

SOME EXAMPLES OF NON-TIDY SPACES

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ABSTRACT. We construct a free \mathbb{Z}_2 -space X_n for a positive integer n such that $w_1(X_n)^n \neq 0$ but there is no \mathbb{Z}_2 -map from S^2 to X_n .

1. INTRODUCTION

All group actions are to be from the right unless otherwise stated. For a group Γ , the orbit space of a Γ -space is denoted by \overline{X} . We shall call a Γ -space X free if the quotient map $X \rightarrow \overline{X}$ is a principal Γ -bundle. By abuse of notation, the image the homomorphism $\pi_1(X, x_0) \rightarrow \pi_1(\overline{X}, \overline{x}_0)$ is again denoted by $\pi_1(X, x_0)$.

We write S_a^n to mean the n -dimensional sphere with the antipodal \mathbb{Z}_2 -action. For a free \mathbb{Z}_2 -space X , set

$$\text{coind}(X) = \sup\{n \geq 0 \mid \text{There is a } \mathbb{Z}_2\text{-map from } S_a^n \text{ to } X.\},$$

$$\text{ind}(X) = \inf\{n \geq 0 \mid \text{There is a } \mathbb{Z}_2\text{-map from } X \text{ to } S_a^n.\},$$

$$h(X) = \sup\{n \geq 0 \mid w_1(X)^n \neq 0\}.$$

and call the coindex, the index, and the (Stiefel-Whitney) height of X respectively¹, where $w_1(X)$ denotes the 1st Stiefel-Whitney class of the double cover $X \rightarrow \overline{X}$. Then we have the inequality

$$\text{coind}(X) \leq h(X) \leq \text{ind}(X)$$

for a free \mathbb{Z}_2 -space X . The \mathbb{Z}_2 -space X is *tidy* if $\text{ind}(X) = \text{coind}(X)$.

The index and coindex of a \mathbb{Z}_2 -space have been researched in algebraic topology (see [1], [5], or [6]), and is recently studied in topological combinatorics (see [3] or [4]). By the definition of non-tidy spaces, the following question naturally arises: for a non-negative integer n , does there exist a \mathbb{Z}_2 -space whose difference between the index and coindex is n ?

According to Dai and Lam [1], Conner mentioned to them that such an example exists for arbitrary n . However, Matoušek pointed out in [3] that he could not find such an example appearing in print. On the other hand, the gap between the index and coindex is known to be arbitrarily large. In fact, the coindex of $\mathbb{R}P^{2m-1}$ is 1 since $h(\mathbb{R}P^{2n-1})$ and the result of Stolz [5] implies that the index of $\mathbb{R}P^{2m-1}$ can be arbitrarily large by choosing n to be sufficiently large. However, there are infinitely many n which is not

¹This terminology is due to [3], and other terms have been used in the literature, see [5] or [6].

represented by $\text{ind}(\mathbb{R}P^{2m-1}) - \text{coind}(\mathbb{R}P^{2m-1})$. Moreover, to determine of the index of $\mathbb{R}P^{2m-1}$ one needs familiarity with algebraic topology.

The aim of this paper is to construct a simple example of a \mathbb{Z}_2 -space X_n which gives an answer to the above question. Let S_b^n denote the \mathbb{Z}_2 - \mathbb{Z}_2 -space whose underlying space is the n -sphere and whose \mathbb{Z}_2 -actions are defined by

$$\begin{aligned}\tau(x_0, \dots, x_n) &= (-x_0, x_1, \dots, x_n), \\ (x_0, \dots, x_n)\tau &= (-x_0, \dots, -x_n),\end{aligned}$$

where τ denotes the generator of \mathbb{Z}_2 . Then X_k is defined inductively by

$$X_1 = S_a^1, X_{k+1} = X_k \times_{\mathbb{Z}_2} S_b^1.$$

Our main task is to prove the following theorem.

Theorem 1.1. *The height of X_n is n , but the coindex of X_n is 1.*

This implies that $\text{ind}(X_n) = n$ since X_n is an n -manifold.

We now review the outline of the proof. The height of X_n is determined by the following theorem, which will be proven in Section 3. Here we should note that Schultz [4] proved the inequality $h(X \times_{\mathbb{Z}_2} S_b^k) \geq h(X) + k$ for a free \mathbb{Z}_2 -space X .

Theorem 1.2. *$h(X \times_{\mathbb{Z}_2} S_b^k) = h(X) + k$ for a free \mathbb{Z}_2 -space X .*

Here we should note that Schultz [4] proved the inequality $h(X \times_{\mathbb{Z}_2} S_b^k) \geq h(X) + k$ for a free \mathbb{Z}_2 -space X .²

Next we shall consider the coindex of X_n . Since X_n is path-connected, there is a \mathbb{Z}_2 -map from S^1 to X_n . Hence to complete the proof, one needs to show that there is no \mathbb{Z}_2 -map from S^2 to X_n . This will be proven in Section 2 by observation of the fundamental groups.

2. EQUIVARIANT MAPS FROM THE 2-SPHERE

In this section we shall characterize the condition $\text{coind}(X) \geq 2$ in terms of fundamental groups (Theorem 2.2), and prove that $\text{coind}(X) = 1$.

Let Γ be a group, and let X be a path-connected free Γ -space with a base point x_0 . By the covering space theory, we have the isomorphism

$$\Phi : \pi_1(\overline{X}, \overline{x}_0) / \pi_1(X, x_0) \cong \Gamma.$$

²After the review of this paper, the author noticed that Theorem 1.1 was already obtained by Dochtermann and Schultz (see the end of Section 3.2 in [2]). They also show that the coindex of X_n is 1, and their proof is almost the same as ours. Using the known inequality $h(X \times S_b^1) \geq h(X) + 1$, they have $h(X_n) \geq n$. This implies $h(X_n) = n$ since \overline{X}_n is an n -manifold, and hence they obtain Theorem 1.1.

However, it was considered that this paper was enough valuable to be published since it contains some interesting results including Theorem 1.2.

Now we recall the construction of this isomorphism. Let $\bar{\varphi}$ be a loop of (\bar{X}, \bar{x}_0) , and let φ be the lift of $\bar{\varphi}$ whose initial point is x_0 . Then Φ is defined by

$$x_0 \cdot (\Phi[\bar{\varphi}]) = \varphi(1).$$

Let $f : X \rightarrow Y$ be a Γ -map between path-connected free Γ -spaces, and set $y_0 = f(x_0)$. Then $\bar{f}_* : \pi_1(\bar{X}, \bar{x}_0) \rightarrow \pi_1(\bar{Y}, \bar{y}_0)$ induces a group homomorphism

$$\bar{f}_* : \pi_1(\bar{X}, \bar{x}_0)/\pi_1(X, x_0) \rightarrow \pi_1(\bar{Y}, \bar{y}_0)/\pi_1(Y, y_0).$$

Proposition 2.1. *The diagram*

$$\begin{array}{ccc} \pi_1(\bar{X}, \bar{x}_0)/\pi_1(X, x_0) & \xrightarrow{\bar{f}_*} & \pi_1(\bar{Y}, \bar{y}_0) \\ \Phi \downarrow & & \downarrow \Phi \\ \Gamma & \xlongequal{\quad} & \Gamma \end{array}$$

is commutative.

Proof. Let $\bar{\varphi}$ be a loop of (\bar{X}, \bar{x}_0) and let φ be the lift of $\bar{\varphi}$ whose initial point is x_0 . Then $f \circ \varphi$ is the lift of $\bar{f} \circ \bar{\varphi}$ whose initial point is $f(x_0) = y_0$. Hence we have

$$y_0 \cdot \Phi(\bar{f}_*[\bar{\varphi}]) = f(\varphi(1)) = f(x_0 \cdot \Phi[\bar{\varphi}]) = y_0 \cdot \Phi[\bar{\varphi}].$$

This completes the proof. \square

Let u be a path in X joining x_0 to another point x'_0 . Then we have an isomorphism

$$\pi_1(\bar{X}, \bar{x}_0)/\pi_1(X, x_0) \rightarrow \pi_1(\bar{X}, \bar{x}'_0)/\pi_1(X, x'_0), [\bar{\varphi}] \mapsto [\bar{u} \cdot \bar{\varphi} \cdot \bar{u}^{-1}]$$

which commutes with the isomorphism Φ . The details are omitted.

From now on, we consider the case $\Gamma = \mathbb{Z}_2$. Let X be a path-connected free \mathbb{Z}_2 -space with a base point x_0 . We call an element α of $\pi_1(\bar{X}, \bar{x}_0)$ odd if it does not belong to $\pi_1(X, x_0)$. Then Proposition 2.1 asserts that for a \mathbb{Z}_2 -map $f : X \rightarrow Y$ between path-connected \mathbb{Z}_2 -spaces, the homomorphism $\bar{f}_* : \pi_1(\bar{X}, \bar{x}_0) \rightarrow \pi_1(\bar{Y}, \bar{y}_0)$ preserves the parity.

The following theorem characterizes the condition $\text{coind}(X) \geq 2$. This theorem was apparently obtained by Živaljević in his unpublished work (see page 101 in [3]).

Theorem 2.2. *For a path-connected free \mathbb{Z}_2 -space X , there is a \mathbb{Z}_2 -map from S_a^2 to X if and only if $\pi_1(\bar{X})$ has an odd element whose order is 2.*

Proof. Let α_0 denote the generator of $\pi_1(\bar{S}_a^2) \cong \mathbb{Z}_2$. If there is a \mathbb{Z}_2 -map $f : S_a^2 \rightarrow X$ then $\bar{f}_*(\alpha)$ is an odd element whose order is 2.

On the other hand, suppose that $\alpha \in \pi_1(\overline{X})$ is an odd element with $\alpha^2 = 1$. Let $\overline{\varphi}$ be a representative of α and let φ be the lift of $\overline{\varphi}$. Since $\overline{\varphi}$ is odd we have $\varphi(1) = \varphi(0)\tau$. Then $\psi = (\varphi\tau) \cdot \varphi$ is regarded as a \mathbb{Z}_2 -map from S_a^1 to X . Denote by p the quotient map $X \rightarrow \overline{X}$. Since

$$p_*[\psi] = [p \circ ((\varphi\tau) \cdot \varphi)] = [\overline{\varphi}]^2 = 1$$

and since $p_* : \pi_1(X, x_0) \rightarrow \pi_1(\overline{X}, \overline{x}_0)$ is injective, we have that ψ is null-homotopic. This implies that ψ can be extended to a \mathbb{Z}_2 -map from S_a^2 to X . \square

Lemma 2.3. *The group $\pi_1(\overline{X}_n)$ has no non-trivial torsion elements.*

Proof. By the definition of X_n , the space \overline{X}_n is the orbit space of a free and isometric $\pi_1(\overline{X}_n)$ -action on \mathbb{R}^n . Remark that for an affine map $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $a_1, \dots, a_m \in \mathbb{R}$ with $\sum_{i=1}^m a_i = 1$ and for $x_1, \dots, x_m \in \mathbb{R}^n$, we have

$$A\left(\sum_{i=1}^m a_i x_i\right) = \sum_{i=1}^m a_i (Ax_i).$$

Let $\alpha \in \pi_1(\overline{X}_n)$ be a torsion element and let k be its order. Let $x \in \mathbb{R}^n$. Then the point

$$y = \frac{1}{k} \sum_{i=1}^k x \alpha^i \in \mathbb{R}^n$$

is fixed by α . Since the action is free, we have $\alpha = 1$. \square

Combining Theorem 2.2 and Lemma 2.3, we have $\text{coind}(X_n) = 1$.

3. HEIGHT OF $X \times_{\mathbb{Z}_2} S_b^k$

The purpose of this section is to prove Theorem 1.2. As was mentioned, Schultz proved the inequality $h(X \times_{\mathbb{Z}_2} S_b^k) \geq h(X) + k$ in [4], applying the following proposition to the inclusion $X \times_{\mathbb{Z}_2} S_b^k \hookrightarrow X \times_{\mathbb{Z}_2} S_b^{k+1}$.

Proposition 3.1 (Lemma 3.1 of [4]). *Suppose that there is a \mathbb{Z}_2 -map $f : X \rightarrow Y$ homotopic to a map $g : X \rightarrow Y$ such that $g(x\tau) = g(x)$ for all $x \in X$. Then $h(Y) \geq h(X) + 1$.*

This is a consequence of the Gysin sequence for double covers. Here we prove Theorem 1.2 from a different viewpoint.

We identify $X \times_{\mathbb{Z}_2} S_b^k$ with the sphere bundle of a certain vector bundle \overline{V}_k over \overline{X} as follows. Let $L = X \times \mathbb{R}$ be the \mathbb{Z}_2 -line bundle over X with the involution $(x, v) \leftrightarrow (x\tau, -v)$, and let $\underline{\mathbb{R}}^k$ be the k -dimensional trivial \mathbb{Z}_2 -vector bundle over X . Put $V_k = L \oplus \underline{\mathbb{R}}^k$. Then V_k induces a vector bundle \overline{V}_k over \overline{X} . The sphere bundle $S(\overline{V}_k)$ is nothing but $X \times_{\mathbb{Z}_2} S_b^k$, and its orbit space is the projective bundle $P(\overline{V}_k)$ of \overline{V}_k .

Let α denote the 1st Stiefel-Whitney class of $S(\bar{V}_k) = X \times_{\mathbb{Z}_2} S_b^k$. It follows from the Leray-Hirsch theorem that $H^*(P(\bar{V}_k); \mathbb{Z}_2)$ is freely generated by $1, \alpha, \dots, \alpha^k$ over $H^*(X; \mathbb{Z}_2)$. Then we have

$$\alpha^{k+1} = \sum_{i=1}^{k+1} w_i(\bar{V}_k) \alpha^{k+1-i}.$$

Note that $w_1(\bar{V}_k) = w_1(X)$ and $w_i(\bar{V}_k) = 0$ if $i \geq 2$ since \bar{V}_k is the direct sum of \bar{L} and the trivial bundle. So we have $\alpha^{k+1} = w_1(X) \alpha^k$ and hence $\alpha^{k+l} = w_1(X)^l \alpha^k$.

Suppose that $w_1(X)^m \neq 0$ and $w_1(X)^{m+1} = 0$. Then we have $\alpha^{k+m} = w_1(X)^m \alpha^k \neq 0$ and $\alpha^{k+m+1} = w_1(X)^{m+1} \alpha^k = 0$. This completes the proof.

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