' \in \mathrm{UP}(m, \mathcal{C})$, then $a \in \mathrm{UP}(m + 1, \mathcal{C})$.*

3 *If $a \in \mathrm{UP}(m + 1, \mathcal{C})$, $a' \in \mathrm{UP}(m, \mathcal{C})$, then $a'' \in \mathrm{UP}(m, \mathcal{C})$.*

□

(3.9) We define ${}^{\perp}\mathcal{C} = {}^{\perp_\infty}\mathcal{C} := \bigcap_{i \geq 0} {}^{\perp_i}\mathcal{C}$ and $\mathrm{UP}(\infty, \mathcal{C}) := \bigcap_{j \geq 0} \mathrm{UP}(j, \mathcal{C})$. Obviously, $\mathcal{C} \subset \mathrm{UP}(\infty, \mathcal{C})$.

Lemma 3.10. *We have*

$$\mathrm{UP}(\infty, \mathcal{C}) = \{a \in \mathcal{A} \mid \textit{There exists some } \mathcal{C}\textit{-exact sequence } 0 \to a \to c^0 \to c^1 \to c^2 \to \cdots \textit{ with } c^i \in \mathcal{C} \textit{ for } i \geq 0\}.$$

Proof. Let $a \in \mathrm{UP}(\infty, \mathcal{C})$, and take any $\mathcal{C}$-exact sequence

$$0 \to a \to c^0 \to a^1 \to 0$$

with $c^0 \in \mathcal{C}$. Then $a^1 \in \mathrm{UP}(\infty, \mathcal{C})$ by Corollary 3.8, and we can continue infinitely. □

(3.11) We define $\mathcal{Y}_\infty := \bigcup_{i \geq 0} \mathcal{Y}_i$. So $a \in \mathcal{Y}_\infty$ if and only if $\mathcal{C}\dim a < \infty$. We also define $\mathcal{X}_{i,j} := {}^{\perp_i}\mathcal{C} \cap \mathrm{UP}(j, \mathcal{C})$ for $0 \leq i, j \leq \infty$.

(3.12) Let $0 \leq i, j \leq \infty$. We say that $a \in \mathcal{A}$ lies in $\mathcal{Z}_{i,j}$ if there is a short exact sequence

$$0 \to y \to x \to a \to 0$$

in $\mathcal{A}$ such that $x \in \mathcal{X}_{i,j}$ and $y \in \mathcal{Y}_i$.

(3.13) We define $\infty \pm r = \infty$ for $r \in \mathbb{R}$.

Lemma 3.14. *Let $0 \leq i, j \leq \infty$ with $j \geq 1$. Assume that $\mathcal{C} \subset {}^{\perp_{i+1}}\mathcal{C}$ (that is, $\mathrm{Ext}^l_{\mathcal{A}}(c, c') = 0$ for $1 \leq l \leq i+1$ and $c, c' \in \mathcal{C}$). Let $0 \to z \xrightarrow{f} x \xrightarrow{g} z' \to 0$ be a short exact sequence in $\mathcal{A}$ with $z \in \mathcal{Z}_{i,j}$ and $x \in \mathcal{X}_{i+1,j-1}$. Then $z' \in \mathcal{Z}_{i+1,j-1}$.*

Proof. By assumption, there is an exact sequence

$$0 \to y \xrightarrow{\iota} x' \xrightarrow{\varphi} z \to 0$$

such that $\mathcal{C}\dim y < i$ and $x' \in \mathcal{X}_{i,j}$. As $j \geq 1$, there is an $\mathcal{C}$-exact sequence

$$0 \to x' \xrightarrow{h} c \to x''' \to 0$$

such that $c \in \mathcal{C}$. Then we have a commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & y & \xrightarrow{h\iota} & c & \longrightarrow & y' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \binom{1}{0}} & & \downarrow & & \\
0 & \longrightarrow & x' & \xrightarrow{\binom{h}{f\varphi}} & c \oplus x & \longrightarrow & x'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle (0\ 1)} & & \downarrow & & \\
0 & \longrightarrow & z & \xrightarrow{f} & x & \xrightarrow{g} & z' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

As the top row is exact, $y \in \mathcal{Y}_i$, and $c \in \mathcal{C}$, $y' \in \mathcal{Y}_{i+1}$. By assumption, $c \in \mathcal{X}_{i+1,\infty}$ and $x \in \mathcal{X}_{i+1,j-1}$. So $c \oplus x \in \mathcal{X}_{i+1,j-1}$. As the middle row is $\mathcal{C}$-exact and $x' \in \mathcal{X}_{i,j}$, we have that $x'' \in \mathcal{X}_{i+1,j-1}$ by Corollary 3.8. The right column shows that $z' \in \mathcal{Z}_{i+1,j-1}$, as desired. $\square$

Lemma 3.15. *Let $0 \leq i, j \leq \infty$, and assume that $i \geq 1$ and $\mathcal{C} \subset {}^{\perp_i}\mathcal{C}$. Let*

$$0 \to z \xrightarrow{f} x \xrightarrow{g} z' \to 0 \tag{3}$$

be a short exact sequence in $\mathcal{A}$ with $z' \in \mathcal{Z}_{i,j}$ and $x \in \mathcal{X}_{i,j+1}$. Then $z \in \mathcal{Z}_{i-1,j+1}$.

Proof. Take an exact sequence $0 \to y' \to x'' \xrightarrow{h} z' \to 0$ such that $x'' \in \mathcal{X}_{i,j}$ and $y' \in \mathcal{Y}_i$. Taking the pull-back of (3) by h, we get a commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & y' & \xrightarrow{1_{y'}} & y' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & z & \xrightarrow{j} & a & \longrightarrow & x'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_z} & & \downarrow & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & z & \xrightarrow{f} & x & \xrightarrow{g} & z' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
.
$$

By induction, we can prove easily that ${}^{\perp_i}\mathcal{C} \subset {}^{\perp_{i+1-l}}\mathcal{Y}_l$. In particular, ${}^{\perp_i}\mathcal{C} \subset {}^{\perp_1}\mathcal{Y}_i$, and $\operatorname{Ext}^1_{\mathcal{A}}(x, y') = 0$. Hence the middle column splits, and we can replace a by $x \oplus y'$. By the definition of $\mathcal{Y}_i$, there is an exact sequence

$$0 \to y \to c \to y' \to 0$$

of $\mathcal{A}$ such that $y \in \mathcal{Y}_{i-1}$ and $c \in \mathcal{C}$. Then adding 1_x to this sequence, we get

$$0 \to y \to x \oplus c \to x \oplus y' \to 0$$

is exact. Pulling back this exact sequence with $j : z \to a = x \oplus y'$, we get a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & . \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & y & \longrightarrow & x' & \longrightarrow & z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_y} & & \downarrow & & \downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & y & \longrightarrow & x \oplus c & \longrightarrow & x \oplus y' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & x'' & \xrightarrow{1_{x''}} & x'' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

As $x'' \in {}^{\perp_1}\mathcal{C}$, the middle column is $\mathcal{C}$-exact. As $x'' \in \mathcal{X}_{i,j}$ and $x \oplus c \in \mathcal{X}_{i,j+1}$, we have that $x' \in \mathcal{X}_{i-1,j+1}$. As the top row shows, $z \in \mathcal{Z}_{i-1,j+1}$, as desired. □

Theorem 3.16. *Let $0 \le n, m \le \infty$, and assume that $\mathcal{C} \subset {}^{\perp_n}\mathcal{C}$ (that is, $\operatorname{Ext}^l_{\mathcal{A}}(c, c') = 0$ for $1 \le l \le n$ and $c, c' \in \mathcal{C}$). For $z \in \mathcal{A}$, the following are equivalent.*

1 *$z \in \mathcal{Z}_{n,m}$.*

2 *There is an exact sequence*

$$0 \to x_n \xrightarrow{d_n} x_{n-1} \xrightarrow{d_{n-1}} x_0 \xrightarrow{\varepsilon} z \to 0 \tag{4}$$

such that $x_i \in \mathcal{X}_{n-i,m+i}$.

If, moreover, for each $a \in \mathcal{A}$, there is a surjection $x \to a$ with $x \in \mathcal{X}_{n,n+m}$, then these conditions are equivalent to the following.

3 *For each exact sequence* (4) *with $x_i \in \mathcal{X}_{n-i,m+i+1}$ for $0 \leq i \leq n-1$, we have that $x_n \in \mathcal{X}_{0,n+m}$.*

Proof. **1⇒2**. There is an exact sequence $0 \to y \to x_0 \xrightarrow{\varepsilon} z \to 0$ with $x_0 \in \mathcal{X}_{n,m}$ and $y \in \mathcal{Y}_n$. So there is an exact sequence

$$0 \to x_n \xrightarrow{d_n} x_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} x_1 \to y \to 0$$

with $x_i \in \mathcal{C}$ for $1 \leq i \leq n$. As $\mathcal{C} \subset \mathcal{X}_{n,\infty}$, we are done.

2⇒1. Let $z_i = \operatorname{Im} d_i$ for $i = 1, \ldots, n$, and $z_0 := z$. Then by descending induction on i, we can prove $z_i \in \mathcal{Z}_{n-i,m+i}$ for $i = n, n-1, \ldots, 0$, using Lemma 3.14 easily.

1⇒3 is also proved easily, using Lemma 3.15.

3⇒2 is trivial. □

4. (n, C)-TF property

(4.1) In the rest of this paper, let Λ be a module-finite R-algebra, which need not be commutative. A Λ-bimodule means a $\Lambda \otimes_R \Lambda^{\mathrm{op}}$-module. Let $C \in \operatorname{mod} \Lambda$ be fixed. Set $\Gamma := \operatorname{End}_{\Lambda^{\mathrm{op}}} C$. Note that Γ is also a module-finite R-algebra. We denote $(?)^\dagger := \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(?, C) : \operatorname{mod} \Lambda \to (\Gamma \operatorname{mod})^{\mathrm{op}}$, and $(?)^\ddagger := \operatorname{Hom}_\Gamma(?, C) : \Gamma \operatorname{mod} \to (\operatorname{mod} \Lambda)^{\mathrm{op}}$.

(4.2) We denote $\operatorname{Syz}_{\operatorname{mod}\Lambda}(n, C)$, $\operatorname{UP}_{\operatorname{mod}\Lambda}(n, C)$, and $C\operatorname{dim}_{\operatorname{mod}\Lambda} M$ respectively by $\operatorname{Syz}_{\Lambda^{\mathrm{op}}}(n, C)$, $\operatorname{UP}_{\Lambda^{\mathrm{op}}}(n, C)$, and $C\operatorname{dim}_{\Lambda^{\mathrm{op}}} M$.

(4.3) Note that for $M \in \operatorname{mod} \Lambda$ and $N \in \Gamma \operatorname{mod}$, we have standard isomorphisms

$$(5) \qquad \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(M, N^\ddagger) \cong \operatorname{Hom}_{\Gamma \otimes_R \Lambda^{\mathrm{op}}}(N \otimes_R M, C) \cong \operatorname{Hom}_\Gamma(N, M^\dagger).$$

The first isomorphism sends $f : M \to N^\ddagger$ to the map $(n \otimes m \mapsto f(m)(n))$. Its inverse is given by $g : N \otimes_R M \to C$ to $(m \mapsto (n \mapsto g(n \otimes m)))$. This shows that $(?)^\dagger$ has $((?)^\ddagger)^{\mathrm{op}} : (\Gamma \operatorname{mod})^{\mathrm{op}} \to \operatorname{mod} \Lambda$ as a right adjoint. Hence $((?)^\dagger)^{\mathrm{op}}$ is right adjoint to $(?)^\ddagger$. We denote the unit of adjunction $\mathrm{Id} \to (?)^{\dagger\ddagger} = (?)^\ddagger(?)^\dagger$ by λ. Note that for $M \in \operatorname{mod} \Lambda$, the map $\lambda_M : M \to M^{\dagger\ddagger}$ is given by $\lambda_M(m)(\psi) = \psi(m)$ for $m \in M$ and $\psi \in M^\dagger = \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(M, C)$. We denote the unit of adjunction $N \to N^{\ddagger\dagger}$ by $\mu = \mu_N$ for $N \in \Gamma\,\mathrm{mod}$. When we view μ as a morphism $N^{\ddagger\dagger} \to N$ (in the opposite category $(\Gamma\,\mathrm{mod})^{\mathrm{op}}$), then it is the counit of adjunction.

Lemma 4.4. $(?)^\dagger$ *and* $(?)^\ddagger$ *give a contravariant equivalence between* $\mathrm{add}\,C \subset \mathrm{mod}\,\Lambda$ *and* $\mathrm{add}\,\Gamma \subset \Gamma\,\mathrm{mod}$.

Proof. It suffices to show that $\lambda : M \to M^{\dagger\ddagger}$ is an isomorphism for $M \in \mathrm{add}\,C$, and $\mu : N \to N^{\ddagger\dagger}$ is an isomorphism for $N \in \mathrm{add}\,\Gamma$. To verify this, we may assume that $M = C$ and $N = \Gamma$. This case is trivial. $\square$

Definition 4.5 (cf. [Tak, (2.2)])**.** Let $M \in \mathrm{mod}\,\Lambda$. We say that M is $(1, C)$-TF or $M \in \mathrm{TF}_{\Lambda^{\mathrm{op}}}(1, C)$ if $\lambda_M : M \to M^{\dagger\ddagger}$ is injective. We say that M is $(2, C)$-TF or $M \in \mathrm{TF}_{\Lambda^{\mathrm{op}}}(2, C)$ if $\lambda_M : M \to M^{\dagger\ddagger}$ is bijective. Let $n \geq 3$. We say that M is (n, C)-TF or $M \in \mathrm{TF}_{\Lambda^{\mathrm{op}}}(n, C)$ if M is $(2, C)$-TF and $\mathrm{Ext}^i_\Gamma(M^\dagger, C) = 0$ for $1 \leq i \leq n - 2$. As a convention, we define that any $M \in \mathrm{mod}\,\Lambda$ is $(0, C)$-TF.

Lemma 4.6. *Let* $\Theta : 0 \to M \to L \to N \to 0$ *be a C-exact sequence in* $\mathrm{mod}\,\Lambda$. *Then for $n \geq 0$, we have the following.*

1 *If* $M \in \mathrm{TF}(n, C)$ *and* $N \in \mathrm{TF}(n, C)$, *then* $L \in \mathrm{TF}(n, C)$.

2 *If* $L \in \mathrm{TF}(n + 1, C)$ *and* $N \in \mathrm{TF}(n, C)$, *then* $M \in \mathrm{TF}(n + 1, C)$.

3 *If* $M \in \mathrm{TF}(n + 1, C)$ *and* $L \in \mathrm{TF}(n, C)$, *then* $N \in \mathrm{TF}(n, C)$.

Proof. We have a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \xrightarrow{h} & L & \longrightarrow & N & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle \lambda_M} & & \downarrow{\scriptstyle \lambda_L} & & \downarrow{\scriptstyle \lambda_N} & & & & \\
0 & \longrightarrow & M^{\dagger\ddagger} & \xrightarrow{h^{\dagger\ddagger}} & L^{\dagger\ddagger} & \longrightarrow & N^{\dagger\ddagger} & \longrightarrow & \mathrm{Ext}^1_\Gamma(M^\dagger, C) & \longrightarrow & \mathrm{Ext}^1_\Gamma(L^\dagger, C) \longrightarrow \cdots
\end{array}$$

with exact rows.

We only prove **3**. We may assume that $n \geq 1$. So λ_M is an isomorphism and λ_L is injective. By the five lemma, λ_N is injective, and the case that $n = 1$ has been done. If $n \geq 2$, then λ_L is also an isomorphism and $\mathrm{Ext}^1_\Gamma(M^\dagger, C) = 0$, and so λ_N is an isomorphism. Moreover, for $1 \leq i \leq n - 2$, $\mathrm{Ext}^i_\Gamma(L^\dagger, C)$ and $\mathrm{Ext}^{i+1}_\Gamma(M^\dagger, C)$ vanish. so $\mathrm{Ext}^i_\Gamma(N^\dagger, C) = 0$ for $1 \leq i \leq n - 2$, and hence $N \in \mathrm{TF}(n, C)$.

1 and **2** are also proved similarly. $\square$

Lemma 4.7 (cf. [Tak, Proposition 3.2]). **1** *For* $n = 0, 1$, $\mathrm{Syz}_{\Lambda^{\mathrm{op}}}(n, C) = \mathrm{UP}_{\Lambda^{\mathrm{op}}}(n, C)$.

2 *For* $n \geq 0$, $\mathrm{TF}_{\Lambda^{\mathrm{op}}}(n, C) = \mathrm{UP}_{\Lambda^{\mathrm{op}}}(n, C)$.

Proof. If $n = 0$, then $\mathrm{Syz}_{\Lambda^{\mathrm{op}}}(n, C) = \mathrm{TF}_{\Lambda^{\mathrm{op}}}(0, C) = \mathrm{UP}_{\Lambda^{\mathrm{op}}}(0, C) = \mathrm{mod}\,\Lambda$. So we may assume that $n \geq 1$.

Let $M \in \mathrm{Syz}_{\Lambda^{\mathrm{op}}}(1, C)$. Then there is an injection $\varphi : M \to N$ with $N \in \mathrm{add}\, C$. Then

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow{\scriptstyle \lambda_M} & & \cong \downarrow{\scriptstyle \lambda_N} \\ M^{\dagger\ddagger} & \xrightarrow{\varphi^{\dagger\ddagger}} & N^{\dagger\ddagger} \end{array}$$

is a commutative diagram. So λ_M is injective, and $M \in \mathrm{TF}_{\Lambda^{\mathrm{op}}}(1, C)$. This shows $\mathrm{UP}_{\Lambda^{\mathrm{op}}}(1, C) \subset \mathrm{Syz}_{\Lambda^{\mathrm{op}}}(1, C) \subset \mathrm{TF}_{\Lambda^{\mathrm{op}}}(1, C)$. So **2**⇒**1**.

We prove **2**. First, we prove $\mathrm{UP}_{\Lambda^{\mathrm{op}}}(n, C) \subset \mathrm{TF}_{\Lambda^{\mathrm{op}}}(n, C)$ for $n \geq 1$. We use induction on n. The case $n = 1$ is already done above.

Let $n \geq 2$ and $M \in \mathrm{UP}_{\Lambda^{\mathrm{op}}}(n, C)$. Then by the definition of $\mathrm{UP}_{\Lambda^{\mathrm{op}}}(n, C)$, there is a C-exact sequence

$$0 \to M \to L \to N \to 0$$

such that $L \in \mathrm{add}\, C$ and $N \in \mathrm{UP}_{\Lambda^{\mathrm{op}}}(n-1, C)$. By induction hypothesis, $N \in \mathrm{TF}_{\Lambda^{\mathrm{op}}}(n-1, C)$. Hence $M \in \mathrm{TF}_{\Lambda^{\mathrm{op}}}(n, C)$ by Lemma 4.6. We have proved that $\mathrm{UP}_{\Lambda^{\mathrm{op}}}(n, C) \subset \mathrm{TF}_{\Lambda^{\mathrm{op}}}(n, C)$.

Next we show that $\mathrm{TF}_{\Lambda^{\mathrm{op}}}(n, C) \subset \mathrm{UP}_{\Lambda^{\mathrm{op}}}(n, C)$ for $n \geq 1$. We use induction on n.

Let $n = 1$. Let $\rho : F \to M^\dagger$ be any surjective Γ-linear map with $F \in \mathrm{add}\, \Gamma$. Then the map $\rho' : M \to F^\ddagger$ which corresponds to ρ by the adjunction (5) is

$$\rho' : M \xrightarrow{\lambda_M} M^{\dagger\ddagger} \xrightarrow{\rho^\ddagger} F^\ddagger,$$

which is injective by assumption. Then ρ is the composite

$$\rho : F \xrightarrow{\mu_F} F^{\ddagger\dagger} \xrightarrow{(\rho')^\dagger} M^\dagger,$$

which is a surjective map by assumption. So $(\rho')^\dagger$ is also surjective, and hence $\rho' : M \to F^\ddagger$ gives a $(1, C)$-universal pushforward.

Now let $n \geq 2$. By what we have proved, M has a $(1, C)$-universal pushforward $h : M \to L$. Let $N = \mathrm{Coker}\, h$. Then we have a C-exact sequence

$$0 \to M \to L \to N \to 0$$

with $L \in \operatorname{add} C$. As $M \in \mathrm{TF}(n, C)$, $N \in \mathrm{TF}(n-1, C)$ by Lemma 4.6. By induction hypothesis, $N \in \mathrm{UP}(n-1, C)$. So by the definition of $\mathrm{UP}(n, C)$, we have that $M \in \mathrm{UP}(n, C)$, as desired. $\square$

Lemma 4.8. *For any $N \in \Gamma \operatorname{mod}$, we have that $N^{\ddagger} \in \mathrm{Syz}(2, C)$.*

Proof. Let

$$F_1 \xrightarrow{h} F_0 \to N \to 0$$

be an exact sequence in $\Gamma \operatorname{mod}$ such that $F_i \in \operatorname{add} \Gamma$. Then

$$0 \to N^{\ddagger} \to F_0^{\ddagger} \xrightarrow{h^{\ddagger}} F_1^{\ddagger}$$

is exact, and $F_i^{\ddagger} \in \operatorname{add} C$. This shows that $N^{\ddagger} \in \mathrm{Syz}(2, C)$. $\square$

(4.9) We denote by $(S_n')_C = (S_n')_C^{\Lambda^{\mathrm{op}}, R}$ the class of $M \in \operatorname{mod} \Lambda$ such that M viewed as an R-module lies in $(S_n')_C^R$, see (2.15).

Lemma 4.10. *Assume that C satisfies (S_n') as an R-module. Then $\mathrm{Syz}(r, C) \subset (S_r')_C^{\Lambda^{\mathrm{op}}, R}$ for $1 \le r \le n$.*

Proof. This follows easily from Corollary 2.9. $\square$

(4.11) For an additive category $\mathcal{C}$ and its additive subcategory $\mathcal{X}$, we denote by $\mathcal{C}/\mathcal{X}$ the quotient of $\mathcal{C}$ divided by the ideal consisting of morphisms which factor through objects of $\mathcal{X}$.

(4.12) For each $M \in \operatorname{mod} \Lambda$, take a presentation

$$\mathbb{F}(M) : F_1(M) \xrightarrow{\partial} F_0(M) \xrightarrow{\varepsilon} M \to 0 \qquad (6)$$

with $F_i \in \operatorname{add} \Lambda_\Lambda$. We denote

$$\operatorname{Coker}(\partial^{\dagger}) = \operatorname{Coker}(1_C \otimes \partial^t) = C \otimes_\Lambda \operatorname{Tr} M$$

by $\operatorname{Tr}_C M$, where $(?)^t = \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(?, \Lambda)$ and Tr is the transpose, see [ASS, (V.2)], and we call it the C-transpose of M. Tr_C is an additive functor from $\underline{\operatorname{mod}}\, \Lambda := \operatorname{mod} \Lambda / \operatorname{add} \Lambda_\Lambda$ to $\Gamma_C\, \underline{\operatorname{mod}} := \Gamma \operatorname{mod} / \operatorname{add} C$.

Proposition 4.13. *Let $n \ge 0$, and assume that Λ_Λ is $(n+2, C)$-TF. Then for $M \in \operatorname{mod} \Lambda$, we have the following.*

0 *For $1 \le i \le n$, $\operatorname{Ext}^i_\Gamma(\operatorname{Tr}_C ?, C)$ is a well-defined additive functor $\underline{\operatorname{mod}}\, \Lambda \to \operatorname{mod} \Lambda$.*

1 *If $n = 1$, there is an exact sequence*

$$0 \to \operatorname{Ext}^1_\Gamma(\operatorname{Tr}_C M, C) \to M \xrightarrow{\lambda_M} M^{\dagger\ddagger} \to \operatorname{Ext}^2_\Gamma(\operatorname{Tr}_C M, C).$$

If $n = 0$, then there is an injective homomorphism $\operatorname{Ker} \lambda_M \hookrightarrow \operatorname{Ext}^1_\Gamma(\operatorname{Tr}_C M, C)$.

2 *If $n \geq 2$, then*

- **i** *There is an exact sequence*

$$0 \to \operatorname{Ext}^1_\Gamma(\operatorname{Tr}_C M, C) \to M \xrightarrow{\lambda_M} M^{\dagger\ddagger} \to \operatorname{Ext}^2_\Gamma(\operatorname{Tr}_C M, C) \to 0.$$

- **ii** *There are isomorphisms* $\operatorname{Ext}^{i+2}\Gamma(\operatorname{Tr}_C M, C) \cong \operatorname{Ext}^i_\Gamma(M^\dagger, C)$ *for* $1 \leq i \leq n-2$.
- **iii** *There is an injective map* $\operatorname{Ext}^{n-1}_\Gamma(M^\dagger, C) \hookrightarrow \operatorname{Ext}^{n+1}_\Gamma(\operatorname{Tr}_C M, C)$.

Proof. **0** is obvious by assumption.

We consider that $\mathbb{F}(M)$ is a complex with M at degree zero. Then consider

$$\mathbb{Q}(M) := \mathbb{F}(M)^\dagger[2] : F_1(M)^\dagger \xleftarrow{\partial^\dagger} F_0(M)^\dagger \xleftarrow{\varepsilon^\dagger} M^\dagger \leftarrow 0$$

where $F_1(M)^\dagger$ is at degree zero. As this complex is quasi-isomorphic to $\operatorname{Tr}_C(M)$, there is a spectral sequence

$$E_1^{p,q} = \operatorname{Ext}^q_\Gamma(\mathbb{Q}(M)^{-p}, C) \Rightarrow \operatorname{Ext}^{p+q}_\Gamma(\operatorname{Tr}_C M, C).$$

In general, $\operatorname{Ker} \lambda_M = E_2^{1,0} \cong E_\infty^{1,0} \subset E^1$. If $n \geq 1$, then $E_1^{0,1} = 0$, and $E_\infty^{1,0} = E^1$. Moreover, as $E_1^{0,1} = 0$, $\operatorname{Coker} \lambda_M \cong E_2^{2,0} \cong E_\infty^{2,0} \subset E^2$. So **1** follows.

If $n \geq 2$, then $E_1^{0,2} = E_1^{1,1} = 0$ by assumption, so $E_\infty^{2,0} = E^2$, and **i** of **2** follows. Note that $E_1^{p,q} = 0$ for $p \geq 3$. Moreover, $E_1^{p,q} = 0$ for $p = 0, 1$ and $1 \leq q \leq n$. So for $1 \leq i \leq n-1$, we have

$$E_1^{2,i} \cong E_\infty^{2,i} \hookrightarrow E^{i+2},$$

and the inclusion is an isomorphism if $1 \leq i \leq n-2$. So **ii** and **iii** of **2** follow. □

Corollary 4.14. *Let $n \geq 1$. If Λ_Λ is $(n+2, C)$-TF, then M is (n, C)-TF if and only if $\operatorname{Ext}^i_\Gamma(\operatorname{Tr}_C M, C) = 0$ for $1 \leq i \leq n$. If Λ_Λ is $(n+1, C)$-TF and $\operatorname{Ext}^i_\Gamma(\operatorname{Tr}_C M, C) = 0$ for $1 \leq i \leq n$, then M is (n, C)-TF.*

5. Canonical module

(5.1) Let $R = (R, \mathfrak{m})$ be semilocal, where $\mathfrak{m}$ is the Jacobson radical of R.

(5.2) We say that a dualizing complex $\mathbb{I}$ over R is *normalized* if for any maximal ideal $\mathfrak{n}$ of R, $\operatorname{Ext}^0_R(R/\mathfrak{n}, \mathbb{I}) \neq 0$. We follow the definition of [Hart].

(5.3) For a left or right Λ-module M, $\dim M$ or $\dim_\Lambda M$ denotes the dimension $\dim_R M$ of M, which is independent of the choice of R. We call $\operatorname{depth}_R(\mathfrak{m}, M)$, which is also independent of R, the global depth, Λ-depth, or depth of M, and denote it by $\operatorname{depth}_\Lambda M$ or $\operatorname{depth} M$. M is called globally Cohen–Macaulay or GCM for short, if $\dim M = \operatorname{depth} M$. M is GCM if and only if it is Cohen–Macaulay as an R-module, and all the maximal ideals of $R/\operatorname{ann}_R M$ have the same height. This notion is independent of R, and depends only on Λ and M. M is called a globally maximal Cohen–Macaulay (GMCM for short) if $\dim \Lambda = \operatorname{depth} M$. We say that the algebra Λ is GCM if the Λ-module Λ is GCM. However, in what follows, if R happens to be local, then GCM and Cohen–Macaulay (resp. GMSM and maximal Cohen–Macaulay) (over R) are the same thing, and used interchangeably.

(5.4) Assume that $(R, \mathfrak{m})$ is complete semilocal, and $\Lambda \neq 0$. Let $\mathbb{I}$ be a normalized dualizing complex of R. The lowest non-vanishing cohomology group $\operatorname{Ext}^{-s}_R(\Lambda, \mathbb{I})$ ($\operatorname{Ext}^i_R(\Lambda, \mathbb{I}) = 0$ for $i < -s$) is denoted by K_Λ, and is called the *canonical module* of Λ. Note that K_Λ is a Λ-bimodule. Hence it is also a Λ^{op}-bimodule. In this sense, $K_\Lambda = K_{\Lambda^{\mathrm{op}}}$. If $\Lambda = 0$, then we define $K_\Lambda = 0$.

(5.5) Let S be the center of Λ. Then S is module-finite over R, and $\mathbb{I}_S = \mathbf{R}\operatorname{Hom}_R(S, \mathbb{I})$ is a normalized dualizing complex of S. This shows that $\mathbf{R}\operatorname{Hom}_R(\Lambda, \mathbb{I}) \cong \mathbf{R}\operatorname{Hom}_S(\Lambda, \mathbb{I}_S)$, and hence the definition of K_Λ is also independent of R.

Lemma 5.6. *The number s in* (5.4) *is nothing but $d := \dim \Lambda$. Moreover,*

$$\operatorname{Ass}_R K_\Lambda = \operatorname{Assh}_R \Lambda := \{P \in \operatorname{Min}_R \Lambda \mid \dim R/P = \dim \Lambda\}.$$

Proof. We may replace R by $R/\operatorname{ann}_R \Lambda$, and may assume that Λ is a faithful module. We may assume that $\mathbb{I}$ is a fundamental dualizing complex of R. That is, for each $P \in \operatorname{Spec} R$, $E(R/P)$, the injective hull of R/P, appears exactly once (at dimension $-\dim R/P$). If $\operatorname{Ext}^{-i}_R(\Lambda, \mathbb{I}) \neq 0$, then there exists some $P \in \operatorname{Spec} R$ such that $\operatorname{Ext}^{-i}_{R_P}(\Lambda_P, \mathbb{I}_P) \neq 0$. Then $P \in \operatorname{Supp}_R \Lambda$ and $\dim R/P \geq i$. On the other hand, $\operatorname{Ext}^{-d}_{R_P}(\Lambda_P, \mathbb{I}_P)$ has length $l(\Lambda_P)$ and is nonzero for $P \in \operatorname{Assh}_R \Lambda$. So $s = d$.

The argument above shows that each $P \in \operatorname{Assh}_R \Lambda = \operatorname{Assh} R$ supports K_Λ. So $\operatorname{Assh}_R \Lambda \subset \operatorname{Min}_R K_\Lambda$. On the other hand, as the complex $\mathbb{I}$ starts at degree $-d$, $K_\Lambda \subset \mathbb{I}^{-d}$, and $\operatorname{Ass} K_\Lambda \subset \operatorname{Ass} \mathbb{I}^{-d} \subset \operatorname{Assh} R = \operatorname{Assh}_R \Lambda$. $\square$

Lemma 5.7. *Let $(R, \mathfrak{m})$ be complete semilocal. Then K_Λ satisfies the $(S_2^\Lambda)^R$-condition.*

Proof. It is easy to see that $(K_\Lambda)_\mathfrak{n}$ is either zero or $K_{\Lambda_\mathfrak{n}}$ for each maximal ideal $\mathfrak{n}$ of R. Hence we may assume that R is local. Replacing R by $R/\operatorname{ann}_R \Lambda$, we may assume that Λ is a faithful R-module, and we are to prove that K_Λ satisfies $(S_2')^R$ by Lemma 2.18. Replacing R by a Noether normalization, we may further assume that R is regular by Lemma 2.20.**1**. Then $K_\Lambda = \operatorname{Hom}_R(\Lambda, R)$. So $K_\Lambda \in \operatorname{Syz}(2, R) \subset (S_2')^R$ by Lemma 4.8 (consider that Λ there is R here, and C there is also R here). $\square$

(5.8) Assume that $(R, \mathfrak{m})$ is semilocal which need not be complete. We say that a finitely generated Λ-bimodule K is a *canonical module* of Λ if $\hat{K}$ is isomorphic to the canonical module $K_{\hat{\Lambda}}$ as a $\hat{\Lambda}$-bimodule, where $\hat{?}$ denotes the $\mathfrak{m}$-adic completion. It is unique up to isomorphisms, and denoted by K_Λ. We say that $K \in \operatorname{mod} \Lambda$ is a right canonical module of Λ if $\hat{K}$ is isomorphic to $K_{\hat{\Lambda}}$ in $\operatorname{mod} \hat{\Lambda}$. If K_Λ exists, then K is a right canonical module if and only if $K \cong K_\Lambda$ in $\operatorname{mod} \Lambda$.

These definitions are independent of R, in the sense that the (right) canonical module over R and that over the center of Λ are the same thing. The right canonical module of Λ^{op} is called the left canonical module. A Λ-bimodule ω is said to be a weakly canonical bimodule if ${}_\Lambda\omega$ is left canonical, and ω_Λ is right canonical. The canonical module $K_{\Lambda^{\mathrm{op}}}$ of Λ^{op} is canonically identified with K_Λ.

(5.9) If R has a normalized dualizing complex $\mathbb{I}$, then $\hat{\mathbb{I}}$ is a normalized dualizing complex of $\hat{R}$, and so it is easy to see that K_Λ exists and agrees with $\operatorname{Ext}^{-d}(\Lambda, \mathbb{I})$, where $d = \dim \Lambda (:= \dim_R \Lambda)$. In this case, for any $P \in \operatorname{Spec} R$, $\mathbb{I}_P$ is a dualizing complex of R_P. So if R has a dualizing complex and $(K_\Lambda)_P \neq 0$, then $(K_\Lambda)_P$, which is the lowest nonzero cohomology group of $\mathbf{R}\operatorname{Hom}_{R_P}(\Lambda_P, \mathbb{I}_P)$, is the R_P-canonical module of Λ_P. See also Theorem 7.5 below.

Lemma 5.10. *Let $(R, \mathfrak{m})$ be local, and assume that K_Λ exists. Then we have the following.*

1 $\operatorname{Ass}_R K_\Lambda = \operatorname{Assh}_R \Lambda$.

2 $K_\Lambda \in (S_2^\Lambda)^R$.

3 *$R/\operatorname{ann} K_\Lambda$ is quasi-unmixed, and hence is universally catenary.*

Proof. All the assertions are proved easily using the case that R is complete. □

(5.11) A Λ-module M is said to be Λ-full over R if $\operatorname{Supp}_R M = \operatorname{Supp}_R \Lambda$.

Lemma 5.12. *Let $(R, \mathfrak{m})$ be local. If K_Λ exists and Λ satisfies the $(S_2)^R$-condition, then $R/\operatorname{ann}_R \Lambda$ is equidimensional, and K_Λ is Λ-full over R.*

Proof. The same as the proof of [Ogo, Lemma 4.1] (use Lemma 5.10.**3**). □

(5.13) Let $(R, \mathfrak{m})$ be local, and $\mathbb{I}$ be a normalized dualizing complex. By the local duality,

$$K_\Lambda^\vee = \operatorname{Ext}^{-d}(\Lambda, \mathbb{I})^\vee \cong H^d_{\mathfrak{m}}(\Lambda)$$

(as Λ-bimodules), where $E_R(R/\mathfrak{m})$ is the injective hull of the R-module $R/\mathfrak{m}$, and $(?)^\vee$ is the Matlis dual $\operatorname{Hom}_R(?, E_R(R/\mathfrak{m}))$.

(5.14) Let $(R, \mathfrak{m})$ be semilocal, and $\mathbb{I}$ be a normalized dualizing complex. Note that $\mathbf{R}\operatorname{Hom}_R(?, \mathbb{I})$ induces a contravariant equivalence between $D_{\mathrm{fg}}(\Lambda^{\mathrm{op}})$ and $D_{\mathrm{fg}}(\Lambda)$. Let $\mathbb{J} \in D_{\mathrm{fg}}(\Lambda \otimes_R \Lambda^{\mathrm{op}})$ be $\mathbf{R}\operatorname{Hom}_R(\Lambda, \mathbb{I})$.

$$\mathbf{R}\operatorname{Hom}_R(?, \mathbb{I}) : D_{\mathrm{fg}}(\Lambda^{\mathrm{op}}) \to D_{\mathrm{fg}}(\Lambda)$$

is identified with

$$\mathbf{R}\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(?, \mathbf{R}\operatorname{Hom}_R({}_\Lambda\Lambda_R, \mathbb{I})) = \mathbf{R}\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(?, \mathbb{J})$$

and similarly,

$$\mathbf{R}\operatorname{Hom}_R(?, \mathbb{I}) : D_{\mathrm{fg}}(\Lambda) \to D_{\mathrm{fg}}(\Lambda^{\mathrm{op}})$$

is identified with $\mathbf{R}\operatorname{Hom}_\Lambda(?, \mathbb{J})$. Note that a left or right Λ-module M is GMCM if and only if $\mathbf{R}\operatorname{Hom}_R(M, \mathbb{I})$ is concentrated in degree $-d$, where $d = \dim \Lambda$.

(5.15) $\mathbb{J}$ above is a dualizing complex of Λ in the sense of Yekutieli [Yek, (3.3)].

(5.16) Λ is GCM if and only if $K_\Lambda[d] \to \mathbb{J}$ is an isomorphism. If so, $M \in \operatorname{mod}\Lambda$ is GMCM if and only if $\mathbf{R}\mathrm{Hom}_R(M, \mathbb{I})$ is concentrated in degree $-d$ if and only if $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(M, K_\Lambda) = 0$ for $i > 0$. Also, in this case, as $K_\Lambda[d]$ is a dualizing complex, it is of finite injective dimension both as a left and a right Λ-module. To prove these, we may take the completion, and may assume that R is complete. All the assertions are independent of R, so taking the Noether normalization, we may assume that R is local. By (5.14), the assertions follow.

(5.17) For any $M \in \operatorname{mod}\Lambda$ which is GMCM,

$$M \cong \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, \mathbb{I}), \mathbb{I}) \cong \mathbf{R}\mathrm{Hom}_R(\mathrm{Ext}^{-d}_{\Lambda^{\mathrm{op}}}(M, K_\Lambda[d]), \mathbb{I})[-d].$$

Hence $M^\dagger := \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(M, K_\Lambda)$ is also a GMCM Λ-module, and hence

$$\mathrm{Hom}_\Lambda(M^\dagger, K_\Lambda) \to \mathbf{R}\mathrm{Hom}_\Lambda(M^\dagger, \mathbb{J}) = \mathbf{R}\mathrm{Hom}_R(M^\dagger, \mathbb{I})$$

is an isomorphism (in other words, $\mathrm{Ext}^i_\Lambda(M^\dagger, K_\Lambda) = 0$ for $i > 0$). So the canonical map

$$(7) \qquad M \to \mathrm{Hom}_\Lambda(\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(M, K_\Lambda), K_\Lambda) = \mathrm{Hom}_\Lambda(M^\dagger, K_\Lambda)$$

$m \mapsto (\varphi \mapsto \varphi m)$ is an isomorphism. This isomorphism is true without assuming that R has a dualizing complex (but assuming the existence of a canonical module), passing to the completion. Note that if $\Lambda = R$, K_R exists, and R is Cohen–Macaulay, then K_R is a dualizing complex of R.

Similarly, for $N \in \Lambda\operatorname{mod}$ which is GMCM,

$$N \to \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_\Lambda(N, K_\Lambda), K_\Lambda)$$

$n \mapsto (\varphi \mapsto \varphi n)$ is an isomorphism.

(5.18) In particular, letting $M = \Lambda$, if Λ is GCM, we have that $K_\Lambda = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda, K_\Lambda)$ is GMCM. Moreover,

$$\Lambda \to \mathrm{End}_{\Lambda^{\mathrm{op}}} K_\Lambda$$

is an R-algebra isomorphism, where $a \in \Lambda$ goes to the left multiplication by a. Similarly,

$$\Lambda \to (\mathrm{End}_\Lambda K_\Lambda)^{\mathrm{op}}$$

is an isomorphism of R-algebras.

(5.19) Let $(R, \mathfrak{m})$ be a d-dimensional complete local ring, and $\dim \Lambda = d$. Then by the local duality,

$$H^d_{\mathfrak{m}}(K_\Lambda)^\vee \cong \operatorname{Ext}^{-d}_R(K_\Lambda, \mathbb{I}) \cong \operatorname{Ext}^{-d}_{\Lambda^{\mathrm{op}}}(K_\Lambda, \mathbb{J}) \cong \operatorname{End}_{\Lambda^{\mathrm{op}}} K_\Lambda,$$

where $\mathbb{J} = \operatorname{Hom}_R(\Lambda, \mathbb{I})$ and $(?)^\vee = \operatorname{Hom}_R(?, E_R(R/\mathfrak{m}))$.

6. n-canonical module

(6.1) We say that ω is an R-semicanonical right Λ-module (resp. R-semicanonical left Λ-module, weakly R-semicanonical Λ-bimodule, R-semicanonical Λ-bimodule) if for any $P \in \operatorname{Spec} R$, $R_P \otimes_R \omega$ is the right canonical module (resp. left canonical module, weakly canonical module, canonical module) of $R_P \otimes_R \Lambda$ for any $P \in \operatorname{Supp}_R \omega$. If we do not mention what R is, then one may take R to be the center of Λ. An R-semicanonical right Λ^{op}-module (resp. R-semicanonical left Λ^{op}-module, weakly R-semicanonical Λ^{op}-bimodule, R-semicanonical Λ^{op}-bimodule) is nothing but an R-semicanonical left Λ-module (resp. R-semicanonical right Λ-module, weakly R-semicanonical Λ-bimodule, R-semicanonical Λ-bimodule).

(6.2) Let $C \in \operatorname{mod} \Lambda$ (resp. $\Lambda \operatorname{mod}$, $(\Lambda \otimes_R \Lambda^{\mathrm{op}}) \operatorname{mod}$, $(\Lambda \otimes_R \Lambda^{\mathrm{op}}) \operatorname{mod}$). We say that C is an n-canonical right Λ-module (resp. n-canonical left Λ-module, weakly n-canonical Λ-bimodule, n-canonical Λ-bimodule) over R if $C \in (S'_n)^R$, and for each $P \in R^{\langle <n \rangle}$, we have that C_P is an R_P-semicanonical right Λ_P-module (resp. R_P-semicanonical left Λ_P-module, weakly R_P-semicanonical Λ_P-bimodule, R_P-semicanonical Λ_P-bimodule). If we do not mention what R is, it may mean R is the center of Λ.

Example 6.3. **0** The zero module 0 is an R-semicanonical Λ-bimodule.

1 If R has a dualizing complex $\mathbb{I}$, then the lowest non-vanishing cohomology group $K := \operatorname{Ext}^{-s}_R(\Lambda, \mathbb{I})$ is an R-semicanonical Λ-bimodule.

2 By Lemma 5.10, any left or right R-semicanonical module K of Λ satisfies the $(S_2^\Lambda)^R$-condition. Thus a (right) semicanonical module is 2-canonical over $R/\operatorname{ann}_R \Lambda$.

3 If K is (right) semicanonical (resp. n-canonical) and L is a projective R-module such that L_P is rank at most one, then $K \otimes_R L$ is again (right) semicanonical (resp. n-canonical).

4 If R is a normal domain and C its rank-one reflexive module of R, then C is a 2-canonical R-module (here $\Lambda = R$).

5 The R-module R is n-canonical if and only if for any prime ideal P of R with $\operatorname{depth} R_P < n$, R_P is Gorenstein. This is equivalent to say that R satisfies $(T_{n-1}) + (S_n)$.

(6.4) As in section 4, let $C \in \operatorname{mod} \Lambda$, and set $\Gamma = \operatorname{End}_{\Lambda^{\mathrm{op}}} C$, $(?)^\dagger = \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(?, C)$, and $(?)^\ddagger = \operatorname{Hom}_\Gamma(?, C)$. Moreover, we set $\Lambda_1 := (\operatorname{End}_\Gamma C)^{\mathrm{op}}$. The R-algebra map $\Psi_1 : \Lambda \to \Lambda_1$ is induced by the right action of Λ on C.

Lemma 6.5. *Let $C \in \operatorname{mod} \Lambda$ be a 1-canonical Λ^{op}-module over R. Let $M \in \operatorname{mod} \Lambda$. Then the following are equivalent.*

1 $M \in \mathrm{TF}(1, C)$.

2 $M \in \mathrm{UP}(1, C)$.

3 $M \in \mathrm{Syz}(1, C)$.

4 $M \in (S'_1)^R_C$.

Proof. **1⇔2** is Lemma 4.7. **2⇒3** is trivial. **3⇒4** follows from Lemma 4.10 immediately.

We prove **4⇒1**. We want to prove that $\lambda_M : M \to M^{\dagger\ddagger}$ is injective. By Example 2.11, localizing at each $P \in R^{\langle 0 \rangle}$, we may assume that $(R, \mathfrak{m})$ is zero-dimensional local. We may assume that M is nonzero. By assumption, C is nonzero, and hence $C = K_\Lambda$ by assumption. As R is zero-dimensional, Λ is GCM, and hence $\Lambda \to \Gamma = \operatorname{End}_{\Lambda^{\mathrm{op}}} K_\Lambda$ is an isomorphism by (5.18). As Λ is GCM and M is GMCM, (7) is an isomorphism. As $\Lambda = \Gamma$, the result follows. □

Lemma 6.6. *Let C be a 1-canonical right Λ-module over R, and $N \in \Gamma \operatorname{mod}$. Then $N^\ddagger \in \mathrm{TF}_{\Lambda^{\mathrm{op}}}(2, C)$. Similarly, for $M \in \operatorname{mod} \Lambda$, we have that $M^\dagger \in \mathrm{TF}_\Gamma(2, C)$.*

Proof. Note that $\lambda_{N^\ddagger} : N^\ddagger \to N^{\ddagger\dagger\ddagger}$ is a split monomorphism. Indeed, $(\mu_N)^\ddagger : N^{\ddagger\dagger\ddagger} \to N^\ddagger$ is the left inverse. Assume that $N^\ddagger \notin \mathrm{TF}(2, C)$, then $W := \operatorname{Coker} \lambda_{N^\ddagger}$ is nonzero. Let $P \in \operatorname{Ass}_R W$. As W is a submodule of $N^{\ddagger\dagger\ddagger}$, $P \in \operatorname{Ass}_R N^{\ddagger\dagger\ddagger} \subset \operatorname{Ass}_R C \subset \operatorname{Min} R$. So C_P is the right canonical module K_{Λ_P}. So $\Gamma_P = \Lambda_P$, and $(\lambda_{N^\ddagger})_P$ is an isomorphism. This shows that $W_P = 0$, and this is a contradiction. The second assertion is proved similarly. □

Lemma 6.7. *Let $(R, \mathfrak{m})$ be local, and assume that K_Λ exists. Let $C := K_\Lambda$. If Λ is GCM, $\Psi_1 : \Lambda \to \Lambda_1$ is an isomorphism.*

Proof. As C possesses a bimodule structure, we have a canonical map $\Lambda \to \Gamma = \operatorname{End}_{\Lambda^{\mathrm{op}}} C$, which is an isomorphism as Λ is GCM by (5.18). So Λ_1 is identified with $\Delta = (\operatorname{End}_\Lambda C)^{\mathrm{op}}$. Then $\Psi_1 : \Lambda \to (\operatorname{End}_\Lambda C)^{\mathrm{op}}$ is an isomorphism again by (5.18). $\square$

Lemma 6.8. *If C satisfies the $(S_1')^R$ condition, then $\Gamma \in (S_1')^R_C$ and $\Lambda_1 \in (S_1')^R_C$. Moreover, $\operatorname{Ass}_R \Gamma = \operatorname{Ass}_R \Lambda_1 = \operatorname{Ass}_R C = \operatorname{Min}_R C$.*

Proof. The first assertion is by $\Gamma = \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(C, C) \in \operatorname{Syz}_\Gamma(2, C)$, and $\Lambda_1 = \operatorname{Hom}_\Gamma(C, C) = \operatorname{Syz}_{\Lambda_1}(2, C)$. We prove the second assertion. $\operatorname{Ass}_R \Gamma \subset \operatorname{Ass}_R \operatorname{End}_R C = \operatorname{Ass}_R C$. $\operatorname{Ass}_R \Lambda_1 \subset \operatorname{Ass}_R \operatorname{End}_R C = \operatorname{Ass}_R C = \operatorname{Min}_R C$. It remains to show that $\operatorname{Supp}_R C = \operatorname{Supp}_R \Gamma = \operatorname{Supp}_R \Lambda_1$. Let $P \in \operatorname{Spec} R$. If $C_P = 0$, then $\Gamma_P = 0$ and $(\Lambda_1)_P = 0$. On the other hand, if $C_P \neq 0$, then the identity map $C_P \to C_P$ is not zero, and hence $\Gamma_P \neq 0$ and $(\Lambda_1)_P \neq 0$. $\square$

(6.9) Let C be a 1-canonical right Λ-module over R. Define $Q := \prod_{P \in \operatorname{Min}_R C} R_P$. If $P \in \operatorname{Min}_R C$, then $C_P = K_{\Lambda_P}$. Hence $\Phi_P : \Lambda_P \to (\Lambda_1)_P$ is an isomorphism by Lemma 6.7. So $1_Q \otimes \Psi_1 : Q \otimes_R \Lambda \to Q \otimes_R \Lambda_1$ is also an isomorphism. As $\operatorname{Ass}_R \Lambda_1 = \operatorname{Min}_R C$, we have that $\Lambda_1 \subset Q \otimes_R \Lambda_1$.

Lemma 6.10. *Let C be a 1-canonical right Λ-module over R. If Λ is commutative, then so are Λ_1 and Γ.*

Proof. As $\Lambda_1 \subset Q \otimes_R \Lambda_1 = Q \otimes_R \Lambda$ and $Q \otimes_R \Lambda$ is commutative, Λ_1 is a commutative ring. We prove that Γ is commutative. As $\operatorname{Ass}_R \Gamma \subset \operatorname{Min}_R C$, Γ is a subring of $Q \otimes \Gamma$. As

$$Q \otimes_R \Gamma \cong \prod_{P \in \operatorname{Min}_R C} \operatorname{End}_{\Lambda_P} C_P \cong \prod_P \operatorname{End}_{\Lambda_P}(K_{\Lambda_P})$$

and $\Lambda_P \to \operatorname{End}_{\Lambda_P}(K_{\Lambda_P})$ is an isomorphism (as Λ_P is zero-dimensional), $Q \otimes_R \Gamma$ is, and hence Γ is also, commutative. $\square$

Lemma 6.11. *Let C be a 1-canonical right Λ-module over R. Let M and N be left (resp. right, bi-) modules of Λ_1, and assume that $N \in (S_1')^{\Lambda_1, R}$. Let $\varphi : M \to N$ be a Λ-homomorphism of left (resp. right, bi-) modules. Then φ is a Λ_1-homomorphism of left (resp. right, bi-) modules.*

Proof. Let $Q = \prod_{P \in \operatorname{Min}_R C} R_P$. Then we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow{\scriptstyle i_M} & & \downarrow{\scriptstyle i_N} \\ Q \otimes_R M & \xrightarrow{1 \otimes \varphi} & Q \otimes_R N \end{array} ,$$

where $i_M(m) = 1 \otimes m$ and $i_N(n) = 1 \otimes n$. Clearly, i_M and i_N are Λ_1-linear. As φ is Λ-linear, $1 \otimes \varphi$ is $Q \otimes \Lambda$-linear. Since $\Lambda_1 \subset Q \otimes \Lambda_1 = Q \otimes \Lambda$, $1 \otimes \varphi$ is Λ_1-linear. As i_N is injective, it is easy to see that φ is Λ_1-linear. ☐

Lemma 6.12. *Let C be a 1-canonical right Λ-module over R. Then the restriction $M \mapsto M$ is a full and faithful functor from $(S_1')^{\Lambda_1,R}$ to $(S_1')^{\Lambda,R}_C$. Similarly, it gives a full and faithful functors $(S_1')^{\Lambda_1^{\mathrm{op}},R} \to (S_1')^{\Lambda^{\mathrm{op}},R}_C$ and $(S_1')^{\Lambda_1 \otimes_R \Lambda_1^{\mathrm{op}},R} \to (S_1')^{\Lambda \otimes_R \Lambda^{\mathrm{op}},R}_C$.*

Proof. We only consider the case of left modules. If $M \in \Lambda_1 \operatorname{mod}$, then it is a homomorphic image of $\Lambda_1 \otimes_R M$. Hence $\operatorname{Supp}_R M \subset \operatorname{Supp}_R \Lambda_1 \subset \operatorname{Supp}_R C$. So the functor is well-defined and obviously faithful. By Lemma 6.11, it is also full, and we are done. ☐

(6.13) Let C be a 1-canonical Λ-bimodule over R. Then the left action of Λ on C induces an R-algebra map $\Phi : \Lambda \to \Gamma = \operatorname{End}_{\Lambda^{\mathrm{op}}} C$. Let $Q = \prod_{P \in \operatorname{Min}_R C} R_P$. Then $\Gamma \subset Q \otimes_R \Gamma = Q \otimes_R \Lambda$. From this we get

Lemma 6.14. *Let C be a 1-canonical Λ-bimodule over R. Let M and N be left (resp. right, bi-) modules of Γ, and assume that $N \in (S_1')^{\Gamma,R}$. Let $\varphi : M \to N$ be a Λ-homomorphism of left (resp. right, bi-) modules. Then φ is a Γ-homomorphism of left (resp. right, bi-) modules.*

Proof. Similar to Lemma 6.11, and left to the reader. ☐

Corollary 6.15. *Let C be as above. $(?)^{\dagger\ddagger} = \operatorname{Hom}_\Gamma(\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(?, C), C)$ is canonically isomorphic to $(?)^{\dagger\star} = \operatorname{Hom}_\Lambda(\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(?, C), C)$, where $(?)^\star = \operatorname{Hom}_\Lambda(?, C)$.*

Proof. This is immediate by Lemma 6.14. ☐

Lemma 6.16. *Let C be a 1-canonical Λ-bimodule over R. Then Φ induces a full and faithful functor $(S_1')^{\Gamma,R} \to (S_1')^{\Lambda,R}_C$. Similarly, $(S_1')^{\Gamma^{\mathrm{op}},R} \to (S_1')^{\Lambda^{\mathrm{op}},R}_C$ and $(S_1')^{\Gamma \otimes_R \Gamma^{\mathrm{op}},R} \to (S_1')^{\Lambda \otimes_R \Lambda^{\mathrm{op}},R}_C$ are also induced.*

Proof. Similar to Lemma 6.12, and left to the reader. ☐

Corollary 6.17. *Let C be a 1-canonical Λ-bimodule. Set $\Delta := (\operatorname{End}_\Lambda C)^{\mathrm{op}}$. Then the canonical map $\Lambda \to \Gamma$ induces an equality*

$$\Lambda_1 = (\operatorname{End}_\Gamma C)^{\mathrm{op}} = (\operatorname{End}_\Lambda C)^{\mathrm{op}} = \Delta.$$

Similarly, we have

$$\Lambda_2 := \operatorname{End}_{\Delta^{\mathrm{op}}} C = \operatorname{End}_{\Lambda^{\mathrm{op}}} C = \Gamma.$$

Proof. As $C \in (S_1')^{\Gamma,R}$, the first assertion follows from Lemma 6.16. The second assertion is proved by left-right symmetry. □

Lemma 6.18. *Let C be a 1-canonical right Λ-module over R. Set $\Lambda_1 := (\operatorname{End}_\Gamma C)^{\mathrm{op}}$. Let $\Psi_1 : \Lambda \to \Lambda_1$ be the canonical map induced by the right action of Λ on C. Then Ψ_1 is injective if and only if Λ satisfies the $(S_1')^R$ condition and C is Λ-full over R.*

Proof. $\Psi_1 : \Lambda \to \Lambda_1$ is nothing but $\lambda_\Lambda : \Lambda \to \Lambda^{\dagger\ddagger}$, and the result follows from Lemma 6.5 immediately. □

Lemma 6.19. *Let C be a 1-canonical Λ-bimodule over R. Then the following are equivalent.*

1 *The canonical map $\Psi : \Lambda \to \Delta$ is injective, where $\Delta = (\operatorname{End}_\Lambda C)^{\mathrm{op}}$, and the map is induced by the right action of Λ on C.*

2 *Λ satisfies the $(S_1')^R$ condition, and C is Λ-full over R.*

3 *The canonical map $\Phi : \Lambda \to \Gamma$ is injective, where the map is induced by the left action of Λ on C.*

Proof. By Corollary 6.17, we have that $\Lambda_1 = (\operatorname{End}_\Gamma C)^{\mathrm{op}} = \Delta$. So **1**$\Leftrightarrow$**2** is a consequence of Lemma 6.18.

Reversing the roles of the left and the right, we get **2**$\Leftrightarrow$**3** immediately. □

Lemma 6.20. *Let C be a 1-canonical right Λ-module over R. Then the canonical map*

$$\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda_1, C) \to \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda, C) \cong C \tag{8}$$

induced by the canonical map $\Psi_1 : \Lambda \to \Lambda_1$ is an isomorphism of $\Gamma \otimes_R \Lambda_1^{\mathrm{op}}$-modules.

Proof. The composite map

$$C \cong \operatorname{Hom}_{\Lambda_1^{\mathrm{op}}}(\Lambda_1, C) = \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda_1, C) \to \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda, C) \cong C$$

is the identity. The map is a $\Gamma \otimes_R \Lambda^{\mathrm{op}}$-homomorphism. It is also Λ_1^{op}-linear by Lemma 6.12. □

(6.21) When $(R, \mathfrak{m})$ is local and $C = K_\Lambda$, then $\Lambda_1 = \Delta$, and the map (8) is an isomorphism of $\Gamma \otimes_R \Delta^{\mathrm{op}}$-modules from K_Δ and K_Λ, where $\Delta = (\operatorname{End}_\Lambda K_\Lambda)^{\mathrm{op}}$. Indeed, to verify this, we may assume that R is complete regular local with $\operatorname{ann}_R \Lambda = 0$, and hence $C = \operatorname{Hom}_R(\Lambda, R)$, and C is a 2-canonical Λ-bimodule over R, see (6.3). So (6.17) and Lemma 6.20 apply. Hence we have

Corollary 6.22. *Let $(R, \mathfrak{m})$ be a local ring with a canonical module K_Λ of Λ. Then $K_\Delta = \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\Delta, K_\Lambda)$ is isomorphic to K_Λ as a $\Gamma \otimes_R \Delta^{\mathrm{op}}$-module, where $\Delta = (\operatorname{End}_\Lambda K_\Lambda)^{\mathrm{op}}$.* □

Lemma 6.23. *Let $n \geq 1$. If C is an n-canonical right Λ-module over R, then*

1 *C is an n-canonical right Λ_1-module over R.*

2 *C is an n-canonical left Γ-module over R.*

Proof. **1**. As the (S'_n)-condition holds, it suffices to prove that for $P \in R^{\langle <n \rangle}$ with $C_P \neq 0$, we have $C_P \cong (K_{\Lambda_1})_P$ as a right $(\Lambda_1)_P$-module. After localization, replacing R by R_P, we may assume that R is local and $C = K_\Lambda$. Then $C \cong K_\Lambda \cong K_{\Lambda_1}$ as right Λ-modules. Both C and K_{Λ_1} are in $(S'_1)^{\Lambda_1^{\mathrm{op}}, R}$, and isomorphic in $\operatorname{mod} \Lambda$. So they are isomorphic in $\operatorname{mod} \Lambda_1$ by Lemma 6.12.

2. Similarly, assuming that R is local and $C = K_\Lambda$, it suffices to show that $C \cong K_\Gamma$ as left Γ-modules. Identifying $\Gamma = \operatorname{End}_{\Delta^{\mathrm{op}}} C = \Lambda_2$ and using the left-right symmetry, this is the same as the proof of **1**. □

Lemma 6.24. *Let $C \in \operatorname{mod} \Lambda$ be a 2-canonical right Λ-module over R. Let $M \in \operatorname{mod} \Lambda$. Then the following are equivalent.*

1 *$M \in \mathrm{TF}(2, C)$.*

2 *$M \in \mathrm{UP}(2, C)$.*

3 *$M \in \mathrm{Syz}(2, C)$.*

4 *$M \in (S'_2)^R_C$.*

Proof. We may assume that Λ is a faithful R-module. **1**⇔**2**⇒**3**⇒**4** is easy. We show **4**⇒**1**. By Example 2.11, localizing at each $P \in R^{\langle \leq 1 \rangle}$, we may assume that R is a Noetherian local ring of dimension at most one. So the formal fibers of R are zero-dimensional, and hence $\hat{M} \in (S'_2)^{\hat{R}}_{\hat{C}}$, where $\hat{?}$ denotes the completion. So we may further assume that $R = (R, \mathfrak{m})$ is complete local.

We may assume that $M \neq 0$ so that $C \neq 0$ and hence $C = K_\Lambda$. The case $\dim R = 0$ is similar to the proof of Lemma 6.5, so we prove the case that $\dim R = 1$. Note that $I = H^0_{\mathfrak{m}}(\Lambda)$ is a two-sided ideal of Λ, and any module in $(S'_1)^{\Lambda^{\mathrm{op}},R}$ is annihilated by I. Replacing Λ by Λ/I, we may assume that Λ is a maximal Cohen–Macaulay R-module. Then (7) is an isomorphism. As $C = K_\Lambda$ and

$$\Lambda \to \operatorname{End}_{\Lambda^{\mathrm{op}}} K_\Lambda = \operatorname{End}_{\Lambda^{\mathrm{op}}} C = \Gamma$$

is an R-algebra isomorphism, we have that $\lambda_M : M \to M^{\dagger\ddagger}$ is identified with the isomorphism (7), as desired. □

Corollary 6.25. *Let C be a 2-canonical right Λ-module over R. Then the canonical map $\Phi : \Lambda \to \Lambda_1$ is an isomorphism if and only if Λ satisfies $(S'_2)^R$ and C is full.*

Proof. Follows immediately by Lemma 6.24 applied to $M = \Lambda$. □

(6.26) Let C be a 2-canonical Λ-bimodule. Let $\Gamma = \operatorname{End}_{\Lambda^{\mathrm{op}}} C$ and $\Delta = (\operatorname{End}_\Lambda C)^{\mathrm{op}}$. Then by the left multiplication, an R-algebra map $\Lambda \to \Gamma$ is induced, while by the right multiplication, an R-algebra map $\Lambda \to \Delta$ is induced. Let $Q = \prod_{P \in \operatorname{Min}_R C} R_P$. Then as $\Gamma \subset Q \otimes_R \Gamma = Q \otimes_R \Lambda = Q \otimes_R \Delta \supset \Delta$, both Γ and Δ are identified with Q-subalgebras of $Q \otimes_R \Lambda$. As $\Delta = \Lambda_1 = \Lambda^{\dagger\ddagger}$, we have a commutative diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\lambda_\Lambda} & \Lambda^{\dagger\ddagger} \qquad = \Delta\,. \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu^{\dagger\ddagger}} \\ \Gamma & \xrightarrow{\lambda_\Gamma} & \Gamma^{\dagger\ddagger} \end{array}$$

As $\Gamma = \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(C, C) = C^\dagger$, $\Gamma \in \operatorname{Syz}_\Lambda(2, C)$ by Lemma 4.8. By Lemma 6.24, we have that $\Gamma \in (S'_2)_C$. Hence by Lemma 6.24 again, $\lambda_\Gamma : \Gamma \to \Gamma^{\dagger\ddagger}$ is an isomorphism. Hence $\Delta \subset \Gamma$. By symmetry $\Delta \supset \Gamma$. So $\Delta = \Gamma$. With this identification, Γ acts on C not only from left, but also from right. As the actions of Γ extend those of Λ, C is a Γ-bimodule. Indeed, for $a \in \Lambda$, the left multiplication $\lambda_a : C \to C$ ($\lambda_a(c) = ac$) is right Γ-linear. So for $b \in \Gamma$, $\rho_b : C \to C$ ($\rho_b(c) = cb$) is left Λ-linear, and hence is left Γ-linear.

Theorem 6.27. *Let the notation be as in (4.1), (6.4), and (4.9). Let C be a 2-canonical right Λ-module. Then the restriction $M \mapsto M$ gives an equivalence $\rho : (S'_2)^{\Lambda_1^{\mathrm{op}},R} \to (S'_2)^{\Lambda^{\mathrm{op}},R}_C$.*

Proof. The functor is obviously well-defined, and is full and faithful by Lemma 6.12. On the other hand, given $M \in (S'_2)_C^{\Lambda^{\mathrm{op}},R}$, we have that $\lambda_M : M \to M^{\dagger\ddagger}$ is an isomorphism. As $M^{\dagger\ddagger}$ has a Λ_1^{op}-module structure which extends the Λ^{op}-module structure of $M \cong M^{\dagger\ddagger}$, we have that ρ is also dense, and hence is an equivalence. □

Corollary 6.28. *Let C be a 2-canonical Λ-bimodule. Then the restriction $M \mapsto M$ gives an equivalence*

$$\rho : (S'_2)_C^{\Gamma\otimes_R\Gamma^{\mathrm{op}},R} \to (S'_2)_C^{\Lambda\otimes_R\Lambda^{\mathrm{op}},R}.$$

Proof. ρ is well-defined, and is obviously faithful. If $h : M \to N$ is a morphism of $(S'_2)_C^{\Lambda\otimes_R\Lambda^{\mathrm{op}},R}$ between objects of $(S'_2)_C^{\Gamma\otimes_R\Gamma^{\mathrm{op}},R}$, then h is Γ-linear Γ^{op}-linear by Theorem 6.27 (note that $\Lambda_1 = \Delta = \Gamma$ here). Hence ρ is full.

Let $M \in (S'_2)_C^{\Lambda\otimes_R\Lambda^{\mathrm{op}},R}$, the left (resp. right) Λ-module structure of M is extendable to that of a left (resp. right) Γ-module structure by Theorem 6.27. It remains to show that these structures make M a Γ-bimodule. Let $a \in \Lambda$. Then $\lambda_a : M \to M$ given by $\lambda_a(m) = am$ is a right Λ-linear, and hence is right Γ-linear. So for $b \in \Gamma$, $\rho_b : M \to M$ given by $\rho_b(m) = mb$ is left Λ-linear, and hence is left Γ-linear, as desired. □

Proposition 6.29. *Let C be a 2-canonical right Λ-module. Then $(?)^\dagger : (S'_2)_C^{\Lambda^{\mathrm{op}},R} \to (S'_2)^{\Gamma,R}$ and $(?)^\ddagger : (S'_2)^{\Gamma,R} \to (S'_2)_C^{\Lambda^{\mathrm{op}},R}$ give a contravariant equivalence.*

Proof. As we know that $(?)^\dagger$ and $(?)^\ddagger$ are contravariant adjoint each other, it suffices to show that the unit $\lambda_M : M \to M^{\dagger\ddagger}$ and the (co-)unit $\mu_N : N \to N^{\ddagger\dagger}$ are isomorphisms. λ_M is an isomorphism by Lemma 6.24. Note that C is a 2-canonical left Γ-module by Lemma 6.23. So μ_N is an isomorphism by Lemma 6.24 applied to the right Γ^{op}-module C. □

Corollary 6.30. *Let C be a 2-canonical Λ-bimodule. Then $(?)^\dagger = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(?, C)$ and $\mathrm{Hom}_\Lambda(?, C)$ give a contravariant equivalence between $(S'_2)_C^{\Lambda^{\mathrm{op}},R}$ and $(S'_2)_C^{\Lambda,R}$. They also give a duality of $(S'_2)_C^{\Lambda\otimes\Lambda^{\mathrm{op}},R}$.*

Proof. The first assertion is immediate by Proposition 6.29 and Theorem 6.27. The second assertion follows easily from the first and Corollary 6.28. □

7. Non-commutative Aoyama's theorem

Lemma 7.1. *Let $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ be a flat local homomorphism between Noetherian local rings.*

1 *Let M be a Λ-bimodule such that $M' := R' \otimes_R M$ is isomorphic to $\Lambda' := R' \otimes_R \Lambda$ as a Λ'-bimodule. Then $M \cong \Lambda$ as a Λ-bimodule.*

2 *Let M be a right Λ module such that $M' := R' \otimes_R M$ is isomorphic to $\Lambda' := R' \otimes_R \Lambda$ as a right Λ'-module. Then $M \cong \Lambda$ as a right Λ-module.*

Proof. Taking the completion, we may assume that both R and R' are complete. Let $1 = e_1 + \cdots + e_r$ be the decomposition of 1 into the mutually orthogonal primitive idempotents of the center S of Λ. Then replacing R by Se_i, Λ by Λe_i, and R' by the local ring of $R' \otimes_R Se_i$ at any maximal ideal, we may further assume that $S = R$. This is equivalent to say that $R \to \operatorname{End}_{\Lambda \otimes_R \Lambda^{\mathrm{op}}} \Lambda$ is isomorphic. So $R' \to \operatorname{End}_{\Lambda' \otimes_{R'} (\Lambda')^{\mathrm{op}}} \Lambda'$ is also isomorphic, and hence the center of Λ' is R'.

1. Let $\psi : M' \to \Lambda'$ be an isomorphism. Then we can write $\psi = \sum_{i=1}^m u_i \psi_i$ with $u_i \in R'$ and $\psi_i \in \operatorname{Hom}_{\Lambda \otimes_R \Lambda^{\mathrm{op}}}(M, \Lambda)$. Also, we can write $\psi_i^{-1} = \sum_{j=1}^n v_j \varphi_j$ with $v_j \in R'$ and $\varphi_j \in \operatorname{Hom}_{\Lambda \otimes_R \Lambda^{\mathrm{op}}}(\Lambda, M)$. As $\sum_{i,j} u_i v_j \psi_i \varphi_j = \psi \psi^{-1} = 1 \in \operatorname{End}_{\Lambda' \otimes_{R'} (\Lambda')^{\mathrm{op}}} \Lambda' \cong R'$ and R' is local, there exists some i, j such that $u_i v_j \psi_i \varphi_j$ is an automorphism of Λ'. Then $\psi_i : M' \to \Lambda'$ is also an isomorphism. By faithful flatness, $\psi_i : M \to \Lambda$ is an isomorphism.

2. It is easy to see that $M \in \operatorname{mod} \Lambda$ is projective. So replacing Λ by Λ/J, where J is the radical of Λ, and changing R and R' as above, we may assume that R is a field and Λ is central simple. Then there is only one simple right Λ-module, and M and Λ are direct sums of copies of it. As $M' \cong \Lambda'$, by dimension counting, the number of copies are equal, and hence M and Λ are isomorphic. □

Lemma 7.2. *Let $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ be a flat local homomorphism between Noetherian local rings.*

1 *Let C be a 2-canonical bimodule of Λ over R. Let M be a Λ-bimodule such that $M' := R' \otimes_R M$ is isomorphic to $C' := R' \otimes_R C$ as a Λ'-bimodule. Then $M \cong C$ as a Λ-bimodule.*

2 *Let C be a 2-canonical right Λ-module over R. Let M be a right Λ-module such that $M' := R' \otimes_R M$ is isomorphic to $C' := R' \otimes_R C$ as a right Λ'-module. Then $M \cong C$ as a right Λ-module.*

Proof. **1**. As $M' \cong C'$ and $C \in (S_2')_C$, it is easy to see that $M \in (S_2')_C$. Hence M is a Γ-bimodule, where $\Gamma = \operatorname{End}_{\Lambda^{\mathrm{op}}} C = \operatorname{End}_\Lambda C$, see (6.26) and Corollary 6.28. Note that $(M^\dagger)' \cong (C^\dagger)' \cong \Gamma'$ as Γ'-bimodules. By Lemma 7.1.**1**, we have that $M^\dagger \cong \Gamma$ as a Γ-bimodule. Hence $M \cong M^{\dagger\ddagger} \cong \Gamma^\ddagger \cong C$.

2. As $(M^\dagger)' \cong (C^\dagger)' \cong \Gamma'$ as Γ'-modules, $M^\dagger \cong \Gamma$ as Γ-modules by Lemma 7.1.**2**. Hence $M \cong M^{\dagger\ddagger} \cong \Gamma^\ddagger \cong C$. □

Proposition 7.3. *Let $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ be a flat local homomorphism between Noetherian local rings. Let M be a right Λ-module. Assume that $R'/\mathfrak{m}R'$ is zero-dimensional, and $M' := R' \otimes_R M$ is the right canonical module of $\Lambda' := R' \otimes_R \Lambda$. If $\Lambda \neq 0$, then $R'/\mathfrak{m}R'$ is Gorenstein.*

Proof. We may assume that both R and R' are complete. Replacing R by $R/\operatorname{ann}_R \Lambda$ and R' by $R' \otimes_R R/\operatorname{ann}_R \Lambda$, we may assume that Λ is a faithful R-module. Let $d = \dim R = \dim R'$. Note that M is a finite R-module.

Then

$$R' \otimes_R H^d_{\mathfrak{m}}(M) \cong H^d_{\mathfrak{m}'}(R' \otimes_R M) \cong H^d_{\mathfrak{m}'}(K_{\Lambda'}) \cong \operatorname{Hom}_{R'}(\Gamma', E'),$$

where $\Lambda' = R' \otimes_R \Lambda$, $E' = E_{R'}(R'/\mathfrak{m}')$ is the injective hull of the residue field, $\Gamma = \operatorname{End}_{\Lambda^{\mathrm{op}}} M$, $\Gamma' = R' \otimes_R \Gamma \cong \operatorname{End}_{\Lambda'} K_{\Lambda'}$, and the isomorphisms are those of Γ'-modules. The last isomorphism is by (5.19). So $R' \otimes_R H^d_{\mathfrak{m}}(M) \in \operatorname{Mod}\Gamma'$ is injective. Considering the spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_{R' \otimes_R (\Gamma \otimes_R k)}(W, \operatorname{Ext}^q_{\Gamma'}(R' \otimes_R (\Gamma \otimes_R k), R' \otimes_R H^d_{\mathfrak{m}}(M))) \Rightarrow \operatorname{Ext}^{p+q}_{\Gamma'}(W, R' \otimes_R H^d_{\mathfrak{m}}(M))$$

for $W \in \operatorname{Mod}(R' \otimes_R (\Gamma \otimes_R k))$, $E_2^{1,0} = E_\infty^{1,0} \subset \operatorname{Ext}^1_{\Gamma'}(W, R' \otimes_R H^d_{\mathfrak{m}}(M)) = 0$ by the injectivity of $R' \otimes_R H^d_{\mathfrak{m}}(M)$. It follows that

$$\operatorname{Hom}_{\Gamma'}(R' \otimes_R (\Gamma \otimes_R k), R' \otimes_R H^d_{\mathfrak{m}}(M)) \cong (R'/\mathfrak{m}R') \otimes_k \operatorname{Hom}_R(k, H^d_{\mathfrak{m}}(M))$$

is an injective $(R'/\mathfrak{m}R') \otimes_k (\Gamma \otimes_R k)$-module. However, as an $R'/\mathfrak{m}R'$-module, this is a free module. Also, this module must be an injective $R'/\mathfrak{m}R'$-module, and hence $R'/\mathfrak{m}R'$ must be Gorenstein. □

Lemma 7.4. *Let $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ be a flat local homomorphism between Noetherian local rings such that $R'/\mathfrak{m}R'$ is Gorenstein. Assume that the canonical module K_Λ of Λ exists. Then $R' \otimes_R K_\Lambda$ is the canonical module of $R' \otimes_R \Lambda$.*

Proof. We may assume that both R and R' are complete and $\dim \Lambda = \dim R$. Let $\mathbb{I}$ be the normalized dualizing complex of R. Then $R' \otimes_R \mathbb{I}[d' - d]$ is a normalized dualizing complex of R', where $d' = \dim R'$ and $d = \dim R$, since $R \to R'$ is a flat local homomorphism with the $d' - d$-dimensional Gorenstein closed fiber, see [AvF, (5.1)] (the definition of a normalized dualizing complex in [AvF] is different from ours. We follow the one in [Hart, Chapter V]). So

$$R' \otimes_R K_\Lambda \cong R' \otimes_R \operatorname{Ext}_R^{-d}(\Lambda, \mathbb{I}) \cong \operatorname{Ext}_R^{-d'}(R' \otimes_R \Lambda, R' \otimes_R \mathbb{I}[d'-d]) \cong K_{\Lambda'}.$$

□

Theorem 7.5 ((Non-commutative Aoyama's theorem) cf. [Aoy, Theorem 4.2])**.** *Let $(R, \mathfrak{m}) \to (R', \mathfrak{m}')$ be a flat local homomorphism between Noetherian local rings, and Λ a module-finite R-algebra.*

1 *If M is a Λ-bimodule and $M' = R' \otimes_R M$ is the canonical module of $\Lambda' = R' \otimes_R \Lambda$, then M is the canonical module of Λ.*

2 *If M is a right Λ-module such that M' is the right canonical module of Λ', then M is the right canonical module of Λ.*

Proof. We may assume that both R and R' are complete. Then the canonical module exists, and the localization of a canonical module is a canonical module, and hence we may localize R' by a minimal element of $\{P \in \operatorname{Spec} R' \mid P \cap R = \mathfrak{m}\}$, and take the completion again, we may further assume that the fiber ring $R'/\mathfrak{m}R'$ is zero-dimensional. Then $R'/\mathfrak{m}R'$ is Gorenstein by Proposition 7.3. Then by Lemma 7.4, $M' \cong K_{\Lambda'} \cong R' \otimes_R K_\Lambda$. By Lemma 7.2, $M \cong K_\Lambda$. In **1**, the isomorphisms are those of bimodules, while in **2**, they are of right modules. The proofs of **1** and **2** are complete. □

Corollary 7.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and assume that K is the canonical* (*resp. right canonical*) *module of Λ. If $P \in \operatorname{Supp}_R K$, then the localization K_P is the canonical* (*resp. right canonical*) *module of Λ_P. In particular, K is a semicanonical bimodule* (*resp. right module*)*, and hence is 2-canonical over $R/\operatorname{ann}_R \Lambda$.*

Proof. Let Q be a prime ideal of $\hat{R}$ lying over P. Then $(\hat{K})_Q \cong \hat{R}_Q \otimes_{R_P} K_P$ is nonzero by assumption, and hence is the canonical (resp. right canonical) module of $\hat{R}_Q \otimes_R \Lambda$. Using Theorem 7.5, K_P is the canonical (resp. right canonical) module of Λ_P. The last assertion follows. □

(7.7) Let $(R, \mathfrak{m})$ be local, and assume that K_Λ exists. Assume that Λ is a faithful R-module. Then it is a 2-canonical Λ-bimodule over R by Corollary 7.6. Letting $\Gamma = \operatorname{End}_{\Lambda^{\mathrm{op}}} K_\Lambda$, $K_\Gamma \cong K_\Lambda$ as Λ-bimodules by Corollary6.22. So by Corollary 6.28, there exists some Γ-bimodule structure of K_Λ such that

$K_\Gamma \cong K_\Lambda$ as Γ-bimodules. As the left Γ-module structure of K_Λ which extends the original left Λ-module structure is unique, and it is the obvious action of $\Gamma = \operatorname{End}_{\Lambda^{\mathrm{op}}} K_\Lambda$. Similarly the right action of Γ is the obvious action of $\Gamma = \Delta = (\operatorname{End}_\Lambda K_\Lambda)^{\mathrm{op}}$, see (6.26).

8. Evans–Griffith's theorem for n-canonical modules

Lemma 8.1 (cf. [Aoy, Proposition 2], [Ogo, Proposition 4.2], [AoyG, Proposition 1.2])**.** *Let $(R, \mathfrak{m})$ be local and assume that Λ has a canonical module K_Λ. Then we have*

1 $\lambda_R : \Lambda \to \operatorname{End}_{\Lambda^{\mathrm{op}}} K_\Lambda$ *is injective if and only if Λ satisfies the $(S_1')^R$ condition and* $\operatorname{Supp}_R \Lambda$ *is equidimensional.*

2 $\lambda_R : \Lambda \to \operatorname{End}_{\Lambda^{\mathrm{op}}} K_\Lambda$ *is bijective if and only if Λ satisfies the $(S_2')^R$ condition.*

Proof. Replacing R by $R/\operatorname{ann}_R \Lambda$, we may assume that Λ is a faithful R-module. Then K_Λ is a 2-canonical Λ-bimodule over R by Corollary 7.6. K_Λ is full if and only if $\operatorname{Supp}_R \Lambda$ is equidimensional by Lemma 5.10.**1**.

Now **1** is a consequence of Lemma 6.19. **2** follows from Corollary 6.25 and Lemma 5.12. □

Proposition 8.2 (cf. [AoyG, (2.3)])**.** *Let $(R, \mathfrak{m})$ be a local ring, and assume that there is an R-canonical module K_Λ of Λ. Assume that $\Lambda \in (S_2)^R$, and K_Λ is a Cohen–Macaulay R-module. Then Λ is Cohen–Macaulay. If, moreover, K_Λ is maximal Cohen–Macaulay, then so is Λ.*

Proof. The second assertion follows from the first. We prove the first assertion. Replacing R by $R/\operatorname{ann}_R \Lambda$, we may assume that Λ is faithful. Let $d = \dim R$. So Λ satisfies (S_2'), and K_Λ is maximal Cohen–Macaulay. As K_Λ is the lowest non-vanishing cohomology of $\mathbb{J} := \mathbf{R}\operatorname{Hom}_R(\Lambda, \mathbb{I})$, there is a natural map $\sigma : K_\Lambda[d] \to \mathbb{J}$ which induces an isomorphism on the $-d$th cohomology groups. Then the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\lambda} & \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(K_\Lambda[d], K_\Lambda[d]) \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\sigma_*} \\ \mathbf{R}\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\mathbb{J}, \mathbb{J}) & \xrightarrow{\sigma^*} & \mathbf{R}\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(K_\Lambda[d], \mathbb{J}) \end{array}$$

is commutative. The top horizontal arrow λ is an isomorphism by Lemma 8.1. Note that

$$\mathbf{R}\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathbb{J},\mathbb{J}) \cong \mathbf{R}\mathrm{Hom}_R(\mathbb{J},\mathbb{I}) = \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(\Lambda,\mathbb{I}),\mathbb{I}) = \Lambda,$$

and the left vertical arrow is an isomorphism. As K_Λ is maximal Cohen–Macaulay, $\mathbf{R}\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(K_\Lambda[d],\mathbb{J})$ is concentrated in degree zero. As $H^i(\mathbb{J}) = 0$ for $i < -d$, we have that the right vertical arrow σ_* is an isomorphism. Thus the bottom horizontal arrow σ^* is an isomorphism. Applying $\mathbf{R}\mathrm{Hom}_\Lambda(?,\mathbb{J})$ to this map, we have that $K_\Lambda[d] \to \mathbb{J}$ is an isomorphism. So Λ is Cohen–Macaulay, as desired. □

Corollary 8.3 (cf. [AoyG, (2.2)]). *Let $(R,\mathfrak{m})$ be a local ring, and assume that there is an R-canonical module K_Λ of Λ. Then K_Λ is a Cohen–Macaulay (resp. maximal Cohen–Macaulay) R-module if and only if $\Gamma = \mathrm{End}_{\Lambda^{\mathrm{op}}} K_\Lambda$ is so.*

Proof. As K_Λ and Γ have the same support, if both of them are Cohen–Macaulay and one of them are maximal Cohen–Macaulay, then the other is also. So it suffices to prove the assertion on the Cohen–Macaulay property. To verify this, we may assume that Λ is a faithful R-module. Note that Γ satisfies (S_2'). By Corollary 6.22, K_Λ is Cohen–Macaulay if and only if K_Γ is. If Γ is Cohen–Macaulay, then K_Γ is Cohen–Macaulay by (5.18). Conversely, if K_Γ is Cohen–Macaulay, then Γ is Cohen–Macaulay by Proposition 8.2. □

Theorem 8.4 (cf. [EvG, (3.8)], [ArI, (3.1)]). *Let R be a Noetherian commutative ring, and Λ a module-finite R-algebra, which need not be commutative. Let $n \geq 1$, and C be a right n-canonical Λ-module. Set $\Gamma = \mathrm{End}_{\Lambda^{\mathrm{op}}} C$. Let $M \in \mathrm{mod}\, C$. Then the following are equivalent.*

1 *$M \in \mathrm{TF}(n,C)$. That is, M is (n,C)-TF.*

2 *$M \in \mathrm{UP}(n,C)$. That is, M has an (n,C)-universal pushforward.*

3 *$M \in \mathrm{Syz}(n,C)$. That is, M is an (n,C)-syzygy.*

4 *$M \in (S_n')_C$. That is, M satisfies the (S_n') condition as an R-module, and $\mathrm{Supp}\, M \subset \mathrm{Supp}\, C$.*

Proof. **1**⇒**2**⇒**3**⇒**4** is easy. We prove **4**⇒**1**. By Lemma 6.5, we may assume that $n \geq 2$. By Lemma 6.24, $M \in \mathrm{TF}(2,C)$. Let

$$\mathbb{F} : 0 \leftarrow M^\dagger \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{n-1}$$

be a resolution of $M^\dagger$ in $\Gamma\,\mathrm{mod}$ with each $F_i \in \operatorname{add}\Gamma$. It suffices to prove its dual

$$\mathbb{F}^\ddagger : 0 \to M \to F_0^\ddagger \to F_1^\ddagger \to \cdots \to F_{n-1}^\ddagger$$

is acyclic. By Lemma 2.10, we may localize at $P \in R^{\langle < n\rangle}$, and may assume that $\dim R < n$. If $M = 0$, then $\mathbb{F}$ is split exact, and so $\mathbb{F}^\ddagger$ is also exact. So we may assume that $M \neq 0$. Then by assumption, $C \cong K_\Lambda$ in $\mathrm{mod}\,\Lambda$, and C is a maximal Cohen–Macaulay R-module. Hence Γ is Cohen–Macaulay by Corollary 8.3. So by (5.16) and Lemma 6.22, $\mathbf{R}\mathrm{Hom}_\Gamma(M^\dagger, C) = \mathbf{R}\mathrm{Hom}_\Gamma(M^\dagger, K_\Gamma) = M$, and we are done. □

Corollary 8.5. *Let the assumptions and notation be as in* Theorem 8.4*. Let* $n \geq 0$*. Assume further that*

1 $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(C, C) = 0$ *for* $1 \leq i \leq n$*;*

2 C *is* Λ*-full.*

3 Λ *satisfies the* $(S'_n)^R$ *condition.*

Then for $0 \leq r \leq n$*,* ${}^{\perp_r}C$ *is contravariantly finite in* $\mathrm{mod}\,\Lambda$*.*

Proof. For any $M \in \mathrm{mod}\,\Lambda$, the nth syzygy module $\Omega^n M$ satisfies the $(S'_n)^R_C$-condition by **2** and **3**. By Theorem 8.4, $\Omega^n M \in \mathrm{TF}_{\Lambda^{\mathrm{op}}}(n, C)$. By Theorem 3.16, $M \in \mathcal{Z}_{r,0}$, and there is a short exact sequence

$$0 \to Y \to X \xrightarrow{g} M \to 0$$

with $X \in \mathcal{X}_{r,0} = {}^{\perp_r}C$ and $Y \in \mathcal{Y}_r$. As $\mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(\mathcal{X}_{r,0}, Y) = 0$, we have that g is a right ${}^{\perp_r}C$-approximation, and hence ${}^{\perp_r}C$ is contravariantly finite. □

Corollary 8.6. *Let the assumptions and notation be as in* Theorem 8.4*. Let* $n \geq 0$*, and* C *a* Λ*-full* $(n+2)$*-canonical* Λ*-bimodule over* R*. Assume that* Λ *satisfies the* $(S'_{n+2})^R$ *condition. Then* ${}^{\perp_n}C$ *is contravariantly finite in* $\mathrm{mod}\,\Lambda$*.*

Proof. By Corollary 8.5, it suffices to show that $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(C, C) = 0$ for $1 \leq i \leq n$. Let $\Delta = (\mathrm{End}_\Lambda C)^{\mathrm{op}}$. Then the canonical map $\Lambda \to \Delta$ is an isomorphism by Lemma 6.25, since C is a Λ-full 2-canonical Λ-bimodule over R. As $\Lambda \in (S'_{n+2})^R$ and C is a Λ-full $(n+2)$-canonical left Λ-module over R, applying Theorem 8.4 to Λ^{op}, we have that $\mathrm{Ext}^i_{\Delta^{\mathrm{op}}}(C, C) = 0$ for $1 \leq i \leq n$. As we have $\Lambda^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ is an isomorphism, we have that $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(C, C) = 0$, as desired. □

References

[Aoy] Y. Aoyama, Some basic results on canonical modules, *J. Math. Kyoto Univ.* **23** (1983), 85–94.

[AoyG] Y. Aoyama and S. Goto, On the endomorphism ring of the canonical module, *J. Math. Kyoto Univ.* **25** (1985), 21–30.

[ArI] T. Araya and K.-i. Iima, Locally Gorensteinness over Cohen–Macaulay rings, `arXiv:1408.3796v1`

[ASS] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras,* Cambridge (2006).

[AvF] L. L. Avramov and H.-B. Foxby, Locally Gorenstein homomorphisms, *Amer. J. Math.* **114** (1992), 1007–1047.

[BS] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications,* Cambridge (1998).

[EvG] E. G. Evans and P. Griffith, *Syzygies,* London Math. Soc. Lecture Note Series **106**, Cambridge (1985).

[Gro] A. Grothendieck, *Eléments de Géométrie Algébrique IV, 3e Partie,* IHES Publ. Math. **28** (1966).

[Hart] R. Hartshorne, *Residues and Duality,* Lecture Notes in Math. **20**, Springer (1966).

[Hart2] R. Hartshorne, Generalized divisors on Gorenstein schemes, *K-Theory* **8** (1994), 287–339.

[Has] M. Hashimoto, Equivariant class group. III. Almost principal fiber bundles, `arXiv:1503.02133v1`

[IW] O. Iyama and M. Wemyss, *On the noncommutative Bondal-Orlov conjecture, J. Reine Angew. Math.* **683** (2013), 119–128.

[LeW] G. J. Leuschke and R. Wiegand, *Cohen–Macaulay Representations,* AMS (2012).

[Mat] H. Matsumura, *Commutative Ring Theory,* First paperback edition, Cambridge (1989).

[Ogo] T. Ogoma, Existence of dualizing complexes, *J. Math. Kyoto Univ.* **24** (1984), 27–48.

[PS] C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale, *Publ. IHES* **42** (1973), 47–119.

[Tak] R. Takahashi, A New approximation theory which unifies spherical and Cohen–Macaulay approximations, *J. Pure Appl. Algebra* **208** (2007), 617–634.

[TT] R. W. Thomason and T. Trobaugh, Higher algebraic K-theory of schemes and of derived categories, in *The Grothendieck Festschrift. III.* (Birkhäuser) (1990), pp. 247–435.

[Yek] A. Yekutieli, Dualizing complexes over noncommutative graded algebras, *J. Algebra* **153** (1992), 41–84.