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CONVEXITY PROPERTIES OF A NEW GENERAL INTEGRAL OPERATOR OF p-VALENT FUNCTIONS

Serap BULUT

Abstract. In this paper, we introduce a new general integral operator and obtain the order of convexity of this integral operator.

1. Introduction and definitions

Let $\mathcal{A}_p$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, \ldots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. In particular, we set $\mathcal{A}_1 \equiv \mathcal{A}$.

A function $f \in \mathcal{A}_p$ is said to be p-valently starlike of order γ $(0 \leq \gamma < p)$ if and only if f satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \tag{1.2}$$

for all $z \in \mathbb{U}$. We say that f is in the class $\mathcal{S}_p^*(\gamma)$ for such functions. In particular, we set $\mathcal{S}_p^*(0) \equiv \mathcal{S}_p^*$ for p-valently starlike functions in $\mathbb{U}$.

On the other hand, a function $f \in \mathcal{A}_p$ is said to be p-valently convex of order γ $(0 \leq \gamma < p)$ if and only if f satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma \tag{1.3}$$

for all $z \in \mathbb{U}$. We say that f is in the class $\mathcal{K}_p(\gamma)$ for such functions. In particular, we set $\mathcal{K}_p(0) \equiv \mathcal{K}_p$ for p-valently convex functions in $\mathbb{U}$.

Also, we note that $\mathcal{S}_1^*(\gamma) \equiv \mathcal{S}^*(\gamma)$ and $\mathcal{K}_1(\gamma) \equiv \mathcal{K}(\gamma)$ are the classes of starlike and convex functions of order γ $(0 \leq \gamma < 1)$, respectively.

A function $f \in \mathcal{A}_p$ is in the class $\mathcal{R}_p(\gamma)$ if it satisfies

$$\Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \gamma$$

for all $z \in \mathbb{U}$.

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A function $f \in \mathcal{A}_p$ is in the class $\mathcal{US}_p(\delta,\gamma)$ if and only if f satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta\left|\frac{zf'(z)}{f(z)}-p\right|+\gamma, \tag{1.4}$$

where $\delta \geq 0$, $\gamma \in [-1,p)$, $\delta+\gamma \geq 0$, $z \in \mathbb{U}$.

Furthermore, a function $f \in \mathcal{A}_p$ is in the class $\mathcal{UK}_p(\delta,\gamma)$ if and only if f satisfies

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta\left|\frac{zf''(z)}{f'(z)}-(p-1)\right|+\gamma, \tag{1.5}$$

where $\delta \geq 0$, $\gamma \in [-1,p)$, $\delta+\gamma \geq 0$, $z \in \mathbb{U}$.

Now we consider following comprehensive class:

A function $f \in \mathcal{A}_p$ is in the class $\mathcal{B}_p(\mu,\gamma)$ if and only if f satisfies

$$\left|\frac{f'(z)}{z^{p-1}}\left(\frac{z^p}{f(z)}\right)^{\mu}-p\right| < p-\gamma,$$

where $\mu \geq 0$, $\gamma \in [0,p)$, $z \in \mathbb{U}$.

Remark 1. This family is a comprehensive class of analytic functions that contains other new classes of analytic functions as well as some very well-known ones. For example,

(**i**) For $\mu = 1$, we have the class

$$\mathcal{B}_p(1,\gamma) \equiv \mathcal{S}_p^*(\gamma).$$

(**ii**) For $\mu = 0$, we have the class

$$\mathcal{B}_p(0,\gamma) \equiv \mathcal{R}_p(\gamma).$$

(**iii**) For $p = 1$, this class studied by Frasin and Jahangiri [11].

(**iv**) For $p = 1$ and $\mu = 2$, this class studied by Frasin and Darus [10].

Let $\alpha_i, \beta_i \in \mathbb{C}$ and $f_i, g_i \in \mathcal{A}_p$ for all $i = 1, 2, \ldots, n$, $n \in \mathbb{N}$. We define the following general integral operator

$$\mathcal{I}_n^{\alpha,\beta}(f,g)(z) = \int_0^z pt^{p-1}\prod_{i=1}^{n}\left(\frac{f_i(t)}{t^p}\right)^{\alpha_i}\left(\frac{g_i'(t)}{pt^{p-1}}\right)^{\beta_i}dt. \tag{1.6}$$

Remark 2. (**i**) For $p = 1$, we have the integral operator

$$\mathcal{K}(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\alpha_i}\left(g_i'(t)\right)^{\beta_i}dt$$

introduced and studied by Ularu [13].

(**ii**) For $\alpha_i > 0,\ \beta_i = 0\ \ (1 \le i \le n)$ and $\alpha_i = 0,\ \beta_i > 0\ \ (1 \le i \le n)$, we have the integral operators

$$\mathcal{F}_p(z) = \int_0^z pt^{p-1} \left(\frac{f_1(t)}{t^p}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t^p}\right)^{\alpha_n} dt$$

and

$$\mathcal{G}_p(z) = \int_0^z pt^{p-1} \left(\frac{g_1'(t)}{pt^{p-1}}\right)^{\beta_1} \cdots \left(\frac{g_n'(t)}{pt^{p-1}}\right)^{\beta_n} dt,$$

respectively. This integral operators are introduced and studied by Frasin [8, 9]. Also for $p = 1$, the integral operators

$$\mathcal{F}_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt$$

and

$$\mathcal{F}_{\beta_1,\ldots,\beta_n}(z) = \int_0^z \left(g_1'(t)\right)^{\beta_1} \cdots \left(g_n'(t)\right)^{\beta_n} dt$$

studied recently by many authors (see [1]-[7]).

The following result will be required in our investigation.

General Schwarz Lemma. [12] *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \le \frac{M}{R^m} |z|^m \quad (z \in \mathbb{U}_R).$$

The equality can hold only if $f(z) = e^{i\theta}(M/R^m)\, z^m$, where θ is constant.

2. Main Results

Theorem 2.1. *Let $\alpha_i, \beta_i \ge 0,\ \delta_i \ge 0,\ \gamma_i \in [-1, p),\ \delta_i + \gamma_i \ge 0$ and $f_i \in \mathcal{US}_p(\delta_i, \gamma_i)$, $g_i \in \mathcal{UK}_p(\delta_i, \gamma_i)$ for all $i = 1, 2, \ldots, n$. If*

$$0 \le p + \sum_{i=1}^{n} (\alpha_i + \beta_i)(\gamma_i - p) \le p, \tag{2.1}$$

then the integral operator $\mathcal{I}_n^{\alpha,\beta}(f, g)$ defined by (1.6) *is p-valently convex of order λ with*

$$\lambda = p + \sum_{i=1}^{n} (\alpha_i + \beta_i)(\gamma_i - p).$$

Proof. From (1.6), it is easy to see that

$$\left(\mathcal{I}_n^{\alpha,\beta}(f,g)\right)'(z) = pz^{p-1}\prod_{i=1}^{n}\left(\frac{f_i(z)}{z^p}\right)^{\alpha_i}\left(\frac{g_i'(z)}{pz^{p-1}}\right)^{\beta_i}. \tag{2.2}$$

Differentiating both sides of (2.2) logarithmically and after some calculus, we obtain

$$\begin{aligned} 1+\frac{z\left(\mathcal{I}_n^{\alpha,\beta}(f,g)\right)''(z)}{\left(\mathcal{I}_n^{\alpha,\beta}(f,g)\right)'(z)} \\ = p+\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}\right)+\sum_{i=1}^{n}\beta_i\left(1+\frac{zg_i''(z)}{g_i'(z)}\right)-\sum_{i=1}^{n}p(\alpha_i+\beta_i). \end{aligned} \tag{2.3}$$

Since $f_i \in \mathcal{US}_p(\delta_i,\gamma_i)$ and $g_i \in \mathcal{UK}_p(\delta_i,\gamma_i)$ for all $i = 1, 2, \ldots, n$, it follows from (1.4) and (1.5) that

$$\begin{aligned} &\Re\left\{1+\frac{z\left(\mathcal{I}_n^{\alpha,\beta}(f,g)\right)''(z)}{\left(\mathcal{I}_n^{\alpha,\beta}(f,g)\right)'(z)}\right\} \\ &= p+\sum_{i=1}^{n}\alpha_i\Re\left\{\frac{zf_i'(z)}{f_i(z)}\right\}+\sum_{i=1}^{n}\beta_i\Re\left\{1+\frac{zg_i''(z)}{g_i'(z)}\right\}-\sum_{i=1}^{n}p(\alpha_i+\beta_i) \\ &> \sum_{i=1}^{n}\alpha_i\delta_i\left|\frac{zf_i'(z)}{f_i(z)}-p\right|+\sum_{i=1}^{n}\beta_i\delta_i\left|\frac{zg_i''(z)}{g_i'(z)}-(p-1)\right| \\ &\quad + p+\sum_{i=1}^{n}(\alpha_i+\beta_i)(\gamma_i-p). \end{aligned} \tag{2.4}$$

Because

$$\alpha_i\delta_i\left|\frac{zf_i'(z)}{f_i(z)}-p\right| \geq 0$$

and

$$\beta_i\delta_i\left|\frac{zg_i''(z)}{g_i'(z)}-(p-1)\right| \geq 0$$

for all $i = 1, 2, \ldots, n$, from (2.4), we obtain

$$\Re\left\{1+\frac{z\left(\mathcal{I}_n^{\alpha,\beta}(f,g)\right)''(z)}{\left(\mathcal{I}_n^{\alpha,\beta}(f,g)\right)'(z)}\right\} > p+\sum_{i=1}^{n}(\alpha_i+\beta_i)(\gamma_i-p).$$

Therefore $\mathcal{I}_n^{\alpha,\beta}(f,g)$ is p-valently convex of order

$$\lambda = p + \sum_{i=1}^{n} (\alpha_i + \beta_i)(\gamma_i - p).$$

□

Remark 3. (**i**) Letting $p = 1$ in Theorem 2.1, we obtain Theorem 2.2 in [13].

(**ii**) Letting $\beta_1 = \cdots = \beta_n = 0$ and $\alpha_1 = \cdots = \alpha_n = 0$ in Theorem 2.1, we obtain Theorem 2.1 and Theorem 3.1 in [9], respectively.

Letting $n = 1$, $p = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\delta_1 = \delta$, $\gamma_1 = \gamma$ and $f_1 = f$, $g_1 = g$ in Theorem 2.1, we have

Corollary 2.2. *Let $\alpha, \beta \geq 0$, $\delta \geq 0$, $\gamma \in [-1,1)$, $\delta + \gamma \geq 0$ and $f \in \mathcal{US}_p(\delta,\gamma)$, $g \in \mathcal{UK}_p(\delta,\gamma)$. If*

$$0 \leq 1 + (\alpha + \beta)(\gamma - 1) \leq 1,$$

then the integral operator

$$\mathcal{I}^{\alpha,\beta}(f,g)(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} \left(g'(t)\right)^{\beta} dt \tag{2.5}$$

is convex of order

$$1 + (\alpha + \beta)(\gamma - 1).$$

Theorem 2.3. *Let $\alpha_i, \beta_i \geq 0$, $\delta_i \geq 0$, $\gamma_i \in [-1,p)$, $\delta_i + \gamma_i \geq 0$ and*

$$\left|\frac{zf_i'(z)}{f_i(z)} - p\right| > -\frac{\frac{p}{2} + \sum_{i=1}^{n} \alpha_i(\gamma_i - p)}{\sum_{i=1}^{n} \alpha_i\delta_i}, \tag{2.6}$$

$$\left|\frac{zg_i''(z)}{g_i'(z)} - (p-1)\right| > -\frac{\frac{p}{2} + \sum_{i=1}^{n} \beta_i(\gamma_i - p)}{\sum_{i=1}^{n} \beta_i\delta_i} \tag{2.7}$$

for all $i = 1, 2, \ldots, n$, then the integral operator $\mathcal{I}_n^{\alpha,\beta}(f,g)$ defined by (1.6) *is p-valently convex in $\mathbb{U}$.*

Proof. From (2.4), (2.6) and (2.7), we easily get $\mathcal{I}_n^{\alpha,\beta}(f,g) \in \mathcal{K}_p$. □

Theorem 2.4. *Let $f_i, g_i \in \mathcal{A}$, where $f_i \in \mathcal{B}(\mu_i,\gamma_i)$ $(\mu_i \geq 0,\ 0 \leq \gamma_i < p)$, $\alpha_i, \beta_i \in \mathbb{C}$ and $M_i \geq 1$ for all $i = 1, \ldots, n$. If*

$$|f_i(z)| < M_i,$$

$$\left|\frac{g_i''(z)}{g_i'(z)}\right| \leq 1$$

for all $i = 1, \ldots, n$, and if

$$0 \leq p - \sum_{i=1}^{n} |\alpha_i| \left(p + (2p - \gamma_i) M_i^{\mu_i - 1}\right) - \sum_{i=1}^{n} |\beta_i| (p+2) < p,$$

then the integral operator $\mathcal{I}_n^{\alpha,\beta}(f,g)$ defined by (1.6) *is p-valently convex of order η with*

$$\eta = p - \sum_{i=1}^{n} |\alpha_i| \left(p + (2p - \gamma_i) M_i^{\mu_i - 1}\right) - \sum_{i=1}^{n} |\beta_i| (p+2).$$

Proof. From (2.3) we have

$$\begin{aligned} & 1 + \frac{z\left(\mathcal{I}_n^{\alpha,\beta}(f,g)\right)''(z)}{\left(\mathcal{I}_n^{\alpha,\beta}(f,g)\right)'(z)} - p \\ & = \sum_{i=1}^{n} \alpha_i \left(\frac{z f_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \beta_i \left(1 + \frac{z g_i''(z)}{g_i'(z)}\right) - \sum_{i=1}^{n} p(\alpha_i + \beta_i). \end{aligned}$$

This implies that

$$\begin{aligned} & \left|1 + \frac{z\left(\mathcal{I}_n^{\alpha,\beta}(f,g)\right)''(z)}{\left(\mathcal{I}_n^{\alpha,\beta}(f,g)\right)'(z)} - p\right| \\ & \leq \sum_{i=1}^{n} |\alpha_i| \left|\frac{z f_i'(z)}{f_i(z)}\right| + \sum_{i=1}^{n} |\beta_i| |z| \left|\frac{g_i''(z)}{g_i'(z)}\right| \\ & \quad + \sum_{i=1}^{n} p|\alpha_i| + \sum_{i=1}^{n} |\beta_i| (p+1) \\ & = \sum_{i=1}^{n} |\alpha_i| \left|\frac{f_i'(z)}{z^{p-1}} \left(\frac{z^p}{f_i(z)}\right)^{\mu_i}\right| \left|\frac{f_i(z)}{z^p}\right|^{\mu_i - 1} \\ & \quad + \sum_{i=1}^{n} |\beta_i| |z| \left|\frac{g_i''(z)}{g_i'(z)}\right| + \sum_{i=1}^{n} p|\alpha_i| + \sum_{i=1}^{n} |\beta_i| (p+1). \end{aligned} \tag{2.8}$$

Since

$$f_i \in \mathcal{B}(\mu_i, \gamma_i),$$

$$|f_i(z)| < M_i$$

and

$$\left|\frac{g_i''(z)}{g_i'(z)}\right| \leq 1$$

for all $i = 1, \ldots, n$, applying General Schwarz Lemma and using (2.8), we obtain

$$\left|1 + \frac{z\left(\mathcal{I}_n^{\alpha,\beta}(f,g)\right)''(z)}{\left(\mathcal{I}_n^{\alpha,\beta}(f,g)\right)'(z)} - p\right|$$

$$\leq \sum_{i=1}^{n} |\alpha_i| \left(\left| \frac{f_i'(z)}{z^{p-1}} \left(\frac{z^p}{f_i(z)}\right)^{\mu_i} - p \right| M_i^{\mu_i - 1} + p M_i^{\mu_i - 1} \right)$$

$$+ \sum_{i=1}^{n} |\beta_i| |z| + \sum_{i=1}^{n} p |\alpha_i| + \sum_{i=1}^{n} |\beta_i| (p+1)$$

$$< \sum_{i=1}^{n} |\alpha_i| \left(p + (2p - \gamma_i) M_i^{\mu_i - 1}\right) + \sum_{i=1}^{n} |\beta_i| (p+2) = p - \eta.$$

This implies that the integral operator $\mathcal{I}_n^{\alpha,\beta}(f,g)$ is p-valently convex of order

$$\eta = p - \sum_{i=1}^{n} |\alpha_i| \left(p + (2p - \gamma_i) M_i^{\mu_i - 1}\right) - \sum_{i=1}^{n} |\beta_i| (p+2).$$

□

Letting $n = 1$, $p = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\mu_1 = \mu$, $\gamma_1 = \gamma$, $M_1 = M$ and $f_1 = f$, $g_1 = g$ in Theorem 2.4, we have

Corollary 2.5. *Let $f, g \in \mathcal{A}$, where $f \in \mathcal{B}(\mu, \gamma)$ $(\mu \geq 0, 0 \leq \gamma < p)$, $\alpha, \beta \in \mathbb{C}$ and $M \geq 1$. If*

$$|f(z)| < M,$$
$$\left|\frac{g''(z)}{g'(z)}\right| \leq 1$$

and if

$$0 \leq 1 - |\alpha| \left(1 + (2 - \gamma) M^{\mu - 1}\right) - 3|\beta| < 1,$$

then the integral operator $\mathcal{I}^{\alpha,\beta}(f,g)$ defined by (2.5) is convex of order

$$1 - |\alpha| \left(1 + (2 - \gamma) M^{\mu - 1}\right) - 3|\beta|.$$

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Serap BULUT
Kocaeli University
Civil Aviation College
Arslanbey Campus
İzmit-Kocaeli, 41285 Turkey
e-mail address: serap.bulut@kocaeli.edu.tr

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