

STEENROD-ČECH HOMOLOGY-COHOMOLOGY THEORIES ASSOCIATED WITH BIVARIANT FUNCTORS

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ABSTRACT. Let \mathbf{NG}_0 denote the category of all pointed numerically generated spaces and continuous maps preserving base-points. In [SYH], we described a passage from bivariant functors $\mathbf{NG}_0^{\text{op}} \times \mathbf{NG}_0 \rightarrow \mathbf{NG}_0$ to generalized homology and cohomology theories. In this paper, we construct a bivariant functor such that the associated cohomology is the Čech cohomology and the homology is the Steenrod homology (at least for compact metric spaces).

1. INTRODUCTION

According to [Du], a topological space X is said to be Δ -generated if it has the final topology with respect to its singular simplexes. CW-complexes are typical examples of such Δ -generated spaces. In [SYH], we showed that the category of Δ -generated spaces is equivalent to the subcategory of the category \mathbf{Diff} of diffeological spaces consisting of those special type of objects which we call numerically generated spaces. Throughout this paper, we use term “numerically generated” instead of “ Δ -generated”. Let \mathbf{NG}_0 be the category of pointed numerically generated spaces and pointed continuous maps. In [SYH], we showed that \mathbf{NG}_0 is a symmetric monoidal closed category with respect to the smash product, and that every bilinear enriched functor $F : \mathbf{NG}_0^{\text{op}} \times \mathbf{NG}_0 \rightarrow \mathbf{NG}_0$ gives rise to a pair of generalized homology and cohomology theories, denoted by $h_\bullet(-, F)$ and $h^\bullet(-, F)$ respectively, such that

$$h_n(X, F) \cong \pi_0 F(S^{n+k}, \Sigma^k X), \quad h^n(X, F) \cong \pi_0 F(\Sigma^k X, S^{n+k})$$

hold whenever k and $n + k$ are non-negative.

As an example, consider the bilinear enriched functor F which assigns to (X, Y) the mapping space from X to the topological free abelian group $AG(Y)$ generated by the points of Y modulo the relation $* \sim 0$. The Dold-Thom theorem says that if X is a CW-complex then the groups $h_n(X, F)$ and $h^n(X, F)$ are, respectively, isomorphic to the singular homology and cohomology groups of X . But this is not the case for general X ; there exists a space X such that $h_n(X, F)$ (resp. $h^n(X, F)$) is not isomorphic to the singular homology (resp. cohomology) group of X .

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The aim of this paper is to construct a bilinear enriched functor such that for any space X the associated cohomology groups are isomorphic to the Čech cohomology groups of X . Interestingly, it turns out that the corresponding homology groups are isomorphic to the Steenrod homology groups for any compact metrizable space X . Thus we obtain a bibariant theory which ties together the Čech cohomology and the Steenrod homology theories.

Let \mathbf{NGC}_0 be the full subcategory of \mathbf{NG}_0 consisting of compact metric spaces. For given a linear enriched functor $T : \mathbf{NG}_0 \rightarrow \mathbf{NG}_0$, let

$$\check{F} : \mathbf{NG}_0^{\text{op}} \times \mathbf{NGC}_0 \rightarrow \mathbf{NG}_0$$

be a bifunctor which maps (X, Y) to the space $\varinjlim_{\lambda} \text{map}_0(X_{\lambda}, \varprojlim_{\mu_i} T(Y_{\mu_i}^{\check{C}}))$. Here λ runs through coverings of X , and X_{λ} is the Vietoris nerve corresponding to λ ([P]). The main results of the paper can be stated as follows.

Theorem 1.1. *The functor \check{F} is a bilinear enriched functor.*

Theorem 1.2. *Let X be a compact metrizable space. Then $h_n(X, \check{F}) = H_n^{\text{st}}(X, \mathbb{S})$ is the Steenrod homology group with coefficients in the spectrum $\mathbb{S} = \{T(S^k)\}$.*

In particular, let T be the functor which assigns to every X the topological abelian group $AG(X)$, and let

$$\check{C} : \mathbf{NG}_0^{\text{op}} \times \mathbf{NGC}_0 \rightarrow \mathbf{NG}_0$$

be the corresponding bifunctor.

Theorem 1.3. *For any pointed space X , $h^n(X, \check{C})$ is the Čech cohomology group of X , and $h_n(X, \check{C})$ is the Steenrod homology group of X if X is a compact metrizable space.*

Recall that the Steenrod homology group is related to the Čech homology group of X by the exact sequence

$$0 \longrightarrow \varprojlim_{\lambda_i}^1 \tilde{H}_{n+1}(X_{\lambda_i}^{\check{C}}) \longrightarrow H_n^{\text{st}}(X) \longrightarrow \tilde{H}_n(X) \longrightarrow 0.$$

According to [KKS], if X is a movable compactum then we have $\varprojlim_{\lambda_i}^1 \tilde{H}_{n+1}(X_{\lambda_i}^{\check{C}}) = 0$, and hence the following corollary follows.

Corollary 1.4. *Let X be a movable compactum. Then $h_n(X, \check{C})$ is the Čech homology group of X .*

The paper is organized as follows. In Section 2 we recall from [SYH] the category \mathbf{NG}_0 and the passage from bilinear enriched functors to generalized homology and cohomology theories. We also recall the definition of Čech

cohomology and Steenrod homology group, and Vietoris and Čech nerves; In Section 3 we prove Theorem 1.1; Finally, in Section 4 we prove Theorems 1.2 and 1.3.

2. PRELIMINARIES

2.1. Homology and cohomology theories via bifunctors. Let \mathbf{NG}_0 be the category of pointed numerically generated topological spaces and pointed continuous maps. In [SYH] we showed that \mathbf{NG}_0 satisfies the following properties:

- (1) It contains pointed CW-complexes;
- (2) It is complete and cocomplete;
- (3) It is monoidally closed in the sense that there is an internal hom Z^Y satisfying a natural bijection $\mathrm{hom}_{\mathbf{NG}_0}(X \wedge Y, Z) \cong \mathrm{hom}_{\mathbf{NG}_0}(X, Z^Y)$;
- (4) There is a coreflector $\nu: \mathrm{Top}_0 \rightarrow \mathbf{NG}_0$ such that the coreflection arrow $\nu X \rightarrow X$ is a weak equivalence;
- (5) The internal hom Z^Y is weakly equivalent to the space of pointed maps from Y to Z equipped with the compact-open topology.

Throughout the paper, we write $\mathrm{map}_0(Y, Z) = Z^Y$ for any $Y, Z \in \mathbf{NG}_0$.

A map $f: X \rightarrow Y$ between topological spaces is said to be numerically continuous if the composite $f \circ \sigma: \Delta^n \rightarrow Y$ is continuous for every singular simplex $\sigma: \Delta^n \rightarrow X$. We have the following.

Proposition 2.1. ([SYH]) *Let $f: X \rightarrow Y$ be a map between numerically generated spaces. Then f is numerically continuous if and only if f is continuous.*

From now on, we assume that \mathbf{C}_0 satisfies the following conditions: (i) \mathbf{C}_0 contains all finite CW-complexes. (ii) \mathbf{C}_0 is closed under finite wedge sum. (iii) If $A \subset X$ is an inclusion of objects in \mathbf{C}_0 then its cofiber $X \cup CA$ belongs to \mathbf{C}_0 ; in particular, \mathbf{C}_0 is closed under the suspension functor $X \mapsto \Sigma X$.

Definition 2.2. Let \mathbf{C}_0 be a full subcategory of \mathbf{NG}_0 . A functor $T: \mathbf{C}_0 \rightarrow \mathbf{NG}_0$ is called *enriched (or continuous)* if the map

$$T: \mathrm{map}_0(X, X') \rightarrow \mathrm{map}_0(T(X), T(X')),$$

which assigns $T(f)$ to every f , is a pointed continuous map.

Note that if f is constant, then so is $T(f)$.

Definition 2.3. An enriched functor T is called *linear* if for any pair of a pointed space X , a sequence

$$T(A) \rightarrow T(X) \rightarrow T(X \cup CA)$$

induced by the cofibration sequence $A \rightarrow X \rightarrow X \cup CA$, is a homotopy fibration sequence.

Example 2.4. Let $AG : CW_0 \rightarrow \mathbf{NG}_0$ be the functor which assigns to a pointed CW-complex (X, x_0) the topological abelian group $AG(X)$ generated by the points of X modulo the relation $x_0 \sim 0$. Then AG is a linear enriched functor. (see [SYH])

Theorem 2.5. ([SYH, Th 6.4]) *A linear enriched functor T defines a generalized homology $\{h_n(X, T)\}$ satisfying*

$$h_n(X, T) = \begin{cases} \pi_n T(X), & n \geq 0 \\ \pi_0 T(\Sigma^{-n} X), & n < 0. \end{cases}$$

Next we introduce the notion of a bilinear enriched functor, and describe a passage from a bilinear enriched functor to generalized cohomology and generalized homology theories. We assume that \mathbf{C}'_0 satisfies the same conditions of \mathbf{C}_0 .

Definition 2.6. Let \mathbf{C}_0 and \mathbf{C}'_0 be full subcategories of \mathbf{NG}_0 . A bifunctor $F : \mathbf{C}_0^{\text{op}} \times \mathbf{C}'_0 \rightarrow \mathbf{NG}_0$ is a function which

- (1) to each objects $X \in \mathbf{C}_0$ and $Y \in \mathbf{C}'_0$ assigns an object $F(X, Y) \in \mathbf{NG}_0$;
- (2) to each $f \in \text{map}_0(X, X')$, $g \in \text{map}_0(Y, Y')$ assigns a continuous map $F(f, g) \in \text{map}_0(F(X', Y), F(X, Y'))$.

F is required to satisfy the following equalities:

- (a) $F(1_X, 1_Y) = 1_{F(X, Y)}$;
- (b) $F(f, g) = F(1_X, g) \circ F(f, 1_Y) = F(f, 1_{Y'}) \circ F(1_{X'}, g)$;
- (c) $F(f' \circ f, 1_Y) = F(f, 1_Y) \circ F(f', 1_Y)$, $F(1_X, g' \circ g) = F(1_X, g') \circ F(1_X, g)$.

Definition 2.7. A bifunctor $F : \mathbf{C}_0^{\text{op}} \times \mathbf{C}_0 \rightarrow \mathbf{NG}_0$ is called *enriched* if the map

$$F : \text{map}_0(X, X') \times \text{map}_0(Y, Y') \rightarrow \text{map}_0(F(X', Y), F(X, Y')),$$

which assigns $F(f, g)$ to every pair (f, g) , is a pointed continuous map.

Note that if either f or g is constant, then so is $F(f, g)$.

Definition 2.8. For any pairs of pointed spaces (X, A) and (Y, B) , F is *bilinear* if the sequences

- (1) $F(X \cup CA, Y) \rightarrow F(X, Y) \rightarrow F(A, Y)$
- (2) $F(X, B) \rightarrow F(X, Y) \rightarrow F(X, Y \cup CB)$,

induced by the cofibration sequences $A \rightarrow X \rightarrow X \cup CA$ and $B \rightarrow Y \rightarrow Y \cup CB$, are homotopy fibration sequences.

Example 2.9. Let $T : \mathbf{NG}_0 \rightarrow \mathbf{NG}_0$ be a linear enriched functor, and let $F(X, Y) = \text{map}_0(X, T(Y))$ for $X, Y \in \mathbf{NG}_0$. Then $F : \mathbf{NG}_0^{\text{op}} \times \mathbf{NG}_0 \rightarrow \mathbf{NG}_0$ is a bilinear enriched functor.

Theorem 2.10. ([SYH, Th 7.4]) *A bilinear enriched functor F defines a generalized cohomology $\{h^n(-, F)\}$ and a generalized homology $\{h_n(-, F)\}$ such that*

$$h_n(Y, F) = \begin{cases} \pi_0 F(S^n, Y) & n \geq 0 \\ \pi_0 F(S^0, \Sigma^{-n}Y) & n < 0, \end{cases} \quad h^n(X, F) = \begin{cases} \pi_0 F(X, S^n) & n \geq 0 \\ \pi_{-n} F(X, S^0) & n < 0, \end{cases}$$

hold for any $X \in \mathbf{C}_0$ and $Y \in \mathbf{C}'_0$.

Proposition 2.11. ([SYH]) *If X is a CW-complex, we have $h_n(X, F) = H_n(X, \mathbb{S})$ and $h^n(X, F) = H^n(X, \mathbb{S})$, the generalized homology and cohomology groups with coefficients in the spectrum $\mathbb{S} = \{F(S^0, S^n) \mid n \geq 0\}$.*

2.2. Čech cohomology and Steenrod homology groups. We recall that the Čech cohomology group of X with coefficients group G is defined to be the colimit of the singular cohomology groups

$$\check{H}^n(X, G) = \varinjlim_{\lambda} H^n(X_{\lambda}^{\check{C}}, G),$$

where λ runs through coverings of X and $X_{\lambda}^{\check{C}}$ is the Čech nerve corresponding to λ , i.e. $v \in X_{\lambda}^{\check{C}}$ is a vertex of $X_{\lambda}^{\check{C}}$ corresponding to an open set $V \in \lambda$. On the other hand, the Steenrod homology group of a compact metric space X is defined as follows. As X is a compact metric space, there is a sequence $\{\lambda_i\}_{i \geq 0}$ of finite open covers of X such that $\lambda_0 = \{X\}$, λ_i is a refinement of λ_{i-1} , and X is the inverse limit $\varprojlim_i X_{\lambda_i}^{\check{C}}$. According to [F], the Steenrod homology group of X with coefficients in the spectrum \mathbb{S} is defined to be the group

$$H_n^{st}(X, \mathbb{S}) = \pi_n \underline{\text{holim}}_{\lambda_i} (X_{\lambda_i}^{\check{C}} \wedge \mathbb{S})$$

where $\underline{\text{holim}}$ denotes the homotopy inverse limit. (See also [KKS] for the definition without using subdivisions.)

2.3. Vietoris and Čech nerves. For each $X \in \mathbf{NG}_0$, let λ be an open covering of X . According to [P], the Vietoris nerve of λ is a simplicial set in which an n -simplex is an ordered $(n+1)$ -tuple (x_0, x_1, \dots, x_n) of points contained in an open set $U \in \lambda$. Face and degeneracy operators are respectively given by

$$d_i(x_0, \dots, x_n) = (x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

and

$$s_i(x_0, x_1, \dots, x_n) = (x_0, x_1, \dots, x_{i-1}, x_i, x_i, x_{i+1}, \dots, x_n), \quad 0 \leq i \leq n.$$

We denote the realization of the Vietoris nerve of λ by X_{λ} . If λ is a refinement of μ , then there is a canonical map $\pi_{\mu}^{\lambda} : X_{\lambda} \rightarrow X_{\mu}$ induced by the identity map of X .

The relation between the Vietoris and the Čech nerves is given by the following Proposition due to Dowker.

Proposition 2.12. ([Do]) *The Čech nerve $X_\lambda^{\check{C}}$ and the Vietoris nerve X_λ have the same homotopy type.*

According to [Do], for arbitrary topological space, the Vietoris and Čech homology groups are isomorphic and the Alexander-Spanier and Čech cohomology groups are isomorphic.

3. PROOF OF THEOREM 1.1

Let T be a linear enriched functor. We define a bifunctor $\check{F} : \mathbf{NG}_0^{\text{op}} \times \mathbf{NGC}_0 \rightarrow \mathbf{NG}_0$ as follows. For $X \in \mathbf{NG}_0$ and $Y \in \mathbf{NGC}_0$, we put

$$\check{F}(X, Y) = \lim_{\rightarrow \lambda} \text{map}_0(X_\lambda, \overleftarrow{\text{holim}}_{\mu_i} T(Y_{\mu_i}^{\check{C}})),$$

where λ is an open covering of X and $\{\mu_i\}_{i \geq 0}$ is a set of finite open covers of Y such that $\mu_0 = \{Y\}$, μ_i is a refinement of μ_{i-1} , and Y is the inverse limit $\overleftarrow{\lim}_i Y_{\mu_i}^{\check{C}}$.

Given based maps $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$, we define a map

$$\check{F}(f, g) \in \text{map}_0(\check{F}(X', Y), \check{F}(X, Y'))$$

as follows. Let ν and γ be open covering of X' and Y' respectively, and let $f^\#\nu = \{f^{-1}(U) \mid U \in \nu\}$ and $g^\#\gamma = \{g^{-1}(V) \mid V \in \gamma\}$. Then $f^\#\nu$ and $g^\#\gamma$ are open coverings of X and Y respectively. By the definition of the nerve, there are natural maps $f_\nu : X_{f^\#\nu} \rightarrow X'_\nu$ and $g_\gamma : Y_{g^\#\gamma}^{\check{C}} \rightarrow (Y')_\gamma^{\check{C}}$. Hence we have the map

$$T(g_\gamma)^{f_\nu} : T(Y_{g^\#\gamma}^{\check{C}})^{X'_\nu} \rightarrow T((Y')_\gamma^{\check{C}})^{X_{f^\#\nu}}$$

induced by f_ν and g_γ . Thus we can define

$$\check{F}(f, g) = \lim_{\rightarrow \nu \leftarrow \gamma} \overleftarrow{\text{holim}}_\gamma T(g_\gamma)^{f_\nu} : \check{F}(X', Y) \rightarrow \check{F}(X, Y').$$

Theorem 1.1. The functor \check{F} is a bilinear enriched functor.

First we prove that the sequence

$$\check{F}(X \cup CA, Z) \rightarrow \check{F}(X, Z) \rightarrow \check{F}(A, Z)$$

induced by the sequence $A \rightarrow X \rightarrow X \cup CA$, is a homotopy fibration sequence. Let λ be an open covering of $X \cup CA$, and let λ_X , λ_{CA} and λ_A be the coverings of X , CA and A consisting of those $U \in \lambda$ such that U intersects with X , CA , and A , respectively. We need the following lemma.

Lemma 3.1. *We have a homotopy equivalence*

$$(X \cup CA)_\lambda^{\check{C}} \simeq X_{\lambda_X}^{\check{C}} \cup C(A_{\lambda_A}^{\check{C}}).$$

Proof. By the definition of the Čech nerve, we have $(X \cup CA)_\lambda^{\check{C}} = X_{\lambda_X}^{\check{C}} \cup (CA)_{\lambda_{CA}}^{\check{C}}$. Since

$$X_{\lambda_X}^{\check{C}} \cup (CA)_{\lambda_{CA}}^{\check{C}} \simeq X_{\lambda_X}^{\check{C}} \cup A_{\lambda_A}^{\check{C}} \times I \cup (CA)_{\lambda_{CA}}^{\check{C}},$$

and since $(CA)_{\lambda_{CA}}^{\check{C}} \simeq *$, we have

$$X_{\lambda_X}^{\check{C}} \cup (CA)_{\lambda_{CA}}^{\check{C}} \simeq X_{\lambda_X}^{\check{C}} \cup C(A_{\lambda_A}^{\check{C}}).$$

Hence we have $(X \cup CA)_\lambda \simeq X_{\lambda_X}^{\check{C}} \cup C(A_{\lambda_A}^{\check{C}})$. \square

By Proposition 2.12 and Lemma 3.1, the sequence

$$A_{\lambda_A} \rightarrow X_{\lambda_X} \rightarrow (X \cup CA)_\lambda$$

is a homotopy cofibration sequence. Hence the sequence

$$[(X \cup CA)_\lambda, Z] \rightarrow [X_{\lambda_X}, Z] \rightarrow [A_{\lambda_A}, Z]$$

is an exact sequence for any λ . Since the nerves of the form λ_X (resp. λ_A) are cofinal in the set of nerves of X (resp. A), we conclude that the sequence

$$\check{F}(X \cup CA, Z) \rightarrow \check{F}(X, Z) \rightarrow \check{F}(A, Z)$$

is a homotopy fibration sequence.

Now we show that the sequence $\check{F}(Z, A) \rightarrow \check{F}(Z, X) \rightarrow \check{F}(Z, X \cup CA)$ is a homotopy fibration sequence. By the linearity of T , the sequence

$$T(A_{\lambda_A}^{\check{C}}) \rightarrow T(X_{\lambda_X}^{\check{C}}) \rightarrow T((X \cup CA)_\lambda^{\check{C}})$$

is a homotopy fibration sequence. Since the fibre $T(A_{\lambda_A}^{\check{C}})$ is homeomorphic to the inverse limit

$$\varprojlim (* \rightarrow T((X \cup CA)_\lambda^{\check{C}}) \leftarrow T(X_{\lambda_X}^{\check{C}})),$$

we have

$$\begin{aligned} & \varprojlim (* \rightarrow \varprojlim_\lambda T((X \cup CA)_\lambda^{\check{C}}) \leftarrow \varprojlim_{\lambda_X} T(X_{\lambda_X}^{\check{C}})) \\ & \simeq \varprojlim \varprojlim_\lambda (* \rightarrow T((X \cup CA)_\lambda^{\check{C}}) \leftarrow T(X_{\lambda_X}^{\check{C}})) \\ & \simeq \varprojlim_\lambda \varprojlim (* \rightarrow T((X \cup CA)_\lambda^{\check{C}}) \leftarrow T(X_{\lambda_X}^{\check{C}})) \\ & \simeq \varprojlim_\lambda T(A_{\lambda_A}^{\check{C}}). \end{aligned}$$

This implies that the sequence

$$\varprojlim_{\lambda_A} T(A_{\lambda_A}^{\check{C}}) \rightarrow \varprojlim_{\lambda_X} T(X_{\lambda_X}^{\check{C}}) \rightarrow \varprojlim_\lambda T((X \cup CA)_\lambda^{\check{C}})$$

is a homotopy fibration sequence, hence so is $\check{F}(Z, A) \rightarrow \check{F}(Z, X) \rightarrow \check{F}(Z, X \cup CA)$.

Next we prove the continuity of \check{F} . Let $F(X, Y) = \text{map}_0(X, \varprojlim_{\mu_i} T(Y_{\mu_i}^{\check{C}}))$, so that we have $\check{F}(X, Y) = \varinjlim_{\lambda} F(X_\lambda, Y)$. We need the following lemma.

Lemma 3.2. *The functor F is an enriched bifunctor.*

Proof. Let $F_1(Y) = \varprojlim_{\mu_i} T(Y_{\mu_i}^{\check{C}})$ and $F_2(X, Z) = \text{map}_0(X, Z)$, so that we have $F(X, Y) = F_2(X, F_1(Y))$. Clearly F_2 is continuous.

Let G_1 be the functor which maps Y to $\varprojlim_{\mu_i} Y_{\mu_i}^{\check{C}}$. Since T is enriched, F_1 is continuous if so is G_1 . It suffices to show that the map $G'_1: \text{map}_0(Y, Y') \times \varprojlim_{\mu_i} Y_{\mu_i}^{\check{C}} \rightarrow \varprojlim_{\lambda_j} (Y')_{\lambda_j}^{\check{C}}$, adjoint to G_1 , is continuous for any Y and Y' . Given an open covering λ of Y' , let p_λ^n be the natural map $\varprojlim_{\lambda} (Y')_{\lambda}^{\check{C}} \rightarrow \text{map}_0(\Delta^n, (Y')_{\lambda}^{\check{C}})$. Then G'_1 is continuous if so is the composite

$$p_\lambda^n \circ G'_1: \text{map}_0(Y, Y') \times \varprojlim_{\mu_i} Y_{\mu_i}^{\check{C}} \rightarrow \text{map}_0(\Delta^n, (Y')_{\lambda}^{\check{C}})$$

for every $\lambda \in \text{Cov}(Y')$ and every n . Here we may assume by [SYH, Proposition 4.3] that $\text{map}_0(\Delta^n, (Y')_{\lambda}^{\check{C}})$ is equipped with the compact open topology. Let $(g, \alpha) \in \text{map}_0(Y, Y') \times \varprojlim_{\mu_i} Y_{\mu_i}^{\check{C}}$, and let $W_{K,U} \subset \text{map}_0(\Delta^n, (Y')_{\lambda}^{\check{C}})$ be an open neighborhood of $p_\lambda^n(G'_1(g, \alpha))$, where K is a compact set of Δ^n and U is an open set of $(Y')_{\lambda}^{\check{C}}$.

Let us choose simplices σ of $Y_{g^\# \lambda}^{\check{C}}$ with vertices $g^{-1}(U(\sigma, k))$, where $U(\sigma, k) \in \lambda$ for $0 \leq k \leq \dim \sigma$. Let

$$O(\sigma) = \bigcap_{0 \leq k \leq \dim \sigma} U(\sigma, k) \subset Y'.$$

Let us choose a point $y_\sigma \in \bigcap_{0 \leq k \leq \dim \sigma} g^{-1}(U(\sigma, k))$, then $g(y_\sigma) \in O(\sigma)$. Let W_1 be the intersection of all $\overline{W}_{y_\sigma, O(\sigma)}$.

There is an integer l such that

$$\mu_l > \overline{\mu}_l > g^\# \lambda$$

where $\overline{\mu}_l$ is a closed covering $\{\overline{V} | V \in \mu_l\}$ of Y . Thus for any $U \in \mu_l$, there is an open set $V_U \in g^\# \lambda$ such that $\overline{U} \subset g^{-1}(V_U)$. Since Y is a compact set, \overline{U} is compact. Let W_2 be the intersection of $W_{\overline{U}, V_U}$, and let $W = W_1 \cap W_2$.

Since $\mu_l > g^\# \lambda$, we have

$$p_\lambda^n(G'_1(g, \alpha)) = (g_\lambda)_* (\pi_{g^\# \lambda}^{\mu_l})_* p_{\mu_l}^n \alpha.$$

where $(g_\lambda)_*$ and $(\pi_{g^\# \lambda}^{\mu_l})_*$ are induced by $g_\lambda: Y_{g^\# \lambda}^{\check{C}} \rightarrow (Y')_{\lambda}^{\check{C}}$ and $\pi_{g^\# \lambda}^{\mu_l}: Y_{\mu_l}^{\check{C}} \rightarrow Y_{g^\# \lambda}^{\check{C}}$, respectively. Let

$$W' = (p_{\mu_l}^n)^{-1}(W_{K, (\pi_{g^\# \lambda}^{\mu_l})^{-1}(g_\lambda)^{-1}(U)}).$$

Then $W \times W'$ is a neighborhood of (g, α) in $\text{map}_0(Y, Y') \times \varprojlim_{\mu_i} Y_{\mu_i}$. To see that $p_\lambda \circ G'_1$ is continuous at (g, α) , we need only show that $W \times W'$ is contained in $(p_\lambda \circ G'_1)^{-1}(U)$. Suppose (h, β) belongs to $W \times W'$. Since W is contained in W_1 , we have

$$y_\sigma \in h^{-1}(O(\sigma)) \subset \bigcap_{0 \leq k \leq \dim \sigma} h^{-1}(U(\sigma, k)).$$

This means that the vertices $h^{-1}(U(\sigma, k)) \in h^\# \lambda$, $0 \leq k \leq \dim \sigma$, determine simplices σ' of $Y_{h^\# \lambda}$ each corresponding to each $\sigma \subset Y_{g^\# \lambda}$. Thus we have an isomorphism

$$s : Y_{h^\# \lambda}^{\check{C}} \rightarrow Y_{g^\# \lambda}^{\check{C}},$$

$$h^{-1}(U(\sigma, k)) \mapsto g^{-1}(U(\sigma, k)).$$

Moreover since W is contained in W_2 , we have $\bar{\mu}_l > h^\# \lambda$.

Since the commutative diagram

$$\begin{array}{ccccc} Y_{\mu_l}^{\check{C}} & \longrightarrow & Y_{g^\# \lambda}^{\check{C}} & \xrightarrow{g_\lambda} & (Y')_{\lambda}^{\check{C}} \\ & \searrow & \uparrow s & \nearrow h_\lambda & \\ & & Y_{h^\# \lambda}^{\check{C}} & & \end{array}$$

is commutative, we have the equation

$$p_\lambda^n \circ G'_1(h, \beta)(K) = h_\lambda \pi_{h^\# \lambda}^{\mu_l}(\beta)(K) = g_\lambda \pi_{g^\# \lambda}^{\mu_l}(\beta)(K)$$

Since $g_\lambda \pi_{g^\# \lambda}^{\mu_l}(\beta)(K)$ is contained in U , so is $p_\lambda^n \circ G'_1(h, \beta)(K)$.

Thus $p_\lambda^n \circ G'_1$ is continuous for all $\lambda \in \text{Cov}(Y')$, and hence so is

$$G'_1 : \text{map}_0(Y, Y') \times \varprojlim_{\mu_i} Y_{\mu_i}^{\check{C}} \rightarrow \varprojlim_{\lambda_j} (Y')_{\lambda_j}^{\check{C}}.$$

□

We are now ready to prove Theorem 1.1. For given pointed spaces X , Y and a covering μ of X , let i_μ denote the natural map $F(X_\mu, Y) \rightarrow \varinjlim_{\mu} F(X_\mu, Y)$. To prove the theorem, it suffices to show that the map

$$\begin{aligned} \check{F}' \circ (1 \times i_\lambda) : \text{map}_0(X, X') \times F(X'_\lambda, Y) &\rightarrow \text{map}_0(X, X') \times \varinjlim_{\lambda} F(X'_\lambda, Y) \\ &\rightarrow \varinjlim_{\mu} F(X_\mu, Y) \end{aligned}$$

which maps (f, α) to $i_{f^\# \lambda}(F(f_\lambda, 1_Y)(\alpha))$, is continuous for every covering λ of X .

Let

$$R_\lambda : \text{map}_0(X, X') \rightarrow \varinjlim_{\mu} \text{map}_0(X_\mu, X'_\lambda)$$

be the map which assigns to $f : X \rightarrow X'$ the image of $\text{map}_0(X, X')$, $f_\lambda \in \text{map}_0(X_{f^\#\lambda}, X'_\lambda)$ in $\varinjlim_{\mu} \text{map}_0(X_\mu, X'_\lambda)$, and let Q_λ be the map

$$\begin{aligned} \varinjlim_{\mu} \text{map}_0(X_\mu, X'_\lambda) \times F(X'_\lambda, Y) &\rightarrow \varinjlim_{\mu} F(X_\mu, Y), \\ [f, \alpha] &\mapsto i_{f^\#\lambda} f_\lambda \circ \alpha = i_{f^\#\lambda}(F(f_\lambda, 1_Y)(\alpha)). \end{aligned}$$

Since we have $\check{F}' \circ (1 \times i_\lambda) = Q_\lambda \circ (R_\lambda \times 1)$, we need only show the continuity of Q_λ and R_λ . Since Q_λ is induced by the maps $\text{map}_0(X_\mu, X'_\lambda) \times F(X'_\lambda, Y) \rightarrow F(X_\mu, Y)$, Q_λ is continuous.

To see that R_λ is continuous, let $W_{K^f, U}$ be a neighborhood of f_λ in $\text{map}_0(X_{f^\#\lambda}, X'_\lambda)$, where K^f is a compact subset of $X_{f^\#\lambda}$ and U is an open subset of X'_λ . Since K^f is compact, there is a finite subcomplex S^f of $X_{f^\#\lambda}$ such that $K^f \subset S^f$. Let τ_i^f , $0 \leq i \leq m$, be simplexes of S^f . By taking a suitable subdivision of $X_{f^\#\lambda}$, we may assume that there is a simplicial neighborhood $N_{\tau_i^f}$ of each τ_i^f , $1 \leq i \leq m$, such that $K^f \subset S^f \subset \cup_i N_{\tau_i^f} \subset f_\lambda^{-1}(U)$.

Let $\{x_k^i\}$ be the set of vertices of τ_i^f and let W be the intersection of all $W_{\{x_k^i\}, U_{(\tau_i^f)'}}$ where $U_{(\tau_i^f)'}$ is an open set of X'_λ containing the set $\{f(x_k^i)\}$. Then W is a neighborhood of f . We need only show that $R_\lambda(W) \subset i_{f^\#\lambda}(W_{K^f, U})$. Suppose that g belongs to W . Since $\{x_k^i\}$ is contained in $g^{-1}(U_{(\tau_i^f)'})$ for any i , a simplex τ_i^g spanned by the vertices is contained in $X_{g^\#\lambda}$. Let S^g be the finite subcomplex of $X_{g^\#\lambda}$ consists of simplexes τ_i^g . By the construction, S^f and S^g are isomorphic. Moreover there is a compact subset K^g of $X_{g^\#\lambda}$ such that K^g and K^f are homeomorphic. On the other hand, since $g(\{x_k^i\}) \subset U_{(\tau_i^f)'}$, there is a simplex of X'_λ having $g_\lambda(\tau_i^g)$ and $(\tau_i^f)'$ as its faces. This means that $g_\lambda(\tau_i^g) \subset f_\lambda(\cup_i N_{\tau_i^f})$. Thus we have $g_\lambda(K^g) = \cup_i g_\lambda(\tau_i^g) \subset f_\lambda(\cup_i N_{\tau_i^f})$.

Let $f^\#\lambda \cap g^\#\lambda$ be an open covering

$$\{f^{-1}(U) \cap g^{-1}(V) \mid U, V \in \lambda\}$$

of X . We regard $X_{f^\#\lambda}$ and $X_{g^\#\lambda}$ as a subcomplex of $X_{f^\#\lambda \cap g^\#\lambda}$. Since $g_\lambda|_{X_{f^\#\lambda \cap g^\#\lambda}}$ is contiguous to $f_\lambda|_{X_{f^\#\lambda \cap g^\#\lambda}}$, we have a homotopy equivalence $g_\lambda|_{X_{f^\#\lambda \cap g^\#\lambda}} \simeq f_\lambda|_{X_{f^\#\lambda \cap g^\#\lambda}}$. By the homotopy extension property of $g_\lambda|_{X_{f^\#\lambda \cap g^\#\lambda}} : X_{f^\#\lambda \cap g^\#\lambda} \rightarrow X'_\lambda$ and $f_\lambda : X_{f^\#\lambda} \rightarrow X'_\lambda$, $g_\lambda|_{X_{f^\#\lambda \cap g^\#\lambda}}$ extends to map $G : X_{f^\#\lambda} \rightarrow X'_\lambda$.

We have the relation $G \sim \pi_{f^\#\lambda}^{f^\#\lambda \cap g^\#\lambda} G = g_\lambda|_{X_{f^\#\lambda \cap g^\#\lambda}} = \pi_{g^\#\lambda}^{f^\#\lambda \cap g^\#\lambda} g_\lambda \sim g_\lambda$, where \sim is the relation of the direct limit. Moreover by $G(K^f) \subset$

$f_\lambda(\cup_i N_{\tau_i^f}) \subset U$, we have $[g_\lambda] = [G] \in i_{f\#\lambda}(W_{K^f, U})$. Hence R_λ is continuous, and so is \check{F}' .

4. PROOFS OF THEOREMS 1.2 AND 1.3

To prove Theorems 1.2 and 1.3, we need several lemmas.

Lemma 4.1. *There exists a sequence $\lambda_1^n < \lambda_2^n < \dots < \lambda_m^n < \dots$ of open coverings of S^n such that :*

- (1) *For each open covering μ of S^n , there is an $m \in \mathbb{N}$ such that λ_m^n is a refinement of μ :*
- (2) *For any m , $S_{\lambda_m^n}^n$ is homotopy equivalent to S^n .*

Proof. We prove by induction on n . For $n = 1$, we define an open covering λ_m^1 of S^1 as follows. For any i with $0 \leq i < 4m$, we put

$$U(i, m) = \left\{ (\cos \theta, \sin \theta) \mid \frac{(4i-3)\pi}{8m} < \theta < \frac{(4i+5)\pi}{8m} \right\}.$$

Let $\lambda_m^1 = \{U(i, m) \mid 0 \leq i < 4m\}$. Then the set λ_m^1 is an open covering of S^1 and is a refinement of λ_{m-1}^1 . Clearly $(S^1)_{\lambda_m^1}^{\check{C}}$ is homeomorphic to S^1 , hence $S_{\lambda_m^1}^1$ is homotopy equivalent to S^1 . Moreover for any open covering μ of S^1 , there exists an m such that λ_m^1 is a refinement of μ . Hence the lemma is true for $n = 1$. Assume now that the lemma is true for $1 \leq k \leq n-1$. Let λ_m^n be the open covering $\lambda_m^{n-1} \times \lambda_m^1$ of $S^{n-1} \times S^1$ and let λ_m^n be the open covering of S^n induced by the natural map $p : S^{n-1} \times S^1 \rightarrow S^{n-1} \times S^1 / S^{n-1} \vee S^1$. Since $S_{\lambda_m^{n-1}}^{n-1}$ is a homotopy equivalence of S^{n-1} , we have

$$S_{\lambda_m^n}^n \approx (S^{n-1} \times S^1 / S^{n-1} \vee S^1)_{\lambda_m^n} \approx (S_{\lambda_m^{n-1}}^{n-1} \times S_{\lambda_m^1}^1) / (S_{\lambda_m^{n-1}}^{n-1} \vee S_{\lambda_m^1}^1) \approx S^n.$$

Thus the sequence $\lambda_1^n < \lambda_2^n < \dots < \lambda_m^n < \dots$ satisfies the required conditions. \square

Lemma 4.2. $h_n(X, \check{F}) \cong \pi_n \mathop{\text{holim}}_{\leftarrow \mu} T(X_\mu^{\check{C}})$ for $n \geq 0$.

Proof. By Lemma 4.1, we have an isomorphism

$$\lim_{\rightarrow \lambda} [S_\lambda^n, \mathop{\text{holim}}_{\leftarrow \mu} T(X_\mu^{\check{C}})] \cong [S^n, \mathop{\text{holim}}_{\leftarrow \mu} T(X_\mu^{\check{C}})].$$

Thus we have

$$\begin{aligned}
h_n(X, \check{F}) &= \pi_0 \check{F}(S^n, X) \\
&= \pi_0 \varinjlim_\lambda \text{map}_0(S_\lambda^n, \varprojlim_\mu T(X_\mu^{\check{C}})) \\
&\cong \varinjlim_\lambda [S^0, \text{map}_0(S_\lambda^n, \varprojlim_\mu T(X_\mu^{\check{C}}))] \\
&\cong \varinjlim_\lambda [S_\lambda^n, \varprojlim_\mu T(X_\mu^{\check{C}})] \\
&\cong [S^n, \varprojlim_\mu T(X_\mu^{\check{C}})] \\
&\cong \pi_n \varprojlim_\mu T(X_\mu^{\check{C}}).
\end{aligned}$$

□

Now we are ready to prove Theorem 1.2. Let X be a compact metric space and let $\mathbb{S} = \{T(S^k) \mid k \geq 0\}$. Since X is a compact metric space, there is a sequence $\{\mu_i\}_{i \geq 0}$ of finite open covers of X with $\mu_0 = X$ and μ_i refining μ_{i-1} such that $X = \varprojlim_i X_{\mu_i}^{\check{C}}$ holds. Let us denote $X_{\mu_i}^{\check{C}} = X_i^{\check{C}}$ and $X_{\mu_i} = X_i$ if there is no possibility of confusion. According to [F], there is a short exact sequence

$$0 \longrightarrow \varprojlim_i^1 H_{n+1}(X_i^{\check{C}}, \mathbb{S}) \longrightarrow H_n^{st}(X, \mathbb{S}) \longrightarrow \varprojlim_i H_n(X_i^{\check{C}}, \mathbb{S}) \longrightarrow 0$$

where $H_n(X, \mathbb{S})$ is the homology group of X with coefficients in the spectrum \mathbb{S} . (This is a special case of the Milnor exact sequence [MI].) On the other hand, by [BK], we have the following.

Lemma 4.3. ([BK]) *There is a natural short exact sequence*

$$0 \longrightarrow \varprojlim_i^1 \pi_{n+1} T(X_i^{\check{C}}) \longrightarrow \pi_n \varprojlim_i T(X_i) \longrightarrow \varprojlim_i \pi_n T(X_i^{\check{C}}) \longrightarrow 0.$$

By Proposition 2.11, we have a diagram

$$\begin{array}{ccccccc}
(4.1) & 0 & \longrightarrow & \varprojlim_i^1 H_{n+1}(X_i^{\check{C}}, \mathbb{S}) & \longrightarrow & H_n^{st}(X, \mathbb{S}) & \longrightarrow & \varprojlim_i H_n(X_i^{\check{C}}, \mathbb{S}) & \longrightarrow & 0 \\
& & & \downarrow \cong & & & & \downarrow \cong & & \\
& 0 & \longrightarrow & \varprojlim_i^1 \pi_{n+1}(T(X_i^{\check{C}})) & \longrightarrow & \pi_n(\varprojlim_i T(X_i^{\check{C}})) & \longrightarrow & \varprojlim_i \pi_n(T(X_i^{\check{C}})) & \longrightarrow & 0.
\end{array}$$

Hence it suffices to construct a natural homomorphism

$$H_n^{st}(X, \mathbb{S}) \rightarrow \pi_n(\varprojlim_i T(X_i^{\check{C}}))$$

making the diagram (4.1) commutative.

Since T is continuous, the identity map $X \wedge S^k \rightarrow X \wedge S^k$ induces a continuous map $i' : X \wedge T(S^k) \rightarrow T(X \wedge S^k)$. Hence we have the composite homomorphism

$$\begin{aligned} H_n^{st}(X, \mathbb{S}) &= \pi_n \mathop{\text{holim}}\limits_i (X_i^{\check{C}} \wedge \mathbb{S}) \\ &\cong \varinjlim_k \pi_{n+k} (\mathop{\text{holim}}\limits_i (X_i^{\check{C}} \wedge T(S^k))) \\ &\xrightarrow{I} \varinjlim_k \pi_{n+k} (\mathop{\text{holim}}\limits_i T(X_i^{\check{C}} \wedge S^k)) \\ &\cong \pi_n (\mathop{\text{holim}}\limits_i T(X_i^{\check{C}})) \end{aligned}$$

in which $I = \varinjlim_k i'_*$ is induced by the homomorphisms

$$i'_* : \pi_{n+k} (\mathop{\text{holim}}\limits_i (X_i^{\check{C}} \wedge T(S^k))) \rightarrow \pi_{n+k} (\mathop{\text{holim}}\limits_i T(X_i^{\check{C}} \wedge S^k)).$$

Clearly resulting the homomorphism $H_n^{st}(X, \mathbb{S}) \rightarrow \pi_n (\mathop{\text{holim}}\limits_i T(X_i^{\check{C}}))$ makes the diagram (4.1) commutative. Thus $h_n(X, \check{F})$ is isomorphic to the Steenrod homology group coefficients in the spectrum \mathbb{S} .

Finally, to prove Theorem 1.3 it suffices to show that $h^n(X, \check{C})$ is isomorphic to the Čech cohomology group of X .

By Lemma 4.1, we have a homotopy commutative diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{=} & AG(S^n) & \xrightarrow{=} & AG(S^n) & \xrightarrow{=} & \cdots \\ & & \downarrow \simeq & & \downarrow \simeq & & \\ \cdots & \longrightarrow & AG(S_{\lambda_{m-1}}^n) & \xrightarrow{\simeq} & AG(S_{\lambda_m}^n) & \longrightarrow & \cdots \end{array}$$

Hence we have $AG(S^n) \simeq \mathop{\text{holim}}\limits_i AG(S_{\lambda_i}^n)$.

Thus we have

$$\begin{aligned} h^n(X, \check{C}) &= \pi_0 \check{C}(X, S^n) \\ &= \pi_0 \varinjlim_{\lambda} \text{map}_0(X_{\lambda}, \mathop{\text{holim}}\limits_{\mu} AG((S^n)_{\mu}^{\check{C}})) \\ &\cong [S^0, \varinjlim_{\lambda} \text{map}_0(X_{\lambda}, AG(S^n))] \\ &\cong \varinjlim_{\lambda} [S^0, \text{map}_0(X_{\lambda}, AG(S^n))] \\ &\cong \varinjlim_{\lambda} [S^0 \wedge X_{\lambda}, AG(S^n)] \\ &\cong \varinjlim_{\lambda} [X_{\lambda}, AG(S^n)]. \end{aligned}$$

Hence $h^n(X, \check{C})$ is isomorphic to the Čech cohomology group of X .

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