# SCATTERING AND SEMI-CLASSICAL ASYMPTOTICS FOR PERIODIC SCHRÖDINGER OPERATORS WITH OSCILLATING DECAYING POTENTIAL

Mouez Dimassi and Anh Tuan Duong

ABSTRACT. In the semi-classical regime (i.e., $h \searrow 0$), we study the effect of an oscillating decaying potential $V(hy, y)$ on the periodic Schrödinger operator $H$. The potential $V(x, y)$ is assumed to be smooth, periodic with respect to $y$ and tends to zero as $|x| \to \infty$. We prove the existence of $\mathcal{O}(h^{-n})$ eigenvalues in each gap of the operator $H + V(hy, y)$. We also establish a Weyl type asymptotics formula of the counting function of eigenvalues with optimal remainder estimate. We give a weak and pointwise asymptotic expansions in powers of $h$ of the spectral shift function corresponding to the pair $(H + V(hy, y), H)$. Finally, under some analytic assumption on the potential $V$ we prove the existence of shape resonances, and we give their asymptotic expansions in powers of $h^{1/2}$. All our results depend on the Floquet eigenvalues corresponding to the periodic Schrödinger operator $H + V(x, y)$, (here $x$ is a parameter).

## 1. INTRODUCTION

Consider the periodic Schrödinger operator in $L^2(\mathbb{R}^n)$

$$H = -\Delta + V_0(y),$$

where $V_0$ is a smooth real-valued potential, periodic with respect to a lattice $\Gamma$ in $\mathbb{R}^n$. It is well known (see e.g. [39]) that the spectrum of $H$ is purely absolutely continuous and has the band-gap structure:

$$\sigma(H) = \sigma_{\mathrm{ess}}(H) = \cup_{j=1}^{\infty} \Lambda_j$$

where $\Lambda_j := [a_j, b_j] = \mu_j(\mathbb{R}^n / \Gamma^*)$ are closed bounded intervals called the bands of the spectrum which may be pairwise disjoint or overlapping with $\lim_{j \to \infty} a_j = +\infty$. Here $(\mu_j(\cdot))_{j \geq 1}$ is the sequence of the Bloch eigenvalues associated to $H$ (see Section 2) and $\Gamma^*$ denotes the dual lattice of $\Gamma$. Introduce the self-adjoint perturbed operator acting on $L^2(\mathbb{R}^n)$

$$H(h) = H + V(hy, y), \ (h \searrow 0),$$

with domain $H^2(\mathbb{R}^n)$- the Sobolev space of order 2, where $h$ is a semi-classical parameter and the external potential $V(x, y)$ is assumed to be smooth, $\Gamma$-periodic with respect to $y$ and tends to zero as $|x| \to \infty$. By

---

the Weyl criterion, the essential spectra of $H(h)$ and $H$ are the same, and discrete eigenvalues with finite multiplicities can arise in the gaps of $H$. Moreover, the potential term may create embedded eigenvalues and resonances.

In the case where the potential $V$ is independent of the periodic variable (i.e., $V(x, y) = V(x)$), a wide literature is devoted to the study of discrete spectrum of $H(1) = H + V(y)$ (see [1, 4, 3, 5, 33, 48] and the references given there). The counting function of eigenvalues accumulating to an edge of a gap of $H$ was studied in [37]. The asymptotic behavior of the discrete spectrum of the operator $H_\mu = -\Delta + V_0(y) + \mu V(y)$ in a gap was well studied for a strong coupling (i.e., $\mu \to \infty$), see [4, 3, 47]. The semi-classical case, $-\Delta + V_0(y) + V(hy)$, $(h \searrow 0)$ was considered in [14, 15].

Recently, there has been a growing interest in studying the Schrödinger operator with decaying oscillating potential (see [38] and the references given there). The asymptotic behaviour of the discrete spectrum of $H(1) = -\Delta + V(y, y)$ near the origin has been studied in [38]. In the one-dimensional case, the existence and the asymptotic behaviour of the eigenvalues of the operator $Q(h) = -\partial_x^2 + V_0(x) + V(x, \frac{x}{h})$, tending to the border of the essential spectrum as $h \searrow 0$, have been established in [8] for $V_0 = 0$, and in [22] for periodic potential $V_0$ (see also [7, 8, 21, 23, 24]). Our problem here is different. In fact, the scaling of $H(h)$ is that of semi-classical analysis. In particular, the number of discrete eigenvalues grows as $h \searrow 0$, and satisfies a Weyl type asymptotics.

In this paper, we investigate the effect of the slowly varying decaying oscillating potential $(V(hy, y), h \searrow 0)$ on the gaps and bands of the non-perturbed Hamiltonian $H$. First, we give a complete asymptotic expansion in powers of $h$ of $\mathrm{tr}(f(H(h)))$ where $f \in C_0^\infty(]a, b[; \mathbb{R})$. In particular, we obtain a Weyl type asymptotics with optimal remainder estimates of the counting function of eigenvalues of $H(h)$ in any closed interval in $[a, b]$.

To investigate the effect of the perturbation on the continuous spectrum of $H$, it is natural to study the spectral shift function (SSF for short) and the resonances of $H(h)$. When $V$ satisfies:

(1.1)
$$\forall \alpha, \beta \in \mathbb{N}^n, \exists C_{\alpha,\beta} > 0, \forall x, y \in \mathbb{R}^n, \quad |\partial_x^\alpha \partial_y^\beta V(x, y)| \le C_{\alpha,\beta}\langle x \rangle^{-\delta}, \quad \text{with } \delta > n,$$

the SSF $\xi(\mu; h)$ is defined as a real-valued function on $\mathbb{R}$ satisfying the Birman-Krein formula:

(1.2)
$$\mathrm{tr}\big[f(H(h)) - f(H)\big] = -\langle \xi'(\cdot; h), f(\cdot) \rangle = \int_{\mathbb{R}} \xi(\mu; h) f'(\mu) d\mu,$$

for any $f \in C_0^\infty(\mathbb{R})$. The function $\xi(\mu; h)$ is fixed up to a constant by the formula, and we normalize $\xi(\mu; h)$ so that $\xi(\mu; h) = 0$ for $\mu < \inf(\sigma(H(h)))$.

The spectral shift function may be considered as a generalization of the eigenvalues counting function. It is one of important physical quantities in scattering theory, and it plays an important role in the study of the location of resonances in various scattering problems. We refer to [45, 9] and references cited there for comprehensive information on related subjects.

Under the assumption (1.1), we give a complete asymptotic expansion in powers of $h$ of the left hand side of (1.2). We also establish a Weyl type asymptotics for $\xi(\mu; h)$, (see Corollary 2.2 and Corollary 2.3).

Finally, under some analytic assumption on the perturbation $V$, we establish the existence of shape resonances near the extremities of the bands, and we give their asymptotic expansions in powers of $h^{1/2}$.

While most of these results are new, similar results have been obtained in the past. As already pointed out, the discrete spectrum of the operator $H(h) = -\Delta + V_0(y) + V(hy), (h \searrow 0)$, has already been studied in [14, 15] and bears many similarities with the present approach. The results in [14, 15] depend only on the Floquet eigenvalues of the non perturbed periodic operator $H = -\Delta + V_0(y)$. However, the oscillating potential affects significantly the results and techniques. In particular, all the results here will depend on the Floquet eigenvalues of the periodic Schrödinger operator $\tilde{H} = -\Delta + V_0(y) + V(x, y)$ (where $x$ is a parameter). Our proofs are based on the effective Hamiltonians and the semi-classical techniques (see Subsection 2.4 for an outline of the proof).

The paper is organized as follows : In the next section, we formulate our main assumptions and results and we give an outline of the proofs. We introduce a class of symbols and the corresponding $h$-Weyl operators (Subsection 3.2). In Subsections 3.1 and 3.3 we recall the effective Hamiltonian method. The proofs of the main results are given in Section 4.

*Notations :* We shall employ the following standard notations. Given a complex function $f_h$ depending on a small positive parameter $h$, the relation $f_h = \mathcal{O}(h^N)$ means that there exists $C_N, h_N > 0$ such that $|f_h| \leq C_N h^N$ for all $h \in ]0, h_N[$. The relation $f_h = \mathcal{O}(h^\infty)$ means that, for all $N \in \mathbb{N} = \{0, 1, 2, \ldots\}$, we have $f_h = \mathcal{O}(h^N)$. We write $f_h \sim \sum_{j=0}^\infty a_j h^j$ if, for each $N \in \mathbb{N}$, we have $f_h - \sum_{j=0}^N a_j h^j = \mathcal{O}(h^{N+1})$.

Let $\mathcal{H}$ be a Hilbert space. The scalar product in $\mathcal{H}$ will be denoted by $\langle \cdot, \cdot \rangle$. The set of linear bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{L}(\mathcal{H}_1)$ in the case where $\mathcal{H}_1 = \mathcal{H}_2$.

## 2. Preliminaries and results

Let $\Gamma = \overset{n}{\underset{i=1}{\oplus}} \mathbb{Z} v_i$ be a lattice generated by the basis $\{v_1, v_2, \cdots, v_n\} \subset \mathbb{R}^n$. The reciprocal lattice $\Gamma^*$ is defined as the lattice generated by the dual basis $\{v_1^*, \cdots, v_n^*\}$ determined by $v_j \cdot v_i^* = 2\pi \delta_{ij}, i, j = 1, \cdots, n$. Let $E$ and $E^*$ be respectively fundamental domains for $\Gamma$ and $\Gamma^*$. If we identify opposite edges of $E$ (resp. $E^*$) then it becomes a flat torus denoted by $\mathbb{T} = \mathbb{R}^n/\Gamma$ (resp. $\mathbb{T}^* = \mathbb{R}^n/\Gamma^*$).

Throughout this paper, we always assume that $V_0$ and $V$ are regular, $\Gamma$-periodic with respect to $y$ and

$$\lim_{|x| \to +\infty} \sup_{y \in \mathbb{T}^*} |V(x, y)| = 0.$$

For $(x, \xi)$ fixed in $\mathbb{R}^{2n}$, we define

$$(2.1) \quad P_1(x, \xi) := (D_y + \xi)^2 + V_0(y) + V(x, y), \quad P_0(\xi) = (D_y + \xi)^2 + V_0(y),$$

as unbounded operators from $L^2(\mathbb{T})$ into $L^2(\mathbb{T})$ with domain $H^2(\mathbb{T})$. The Hamiltonian $P_j$ is semibounded and self-adjoint. Since the resolvent of $(D_y + \xi)^2$ is compact, the resolvent of $P_j$ is also compact, and therefore $P_j$ has a complete set of (normalized) eigenfunctions. The corresponding eigenvalues accumulate at infinity and we enumerate them according to their multiplicities,

$$(2.2) \qquad \lambda_1(x, \xi) \leq \lambda_2(x, \xi) \leq \cdots \quad \text{and} \quad \mu_1(\xi) \leq \mu_2(\xi) \leq \cdots$$

Since $e^{-iy \cdot \gamma^*} P_j e^{iy \cdot \gamma^*} = P_j(\cdot, \xi + \gamma^*)$, it follows that $\xi \mapsto \lambda_m(x, \xi)$, $\mu_m(\xi)$ are $\Gamma^*$−periodic and are called the band functions. Standard perturbation theory shows that they are real continuous functions. The spectrum of $H$ is absolutely continuous (see [39, 43]) and consists of the bands $\Lambda_m$, $m = 1, 2, \cdots$. Indeed, $\sigma(H) = \sigma_{\mathrm{ac}}(H) = \underset{m \geq 1}{\cup} \Lambda_m$, $\Lambda_m = \mu_m(\mathbb{T}^*)$ (see also [46]).

Let us now introduce the densities of states $\rho(\cdot, x)$ and $\rho_0(\cdot)$ associated with $P_1$ and $P_0$ (see [46])

$$(2.3) \quad \rho(t, x) := \frac{1}{(2\pi)^n} \sum_{m \geq 1} \int_{\{\xi \in E^*; \ \lambda_m(x, \xi) \leq t\}} d\xi, \quad \text{(here } x \text{ is a parameter)},$$

$$(2.4) \qquad \rho_0(t) := \frac{1}{(2\pi)^n} \sum_{m \geq 1} \int_{\{\xi \in E^*; \ \mu_m(\xi) \leq t\}} d\xi.$$

2.1. **Discrete Spectrum.** In this subsection we shall discuss the asymptotic behaviour of the discrete spectrum of $H(h)$ in the gaps of $H$. The

results of this subsection extend those obtained in [18] where the case $V_0 = 0$ was considered. Fix $[a, b]$ such that

$$\sigma(H) \cap [a, b] = \emptyset.$$

**Theorem 2.1.** *For $f \in C_0^\infty(]a, b[; \mathbb{R})$, the operator $f(H(h))$ is of trace class and, there exists a sequence of real numbers $(a_j)_{j \in \mathbb{N}}$ such that*

$$(2.5) \qquad \mathrm{tr}\Big[f(H(h))\Big] \sim \sum_{j=0}^{\infty} a_j h^{j-n}, \ h \searrow 0,$$

*with*
(2.6)
$$a_0 = (2\pi)^{-n} \sum_{k \geq 1} \iint_{\mathbb{R}_x^n \times E^*} f(\lambda_k(x, \xi)) dx d\xi = -\int_{\mathbb{R}_x^n} \int_{\mathbb{R}} f'(t)\rho(t, x) dt dx.$$

Let $N([a, b]; h)$ denote the number of eigenvalues of $H(h)$ in $[a, b]$ (counted with their multiplicity).

**Corollary 2.2.** *There holds*

$$(2.7) \qquad \lim_{h \searrow 0}\Big[h^n N([a, b]; h)\Big] = \int_{\mathbb{R}^n} \Big[\rho(b, x) - \rho(a, x)\Big] dx.$$

By adding an additional assumption, we shall improve the above corollary. For $\tau \in \{a, b\}$, we let

$$\Sigma_\tau = \cup_{j=1}^{\infty}\{(x, \xi) \in \mathbb{R}^n \times E^*; \ \lambda_j(x, \xi) = \tau\}.$$

We make the following assumption:

$\mathbf{H}$ : for $(x_0, \xi_0) \in \Sigma_\tau$, $\lambda_j(x_0, \xi_0) = \tau$ is simple (i.e., $\lambda_{j-1}(x_0, \xi_0) < \lambda_j(x_0, \xi_0) < \lambda_{j+1}(x_0, \xi_0)$), and $\nabla_{x, \xi}\lambda_j(x_0, \xi_0) \neq 0$.

**Theorem 2.3.** *Under the assumption $\mathbf{H}$, we have*

$$(2.8) \qquad h^n N([a, b]; h) = \int_{\mathbb{R}^n}\Big[\rho(b, x) - \rho(a, x)\Big] dx + \mathcal{O}(h), \ (h \searrow 0).$$

2.2. **Spectral Shift Function.**

**Theorem 2.4.** *(Weak asymptotics) Assume (1.1), and let $f \in C_0^\infty(\mathbb{R})$. The operator $\Big[f(H(h)) - f(H)\Big]$ is of trace class and, there exists a sequence of real numbers $(a_j)_{j \in \mathbb{N}}$ such that*

$$(2.9) \qquad \mathrm{tr}\Big[f(H(h)) - f(H)\Big] \sim \sum_{j=0}^{\infty} a_j h^{j-n}, \ h \searrow 0,$$

*with*

$$(2.10) \qquad a_0 = \int_{\mathbb{R}_x^n} \int_{\mathbb{R}} f'(t)\Big[\rho_0(t) - \rho(t, x)\Big] dt dx.$$

**Corollary 2.5.** ( *Pointwise asymptotics) Suppose that* $V_0 = 0$, *and fix* $\lambda > 0$. *Under the hypothesis of Theorem 2.4, we have*

$$(2.11) \qquad \lim_{h \searrow 0} \left[ (2\pi h)^n \xi(\lambda; h) \right] = \int_{\mathbb{R}^n} \left[ \rho(\lambda, x) - c_n (2\pi)^{-n} \lambda^{n/2} \right] dx.$$

*Here* $c_n$ *is the volume of the unit ball in* $\mathbb{R}^n$.

2.3. **Resonances.** In this subsection, we first recall the definition of resonances which can be found in [29] . Fix a point $\lambda_0$ in the interior of the spectrum of $H$, and assume that there exists a dense subset $\mathcal{A}$ in $L^2(\mathbb{R}^n)$ such that for all $\Phi, \Psi \in \mathcal{A}$ the function

$$\langle (z - H)^{-1} \Phi, \Psi \rangle, \ (\text{resp.}, \ K_{\Phi,\Psi} := \langle (z - H(h))^{-1} \Phi, \Psi \rangle),$$

has a holomorphic (resp. meromorphic) continuation from the upper half plane $\mathbb{C}^+ := \{z \in \mathbb{C}; \text{Im} z > 0\}$ to a complex disc around $\lambda_0$. The poles of $K_{\Phi,\Psi}$ are called resonances of $H(h)$. To state our resonance result, we need to introduce the following additional assumptions.

   (**H1**) There exist positive constants $\delta_0, \delta_1 > 0$ such that $x \mapsto V(x, y)$ extends analytically to $D(\delta_0) = \{z \in \mathbb{C}^n; |\text{Im} z| \leq \delta_0 \langle \text{Re} z \rangle \}$ and

$$(2.12) \qquad\qquad\qquad |V(z, y)| \leq C \langle z \rangle^{-\delta_1}$$

uniformly in $z \in D(\delta_0)$. Here $\langle z \rangle = (1 + |z|^2)^{1/2}$.

   Fix $\lambda_0 \in \sigma(H) = \cup_{j=1}^\infty \Lambda_j = \mu_j(\mathbb{T}^*)$. We assume that

   (**H2**)  There exists $m \geq 1$ such that

$$\begin{aligned} \Sigma_{\lambda_0} : &= \cup_{j=1}^\infty \{(x, \xi) \in \mathbb{R}^n \times E^*; \ \lambda_j(x, \xi) = \lambda_0\} \\ &= \{(x, \xi) \in \mathbb{R}^n \times E^*; \ \lambda_m(x, \xi) = \lambda_0\}, \end{aligned}$$

$$\Pi_{\lambda_0} := \cup_{j=1}^\infty \{\xi \in E^*; \ \mu_j(\xi) = \lambda_0\} = \{\xi \in E^*; \ \mu_m(\xi) = \lambda_0\}.$$

   (**H3**) If $\mu_m(\xi) = \lambda_0$, then $\nabla_\xi \mu_m(\xi) \neq 0$.
   (**H4**)  $\Sigma_{\lambda_0} = (x_0, \xi_0) \cup \Omega$, where $\Omega$ is a connected component, $(x_0, \xi_0) \notin \Omega$ and $\lambda_m$ has a local non degenerate extremum (local minimum or local maximum) at $(x_0, \xi_0)$. We assume in addition that near $\Omega$, $G(x, \xi) = x \cdot \mu_m(\xi)$ is an escape function for the classical Hamiltonian $\lambda_m$ i.e.,

$$(2.13) \qquad \{\lambda_m, G\} := \frac{\partial \lambda_m}{\partial \xi} \frac{\partial G}{\partial x} - \frac{\partial G}{\partial \xi} \frac{\partial \lambda_m}{\partial x} > c_0 > 0, \ \forall (x, \xi) \in \Omega.$$

   Notice that the  assumption (**H2**) implies that the eigenvalue $\lambda_m(x', \xi')$ (resp. $\mu_m(\xi')$) is simple if $\lambda_j(x', \xi') = \lambda_0$ (resp. $\mu_m(\xi') = \lambda_0$). In particular, it follows from (**H1** $-$ **2**) that the function $(x, \xi) \mapsto \lambda_m(x, \xi)$ (resp. $\xi \mapsto \mu_m(\xi)$) is analytic near $(x', \xi')$ (resp. $\xi'$) (see [29]).

   Without any loss of generality, we may assume that $(x_0, \xi_0) = (0, 0)$ and $\lambda_m''(0, 0)$ is strictly positive, (i.e., $(0, 0)$ is a local minimum of $\lambda_m$). Let

$\kappa_1, \cdots, \kappa_n, \zeta_1, \cdots, \zeta_n$ be the eigenvalues of the square Hermitian matrix $\frac{1}{2}\lambda_m''(0,0)$, and put

$$\mathcal{R} := \left\{ \sum_{j=1}^{n} (2m_j + 1)\zeta_j \kappa_j; \quad (m_1, \cdots, m_n) \in \mathbb{N}^n \right\}.$$

Let $e_1 < e_2 \leq e_3 \leq \cdots$ be the elements of $\mathcal{R}$ listed in increasing size counting multiplicity. Fix $0 < C_0 \notin \mathcal{R}$, and let $N_0$ be the number of $e_j \in [0, C_0]$, so that $e_1 < e_2 \leq \cdots \leq e_{N_0} < C_0 < e_{N_0+1}$. Our result concerning resonances is the following:

**Theorem 2.6.** *Let $C_0$ be as above and assume that (**H1 − H4**) hold. Then, there exists $h_0 > 0$ small enough such that $H(h)$ has exactly $N_0$ resonances $\left( e_j(h) \right)_{1 \leq j \leq N_0}$ in the disk $D(\lambda_0, C_0 h) = \{z \in \mathbb{C}; |z - \lambda_0| < C_0 h\}$ (counted with their algebraic multiplicity). Moreover, the following asymptotics holds:*

$$(2.14) \qquad e_j(h) \sim \lambda_0 + he_j + \sum_{k=1}^{\infty} e_{j,k} h^{1+\frac{k}{2}}, \quad e_{j,k} \in \mathbb{R}, \quad (h \searrow 0).$$

## 2.4. Comments and outline of the proofs.

By the change of variable $z = hy$, the operator $H(h)$ is unitarily equivalent to

$$(2.15) \qquad \tilde{H}(h) = -h^2 \Delta_z + V(z, \frac{z}{h}).$$

In the case where $V(x, y) = V(x)$ is independent of the periodic variable, the operator $\tilde{H}(h)$ is still the semi-classical Schrödinger one, and then all our results are well known in this case (see [20, 40] and the reference given there).

However, in our case, there are two spatial scales in the potential $V(hy, y)$, $y$ and $x = hy$, which are completely different when $h$ tends to zero. So $H(h)$ cannot be identified to the semi-classical Schrödinger operator. Here, as in [14, 15, 18], we use the effective Hamiltonian method, which allows us to reduce the spectral study of $H(h)$ to the one of a system of $h$-pseudodifferential operators $E_{-+}^{1,w}(x, hD_x, z; h) = E_{0,-+}^{1,w}(x, hD_x, z) + hE_{1,-+}^{1,w}(x, hD_x, z) + \cdots$, acting on $L^2(\mathbb{T}^*; \mathbb{C}^N)$ (see Proposition 3.3).

Thus we establish some trace formula involving the effective Hamiltonian $E_{-+}^{1,w}(x, hD_x, z; h)$ (see (4.7) and (4.12)). After that, using some standard results on $h$-pseudodifferential calculus we prove Theorem 2.1, Corollary 2.2, Theorem 2.3, Theorem 2.4 and Corollary 2.5.

Let us recall that, in the study of the spectral properties of a system of $h$-pseudodifferential operator $E_{-+}^{1,w}(x, hD_x, z; h)$, the characteristic set $\Sigma_z := \{(x, \xi); \det E_{0,-+}^1(x, \xi, z) = 0\}$ plays a crucial role. Here $E_{0,-+}^1(x, \xi, z)$ is

the principal symbol of $E^{1,w}_{-+}(x, hD_x, z; h)$. On the other hand, according to (3.16) we have

$$\Sigma_z = \cup_{j=1}^\infty \{(x,\xi); \lambda_j(x,\xi) = z\}.$$

Thus "microlocally" we will be only concerned with the behavior of $E^1_{0,-+}(x,\xi,z)$ near $\Sigma_z$. In particular, this explains why all our results only depend on $\lambda_j(x,\xi)$.

To prove Theorem 2.6, we notice that the assumption (**H2**) allows us to chose $E^1_{-+}(x,\xi,z;h)$ scalar valued with

$$E^1_{0,-+}(x,\xi,z) = z - \lambda_j(x,\xi).$$

On the other hand, the assumption (2.13) (which is a standard non-trapping condition) implies that $E^1_{-+}(x,\xi,z;h)$ restricted to $\Omega$ does not produce resonances near $\lambda_0$. Thus it suffices to study $E^1_{-+}(x,\xi,z;h)$ for $(x,\xi)$ near $(x_0,\xi_0)$. By a change of variable we may assume that $(x_0,\xi_0) = (0,0)$. Next, assumption (**H4**) tells us that

$$E^1_{0,-+}(x,\xi,\lambda_0) = \lambda_0 - \sum_{j=1}^n (\kappa_j x_j^2 + \zeta_j \xi_j^2) + \mathcal{O}(|(x,\xi)|^3),$$

for $(x,\xi)$ near $(0,0)$. Thus, modulo $\mathcal{O}(h^{3/2})$, the resonances of $H(h)$ near $\lambda_0$ coincide with the eigenvalues of the semi-classical harmonic oscillator

$$\sum_{j=1}^n (\kappa_j x_j^2 + \zeta_j h^2 D_{x_j}^2).$$

## 3. Effective Hamiltonian method

3.1. **Grushin problem: brief description.** In this paragraph, we review some of the standard facts on Grushin problem. Let $H_1, H_2$ and $H_3$ be three Hilbert spaces, and let $P \in \mathcal{L}(H_1, H_3)$. Assume that there exist $R_+ \in \mathcal{L}(H_1, H_2)$ and $R_- \in \mathcal{L}(H_2, H_3)$ such that the following operator

$$\mathcal{P}(z) = \begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : H_1 \times H_2 \to H_3 \times H_2$$

is bijective for $z \in \Omega$. Here $\Omega$ is an open bounded set in $\mathbb{C}$. Let

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}$$

be its inverse. We refer to the problem $\mathcal{P}(z)$ as a Grushin problem and the operator $E_{-+}(z)$ is called effective Hamiltonian. The following properties are consequences of the identities $\mathcal{E} \circ \mathcal{P} = I$ and $\mathcal{P} \circ \mathcal{E} = I$:

(3.1)        $(P - z)$ is invertible if and only if $E_{-+}(z)$ is invertible,

$$(3.2) \qquad \dim \ker(P - z) = \dim \ker(E_{-+}(z)),$$

$$(3.3) \qquad (P - z)^{-1} = E(z) - E_+(z)E_{-+}^{-1}(z)E_-(z),$$

$$(3.4) \qquad E_{-+}^{-1}(z) = -R_+(P - z)^{-1}R_-.$$

On the other hand, since $z \mapsto (P - z)$ is holomorphic, it follows that the operators $E(z), E_\pm(z)$ and $E_{-+}(z)$ are also holomorphic in $z \in \Omega$. Moreover, we have

$$(3.5) \qquad \partial_z E_{-+}(z) = E_-(z)E_+(z).$$

This identity comes from the fact that $R_\pm$ are independent of $z$.

### 3.2. Classes of symbols and notations.

For $(m, N) \in \mathbb{R} \times \mathbb{N}$ we denote by $S^m(\mathbb{T}^* \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C}))$ the space of $P \in C^\infty(\mathbb{R}^{2n}; \mathcal{M}_N(\mathbb{C}))$, $\Gamma^*$-periodic with respect to $x$, such that for all $\alpha$ and $\beta$ in $\mathbb{N}^n$ there exists $C_{\alpha,\beta} > 0$ such that

$$(3.6) \qquad \|\partial_\xi^\alpha \partial_x^\beta P(x,\xi)\|_{\mathcal{M}_N(\mathbb{C})} \le C_{\alpha,\beta}\langle\xi\rangle^{-m-|\alpha|}, \quad \langle\xi\rangle = \left(1 + |\xi|^2\right)^{\frac{1}{2}},$$

where $\mathcal{M}_N(\mathbb{C})$ is the set of $N \times N$-matrices. In the special cases when $N = 1$ (i.e., $P$ is real-valued) or $m = 0$, we will write $S^m(\mathbb{T}^* \times \mathbb{R}^n)$ or $S(\mathbb{T}^* \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C}))$ instead of $S^m(\mathbb{T}^* \times \mathbb{R}^n; \mathcal{M}_1(\mathbb{C}))$ and $S^0(\mathbb{T}^* \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C}))$.

If $P$ depends on a semi-classical parameter $h \in ]0, h_0]$ and possibly on other parameters as well, we require (3.6) to hold uniformly with respect to these parameters. For $h$- dependent symbols, we say that $P(x, \xi; h) \in S^m(\mathbb{T}^* \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C}))$ has an asymptotic expansion in powers of $h$, and we write

$$P(x, \xi; h) \sim \sum_{j=0}^\infty P_j(x, \xi)h^j,$$

if for every $k \in \mathbb{N}$, $h^{-(k+1)}\left(P - \sum_{j=0}^k P_j h^j\right) \in S^m(\mathbb{T}^* \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C}))$.

For $P \in S^m(\mathbb{T}^* \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C}))$, the $h$-Weyl operator $P^w(x, hD_x; h)$ is defined by:

$$P^w(x, hD_x; h)u(x) = (2\pi h)^{-n}\int_{\mathbb{T}^*}\int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\xi}P(\frac{x+y}{2}, \xi; h)u(y)\, dy\, d\xi.$$

$$\text{Here } D_x = \frac{1}{i}\frac{\partial}{\partial x}.$$

If $m \ge 0$, then $P^w(x, hD_x; h)$ is well defined and bounded from $L^2(\mathbb{T}^*)$ into $L^2(\mathbb{T}^*)$. In particular, we have a global $h$-pseudodifferential calculus on the torus in analogy to the one in Euclidean space. In an appendix, we recall some well known results on the $h$-pseudodifferential calculus.

3.3. **Effective Hamiltonian.** In this subsection, we recall the effective Hamiltonian method. More precisely, we will construct a suitable auxiliary (so-called Grushin) problem associated with the operator $\big(P(h)-z\big)$ for $z$ in a small complex neighborhood of some bounded interval $I = [a,b] \subset \mathbb{R}$. The reader can find more details and the proofs of the results of this subsection in [26] (see also [20, 19]). For the reader convenience, let us point out the main change in our situation and fix the notations.

Denote by $T_\Gamma$ the distribution in $\mathcal{S}'(\mathbb{R}^{2n})$ defined by

$$T_\Gamma(x,y) = \frac{1}{\mathrm{vol}(E)h^n} \sum_{\beta^* \in \Gamma^*} e^{i(x-hy)\frac{\beta^*}{h}}.$$

We recall that $E$ is a fundamental domain of $\Gamma$.

For $m \in \mathbb{N}$, we introduce the following Hilbert space with their natural norms

$$\mathbb{L}^m := \{u(x)T_\Gamma(x,y);\ \partial_x^\alpha u \in L^2(\mathbb{R}^n),\ \forall \alpha,\ |\alpha| \leq m\},$$

$$K_0 = L^2(\mathbb{T}),\ K_{m,\xi} = \{u \in K_0; (D_y + \xi)^\alpha u \in K_0, \forall |\alpha| \leq m\}.$$

It was shown in [20, Chapter 13, Proposition 13.5] that the operator $H(h)$ acting on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$ is unitarily equivalent to

$$(3.7) \qquad \mathbb{H}^1(h) := \big(D_y + hD_x\big)^2 + V_0(y) + V(x,y),$$

acting on $\mathbb{L}^0$ with domain $\mathbb{L}^2$, and the following propositions hold ( see [26]).

**Proposition 3.1.** ([26], *Proposition 2.1) There exist $N \in \mathbb{N}$, a complex neighborhood $\Omega$ of $I$, and a bounded operator $r_+$ in $\mathcal{L}\big(L^2(\mathbb{T}); \mathbb{C}^N\big)$ such that for all $z \in \Omega$ and $0 < h < h_0$ small enough, the operator*

$$(3.8) \qquad \mathcal{P}(x,\xi,z) := \begin{pmatrix} P_1(x,\xi) - z & r_+^* \\ r_+ & 0 \end{pmatrix} : H^2(\mathbb{T}) \times \mathbb{C}^N \to L^2(\mathbb{T}) \times \mathbb{C}^N,$$

*is bijective with bounded two-sided inverse*

$$(3.9) \qquad \mathcal{E}(x,\xi,z) := \begin{pmatrix} e(x,\xi,z) & e_+(x,\xi,z) \\ e_-(x,\xi,z) & e_{-+}(x,\xi,z) \end{pmatrix}.$$

*Here $e_{-+} \in S(\mathbb{R}^{2d}_{x,\xi}; \mathcal{M}_n(\mathbb{C}))$ is $\Gamma^*$-periodic in $\xi$.*

**Theorem 3.2.** ([26], *Theorem 2.3) For $h$ sufficiently small, the operator $\mathcal{P}^1(z,h) := \mathcal{P}^w(x, hD_x, z)$ has a uniformly bounded inverse of the form $\mathcal{E}^1(z,h) = \mathcal{E}^w(x, hD_x, z; h)$, where*

$$\mathcal{E}(x,\xi,z;h) \in S^0(\mathbb{R}^{2n}; \mathcal{L}(K_0 \oplus \mathbb{C}^N, K_{2,\xi} \oplus \mathbb{C}^N)).$$

**Proposition 3.3.** ([26], *Theorem 3.7, Remark 3.9) There exist $N \in \mathbb{N}$, a complex neighborhood $\Omega$ of $I$, and a bounded operator $R_+$*

in $\mathcal{L}\big(\mathbb{L}^0; L^2(\mathbb{T}^*; \mathbb{C}^N)\big)$ such that for all $z \in \Omega$ and $0 < h < h_0$ small enough, the operator

(3.10)
$$\mathcal{P}^1(z, h) := \begin{pmatrix} \mathbb{H}^1(h) - z & R_+^* \\ R_+ & 0 \end{pmatrix} : \mathbb{L}^2 \times L^2(\mathbb{T}^*; \mathbb{C}^N) \to \mathbb{L}^0 \times L^2(\mathbb{T}^*; \mathbb{C}^N),$$

is bijective with bounded two-sided inverse

(3.11)
$$\mathcal{E}^1(z, h) := \begin{pmatrix} E^1(z, h) & E_+^1(z, h) \\ E_-^1(z, h) & E_{-+}^1(z, h) \end{pmatrix}.$$

Here $E_{-+}^1(z, h) = E_{-+}^{1,w}(x, hD_x, z; h)$ is an $h-$pseudodifferential operator with symbol

(3.12) $\quad E_{-+}^1(x, \xi, z; h) \sim \sum_{l \geq 0} E_{l,-+}^1(x, \xi, z)\, h^l, \ in\ S^0\Big(\mathbb{T}^* \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C})\Big)$

where $E_{0,-+}^1(x, \xi, z) = e_{-+}(\xi, -x, z)$ is given in the above proposition.

**Remark 3.4.** (1) We denote by

$$\mathcal{P}^0(z, h) = \begin{pmatrix} \mathbb{H}^0(h) - z & R_+^* \\ R_+ & 0 \end{pmatrix}, \ \ \mathcal{E}^0(z, h) := \begin{pmatrix} E^0(z, h) & E_+^0(z, h) \\ E_-^0(z, h) & E_{-+}^0(z, h) \end{pmatrix}$$

the operators given by Proposition 3.1 when $V(x, y) = 0$.

(2) Note that, $R_+$ depends only on the non-perturbed periodic Schrödinger operator $H$. (see [26, Proposition 2.1] and [20, Chapter 13]). Therefore, we may take the same $R_+$ for $\mathcal{P}^1(z, h)$ and $\mathcal{P}^0(z, h)$.

For simplicity of notation we ignore the dependence of $E^j, E_\pm^j, E_{-+}^j$ on $(z, h)$.

The following formulas are consequences of (3.1), (3.2), (3.3), (3.4), and Propositions 3.1-3.3.

(3.13)
$$\big(\mathbb{H}^j(h) - z\big)^{-1} = E^j - E_+^j(E_{-+}^j)^{-1}E_-^j,$$

(3.14)
$$(E_{-+}^j)^{-1} = -R_+\big(\mathbb{H}^j(h) - z\big)^{-1}R_+^*,$$

and

(3.15)
$$\partial_z E_{-+}^j = E_-^j E_+^j,$$

(3.16)
$$\det\big(e_{-+}(x, \xi, z)\big) = 0 \iff \exists k \in \mathbb{N} \text{ such that } z = \lambda_k(x; \xi),$$

(3.17)
$$\|(e_{-+}(x, \xi, z))^{-1}\|_{\mathcal{L}(\mathcal{M}_N(\mathbb{C}))} \leq \frac{C}{|\mathrm{Im}z|},$$

$$\text{(3.18)} \qquad \dim \ker(P(x,\xi) - z) = \dim \ker(e_{-+}(x,\xi,z)).$$

**Remark 3.5.** *Let $z_0 \in \mathbb{R}$, $d = \dim \ker (e_{-+}(x,\xi,z_0))$ for a fixed $(x,\xi)$. By ordinary perturbation theory (see Kato [30]) we can reorder the eigenvalues $(\lambda_j(z))_{1 \le j \le N}$ of $e_{-+}(x,\xi,z)$ to be holomorphic in a neighborhood of $z_0 \in \mathbb{R}$ and $\lambda_1(z_0) = \cdots = \lambda_d(z_0) = 0$. Using (3.17) we see that*

$$|\lambda_j(z)| \ge C_j |\mathrm{Im} z|,$$

*so $\lambda'_j(z_0) \ne 0$ for all $1 \le j \le N$. Hence, $z \mapsto \det e_{-+}(x,\xi,z)$ has a root $z_0$ of multiplicity $d$.*

## 4. Proof of the results

4.1. **Proof of Theorem 2.1.** Fix $a < b$ such that $[a,b] \cap \sigma(H) = \emptyset$. Let $f \in C_0^\infty(]a,b[;\mathbb{R})$ and let $\varphi(x) \in C^\infty(\mathbb{R}_x^n; [0,1])$ be equal to one for $|x| > 2R$ and $\varphi(x) = 0$ for $|x| < R$. Since

$$\lim_{|x| \to +\infty} \sup_{y \in \mathbb{T}^*} |V(x,y)| = 0,$$

we choose $R$ large enough such that

$$\text{(4.1)} \qquad \sup_{(x,y) \in \mathbb{R}^{2n}} |\varphi(x) V(x,y)| \le \frac{\mathrm{dis}(\sigma(H), [a,b])}{2}.$$

Let $\widehat{e}_{-+}(x,\xi,z)$ be the effective Hamiltonian given by Proposition 3.3 associated with

$$\widehat{P}(x,\xi) = (D_y + \xi)^2 + V_0(y) + \varphi(x) V(x,y),$$

and put

$$\text{(4.2)} \quad \widehat{E}_{-+}(x,\xi,z;h) = \widehat{e}_{-+}(\xi, -x, z) + E_{-+}^1(x,\xi,z;h) - E_{0,-+}^1(x,\xi,z).$$

$$= \widehat{e}_{-+}(\xi, -x, z) + \sum_{j \ge 1} h^j E_{j,-+}^1(x,\xi,z).$$

Making use of (3.16) and (4.1) we deduce that

$$|\det \widehat{e}_{-+}(x,\xi,z)| \ge \frac{1}{C} \text{ uniformly on } (x,\xi,z) \in \mathbb{R}^n \times \mathbb{T}^* \times [a,b]$$

which together with (4.2) yield, for $h$ small enough,

$$\text{(4.3)} \quad |\det \widehat{E}_{-+}(x,\xi,z;h)| \ge \frac{1}{2C} \text{ uniformly on } (x,\xi,z) \in \mathbb{T}^* \times \mathbb{R}^n \times [a,b].$$

On the other hand, from the properties of $\varphi$ we have

$$E_{-+}^1(x,\xi,z;h) = \widehat{E}_{-+}(x,\xi,z;h) \text{ for large } |\xi|.$$

It follows from (4.3) and Proposition 5.4 that for $h$ small enough $(\widehat{E}_{-+})^{-1}$ is well defined and is holomorphic in $z$ near $[a, b]$ and

$$(4.4) \qquad \|(\widehat{E}_{-+})^{-1}\|_{\mathcal{L}(L^2(\mathbb{T}^*;\mathbb{C}^N))} = \mathcal{O}(1).$$

Let $\widetilde{f} \in C_0^\infty((a, b) + i[-1, 1])$ be an almost analytic extension of $f$, i.e., $\widetilde{f} = f$ on $\mathbb{R}$ and $\overline{\partial}_z \widetilde{f}$ vanishes on $\mathbb{R}$ to infinite order, i.e. $\overline{\partial}_z \widetilde{f}(z) = \mathcal{O}_N(|\text{Im } z|^N)$ for all $N \in \mathbb{N}$. Then the functional calculus due to Helffer-Sjöstrand (see e.g. [20, Chapter 8]) yields

$$f(\mathbb{H}^1(h)) = -\frac{1}{\pi} \int \overline{\partial}_z \widetilde{f}(z)(z - \mathbb{H}^1(h))^{-1} L(dz).$$

Here $L(dz) = dxdy$ is the Lebesgue measure on the complex plane $\mathbb{C} \sim \mathbb{R}^2_{x,y}$. The identity

$$(E^1_{-+})^{-1} = \widehat{E}^{-1}_{-+} - (E^1_{-+})^{-1}(E^1_{-+} - \widehat{E}_{-+})\widehat{E}^{-1}_{-+},$$

combined with (3.13) and the fact that $\widehat{E}^{-1}_{-+}, E^1, E^1_+, E^1_-$ are holomorphic in $z$ near $[a, b]$, give

$$(4.5) \quad f(\mathbb{H}^1(h)) = -\frac{1}{\pi} \int \overline{\partial}_z \widetilde{f}(z)\Big(E^1_+(E^1_{-+})^{-1}(\widehat{E}_{-+} - E^1_{-+})\widehat{E}^{-1}_{-+}E^1_-\Big) L(dz).$$

In the above equality we have used the fact that $\int \overline{\partial}_z \widetilde{f}(z) K(z) L(dz) = 0$ provided that $K(z)$ is holomorphic in a neighborhood of $\text{supp} \widetilde{f}$.

By Proposition 5.3, $(E^1_{-+} - \widehat{E}_{-+})$ is of trace class and we can take the trace and permute integration and the operator 'tr' in (4.5). The identity $\partial_z E^1_{-+} = E^1_- E^1_+$ shows that for $\text{Im } z \neq 0$,

$$(4.6) \quad \begin{aligned} &\text{tr}\,(E^1_+(E^1_{-+})^{-1}(\widehat{E}_{-+} - E^1_{-+})\widehat{E}^{-1}_{-+}E^1_-) \\ &= \text{tr}\,((E^1_{-+})^{-1}(\widehat{E}_{-+} - E^1_{-+})\widehat{E}^{-1}_{-+}\partial_z E^1_{-+}). \end{aligned}$$

Let $\chi \in C_0^\infty(\mathbb{R}^n_\xi)$ be equal to 1 in a neighborhood of

$$\mathbb{S} := \{\xi \in \mathbb{R}^n; (x, \xi) \in \text{supp}\,(E^1_{0,-+}(x, \xi, z) - \widehat{e}_{-+}(\xi, -x, z))\},$$

and denote by $\widehat{\chi} = \chi^w(hD_x)$ the corresponding operator on $L^2(\mathbb{T}^*; \mathbb{C}^N)$. Since

$$\mathbb{S} \cap \text{supp}\,(1 - \chi) = \emptyset,$$

it follows from Propositions 5.1 and 5.3 that:

$$\|(\widehat{E}_{-+} - E^1_{-+})\widehat{E}^{-1}_{-+}\partial_z E^1_{-+}(1 - \widehat{\chi})\|_{\text{tr}} = \mathcal{O}(h^\infty).$$

On the other hand, (3.14) yields

$$\|(E^1_{-+})^{-1}\| = \mathcal{O}(|\text{Im}z|^{-1}).$$

Hence
$$\|(E_{-+}^1)^{-1}(\widehat{E}_{-+} - E_{-+}^1)\widehat{E}_{-+}^{-1}\partial_z E_{-+}^1(1 - \widehat{\chi})\|_{\mathrm{tr}} = \mathcal{O}(h^\infty|\mathrm{Im}\, z|^{-1}).$$
Combining this equality with (4.5) and (4.6) we obtain

$$\mathrm{tr}\left[f(\mathbb{H}^1(h))\right]$$
$$= -\frac{1}{\pi}\mathrm{tr}\left[\int \overline{\partial}_z\widetilde{f}(z)(E_{-+}^1)^{-1}(\widehat{E}_{-+} - E_{-+}^1)\widehat{E}_{-+}^{-1}\partial_z E_{-+}^1\widehat{\chi}L(dz)\right] + \mathcal{O}(h^\infty).$$

Splitting the integral into two terms and using the fact that $\widehat{E}_{-+}^{-1}\partial_z\widehat{E}_{-+}$ is holomorphic in $z$, we get

$$(4.7)\quad \mathrm{tr}\left[f(\mathbb{H}^1(h))\right] = -\frac{1}{\pi}\mathrm{tr}\left[\int \overline{\partial}_z\widetilde{f}(z)(E_{-+}^1)^{-1}\partial_z E_{-+}^1\widehat{\chi}L(dz)\right] + \mathcal{O}(h^\infty).$$

The proof of the following lemma is similar to the one in [15].

**Lemma 4.1.** *There exists $r^1(x, \xi; h) \in S^0(\mathbb{R}^{2n}, \mathcal{L}(\mathbb{C}^N))$ such that $r^1(x, \xi; h) \sim \sum_{j\geq 0} h^j r_j(x, \xi)$ and*

$$(4.8)\qquad \mathrm{Op}_h^w(r^1(x, \xi; h)) = -\frac{1}{\pi}\int_{|\mathrm{Im}\, z|\geq h^\delta} \overline{\partial}_z\widetilde{f}(z)(E_{-+}^1)^{-1}\partial_z E_{-+}^1 L(dz).$$

*Moreover, $r_j$ is $\Gamma^*$-periodic in $x$ for all $j \geq 0$ and*

$$r_0(x, \xi) = -\frac{1}{\pi}\int \overline{\partial}_z\widetilde{f}(z)(E_{0,-+}^1(x, \xi, z))^{-1}\partial_z E_{0,-+}^1(x, \xi, z)L(dz).$$

**End of the proof of Theorem 2.1.** If we restrict the integral in the right hand side of (4.7) to the domain $|\mathrm{Im}\, z| \leq h^\delta$ then we get a term $\mathcal{O}(h^\infty)$ in trace norm. Here we have used the fact that $\overline{\partial}_z\widetilde{f}(z) = \mathcal{O}_N(|\mathrm{Im}\, z|^N)$ for all $N \in \mathbb{N}$. If we restrict our attention to the domain $|\mathrm{Im}\, z| \geq h^\delta$ then by Lemma 4.1 we get (2.5). To finish the proof let us compute $a_0$. We have

$$a_0 = \iint_{E^*\times\mathbb{R}^n} \widehat{\mathrm{tr}}\,[r_0(x, \xi)]dxd\xi = \iint_{E^*\times\mathbb{R}^n} \widehat{\mathrm{tr}}\,[r_0(x, \xi)]dxd\xi$$

$$= \iint_{E^*\times\mathbb{R}^n}(-\frac{1}{\pi}\int \overline{\partial}_z\widetilde{f}(z)\widehat{\mathrm{tr}}\left[(E_{0,-+}^1(x, \xi, z))^{-1}\partial_z E_{0,-+}^1(x, \xi, z)\right]L(dz))dxd\xi.$$

Here $\widehat{\mathrm{tr}}$ denotes the trace of square matrices. Thanks to Liouville's formula (i.e. $\widehat{\mathrm{tr}}\,(\partial_z A(z)A^{-1}(z)) = \frac{\partial_z\det A(z)}{\det A(z)}$ in the sense of matrices), we get

$$a_0 = \iint_{E^*\times\mathbb{R}^n}\left(-\frac{1}{\pi}\int \overline{\partial}_z\widetilde{f}(z)\frac{\partial_z\det E_{0,-+}^1(x, \xi, z)}{\det E_{0,-+}^1(x, \xi, z)}L(dz)\right)dxd\xi.$$

To prove (2.6) we use Remark 3.5 and the following lemma:

**Lemma 4.2.** *Let $g$ be an analytic function. Let $(z_k)_{k\geq 1}$ be the roots (counted with their multiplicity) of $g$ in $\operatorname{supp}(\widetilde{f})$. We have:*

$$\frac{-1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)\frac{g'(z)}{g(z)}L(dz) = \sum_{k\geq 1} f(z_k).$$

**Proof.** This follows from the formula $\frac{1}{\pi}\overline{\partial}_z(\frac{1}{z-z_0}) = \delta(\cdot - z_0)$ and the fact that $\frac{g'(z)}{g(z)} = \sum_{k\geq 1}\frac{1}{z-z_k} + k(z)$, where $k$ is holomorphic for $z$ in a small neighborhood of $\operatorname{supp}\widetilde{f}$. Applying the above lemma to $g(z) = \frac{\partial_z \det E^1_{0,-+}(x,\xi,z)}{\det E^1_{0,-+}(x,\xi,z)}$ and using Remark 3.5 we get

$$(4.9) \qquad a_0 = (2\pi)^{-n}\sum_{k\geq 1}\iint_{\mathbb{R}^n_x \times E^*} f(\lambda_k(x,\xi))dxd\xi.$$

The last equality in (2.6) follows from (2.3) and (4.9) by integrating by parts. This ends the proof of Theorem 2.1.

4.2. **Proof of Corollary 2.2.** For every small $\epsilon > 0$, choose $\overline{f_\epsilon}, \underline{f_\epsilon} \in C^\infty_0(\mathbb{R}; [0,1])$ with

$$1_{[a+\epsilon, b-\epsilon]} \leq \underline{f_\epsilon} \leq 1_{[a,b]} \leq \overline{f_\epsilon} \leq 1_{[a-\epsilon, b+\epsilon]}.$$

It then suffices to observe that

$$\operatorname{tr}\left[\underline{f_\epsilon}(H(h))\right] \leq N([a,b];h) \leq \operatorname{tr}\left[\overline{f_\epsilon}(H(h))\right],$$

which yields

$$\lim_{\epsilon \searrow 0}\lim_{h \searrow 0}\left((2\pi h)^n \operatorname{tr}\left[\underline{f_\epsilon}(H(h))\right]\right) \leq \lim_{h \searrow 0}(2\pi h)^n N([a,b];h)$$

$$\leq \lim_{\epsilon \searrow 0}\lim_{h \searrow 0}\left((2\pi h)^n \operatorname{tr}\left[\overline{f_\epsilon}(H(h))\right]\right),$$

and to apply Theorem 2.1.

4.3. **Proof of Theorem 2.3.** To prove this theorem one needs a refinement of Theorem 2.1. Let $\theta \in C^\infty_0(\mathbb{R})$ and put

$$\breve{\theta}_h(\tau) := \frac{1}{2\pi h}\int e^{it\tau/h}\theta(t)dt.$$

Analysis similar to that in the proof of (4.7) shows that

$$(4.10) \qquad \operatorname{tr}\left[f(H(h))\breve{\theta}_h(t - H(h))\right] =$$

$$\operatorname{tr}\left[-\frac{1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)\breve{\theta}_h(t - z)(E^1_{-+})^{-1}\partial_z E^1_{-+}\widehat{\chi}\, L(dz)\right] + \mathcal{O}(h^\infty),$$

In the first equality we have used the fact that $\widetilde{f}(z)\breve{\theta}_h(t-z)$ is an almost analytic extension of $f(x)\breve{\theta}_h(t-x)$, since $z \mapsto \breve{\theta}_h(t-z)$ is analytic. Here the support of $\tilde{f}$ is in a small neighborhood of $\tau \in \{a, b\}$. Trace formulas involving effective Hamiltonian like (4.10) were studied in [15].

According to the definition of $\Sigma_\tau$ before the assumption (**H**) and (3.16) we have

$$\Sigma_\tau = \{(x, \xi) \in \mathbb{R}^{2n};\ \det(e_{-+}(x, \xi, \tau)) = 0\}.$$

Fix $(x_0, \xi_0) \in \Sigma_\tau$. Under the assumption (**H**) we may choose

$$e_{-+}(x, \xi, z) = \begin{pmatrix} \lambda_j(x, \xi) - z & 0 & \cdot & & \cdot & \cdot & 0 \\ & 0 & & \cdot & \cdot & \cdot & & \cdot & \cdot \\ & \cdot & & \cdot & \cdot & & \cdot & & \cdot & \cdot \\ & \cdot & & & \cdot & \cdot & g(x, \xi, z) & \cdot & \cdot \\ & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\ & 0 & & \cdot & \cdot & & \cdot & & \cdot & \cdot \end{pmatrix},$$

where $\det(g(x, \xi, z)) \neq 0$ for all $(x, \xi, z)$ in a small neighborhood $\mathcal{W}$ of $(x_0, \xi_0, \tau)$.

The assumption (**H**) implies that the principal symbol $e_{-+}(\xi, -x, z)$ of $E^1_{-+}$ is micro-hyperbolic (in the sense of [15, 20]) at every point $(x, \xi) \in \Sigma_\tau$. Thus, applying Theorem 1.8 in [15] to the left hand side of (4.10), we obtain

$$(4.11) \qquad \mathrm{tr}\Big[f(H(h))\breve{\theta}_h(t - H(h))\Big] \sim \sum_{j=0}^{\infty} \beta_j h^{j-n},\ (h \searrow 0).$$

Theorem 2.3 now follows from Theorem 2.1, (4.11) and tauberian arguments (see Theorem V-13 in [40]).

4.4. **Proof of Theorem 2.4.** In what follows, we write $[a^j]_{j=0}^1 = a^1 - a^0$. The proof of Theorem 2.4 is based on the following proposition.

**Proposition 4.3.** *Assume that (1.1) holds. Let $f \in C_0^\infty(\mathbb{R})$ and let $\widetilde{f}$ be an almost analytic extension of $f$. Then the operator $[f(H(h)) - f(H)]$ is of trace class as an operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and*

$$(4.12)\ \ \mathrm{tr}[f(H(h)) - f(H)] = \mathrm{tr}[f(\mathbb{H}^1(h)) - f(\mathbb{H}^0(h))] =$$

$$\mathrm{tr}\Big(-\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)\big[(E^j_{-+})^{-1}\partial_z E^j_{-+}\big]_{j=0}^1 L(dz)\Big).$$

*Here $\overline{\partial} = \dfrac{\partial}{\partial\overline{z}}$ and $L(dz) = dxdy$ denotes the Lebesgue measure on $\mathbb{C}$.*

Now, analysis similar to that in the proof of Theorem 2.1 with (4.7) replaced by (4.12) gives Theorem 2.4.

The remainder of this subsection will be devoted to the proof of Proposition 4.3.

**Lemma 4.4.** *We have*

(4.13)
$$\left[E_+^j\right]_{j=0}^1 = E^1 V E_+^0,$$

(4.14)
$$\left[E_-^j\right]_{j=0}^1 = E_-^0 V E^1,$$

*and*

(4.15)
$$\left[E_{-+}^j\right]_{j=0}^1 = E_-^1 V E_+^0.$$

*In particular, if* (1.1) *is satisfied then*

(4.16)
$$\left[E_{-+}^j\left(k, r, z; h\right)\right]_{j=0}^1 \in S^\delta\left(\mathbb{T}^* \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C})\right).$$

*Proof.* Identities (4.13)-(4.15) follow from the first resolvent equation

$$\left[\mathcal{E}^j(z, h)\right]_{j=0}^1 = \mathcal{E}^1(z, h)\left[\mathcal{P}^0(z, h) - \mathcal{P}^1(z, h)\right]\mathcal{E}^0(z, h)$$
$$= -\mathcal{E}^0(z, h)\left[\mathcal{P}^1(z, h) - \mathcal{P}^0(z, h)\right]\mathcal{E}^1(z, h)$$

and the fact that $\left[\mathcal{P}^j(z, h)\right]_{j=0}^1 = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix}$.

Formula (4.16) is a simple consequence of (4.15) and standard $h$-pseudodifferential calculus of operator-valued symbols. $\qquad\square$

**Lemma 4.5.** *Under the assumption* (1.1), *the operators*

(4.17)
$$V E_+^0 : L^2(\mathbb{T}^*; \mathbb{C}^N) \to \mathbb{L}^0,$$

*and*

(4.18)
$$E_-^0 V : \mathbb{L}^0 \to L^2(\mathbb{T}^*; \mathbb{C}^N),$$

*are of trace class.*

*Proof.* We observe that $\mathcal{P}^j(z, h)^* = \mathcal{P}^j(\bar{z}, h)$, which implies that $\mathcal{E}^j(z, h)^* = \mathcal{E}^j(\bar{z}, h)$. From this, we deduce the following identity:

(4.19)
$$E_-^0(z, h)^* = E_+^0(\bar{z}, h), \quad j = 0, 1.$$

Since $\left(E_-^0 V\right)^* = V E_+^0(\bar{z}, h)$ it suffices to prove (4.17).

Consider the operator $A = \left(\mathrm{Id} - h^2\Delta_{\mathbb{T}^*}\right)^{-\frac{\delta}{2}}$ on $L^2(\mathbb{T}^*; \mathbb{C}^N)$. Set $B = V E_+^0(z, h)$, $C = B^* B$ and $D = A^{-1} C A^{-1}$.

According to the $h$-pseudodifferential calculus of operator-valued symbols (see Proposition 5.7, Proposition 5.6), it follows from the assumption (1.1)

and Theorem 3.2 that $D \in S^0(\mathbb{T}_k^* \times \mathbb{R}_r^n; \mathcal{M}_N(\mathbb{C}))$. In particular, by construction of $r_+(x, \xi, z), e_+(x, \xi, z)$ and $E_+^0(z, h)$ (see [26]) the principal symbol of $C$ is given by $C_0(k, r, z) = (c_{ij}(k, r))_{1 \leq i,j \leq N}$ where

$$c_{i,j}(k, r) = \int_E \Psi_i(y, k) V(r, y) \overline{\Psi_j(y, k)} dy.$$

Here $\Psi_j(y, k)$ are smooth functions, $\Gamma^*$-periodic with respect to $k$ and satisfying $\Psi_j(y + \gamma, k) = e^{i\gamma \cdot k} \Psi_j(y, k)$ for all $\gamma$ in $\Gamma$. Therefore, $D$ extends to a bounded operator from $L^2(\mathbb{T}^*; \mathbb{C}^N)$ into $L^2(\mathbb{T}^*; \mathbb{C}^N)$, (see Proposition 5.2). Combining this with the fact that $C$ is non-negative, we get:

$$0 \leq C = ADA \leq \|D\| A^2,$$

which implies

$$0 \leq C^{\frac{1}{2}} \leq \sqrt{\|D\|} A.$$

Since $\delta > n$, it follows from Proposition 5.3 that $A : L^2(\mathbb{T}^*; \mathbb{C}^N) \to L^2(\mathbb{T}^*; \mathbb{C}^N)$ is of trace class and the lemma follows from the above inequality. $\qquad\square$

**Proposition 4.6.** *Suppose that* (1.1) *holds. For $z \in \Omega$ such that $\mathrm{Im}(z) \neq 0$, the operator*

$$\left[ E_+^j (E_{-+}^j)^{-1} E_-^j \right]_{j=0}^1$$

*is of trace class from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and*

$$(4.20) \qquad \mathrm{tr}\left( \left[ E_+^j (E_{-+}^j)^{-1} E_-^j \right]_{j=0}^1 \right) = \mathrm{tr}\left( \left[ (E_{-+}^j)^{-1} \partial_z E_{-+}^j \right]_{j=0}^1 \right).$$

*Here the operator in the right hand side of* (4.20) *is defined on $L^2(\mathbb{T}^*; \mathbb{C}^N)$.*

*Proof.* Let $z \in \Omega$ such that $\mathrm{Im}(z) \neq 0$, we have the following identity:
(4.21)
$$\left[ E_+^j (E_{-+}^j)^{-1} E_-^j \right]_{j=0}^1 = \left[ ([E_+^j]_{j=0}^1)(E_{-+}^1)^{-1} E_-^1 \right] + \left[ E_+^0 (E_{-+}^0)^{-1} ([E_-^j]_{j=0}^1) \right] -$$

$$\left[ E_+^0 (E_{-+}^1)^{-1} ([E_{-+}^j]_{j=0}^1)(E_{-+}^0)^{-1} E_-^1 \right].$$

According to Lemmas 4.4 and 4.5, all the terms of the rhs in the last equality are of trace class. Using the cyclicity of the trace and identity (3.15), we obtain

$$(4.22) \qquad \mathrm{tr}\left( \left[ E_+^j (E_{-+}^j)^{-1} E_-^j \right]_{j=0}^1 \right) = \mathrm{tr}\left( (E_{-+}^1)^{-1} \left( \partial_z E_{-+}^1 - E_-^1 E_+^0 \right) \right) +$$

$$\mathrm{tr}\left( (E_{-+}^0)^{-1} \left( E_-^1 E_+^0 - \partial_z E_{-+}^0 \right) \right) - \mathrm{tr}\left( (E_{-+}^0)^{-1} E_-^1 E_+^0 - (E_{-+}^1)^{-1} E_-^1 E_+^0 \right),$$

which yields the proposition. $\qquad\square$

**Proof of Proposition 4.3.** By the Helffer-Sjöstrand formula (see [20]), we have

$$f(\mathbb{H}^1(h)) - f(\mathbb{H}^0(h)) = -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)\Big[(z - \mathbb{H}^1(h))^{-1} - (z - \mathbb{H}^0(h))^{-1}\Big] L(dz).$$

Combining this with (3.13), we obtain

$$(4.23) \qquad f(\mathbb{H}^1(h)) - f(\mathbb{H}^0(h)) = \frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)[E^j]_{j=0}^1 \, L(dz)$$

$$- \frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)\Big[E_+^j (E_{-+}^j)^{-1} E_-^j\Big]_{j=0}^1 L(dz).$$

Since $E^j$, $j = 0, 1$ are holomorphic in a neighborhood of $\mathrm{supp}(\widetilde{f})$, the first term in the rhs of (4.23) vanishes. Consequently,

$$f(\mathbb{H}^1(h)) - f(\mathbb{H}^0(h)) = -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)\Big[E_+^j (E_{-+}^j)^{-1} E_-^j\Big]_{j=0}^1 L(dz).$$

Using Proposition 4.6, we conclude that $[f(\mathbb{H}^1(h)) - f(\mathbb{H}^0(h))]$ is of trace class and applying (4.20), we obtain the second equality of (4.12). The first equality follows from the fact that $\mathbb{H}^1(h)$ (resp. $\mathbb{H}^0(h)$) is unitarily equivalent to $H(h)$ (resp. $H$).

4.5. **Proof of Corollary 2.5.** Assume for instance that $\mu \mapsto \xi(\mu; h)$ is monotone (i.e., $\xi'(\cdot; h)$ is positive in the sense of distributions). For small $\epsilon > 0$, choose $\overline{f_\epsilon}, \underline{f_\epsilon} \in C_0^\infty(\mathbb{R}; [0, 1])$ satisfying

$$1_{[C+\epsilon, \lambda-\epsilon]} \leq \underline{f_\epsilon} \leq 1_{[C,\lambda]} \leq \overline{f_\epsilon} \leq 1_{[C-\epsilon, \lambda+\epsilon]}.$$

According to (1.2), we have

$$\mathrm{tr}\Big[\underline{f_\epsilon}(H(h)) - \underline{f_\epsilon}(H)\Big] \leq \xi(\lambda; h) - \xi(C, h) \leq \mathrm{tr}\Big[\overline{f_\epsilon}(H(h)) - \overline{f_\epsilon}(H)\Big].$$

Multiplying these inequalities by $(2\pi h)^n$, applying Theorem 2.4 and letting $(h \searrow 0)$ and $(\epsilon \searrow 0)$ we obtain (2.11). We recall that $\xi(C; h) = 0$ for $C << -1$, and $\rho_0(\lambda) = c_n (2\pi)^{-n} \lambda^{n/2}$ in the case where $V_0 = 0$. Here $c_n$ is the volume of the unit ball in $\mathbb{R}^n$.

For the general case we use a trick due to D.Robert [41], which consists to write $\xi(\mu; h) = \xi_1(\mu; h) - \xi_2(\mu; h)$ where $\mu \mapsto \xi_i(\mu; h)$, $i = 1, 2$ are monotonic. Now, it suffices to apply the above argument for each $\xi_i(\mu; h)$.

Notice that, Robert's method holds for the Schrödinger operator $-\Delta_x + W(x)$ under the assumption that $|W(x) - x \cdot \nabla W(x)| = \mathcal{O}(\langle x \rangle^{-\delta})$, with $\delta > n$, and near a strict positive energy (i.e., $\lambda > 0$).

4.6. **Proof of Theorem 2.6.** Here we deal with $E^j, E^j_{-+}, E^j_{\pm}$ when $j = 1$. For that we omit the index $j$. Since $E, E_{\pm}$ are holomorphic in $z$, it follows from (3.13) and (3.14) that the poles of the meromorphic extension of $(z - \mathbb{H})^{-1}$ are the zeros of the meromorphic extension of $E^{-1}_{-+}$ in a suitable space with the same algebraic multiplicity. Hence, we are reduced to the study of the meromorphic extension of $E^{-1}_{-+}$. Now let $\Omega$ be a small complex neighborhood of $\lambda_0$. We recall that $\lambda_0 \in \mu_m(\mathbb{T}^*)$ is a fixed level. Under the assumption (**H2**), the effective Hamiltonian in Proposition 3.1 can be chosen real-valued with

$$e_{-+}(x, \xi, z) = z - \lambda_m(x, \xi),$$

for $z$ in $\Omega$. In particular, due to the assumptions (**H1 − 2**), for all $N \in \mathbb{N}$ we have

$$E_{-+}(x, \xi, z; h) = e_{-+}(\xi, -x, z) + \sum_{j=1}^{N} h^j E_{j,-+}(x, \xi, z) + h^{N+1} F_N(x, \xi, z; h),$$

where $E_{j,-+}(x, \xi, z), F_N(x, \xi, z; h)$ are holomorphic in $(x, \xi)$ on a neighborhood of $E^* \times \mathbb{R}^n$. In the following, we follow quite closely the exposition and the proof in [16, 35]. For this reason we omit the details.

Let $t_0$ be a small positive constant. For $t \in D(t_0) := \{t \in \mathbb{C}; |t| < t_0\}$, set

$$\nu_t(x) = x - t\nabla\mu_m(x).$$

We denote by $J_t(x) = \det[D\nu_t(x)]$ the Jacobian of $\nu_t(x)$. Since $\mu_m(x)$ is bounded with all its derivatives, there exists $t_0$ small enough such that $\nu_t$ is invertible for all $t \in D(t_0)$. We define the following map on the Schwartz space of rapidly decreasing functions:

$$\mathcal{U}_t u(x) = \{J_t^{\frac{1}{2}}(x)(\mathcal{F}_h u)(\nu_t(x))\}, \ u \in S(\mathbb{R}^n),$$

where $\mathcal{F}$ is the semi-classical Fourier transform

$$\mathcal{F}_h u(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi/h} u(x) dx.$$

For $t \in D(t_0)$, we define

(4.24)          $E^{w,t}_{-+}(x, hD_x, z; h) := \mathcal{U}_t E^w_{-+}(x, hD_x, z; h)\mathcal{U}_t^{-1}.$

According to Theorem 2.3 and Corollary 2.5 in [16], the operator $E^{w,t}_{-+}(x, hD_x, z; h)$ is well defined from $L^2(\mathbb{T}^*)$ into $L^2(\mathbb{T}^*)$ and satisfies the following properties:

1) $E^t_{-+}(x, \xi, z; h) \sim \sum_{l\geq 0} E^t_{l,-+}(x, \xi, z) h^l$ in $S^{\delta_1}(\mathbb{T}^* \times \mathbb{R}^n)$.

2) For $t \in ]-t_0, t_0[$, the operator $E^{w,t}_{-+}(x, hD_x, z; h)$ is unitarily equivalent to $E^w_{-+}(x, hD_x, z; h)$.

3) $E_{0,-+}^t(x,\xi,z) = z - \lambda_m\Big((1-tM(x))^{-1}\xi, \nu_t(x)\Big)$, where $M(x) = \Big(\frac{\partial \nu_j}{\partial x_i}(x)\Big)_{1\leq i,j\leq n}$.

4) For fixed $t = i\mathrm{Im}t$ with $\mathrm{Im}t > 0$, the operator $E_{-+}^{w,t}(x,hD_x,z;h)$ has a meromorphic

extension from $\{z \in \mathbb{C}; \mathrm{Im}z > 0, |z-\lambda_0| < \epsilon\}$ to $\{z \in \mathbb{C}; \mathrm{Im}z > -\eta h, |z-\lambda_0| < \epsilon\}$. Here $\epsilon$

and $\eta$ are small positive constants.

Now to prove Theorem 2.6 it suffices to show that $(E_{-+}^{w,t}(x,hD_x,z;h))^{-1}$ has exactly $N$ poles $e_1(h), \cdots e_N(h)$ (counted with their algebraic multiplicity) in the disc $D(\lambda_0, C_0 h)$.

Notice that, the assumption (2.13) implies that $E_{-+}^t(x,\xi,\lambda_0;h)$ is elliptic except at $(x,\xi) = (x_0,\xi_0)$. We recall that $(x_0,\xi_0)$ is a local non-degenerate extremum of the function $(x,\xi) \mapsto \lambda_m(x,\xi)$. Thus, to determine the spectrum of $E_{-+}^{w,t}(x,hD_x,z;h)$ near $\lambda_0$ we have reduced to study microlocally $E_{-+}^t(x,\xi,z;h)$ near $(x_0,\xi_0)$. Next, by the WKB method (see [20], Chapter 3 and the end of Chapter 4 and Chapter 14), we construct $z_1(h), \cdots z_N(h)$ satisfying (2.14) and $g_1(x,h), \cdots g_N(x,h)$ such that

$$E_{-+}^{t,w}(x,hD_x,z_i(h);h)g_j(x,h) = \mathcal{O}(h^\infty).$$

From this we deduce Theorem 2.6. For the details we refer to [16] Section 5.

## 5.  APPENDIX

In this appendix, we recall some well known results on the scalar and operator-valued $h$-pseudodifferential calculus. For the proof we refer to [20].

### 5.1. $h$-**pseudodifferential operator.** By $X$ we denote either $\mathbb{R}^{2n}$ or $\mathbb{T}^* \times \mathbb{R}^n$. We recall that $S(\mathbb{T}^* \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C})) = \{P \in S(\mathbb{R}^{2n}; \mathcal{M}_N(\mathbb{C})); \Gamma^* -$ periodic in $x\}$. Put $Y = \Pi_x X$ (i.e., $Y = \mathbb{R}^n$ (resp, $\mathbb{T}^*$) for $X = \mathbb{R}^{2n}$ (resp. $\mathbb{T}^* \times \mathbb{R}^n$)).

**Proposition 5.1. (Composition formula)** *Let* $a_i \in S(X; \mathcal{M}_N(\mathbb{C}))$, $i = 1,2$. *Then* $b^w(y,hD_y;h) = a_1^w(y,hD_y) \circ a_2^w(y,hD_y)$ *is an $h$-pseudo-differential operator, and*

$$b(y,\eta;h) \sim \sum_{j=0}^{\infty} b_j(y,\eta)h^j, \text{ in } S(X; \mathcal{M}_N(\mathbb{C})).$$

**Proposition 5.2. ($L^2-$ boundedness)** *Let* $a = a(x,\xi;h) \in S(X; \mathcal{M}_N(\mathbb{C}))$. *Then* $a^w(x,hD_x;h)$ *is bounded* : $L^2(Y;\mathbb{C}^N) \to L^2(Y;\mathbb{C}^N)$, *and there is a*

*constant $C$ independent of $h$ such that*

$$\|a^w(x, hD_x; h)\| \leq C.$$

**Proposition 5.3. (trace)** *Let $a = a(x, \xi; h) \in S(X; \mathcal{M}_N(\mathbb{C}))$. We assume that $\partial_x^\alpha \partial_\xi^\beta a \in L^1(X)$, for all $|\alpha| + |\beta| \leq 2n + 2$. Then $a^w(x, hD_x; h)$ is trace class operator and*

$$\mathrm{tr}(a^w(x, hD_x; h)) = \frac{1}{(2\pi h)^n} \iint_Y \widehat{\mathrm{tr}}\big(a(x, \xi; h)\big) dx d\xi.$$

**Proposition 5.4. (invertibility)** *Let $a = a(x, \xi; h) \in S(X; \mathcal{M}_N(\mathbb{C}))$. We assume that there exists $C > 0$ (independent of $h$) such that*

$$|\det a(x, \xi; h)| \geq C.$$

*Then, for $h$ small enough, the operator $a^w(x, hD_x; h) : L^2(Y) \to L^2(Y)$ is invertible with uniformly bounded inverse.*

### 5.2. Operator valued $h$-pseudodifferential operator.

Our main reference is here the unpublished work of Balazard–Konlein and [20]. We shall consider a family of Hilbert spaces $\mathcal{A}_X$, $X = (x, \xi) \in \mathbb{R}^{2n}$ satisfying:

(5.1)           $\mathcal{A}_X = \mathcal{A}_Y$ as vector spaces for all $X, Y \in \mathbb{R}^{2n}$,

there exist $N_0, C_0 > 0$ such that

(5.2)                          $\|u\|_{\mathcal{A}_X} \leq C\langle X - Y \rangle^{N_0} \|u\|_{\mathcal{A}_Y}$

for all $u \in \mathcal{A}_0, X, Y \in \mathbb{R}^{2n}$.

Let $\mathcal{B}_X, X \in \mathbb{R}^{2n}$ be a second family with the same properties. We say that $p \in C^\infty(\mathbb{R}^{2n}; \mathcal{L}(\mathcal{A}_0, \mathcal{B}_0))$ belongs to $S^0(\mathbb{R}^{2n}; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$ if for every $\alpha \in \mathbb{N}^{2n}$, there is a constant $C_\alpha$ such that

(5.3)                          $\|\partial_X^\alpha p\|_{\mathcal{L}(\mathcal{A}_X; \mathcal{B}_X)} \leq C_\alpha$, for all $X \in \mathbb{R}^{2n}$.

We can then associate with $p$ the operator $p^w(x, hD_x)$. In particular, the following propositions hold.

**Proposition 5.5.** *. Let $p \in S^0(\mathbb{R}^{2n}; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$, where $\mathcal{A}_X, \mathcal{B}_X$ satisfy (5.1), (5.2). Then $\mathrm{Op}_h^w(p) = p^w(x, hD_x)$ is uniformly continuous from $\mathcal{S}(\mathbb{R}^n; \mathcal{A}_0)$ to $\mathcal{S}(\mathbb{R}^n; \mathcal{B}_0)$.*

**Proposition 5.6.** *. Assume $\mathcal{A}_X = \mathcal{A}_0$, $\mathcal{B}_X = \mathcal{B}_0, \forall X \in \mathbb{R}^{2n}$. If $p \in S^0(\mathbb{R}^{2n}; \mathcal{L}(\mathcal{A}_0, \mathcal{B}_0))$ (i.e. $\|\partial_X^\alpha p\|_{\mathcal{L}(\mathcal{A}_0; \mathcal{B}_0)} \leq C_\alpha$, for all $X \in \mathbb{R}^{2n}$) then $\mathrm{Op}_h^w(p)$ is uniformly bounded from $L^2(\mathbb{R}^n; \mathcal{A}_0)$ to $L^2(\mathbb{R}^n; \mathcal{B}_0)$.*

Let $\mathcal{C}_X$ be a third family of Hilbert spaces which also satisfies (5.1), (5.2).

**Proposition 5.7.** *Let $p \in S^0(\mathbb{R}^{2n}; \mathcal{L}(\mathcal{B}_X, \mathcal{C}_X)), q \in S^0(\mathbb{R}^{2n}; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$. Then $\mathrm{Op}_h^w(p) \circ \mathrm{Op}_h^w(q) = \mathrm{Op}_h^w(r)$, where $r \in S^0(\mathbb{R}^{2n}; \mathcal{L}(\mathcal{A}_X, \mathcal{C}_X))$ is given by*

$$r = \exp\left(\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\right)(p(x,\xi)q(y,\eta))_{\big|_{x=y,\xi=\eta}},$$

*where $\sigma$ is the usual symplectic 2-form. We have the asymptotic formula:*

$$r \sim \sum_{k=0}^\infty \frac{1}{k!}(\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta))^k p(x,\xi)q(y,\eta)_{|x=y,\xi=\eta}.$$

## REFERENCES

[1] S. Alama, P. Deift, R. Hempel, *Eigenvalues branches of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$*. Comm. Math. Phys., **121**, (1989), 291–321.

[2] E. Bardos, J.-C. Guillot, J. Ralston, *La relation de Poisson pour l'équation des ondes dans un ouvert non-borné* , Comm. Partial Differential Equations **7**, (1982), 905–958.

[3] M. Birman, *Discrete spectrum in the gaps of a continuous one for perturbations with large coupling limit.* Adv. in Soviet Math., **7**, (1991), 57–73.

[4] M. Birman, *On the discrete spectrum in the gaps of a perturbed periodic second order operator.* Funct. Ana. Appl., **25**, (1991), 158–161.

[5] M. Sh. Birman, A. Laptev, and T. A. Suslina, *Discrete spectrum of the two-dimensional periodic elliptic second order operator perturbed by a decreasing potential. I. Semi-in nite gap.* St. Petersburg Math. J. **12**, (2001), 535–567.

[6] M-Sh. Birman and D-R. Yafaev, *The spectral shift function. the papers of M-G. Kreĭn and their further development*, St. Petersburg Math. J. **4**, (1993), no. 5, 833–870.

[7] D.I. Borisov and R. R. Gadyl'shin, *On the spectrum of the Schrödinger operator with a rapidly oscillating compactly supported potential.* (Russian) Teoret. Mat. Fiz. 147 (2006), no. 1, 58–63; translation in Theoret. and Math. Phys. **147**, (2006), no. 1, 496–500.

[8] D.I. Borisov and R. R. Gadyl'shin, *On the spectrum of a selfadjoint differential operator with rapidly oscillating coefficients on the axis.* (Russian) Mat. Sb. 198 (2007), no. 8, 3–34; translation in Sb. Math. **198**, (2007), no. 7-8, 1063–1093.

[9] V. Bruneau, V. Petkov, *Meromorphic continuation of the spectral shift function,* Duke Math journal, **116**, (2003), 389–430.

[10] V-S. Buslaev, *Semi-classical approximation for equations with periodic coefficients*, Russian. Math. Surveys **42**, (1987), no. 6, 97–125.

[11] M. Christ and A. Kiselev, *Absolutely continuous spectrum for one-dimensional Schrödinger operators with slowly decaying potentials: some optimal results.* J. Am. Math. Soc. **11**, (1998), 771–797.

[12] A. Devinatz and P. Rejto *A limiting absorption principle for Schrödinger operators with oscillating potentials, Part I,* J. Differential equation (1983), 29–84.

[13] S. Denisov, *Absolutely continuous spectrum of multidimensional Schrödinger operator,* Int. Math. Res. Not. (2004), 3963–3982.

[14] M. Dimassi, *Développements asymptotiques des perturbations lentes de l'opérateur de Schrödinger périodique,* Comm. Partial Differential Equations **18**, (1993), no. 5-6, 771–803.

[15] _____, *Trace asymptotics formulas and some applications,* Asymptot. Anal. **18**, (1998), no. 1-2, 1–32.

[16] _____, *Resonances for a slowly varying perturbation of a periodic Schrödinger operator,* Canad. J. Math. **54**, (2002), no. 5, 998–1037.

[17] _____, *Spectral shift function and resonances for slowly varying perturbations of periodic Schrödinger operators,* J. Funct. Anal. **225**, (2005), no. 1, 193–228.

[18] _____, *Semi-classical asymptotics for Schrödinger operator with oscillating decaying potential.* Canadian Mathematical Bulletin http://dx.doi.org/10.4153/CMB-2016-022-8.

[19] M. Dimassi and M. Zerzeri, *A local trace formula for resonances of perturbed periodic Schrödinger operators,* J. Funct. Anal. **198**, (2003), no. 3, 142–159.

[20] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit,* London Mathematical Society Lecture Note Series, vol. **268**, Cambridge University Press, Cambridge, 1999.

[21] V. Duchênes, I. Vukicevic and M. Weinstein, *Scattering and Localization Properties of Highly Oscillatory Potentials.* Commun. Pure Appl. Math. volume **67**, 0083–0128 (2014).

[22] V. Duchênes, I. Vukicevic and M. Weinstein, *Oscillatory and localized perturbations of periodic structures and the bifurcation of defect modes.* SIAM J. Math. Anal. **47**, (2015), no. 5, 3832–3883. http://dx.doi.org/10.1137/140980302.

[23] V. Duchênes, I. Vukicevic and M. Weinstein, *Homogenized description of defect modes in periodic structures with localized defects.* Commun. Math. Sci. Vol. **13**, No. 3, pp. 777–823 (2015).

[24] V. Duchênes and M. Weinstein *Scattering, Homogenization, and Interface Effects for Oscillatory Potentials with Strong Singularities,* Multiscale Model. Simul., **9**, (2011), 1017–1063.

[25] C. Gérard, *Resonances theory for periodic Schrödinger operators,* Bull.Soc.Math. France **118**, (1990), no. 1, 27–54.

[26] C. Gérard, A. Martinez, and J. Sjöstrand, *A mathematical approach to the effective hamiltonian in perturbed periodic problems,* Comm. Math. Phys. **142**, (1991), 217–244.

[27] C. Gérard and F. Nier, *Scattering theory for the perturbations of periodic Schrödinger operators,* J. Math. Kyoto Univ. **38**, (1998), 595–634.

[28] J-C. Guillot, J. Ralston, and E. Trubowitz, *Semi-classical methods in solid state physics,* Comm. Math. Phys. **116**, (1988), 401–415.

[29] P. D. Hislop and I. M. Sigal. *Introduction to spectral theory,* volume 113 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1996. With applications to Schrödinger operators.

[30] T. Kato, *Perturbation theory,* Springer-Verlag, New York, (1966).

[31] M-G. Kreĭn, *On the trace formula in perturbation theory. (Russian),* Mat. Sbornik N.S. **33**, (1953), no. 75, 597–626.

[32] A. Laptev, S. Naboko and O. Safronov, *Absolutely continuous spectrum of Schrödinger operators with slowly decaying and oscillating potentials.* Comm. Math. Phys. **253**, (2005), 611–631.

[33] S. Z. Levendorskii, *Lower bounds for the number of eigenvalue branches for the Schrödinger operator $H - \lambda W$ in a gap of $H$: The case of indefinite $W$*. Comm. in Partial Differential Equations, **20**,(1995), 827–854.

[34] I-M. Lifshits, *On a problem in perturbation theory. (Russian)*, Uspekhi Mat. Nauk **7** (1952), no. 1, 171–180.

[35] A. Martinez, *Résonances dans l'approximation de Born-Oppenheimer I.* J. Differential Equations **91**, (1991), 204–230.

[36] R.-B. Melrose, *Scattering theory and the trace of wave group,* J. Funct. Anal. **45**, (1982) 429–440.

[37] G. D. Raikov, *Eigenvalue asymptotics for the Schrödinger operator with perturbed periodic potential.* Inventiones mathematicae December 1992, Volume **110**, Issue 1, pp 75–93.

[38] G. D. Raikov, *Discrete spectrum for Schrödinger operators with oscillating decaying potentials*, Preprint (2015).

[39] M. Reed and B. Simon, *Methods of modern mathematical physics. IV*, Academic Press, New York, 1978, Analysis of operators.

[40] D. Robert, *Autour de l'approximation semi-classique*, Progress in Mathematics, no. **68**, Birkhäuser Boston, Inc., Boston, MA., 1987.

[41] ———, *Semiclassical asymptotics for the spectral shift function*, Amer. Math. Soc. Transl. **189** (1999), no. 2, 187–203.

[42] D. Robert and H. Tamura, *Semi-classical Bounds for Resolvents of Schrödinger Operators and Asymptotics for Scattering Phases*, Comm. Partial Differential Equations **9**, (1984), no. 10, 1017–1058.

[43] M-M. Skriganov, *The spectrum band structure of the three-dimensional Schrödinger operator with periodic potential*, Invent. Math. **80**, (1985), no. 1, 107–121.

[44] ———, *Geometric and arithmetic methods in the spectral theory of multidimensional periodic operators*, Russian A translation of Trudy Mat. Inst. Steklov. **171** (1985), Proc. Steklov Inst. Math. **171**, (1987), no. 2, vi+121 pp.

[45] J. Sjöstrand, *A trace formula and review of somme estimates for resonances, in Microlocal Analysis and Spectral Theory,* NATO ASI Series C, Vol. 490, Kluwer, Dordrecht, 1997, pp. 337–440.

[46] M-A. Shubin, *The spectral theory and the index of elliptic operators with almost periodic coefficients*, Russian Math. Surveys **34**, (1979), no. 2, 109–158.

[47] V.A. Sloushch, *Discrete spectrum of a periodic Schrödinge operator with variable metric perturbed by a nonnegative rapidly decaying potential.* St Petersbourg Math Journal vol **27**, (2016), 317–326.

[48] *T. A. Suslina, Discrete spectrum of a two-dimensional periodic elliptic second order operator perturbed by a decaying potential. II. Internal gaps.* Algebra i Analiz 15(2003), no. 2, 228–289; St. Petersburg Math. J. **15**, (2004), 249–287.

[49] D. Yafaev, *Scattering theory: some old and new problems*, Scattering theory: some old and new problems, Lecture Notes in Math., vol. **1737**, Springer, Berlin, 2000, pp. xvi+169 pp.

MOUEZ DIMASSI
UNIVERSITÉ BORDEAUX I,
INSTITUT DE MATHÉMATIQUES DE BORDEAUX,
351, COURS DE LA LIBÉRATION, 33405 TALENCE, FRANCE

*e-mail address*: mouez.dimassi@math.u-bordeaux1.fr

Anh Tuan Duong

Department of Mathematics,

Hanoi National University of Education,

136 Xuan Thuy, Cau Giay, Hanoi, Vietnam

*e-mail address*: tuanda@hnue.edu.vn