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A MODEL FOR THE WHITEHEAD PRODUCT IN RATIONAL MAPPING SPACES

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Abstract. We describe the Whitehead products in the rational homotopy group of a connected component of a mapping space in terms of the André-Quillen cohomology. As a consequence, an upper bound for the Whitehead length of a mapping space is given.

1. Introduction

We assume that all spaces in this paper are path connected CW-complexes with a nondegenerate base point $*$. Let X and Y be simply-connected spaces and $\mathrm{map}(X,Y;f)$ the path component of the space of free maps from X to Y containing the based map $f : X \to Y$. We denote by ΛV and B a minimal Sullivan model for Y and a CDGA model for X, respectively. Let $\overline{f} : \Lambda V \to B$ be a model for f and $\mathrm{Der}^*(\Lambda V, B; \overline{f})$ the complex of $\overline{f}$-derivations; see next section for precise definitions and details. The cohomology of $\mathrm{Der}^*(\Lambda V, B; \overline{f})$ is called the André-Quillen cohomology of ΛV with coefficients in B, denoted by $H^*_{\mathrm{AQ}}(\Lambda V, B; \overline{f})$ [2].

Suppose that X is a finite CW-complex. The n-th rational homotopy group of $\mathrm{map}(X,Y;f)$ is isomorphic to $H^{-n}_{\mathrm{AQ}}(\Lambda V, B; \overline{f})$ as abelian groups for $n \geq 2$. This fact has been proved by Block and Lazarev [2], Buijs and Murillo [4], Lupton and Smith [12]. Moreover Buijs and Murillo [4] defined a bracket in the André-Quillen cohomology $H^*_{\mathrm{AQ}}(\Lambda V, B; \overline{f})$ which coincides with the Whitehead product in $\pi_*(\mathrm{map}(X,Y;f)) \otimes \mathbb{Q}$. We mention that the isomorphism due to Buijs and Murillo is constructed relying on the Sullivan model for $\mathrm{map}(X,Y;f)$ due to Haefliger [7] and Brown and Szczarba [5]. To this end, the finiteness of a model B for the source space X is assumed in the result [5, Theorem 1.3] and also [7, §3].

On the other hand, the finiteness hypothesis on X assures that $\pi_n(\mathrm{map}(X,Y;f)) \otimes \mathbb{Q}$ is isomorphic to $\pi_n(\mathrm{map}(X, Y_{\mathbb{Q}}; lf))$, where $l : Y \to Y_{\mathbb{Q}}$ the localization map; see [9, II. Theorem 3.11] and [14, Theorem 2.3]. Then the isomorphism constructed in [2] and [12] factors as follows:

$$\pi_n(\mathrm{map}(X,Y;f)) \otimes \mathbb{Q} \xrightarrow[\cong]{} \pi_n(\mathrm{map}(X, Y_{\mathbb{Q}}; lf)) \xrightarrow[\cong]{\Phi} H^{-n}_{\mathrm{AQ}}(\Lambda V, B; \overline{f}).$$

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The precise definition of Φ is described in Section 2. By the proof of [12, Theorem 2.1], we see that the second map Φ is an isomorphism without a finiteness hypothesis on X. Also the assertion of [2, Theorem 3.8] is that the map Φ is an isomorphism. In this paper, we introduce a bracket in the André-Quillen cohomology which coincides with the Whitehead product in $\pi_*(\mathrm{map}(X, Y_{\mathbb{Q}}; f))$ up to the isomorphism Φ without assuming that X has a finite dimensional commutative model.

Let X be a simply-connected space with a CDGA model B and Y be a $\mathbb{Q}$-local, simply-connected space of finite type. Then we have a model $\overline{f} : \Lambda V \to B$ for a based map $f : X \to Y$. Now, we define a bracket in $H^*_{\mathrm{AQ}}(\Lambda V, B; \overline{f})$

$$[\ ,\ ] : H^n_{\mathrm{AQ}}(\Lambda V, B; \overline{f}) \otimes H^m_{\mathrm{AQ}}(\Lambda V, B; \overline{f}) \longrightarrow H^{n+m+1}_{\mathrm{AQ}}(\Lambda V, B; \overline{f})$$

by

$$\begin{aligned}(1.1)\quad [\varphi, \psi](v) &= (-1)^{n+m-1} \\ &\times \sum\Big(\sum_{i \neq j} (-1)^{\varepsilon_{ij}} \overline{f}(v_1 \cdots v_{i-1}) \varphi(v_i) \overline{f}(v_{i+1} \cdots v_{j-1}) \psi(v_j) \overline{f}(v_{j+1} \cdots v_s)\Big),\end{aligned}$$

where v is a basis of V, $dv = \sum v_1 v_2 \cdots v_s$ and

$$\varepsilon_{ij} = \begin{cases} |\varphi|(\sum_{k=1}^{i-1} |v_k|) + |\psi|(\sum_{k=1}^{j-1} |v_k|) + |\varphi||\psi| & (i < j) \\ |\varphi|(\sum_{k=1}^{i-1} |v_k|) + |\psi|(\sum_{k=1}^{j-1} |v_k|) & (j < i). \end{cases}$$

The following is our main result of this paper.

Theorem 1.1. *The isomorphism* $\Phi : \pi_n(\mathrm{map}(X, Y; f)) \to H^{-n}_{\mathrm{AQ}}(\Lambda V, B; \overline{f})$ *is compatible with the Whitehead product in* $\pi_n(\mathrm{map}(X, Y; f))$ *and the bracket in* $H^{-n}_{\mathrm{AQ}}(\Lambda V, B; \overline{f})$ *defined by the formula* (1.1).

If X is finite, then the bracket in $H^*_{\mathrm{AQ}}(\Lambda V, B; \overline{f})$ coincides with that due to Buijs and Murillo [4] up to sign. Thus Theorem 1.1 is regarded as a generalization of [4, Theorem 2]. Let $\mathrm{map}_*(X, Y; f)$ be the path-component of the space of based maps from X to Y containing the based map $f : X \to Y$. We apply the same argument to the case of the based mapping space $\mathrm{map}_*(X, Y; f)$; see the last of Section 3 for details.

As an application of the main result, we study the Whitehead length of a mapping space. The *Whitehead length* of a space Z, written $\mathrm{WL}(Z)$, is the length of non-zero iterated Whitehead products in $\pi_{\geq 2}(Z)$. By the definition, $\mathrm{WL}(Z) = 1$ means that all Whitehead products vanish. In [13], Lupton and Smith give some results and examples related to a Whitehead length of mapping spaces $\mathrm{map}(X, Y; f)$ using a Quillen model. We will give another proof of their results using the bracket in the André-Quillen cohomology; see Proposition 4.1. To give an upper bound for the Whitehead length of $\mathrm{map}_*(X, Y; f)$, we introduce a numerical invariant.

Definition 1.2 ([6, p315])**.** The *product length* of a connected graded algebra A, written $\mathrm{nil}A$, is the greatest integer n such that $A^+A^+ \cdots A^+ \neq 0$ (n factors).

In [3], Buijs proved the following theorem.

Theorem 1.3 ([3, Theorem 0.3])**.** *Let X and Y be simply-connected spaces with finite type over $\mathbb{Q}$ and B a CDGA model for X. If $\mathrm{WL}(Y_{\mathbb{Q}}) = 1$, then*

$$\mathrm{WL}(\mathrm{map}_*(X, Y; f)_{\mathbb{Q}}) \leq \mathrm{nil}B - 1.$$

Using the bracket in the André-Quillen cohomology, we can prove the following proposition, which refines the above result; see Remark 4.4.

Proposition 1.4. *Let X and Y be simply-connected spaces with finite type over $\mathbb{Q}$, ΛV a minimal Sullivan model for Y and B a CDGA model for X. Assume further that Y is $\mathbb{Q}$-local and the differential of ΛV is not zero. If $\mathrm{WL}(Y) = 1$ and $\mathrm{nil}B \geq 2$, then*

$$\mathrm{WL}(\mathrm{map}_*(X, Y; f)) \leq \frac{1}{\omega - 1}(\mathrm{nil}B - 1) + 1,$$

where $\omega = \min\{n \geq 2 \mid d(V) \subset \Lambda^{\geq n}V\}$.

We here remark that the equation $\mathrm{WL}(Y) = 1$ implies that $\omega \geq 3$. Furthermore, ω is the largest number such that all Whitehead products of order less than ω vanish in Y [1, Proposition 6.4]. If Y has a minimal Sullivan model with a zero differential, we readily see that $\mathrm{WL}(\mathrm{map}_*(X, Y; f)) = 1$ by the bracket (1.1). As computational examples, we will compute the Whitehead length of mapping spaces $\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^m, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^n_{\mathbb{Q}}; f)$.

The organization of this paper is as follows. In Section 2, we will recall several fundamental results on rational homotopy theory. The isomorphism Φ in [2] and [12] is also described. In Section 3, we prove Theorem 1.1. To this end, a model for the Whitehead product of mapping spaces will be constructed in the section. The Whitehead length of mapping spaces is considered in Section 4. A computational example of the Whitehead length is presented in Section 5.

2. Preliminaries

We refer the reader to the book [6] for the fundamental facts on rational homotopy theory. A *Sullivan algebra* is a free commutative differential graded algebra over the field of rational numbers $\mathbb{Q}$ (or simply CDGA in this paper), $(\Lambda V, d)$, with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i\geq 1} V^i$ where V has an increasing sequence of subspaces $V(0) \subset V(1) \subset \cdots$ which satisfy the conditions that $V = \bigcup_{i\geq 0} V(i)$, $d = 0$ in $V(0)$ and $d : V(i) \to \Lambda V(i-1)$ for any $i \geq 1$.

We recall a *minimal Sullivan model* for a simply-connected space X with finite type. It is a Sullivan algebra of the form $(\Lambda V, d)$ with $V = \bigoplus_{i\geq 2} V^i$ where each V^i is of finite dimension and d is decomposable; that is, $d(V) \subset \Lambda^{\geq 2} V$. Moreover, $(\Lambda V, d)$ is equipped with a quasi-isomorphism $(\Lambda V, d) \xrightarrow{\simeq} A_{\mathrm{PL}}(X)$ to the CDGA $A_{\mathrm{PL}}(X)$ of differential polynomial forms on X. Observe that, as algebras, $H^*(\Lambda V, d) \cong H^*(A_{\mathrm{PL}}(X)) \cong H^*(X; \mathbb{Q})$. For instance, a minimal Sullivan model for the n-sphere S^n, $M(S^n)$, is the form $(\Lambda(e_n), 0)$ if n is odd and $(\Lambda(e_n, e_{2n-1}), de_{2n-1} = e_n^2)$ if n is even, where $|e_n| = n$ and $|e_{2n-1}| = 2n-1$.

A *CDGA model* for a space X is a connected CDGA (B, d) such that there is a quasi-isomorphism from a minimal Sullivan model for X to B. The two maps of CDGA φ_1 and φ_2 from a Sullivan algebra ΛV to a CDGA A are *homotopic* if there exists a CDGA map $H : \Lambda V \to A \otimes \Lambda(t, dt)$ such that $(1 \cdot \varepsilon_i)H = \varphi_i$ for $i = 0, 1$. Here, $\Lambda(t, dt)$ is the free CDGA with $|t| = 0$, $|dt| = 1$ and the differential d of $\Lambda(t, dt)$ sends t to dt. The map $\varepsilon_i : \Lambda(t, dt) \to \mathbb{Q}$ defined by $\varepsilon_i(t) = i$. Denote $[\Lambda V, A]$ by the set of homotopy classes of CDGA maps from ΛV to A.

Let $f : X \to Y$ be a map between spaces of finite type. Then there exists a CDGA map $\widetilde{f}$ from a minimal Sullivan model $(\Lambda V_Y, d)$ for Y to a minimal Sullivan model $(\Lambda V_X, d)$ for X which makes the diagram

$$\begin{tikzcd} A_{\mathrm{PL}}(Y) \arrow[r, "A_{\mathrm{PL}}(f)"] & A_{\mathrm{PL}}(Y) \\ \Lambda V_Y \arrow[u, "\simeq"] \arrow[r, "\widetilde{f}"'] & \Lambda V_X \arrow[u, "\simeq"'] \end{tikzcd}$$

commutative up to homotopy. Let $\rho : \Lambda V_X \xrightarrow{\simeq} B$ a CDGA model for X, we call $\rho\widetilde{f}$ a *model* for f associated with models ΛV_Y and B and denote it by $\overline{f}$.

We use the following result when constructing a model for the Whitehead product of a mapping space.

Proposition 2.1 ([6, Proposition 12.9]). *Let A and C be CDGAs, ΛV a Sullivan algebra and $\pi : A \to C$ a quasi-isomorphism. Then the map*

$$\pi_* : [\Lambda V, A] \longrightarrow [\Lambda V, C]$$

induced by π is bijective.

Remark 2.2. If π is a surjective quasi-isomorphism and ΛV is a minimal Sullivan model, we can construct a CDGA map $\phi : \Lambda V \to A$ such that $\pi\phi = \psi$ for any CDGA map $\psi : \Lambda V \to C$ by induction on a degree of V [6, Lemma 12.4]. Let v be a basis of V and assume that ϕ is constructed in $\Lambda V^{<|v|}$. Then $\phi d(v)$ is defined. Since π is a surjective quasi-isomorphism and $\pi\phi d(v) = d\psi(v)$, we can find $a \in A$ such that $d(a) = \phi d(v)$ and $\pi(a) = \psi(v)$. Then, we extend ϕ with $\phi(v) = a$.

We next recall the definition of the Whitehead product. Let $\alpha \in \pi_n(X)$ and $\beta \in \pi_m(X)$ be elements represented by $a : S^n \to X$ and $b : S^m \to X$, respectively. Then the Whitehead product $[\alpha, \beta]_w$ is defined to be the homotopy class of composite

$$S^{n+m-1} \xrightarrow{\eta} S^n \vee S^m \xrightarrow{\nabla(a\vee b)} X$$

where η is the universal example and $\nabla : X \vee X \to X$ is the folding map. Recall that the differential d of ΛV can be written by $d = \sum_{i\geq 1} d_i$ with $d_i(V) \subset \Lambda^{i+1}V$. The map d_1 is called the *quadratic part* of d. We see that the quadratic part d_1 is related with the Whitehead products in $\pi_*(X)$. We denote by $Q(g)^n : V^n \to \mathbb{Q}e_n$ the linear part of a model $\overline{g}$ for g, where $\overline{g} : \Lambda V \to M(S^n)$. Define a paring and a trilinear map

$$\langle\ ;\ \rangle : V \times \pi_*(X) \longrightarrow \mathbb{Q},$$
$$\langle\ ;\ ,\ \rangle : \Lambda^2 V \times \pi_*(X) \times \pi_*(X) \longrightarrow \mathbb{Q}$$

by

$$\langle v; \alpha\rangle e_n = \begin{cases} Q(a)^n v & (|v| = n) \\ 0 & (|v| \neq n) \end{cases}$$

and

$$\langle vw; \alpha, \beta\rangle = \langle v; \alpha\rangle\langle w; \beta\rangle + (-1)^{|w||\alpha|}\langle w; \alpha\rangle\langle v; \beta\rangle,$$

respectively.

Proposition 2.3 ([6, Proposition 13.16])**.** *The following holds*

$$\langle d_1 v; \alpha, \beta\rangle = (-1)^{n+m-1}\langle v; [\alpha, \beta]_w\rangle,$$

where $v \in V$, $\alpha \in \pi_n(X)$, $\beta \in \pi_m(X)$.

We conclude this section by recalling the isomorphism Φ defined in [2] and [12] from $\pi_n(\mathrm{map}(X, Y; f))$ to $H^{-n}_{\mathrm{AQ}}(\Lambda V, B; \overline{f})$ in the setting of a simply-connected space X and a $\mathbb{Q}$-local, simply-connected space Y with finite type. We here recall the complex of $\overline{f}$-derivations from ΛV to B which denoted by $\mathrm{Der}^*(\Lambda V, B; \overline{f})$. An element $\theta \in \mathrm{Der}^n(\Lambda V, B; \overline{f})$ is a $\mathbb{Q}$-linear map of degree n with $\theta(xy) = \theta(x)\overline{f}(y) + (-1)^{n|x|}\overline{f}(x)\theta(y)$ for any $x, y \in \Lambda V$.

The differentials $\partial : \mathrm{Der}^n(\Lambda V, B; \overline{f}) \to \mathrm{Der}^{n+1}(\Lambda V, B; \overline{f})$ are defined by $\partial(\theta) = d\theta - (-1)^n \theta d$.

Let $\alpha \in \pi_n(\mathrm{map}(X, Y; f))$ and $g : S^n \times X \to Y$ the adjoint of α. We note that g satisfy $g|_X = f$. Then there exists a model $\overline{g} : \Lambda V \to M(S^n) \otimes B$ for g such that the following diagram is strictly commutative;

$$\begin{tikzcd} \Lambda V \arrow[rr, "\overline{g}"] \arrow[dr, "\overline{f}"'] & & M(S^n) \otimes B \arrow[dl, "\varepsilon \cdot 1"] \\ & B, & \end{tikzcd}$$

where $\varepsilon : M(S^n) \to \mathbb{Q}$ is the augmentation; see Lemma 3.1. Since S^n is formal, there is a quasi-isomorphism $\phi : M(S^n) \to (H^*(S^n; \mathbb{Q}), 0)$ and, for any $v \in \Lambda V$, we may write

$$(\phi \otimes 1)\overline{g}(v) = 1 \otimes \overline{f}(v) + e_n \otimes \theta(v).$$

Then we see that θ is a $\overline{f}$-derivation of degree $-n$ and also a cycle in $\mathrm{Der}^*(\Lambda V, B; \overline{f})$. Put $\Phi(\alpha) = \theta$.

Theorem 2.4 ([2, Theorem 3.8] [12, Theorem 2.1]). *The map*

$$\Phi : \pi_n(\mathrm{map}(X, Y; f)) \longrightarrow H^{-n}_{\mathrm{AQ}}(\Lambda V, B; \overline{f})$$

is an isomorphism of abelian groups for $n \geq 2$.

3. A model for the adjoint of the Whitehead product

We retain the notation and terminology described in the previous section. In order to consider the image of the Whitehead product in $\pi_*(\mathrm{map}(X, Y; f))$ by the isomorphism Φ, we construct an appropriate model for the adjoint of the Whitehead product. This is the key to proving Theorem 1.1. Let X be a simply-connected space, Y a $\mathbb{Q}$-local, simply-connected space of finite type and $f : X \to Y$ a based map. We denote by $(\Lambda V, d)$ and (B, d) a minimal Sullivan model for Y and a CDGA model for X, respectively. Let $\overline{f} : \Lambda V \to B$ be a model for f associated with such the models.

We prepare for proving Theorem 1.1. We see that a minimal Sullivan model for $S^n \vee S^m$ has the form

$$M(S^n \vee S^m) = (M(S^n) \otimes M(S^m) \otimes \Lambda(\iota_{n+m-1}, x_1, x_2, \cdots), d)$$

in which $d\iota_{n+m-1} = e_n e_m$ and $|\iota_{n+m-1}| = n + m - 1 < |x_i|$ for any $i \geq 1$; see [6, p177].

Lemma 3.1. *Let $g : S^n \times X \longrightarrow Y$ be a map with $g|_X = f$. Then there exists a model $\overline{g}$ for g such that the diagram is strictly commutative:*

$$\begin{tikzcd} \Lambda V \arrow[rr, "\overline{g}"] \arrow[dr, "\overline{f}"'] & & M(S^n) \otimes B \arrow[dl, "\varepsilon\cdot 1"] \\ & B, & \end{tikzcd}$$

where $\varepsilon : M(S^n) \to \mathbb{Q}$ is the augmentation. Moreover, if g satisfy $g|_X = f$ and $g|_{S^n} = *$, where $* : S^n \to Y$ is the constant map to the base point, then there is a model $\overline{g}$ for g such that the following diagram commute strictly:

$$\begin{tikzcd} & M(S^n) & \\ \Lambda V \arrow[ur, "u\varepsilon"] \arrow[rr, "\overline{g}"] \arrow[dr, "\overline{f}"'] & & M(S^n) \otimes B \arrow[ul, "1\cdot\varepsilon"'] \arrow[dl, "\varepsilon\cdot 1"] \\ & B, & \end{tikzcd}$$

where $u : \mathbb{Q} \to M(S^n)$ is the unit map.

Proof. Let $\overline{g}'$ be a model for g. We define the map $\overline{g} : \Lambda V \to M(S_n) \otimes B$ by

$$\overline{g}(v) = 1 \otimes (\overline{f} - (\varepsilon \cdot 1)\overline{g}')(v) + \overline{g}'(v).$$

Then $\overline{g}$ and $\overline{g}'$ are homotopic. Indeed, $\overline{f}$ and $(\varepsilon \cdot 1) \circ \overline{g}'$ are homotopic and let $H : \Lambda V \longrightarrow B \otimes \Lambda(t, dt)$ be a its homotopy. Then, the map $\overline{H} : \Lambda V \longrightarrow M(S^n) \otimes B \otimes \Lambda(t, dt)$ defined by

$$\overline{H}(v) = 1 \otimes H(v) + \overline{g}'(v) \otimes 1 - 1 \otimes (\varepsilon \cdot 1)\overline{g}'(v) \otimes 1$$

is a homotopy from $\overline{g}'$ to $\overline{g}$. A similar argument shows the second assertion. $\square$

Given $\alpha \in \pi_n(\mathrm{map}(X, Y; f))$ and $\beta \in \pi_m(\mathrm{map}(X, Y; f))$. Let $g : S^n \times X \to Y$ and $h : S^m \times X \to Y$ be the adjoint maps of α and β, respectively. In order to consider the image of $[\alpha, \beta]_w$ by Φ, we construct a model for the adjoint of $[\alpha, \beta]_w$

$$ad([\alpha, \beta]_w) : S^{n+m-1} \times X \xrightarrow{\eta\times 1} (S^n \vee S^m) \times X \xrightarrow{(g|h)} Y,$$

where $(g|h)$ is a map defined by $(g|h)(u_n, x) = g(u_n, x)$ and $(g|h)(u_m, x) = h(u_m, x)$ for any $u_n \in S^n$, $u_m \in S^m$ and $x \in X$. It is readily seen that the canonical map

$$\pi : M(S^n \vee S^m) \longrightarrow M(S^n) \times_{\mathbb{Q}} M(S^m)$$

is a surjective quasi-isomorphism, where $M(S^n) \times_{\mathbb{Q}} M(S^m)$ is the pull-back of the augmentations $M(S^n) \to \mathbb{Q}$ and $M(S^m) \to \mathbb{Q}$. By Proposition 2.1, we have the following homotopy commutative square

$$\begin{tikzcd} A_{\mathrm{PL}}(S^n \vee S^m) \arrow[r, "(A_{\mathrm{PL}}(i_1),A_{\mathrm{PL}}(i_2))"] & A_{\mathrm{PL}}(S^n) \times_{\mathbb{Q}} A_{\mathrm{PL}}(S^m) \\ M(S^n \vee S^m) \arrow[u, "\simeq"] \arrow[r, "\pi"'] & M(S^n) \times_{\mathbb{Q}} M(S^m), \arrow[u, "\simeq"'] \end{tikzcd}$$

where $i_1 : S^n \to S^n \vee S^m$ and $i_2 : S^m \to S^n \vee S^m$ are the inclusions. The commutative diagram

$$\begin{tikzcd} S^n \times X \arrow[r, "i_1 \times 1"] \arrow[dr, "g"'] & (S^n \vee S^m) \times X \arrow[d, "(g|h)"'] & S^m \times X \arrow[l, "i_2 \times 1"'] \arrow[dl, "h"] \\ & Y & \end{tikzcd} \tag{3.1}$$

enables us to give the following homotopy commutative diagram:

$$\begin{tikzcd} & M(S^n \vee S^m) \otimes B \arrow[d, "\pi \otimes 1"] \\ \Lambda V \arrow[ur, "\overline{(g|h)}"] \arrow[r, "(\overline{g},\overline{h})"'] & (M(S^n) \times_{\mathbb{Q}} M(S^m)) \otimes B, \end{tikzcd} \tag{3.2}$$

where $(\overline{g},\overline{h})$ is the map defined by $(\overline{g},\overline{h})(v) = -1 \otimes \overline{f}(v) + (j_1 \otimes 1)\overline{g}(v) + (j_2 \otimes 1)\overline{h}(v)$ for any $v \in V$ and $j_1 : M(S^n) \to M(S^n) \times_{\mathbb{Q}} M(S^m)$ and $j_2 : M(S^m) \to M(S^n) \times_{\mathbb{Q}} M(S^m)$ are the inclusion. Indeed, by the diagram (3.1), we see that the diagram

$$\begin{tikzcd} M(S^n) \otimes B & (M(S^n) \times_{\mathbb{Q}} M(S^m)) \otimes B \arrow[l, "p_1 \otimes 1"'] \arrow[r, "p_2 \otimes 1"] & M(S^m) \otimes B \\ & \Lambda V \arrow[ul, "\overline{g}"] \arrow[u, "(\pi\otimes 1)\overline{(g|h)}"] \arrow[ur, "\overline{h}"'] & \end{tikzcd}$$

is homotopy commutative, where p_1 and p_2 are the projection. Let H_1 and H_2 be homotopies from $(p_1\pi \otimes 1)\overline{(g|h)}$ to $\overline{g}$ and from $(p_2\pi \otimes 1)\overline{(g|h)}$ to $\overline{h}$, respectively. Then, a CDGA map $H : \Lambda V \to (M(S^n) \times_{\mathbb{Q}} M(S^m)) \otimes B \otimes \Lambda(t, dt)$ defined by

$$H(v) = -1 \otimes \overline{f}(v) \otimes 1 + (j_1 \otimes 1 \otimes 1)H_1(v) + (j_2 \otimes 1 \otimes 1)H_2(v)$$

for any $v \in V$ is a homotopy from $(\pi \otimes 1)\overline{(g|h)}$ to $(\overline{g},\overline{h})$. If there is a map $\phi : \Lambda V \to M(S^n \vee S^m) \otimes B$ such that $(\pi \otimes 1)\phi = (\overline{g},\overline{h})$, ϕ and $\overline{(g|h)}$ is homotopic by Proposition 2.1. Therefore, it is only necessary to construct of a lift ϕ of the diagram (3.2) for getting a model for $(g|h)$.

Lemma 3.2. *There is a model ϕ for $(g|h)$ such that for any $v \in V$, $\phi(v)$ has no term of the form $e_n e_m \otimes u$ for some $u \in B$ and the following diagram commutes strictly:*

$$\begin{tikzcd} \Lambda V \arrow[rr, "\phi"] \arrow[dr, "\overline{f}"'] & & M(S^n \vee S^m) \otimes B \arrow[dl, "\varepsilon\cdot 1"] \\ & B. & \end{tikzcd}$$

Proof. First, we recall the construction of a lift ϕ' in Remark 2.2. For any basis v of V, we can find $a \in M(S^n \vee S^m) \otimes B$ so that $da = \phi' dv$ and $(\pi \otimes 1)a = (\overline{g}, \overline{h})v$. We may write

$$a = 1 \otimes \overline{f}(a) + e_n \otimes a_2 + e_m \otimes a_3 + \iota_{n+m-1} \otimes a_4 + e_n e_m \otimes a_5 + \mathcal{O}_a,$$

where $a_i \in B$ and $\mathcal{O}_a$ denote other terms. We put

$$a' = 1 \otimes \overline{f}(a) + e_n \otimes a_2 + e_m \otimes a_3 + \iota_{n+m-1} \otimes (a_4 + da_5) + \mathcal{O}_a. \tag{3.3}$$

Then it follows that $d(a) = d(a')$ and $(\pi \otimes 1)(a) = (\pi \otimes 1)(a')$. Hence, the map ϕ defined by

$$\phi(v) = a'$$

satisfies the condition that $(\pi \otimes 1)\phi = (\overline{g}, \overline{h})$. Thus we see that ϕ is a model for $(g|h)$. The second assertion is shown using the equation (3.3). ☐

Combining these results we prove our main result.

Proof of Theorem 1.1. Given two elements $\alpha \in \pi_n(\mathrm{map}(X, Y; f))$ and $\beta \in \pi_m(\mathrm{map}(X, Y; f))$. Let $g : S^n \times X \to Y$ and $h : S^m \times X \to Y$ be the adjoint maps of α and β, respectively. First, by the proof of Proposition 2.3, we see that a model $\overline{\eta}$ for the universal example η sends $\iota_{n+m-1} \in M(S^n \vee S^m)$ to $(-1)^{n+m-1} e_{n+m-1} \in M(S^{n+m-1})$. We choose a model ϕ for the map $(g|h)$ as in Lemma 3.2. We may write, modulo the ideal generated by elements of $M(S^n \vee S^m)$ of degree greater than $n+m-1$ and generators e_{2n-1} and e_{2m-1} if there exists,

$$\phi(v) \equiv 1 \otimes \overline{f}(v) + e_n \otimes u_2 + e_m \otimes u_3 + \iota_{n+m-1} \otimes u_4,$$
$$\phi(v_i) \equiv 1 \otimes \overline{f}(v_i) + e_n \otimes u_{i2} + e_m \otimes u_{i3} + \iota_{n+m-1} \otimes u_{i4}$$

for any $v \in V$ and $dv = \sum v_1 v_2 \cdots v_s$. Since, $(\overline{\eta} \otimes 1)\phi(v) = 1 \otimes \overline{f}(v) + e_{n+m-1} \otimes (-1)^{n+m-1} u_4$, it follows that $\Phi([\alpha, \beta]_w)(v) = (-1)^{n+m-1} u_4$. On the other hand, ϕ is a CDGA map and satisfies the condition of Lemma 3.2. We then have

$$e_n e_m \otimes u_4 = \\ e_n e_m \otimes \sum \Big(\sum_{i \neq j} (-1)^{\varepsilon_{ij}} \overline{f}(v_1 \cdots v_{i-1}) u_{i2} \overline{f}(v_{i+1} \cdots v_{j-1}) u_{j3} \overline{f}(v_{j+1} \cdots v_s) \Big).$$

By commutativity of the diagram (3.2) and the definition of Φ, we see that $u_{i2} = \Phi(\alpha)(v_i)$ and $u_{j3} = \Phi(\beta)(v_j)$. Therefore,

$$\Phi([\alpha, \beta]_w)(v) = (-1)^{n+m-1}u_4 = [\Phi(\alpha), \Phi(\beta)](v).$$

This completes the proof. $\square$

In the rest of this section, we also consider the Whitehead product in a based mapping space $\mathrm{map}_*(X, Y; f)$. Given $\alpha \in \pi_n(\mathrm{map}_*(X, Y; f))$ and let $g : S^n \times X \to Y$ be the adjoint map of α. Since g satisfy $g|_X = f$ and $g|_{S^n} = *$, by Lemma 3.1, there exists a model for g, $\overline{g}$, such that $(\varepsilon \cdot 1)\overline{g} = \overline{f}$ and $(1 \cdot \varepsilon)\overline{g} = u\varepsilon$. The second equation shows that $\Phi(\alpha)$ is a $\overline{f}$-derivation of degree $-n$ from ΛV to the augmentation ideal B^+ of B. We then get the map of abelian groups

$$\Phi' : \pi_n(\mathrm{map}_*(X, Y; f)) \longrightarrow H^{-n}_{\mathrm{AQ}}(\Lambda V, B^+; \overline{f});\ \Phi'(\alpha) = \Phi(\alpha)$$

for $n \geq 2$ and a straight-forward modification of Theorem 2.4 shows the following proposition:

Proposition 3.3. *The map $\Phi' : \pi_n(\mathrm{map}_*(X, Y; f)) \to H^{-n}_{\mathrm{AQ}}(\Lambda V, B^+; \overline{f})$ is an isomorphism for $n \geq 2$.*

This proposition also enables us to get the following corollary.

Corollary 3.4. *The restriction of the bracket defined by the formula* (1.1) *in $H^*_{\mathrm{AQ}}(\Lambda V, B; \overline{f})$ to $H^*_{\mathrm{AQ}}(\Lambda V, B^+; \overline{f})$ corresponds the Whitehead product in $\pi_*(\mathrm{map}_*(X, Y; f))$ via the isomorphism Φ' from $\pi_n(\mathrm{map}_*(X, Y; f))$ to $H^{-n}_{\mathrm{AQ}}(\Lambda V, B^+; \overline{f})$.*

Proof. Given $\alpha \in \pi_n(\mathrm{map}_*(X, Y; f))$ and $\beta \in \pi_m(\mathrm{map}_*(X, Y; f))$. Since $\varepsilon\Phi'(\alpha) = 0$ and $\varepsilon\Phi'(\beta) = 0$, it follows that $\varepsilon\Phi'([\alpha, \beta]_w) = 0$ by the formula (1.1). $\square$

4. The Whitehead length of mapping spaces

In this section, we consider the Whitehead length of mapping spaces. We recall the definition of the Whitehead length; see Section 1. Now we consider a upper bound of $\mathrm{WL}(\mathrm{map}(X, Y; f))$. The following result is proved by Lupton and Smith.

Proposition 4.1 ([13, Theorem 6.4])**.** *Let X and Y be $\mathbb{Q}$-local, simply-connected spaces with finite type. If Y is coformal; that, is a minimal Sullivan model for Y of the form $(\Lambda V, d_1)$, then*

$$\mathrm{WL}(\mathrm{map}(X, Y; f)) \leq \mathrm{WL}(Y).$$

We give another proof of Proposition 4.1 using the bracket defined by Theorem 1.1. Before proving the proposition, we introduce a numerical invariant which is called the d_1-depth for a simply-connected space Z and recall the relationship between the Whitehead length and the d_1-depth.

Definition 4.2. Let $(\Lambda V, d)$ be a minimal Sullivan model for a simply-connected space Z and d_1 the quadratic part of d. The d_1*-depth* of Z, denoted by d_1-depth(Z), is the greatest integer n such that V_{n-1} is a proper subspace of V_n with

$$V_{-1} = 0,\ V_0 = \{v \in V \mid d_1 v = 0\} \text{ and } V_i = \{v \in V \mid d_1 v \in \Lambda V_{i-1}\}\ (i \geq 1).$$

Theorem 4.3 ([10, Theorem 4.15][11, Theorem 2.5])**.** *Let Y be a $\mathbb{Q}$-local, simply-connected space. Then* d_1-depth$(Y) + 1 = \mathrm{WL}(Y)$.

Proof of Proposition 4.1. Let ΛV be a minimal Sullivan model for Y and $m = d_1$-depth(Y). For any $v \in V$, we may write $d_1(v) = \sum_{j=1}^{n} u_{j1}u_{j2}\cdots u_{jk_j}$ where u_{ji} are basis of V. Then, put

$$T'_{d_1}(v) = \{u_{j1}u_{j2}\cdots u_{jk_j} \mid j = 1 \ldots n\}$$

and

$$T_{d_1}(u_1u_2\cdots u_s) = \bigcup_{i=1\ldots s} \{u_1\cdots u_{i-1}u'u_{i+1}\cdots u_s \mid u' \in T'_{d_1}(u_i)\}.$$

We also set

$$T_{d_1}(U) = \bigcup_{u\in U} T_{d_1}(u)$$

where U is a set of terms of ΛV. By the definition of d_1-depth, $T_{d_1}^{(m+1)}(v) = \{0\}$ and it follows that

$$[\varphi_1, [\varphi_2, \cdots [\varphi_{m+1}, \varphi_{m+2}]\cdots]](v) = 0$$

for any $\varphi_1, \varphi_2, \ldots, \varphi_{m+2} \in H_{\mathrm{AQ}}^{\leq -2}(\Lambda V, B; \overline{f})$. Hence, by Theorem 1.1 and Theorem 4.3, we have $\mathrm{WL}(\mathrm{map}(X, Y; f)) \leq m + 1 = \mathrm{WL}(Y)$. $\square$

We next prove Proposition 1.4.

Proof of Proposition 1.4. Let $m = \mathrm{WL}(\mathrm{map}_*(X, Y; f))$. If $m = 1$, then the assertion is trivial and so we may assume that $m \geq 2$. By Corollary 3.4, there are elements $\varphi_1, \varphi_2, \cdots, \varphi_m$ in $H_{\mathrm{AQ}}^{\leq -2}(\Lambda V, B^+; \overline{f})$ such that

$$[\varphi_1, [\varphi_2, \cdots, [\varphi_{m-1}, \varphi_m]\cdots]](v) \neq 0 \tag{4.1}$$

for some $v \in V$. For any element $u_1u_2\cdots u_s \in T_{d_1}^m(v)$, the length s of $u_1u_2\cdots u_s$ is greater than or equal to $(m-2)(\omega - 1) + \omega$ by the definition of ω. Therefore, the equation (4.1) implies that

$$\mathrm{nil}B \geq (m-2)(\omega - 1) + \omega$$

and hence we have

$$m \leq \frac{1}{\omega - 1}(\mathrm{nil}B - 1) + 1.$$

$\square$

Remark 4.4. Suppose that $\mathrm{WL}(Y) = 1$ and $\mathrm{WL}(\mathrm{map}_*(X, Y; f)) > 1$. The proof of Proposition 1.4 enables us to conclude that $\mathrm{nil}B \geq \omega$ and that $\omega \geq 3$ since $V = \mathrm{Ker}d_1$. Moreover we have

$$\mathrm{WL}(\mathrm{map}_*(X, Y; f)) \leq \frac{1}{\omega - 1}(\mathrm{nil}B - 1) + 1 \leq \mathrm{nil}B - 1.$$

Thus our upper bound of the Whitehead length of the mapping space may be less than that described in Theorem 1.3.

5. Computational examples

We shall determine the Whitehead length of the mapping space from $\mathbb{C}P^\infty \times \mathbb{C}P^n$ to $\mathbb{C}P^\infty_\mathbb{Q} \times \mathbb{C}P^m_\mathbb{Q}$. For this, we first compute the homotopy group of the mapping space. Recall that the CDGAs $(\Lambda(x_2, x'_{2n+1}),\ dx'_{2n+1} = x_2^{n+1})$ and $(\mathbb{Q}[z_2], 0)$ are minimal Sullivan models for $\mathbb{C}P^n$ and $\mathbb{C}P^\infty$, respectively. Here, $|x_2| = |z_2| = 2$ and $|x'_{2n+1}| = 2n + 1$. Since $\mathbb{C}P^n$ is formal, that is the CDGA map ρ

$$(\Lambda(x_2, x'_{2n+1}),\ dx'_{2n+1} = x_2^{n+1}) \longrightarrow (\mathbb{Q}[x_2]/(x_2^{n+1}), 0) = H^*(\mathbb{C}P^n; \mathbb{Q})$$

defined by $\rho(x_2) = x_2,\ \rho(x'_{2n+1}) = 0$ is a quasi-isomorphism, the CDGA $(\mathbb{Q}[z_2] \otimes \mathbb{Q}[x_2]/(x_2^{n+1}), 0)$ is a CDGA model for $\mathbb{C}P^\infty \times \mathbb{C}P^n$.

Proposition 5.1. *Let $k \geq 2$ and $m < n$. Then*

$$\pi_k(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_\mathbb{Q} \times \mathbb{C}P^m_\mathbb{Q}; f))$$
$$= \begin{cases} \mathbb{Q} & (k = 2 \text{ and } q_2 \neq 0) \\ \mathbb{Q} \oplus \mathbb{Q} & (k = 2 \text{ and } q_2 = 0) \\ \displaystyle\bigoplus_{0 \leq i = n-m-l+1}^{n-l+1} \mathbb{Q} & (k = 2l - 1,\ 2 \leq l \leq n + 1) \\ 0 & (\text{otherwise}). \end{cases}$$

Here, the map f is the realization of the CDGA map $\overline{f}$

$$M(\mathbb{C}P^\infty \times \mathbb{C}P^n) = \mathbb{Q}[z_2] \otimes \Lambda(x_2, x'_{2n+1})$$
$$\longrightarrow \mathbb{Q}[w_2] \otimes \Lambda(y_2, y'_{2m+1}) = M(\mathbb{C}P^\infty \times \mathbb{C}P^m)$$

defined by $\overline{f}(z_2) = q_1(w_2 \otimes 1)$, $\overline{f}(x_2) = q_2(w_2 \otimes 1) + q_3(1 \otimes y_2)$ and $\overline{f}(x'_{2n+1}) = 0$ for some $q_1, q_2, q_3 \in \mathbb{Q}$.

Proof. We put $\mathrm{Der}^n = \mathrm{Der}^n(\mathbb{Q}[z_2]\otimes\Lambda(x_2, x'_{2n+1}),\ \mathbb{Q}[w_2]\otimes\mathbb{Q}[y_2]/(y_2^{m+1}); \rho\overline{f})$ for convenience. For any elements $\theta_{r,s} \in \mathrm{Der}^{-2}$, we may write

$$\theta_{r,s}(z_2) = r,\ \theta_{r,s}(x_2) = s \text{ and } \theta_{r,s}(x'_{2n+1}) = 0$$

for some $r, s \in \mathbb{Q}$. Then,

$$\partial\theta_{r,s}(z_2) = \partial\theta_{r,s}(x_2) = 0,\ \partial\theta_{r,s}(x'_{2n+1}) = -ns\Big(\sum_{i+j=n} q_2^i q_3^j w_2^i \otimes y_2^j\Big).$$

When $q_2 \neq 0$, we see that $\theta_{r,s}$ is a cycle if and only if $s = 0$, that is all cycles of Der^{-2} generated by $\theta_{1,0}$. When $q_2 = 0$, $\theta_{r,s}(x'_{2n+1}) = 0$ since $y_2^n = 0$. Hence, $\theta_{1,0}$ and $\theta_{0,1}$ are generators of all cycles of Der^{-2}. In general, $\mathrm{Der}^{-2l} = 0$ for $l \geq 2$ by degree reasons. It follows that

$$\pi_{2l}(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f)) \cong H^{-2l}(\mathrm{Der}^*) = 0\ (l \geq 2).$$

For any $\theta \in \mathrm{Der}^{-2l+1}$, we may write

$$\theta(z_2) = 0,\ \theta(x_2) = 0 \text{ and } \theta(x'_{2n+1}) = \sum_{i=0}^{n-l+1} r_i w_2^i \otimes y_2^{n-l+1-i}.$$

Note that if $l > n+1$, $\mathrm{Der}^{-2l+1} = 0$ by degree reasons. It is easily seen that all elements of Der^{-2l+1} are cycles. Moreover, we see that $y_2^{n-l+1-i} = 0$ if and only if $0 \leq i \leq n-m-l$. Therefore, we have

$$\begin{aligned}&\pi_2(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f))\\ &\cong H^{-2}(\mathrm{Der}^*) \cong \begin{cases}\mathbb{Q} & (k = 2 \text{ and } q_2 \neq 0)\\ \mathbb{Q}\oplus\mathbb{Q} & (k = 2 \text{ and } q_2 = 0)\end{cases}\end{aligned}$$

and

$$\begin{aligned}&\pi_{2l-1}(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f))\\ &\cong H^{-2l+1}(\mathrm{Der}^*) \cong \bigoplus_{0\leq i=n-m-l+1}^{n-l+1}\mathbb{Q} \quad (2 \leq l \leq n+1),\end{aligned}$$

$$\pi_{2l-1}(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f)) \cong H^{-2l+1}(\mathrm{Der}^*) = 0\ (l > n+1).$$

□

Proposition 5.2. *Let $m < n$. Then one has*

$$\mathrm{WL}(\mathrm{map}(\mathbb{C}P^\infty\times\mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}}\times\mathbb{C}P^m_{\mathbb{Q}}; f)) = \begin{cases}2 & (n-m=1,\ q_2 = 0,\ q_3 \neq 0)\\ 1 & (\textit{otherwise}).\end{cases}$$

Proof. By the definition of the bracket in $H^*(\mathrm{Der}^*)$, we see that if $\varphi, \psi \in H^{\leq -3}(\mathrm{Der}^*)$, then $[\varphi, \psi] = 0$ since $\varphi(x_2) = 0$ and $\psi(x_2) = 0$. That is $[\varphi', \psi'] \neq 0$ means $|\varphi'| = |\psi'| = -2$. It shows that

$$\mathrm{WL}(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f)) \leq 2.$$

If $q_2 \neq 0$, by Proposition 5.1, $H^{-2}(\mathrm{Der}^*)$ is generated by $\theta_{1,0}$. The equality $[\theta_{1,0}, \theta_{1,0}] = 0$ shows that $\mathrm{WL}(\mathrm{map}(\mathbb{C}P^\infty \times \mathbb{C}P^n, \mathbb{C}P^\infty_{\mathbb{Q}} \times \mathbb{C}P^m_{\mathbb{Q}}; f)) = 1$. On the other hand, if $q_2 = 0$, $\theta_{0,1}$ is a generator of $H^{-2}(\mathrm{Der}^*)$ and

$$[\theta_{0,1}, \theta_{0,1}](x'_{2n+1}) = q_3^{n-1} y_2^{n-1}.$$

This completes the proof. $\square$

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