Title of Thesis

A homotopy theory of diffeological and numerically generated spaces

2013, 3

Tadayuki Haraguchi

Graduate School of
Natural Science and Technology
(Doctor's Course)
OKAYAMA UNIVERSITY

Preface

This is a thesis submitted to the Graduate School of Natural Science and Technology, Okayama University in partial fulfillment of the requirements for the academic degree of Doctor of Science in the field of Mathematics at Okayama University.

The thesis consists of five sections. To discuss a model structure of some categories are main contents in sections 2, 4 and 5.

Acknowledgements

First and foremost, I would like to thank my supervisor Kazuhisa Shimakawa. He introduced me to the project of building a homotopy theory on the categories **Diff** and **NG** of diffeological and numerically generated spaces. He gave me the knowledge of his mathematics unstintingly, and spent much time on me. In the course we prove the main theorem (cf. Theorem 4.3.1) to be, he solved the important problem (cf. Theorem 4.2.7). While I was on the register in Okayama University, I got knowledge for becoming a mathematician under him. From now on, I would like to try hard to collaborate with him.

Next I would like to thank Takeshi Torii. When I was troubled by the proof of Theorem 4.3.8, he gave me various ideas. I was able to solve this problem through his help.

Finally, I would like to be thankful to many people supporting me.

Contents

1 Introduction **4**

2 Model categories **8**
2.1 Categories . . . 8
2.2 Model structure . . . 10
2.3 Cofibrantly generated model structure . . . 12
2.4 On model structure for coreflective subcategories of a model category . . . 14

3 Diffeological spaces **18**
3.1 Diffeological spaces . . . 18
3.2 Functional diffeological spaces . . . 23
3.3 D-Topology and T-Diffeology . . . 26
3.4 Weak diffeology . . . 28

4 Homotopy theory of diffeological spaces **31**
4.1 Diffeological homotopy groups . . . 31
4.2 Diffeological CW complexes . . . 33
4.3 Model structure of diffeological spaces . . . 40
4.4 Quillen adjunction between Diff and Top . . . 47

5 Numerically generated spaces **49**
5.1 Numerically generated spaces . . . 49
5.2 Exponentials in NG . . . 51
5.3 Model structure of NG . . . 54

1 Introduction

In this thesis we discuss model structure on the categories of topological and diffeological spaces. We first show that a subcategory of a model category satisfying certain conditions inherits a model structure. As an application, we prove that well-known convenient categories of topological spaces, such as k-spaces, compactly generated spaces, and Δ-generated spaces (called numerically generated in [13]) admit a finitely generated model structure which is Quillen equivalent to the standard model structure on the category **Top** of topological spaces.

We next turn to the category **Diff** of diffeological spaces. In [14], J.-M. Souriau introduced diffeological spaces as a generalization of the notion of topological smooth manifolds. Moreover, P. Iglesias-Zemmour developed it in his book [8]. We further develop the diffeological space theory from several aspects. Our main concern is to build a model structure on **Diff**.

Our strategy is as follows. In section 2, we discuss the basic theory of categories and model categories. Model categories were introduced by Quillen in [11], as an abstraction of usual situation in topological spaces. This is where the terminology came from as well. Quillen′s definitions have been modified over the years, by Quillen himself in [12] and, more recently, by Dwyer, Hirschhorn, and Kan.

In subsection 2.1 we introduce the notion of categories theory. In subsection 2.2 we define model categories, Quillen adjunctions and Quillen equivalences. In subsection 2.3 we introduce the notion of cofibrantly generated model categories. We give an example of cofibrantly generated model categories. The category **Top** is a finitely generated model category. In subsection 2.4 we construct a model structure on subcategories of a model category. Let $\mathbf{C}$ be a coreflective subcategory of a cofibrantly generated model category $\mathbf{D}$. In this paper we shall show that under suitable conditions $\mathbf{C}$ admits a cofibrantly generated model structure and the inclusion functor $i\colon \mathbf{C} \to \mathbf{D}$ is a Quillen adjunction. Moreover $i\colon \mathbf{C} \to \mathbf{D}$ is a Quillen equivalence if the coreflection arrow $\epsilon\colon GY \to Y$ is a weak equivalence for any fibrant object $Y \in \mathbf{D}$.

In section 3 we discuss the basic theory of diffeological spaces. In [8], P.Iglesias-Zemmour developed this theory, and used it to simplify and unify several important concepts and constructions in mathematics and physics. In subsection 3.1 we introduce the define of diffeological spaces and smooth maps between diffeological spaces. Clearly, the class of diffeological spaces and smooth maps form a category **Diff**. Moreover we can construct subspaces, product spaces, coproduct spaces and quotient spaces of diffeological spaces, respectively. Therefore the category **Diff** is complete, cocomplete, and cartesian closed. In subsection 3.2 we shall show that the category **Diff** is cartesian closed. Let $C^\infty(X, Y)$ be a set consisting of all smooth maps between diffeological spaces X and Y. Let ev be the evaluation map, defined by

$$\mathrm{ev}\colon C^\infty(X, Y) \times X \to Y, \text{ and } \mathrm{ev}(f, x) = f(x).$$

Then there exists a coarsest diffeology of $C^\infty(X, Y)$ such that the map ev is smooth. Now we have the following condition:

$$C^\infty(X \times Y, Z) \cong C^\infty(X, C^\infty(Y, Z)),$$

where X, Y and Z are diffeological spaces. In subsection 3.3 we study D-Topology and T-Diffeology. There are the following functors:

1. A functor $T\colon \mathbf{Diff} \to \mathbf{Top}$ which maps a diffeological space X to the D-topological space TX.
2. A functor $D\colon \mathbf{Top} \to \mathbf{Diff}$ which maps a topological space Y to the T-diffeological space DX.

Then T is a left adjoint to D. In subsection 3.4 we define the notion of the weak diffeology. This notion is my original theory. We say that a diffeological space X has a weak diffeology with respect to a cover $\{X_\lambda\}_{\lambda\in\Lambda}$ of subspaces of X if for any plot $P\colon U \to X$ of X, there exists a plot of X_λ which lifts locally, at any point $r \in U$, along the inclusion map $i_\lambda\colon X_\lambda \to X$. It turned out that properties of weak diffeology has the properties similar to weak topology. For example, a map $f\colon X \to Y$ between diffeological spaces is smooth if and only if for any $\lambda \in \Lambda$, the restriction map $f|X_\lambda\colon X_\lambda \to Y$ is smooth. Moreover if two diffeological spaces X and Y have the weak diffeology with respect to covers $\{X_\lambda\}_{\lambda\in\Lambda}$ and $\{Y_{\lambda'}\}_{\lambda'\in\Lambda'}$ of subspaces of X and Y, respectively, $X \times Y$ has the weak diffeology with respect to a cover $\{X_\lambda \times Y_{\lambda'}\}_{(\lambda,\lambda')\in\Lambda\times\Lambda'}$ of $X \times Y$.

In section 4 we develop the diffeological space theory from several aspects: diffeological homotopy groups and diffeological CW complexes. Our main concern is to build a suitable homotopy theory (also called a model category structure) on the category **Diff** of diffeological spaces. In subsection 4.1 we construct diffeological homotopy groups. Let I be the unit interval, and let $\lambda\colon \mathbf{R} \to I$ be the smashing function, that is, a smooth function such that $\lambda(t) = 0$ for $t \geq 0$ while $\lambda(t) = 1$ for $t \geq 1$. Let $\tilde{I}$ denote the unit interval equipped with the quotient diffeology $\lambda_*(D_{\mathbf{R}})$, where

$D_{\mathbf{R}}$ is the standard diffeology of $\mathbf{R}$. Two smooth maps $f,\ g\colon X \to Y$ between diffeological spaces is called homotopic if there exists a homotopy $F\colon X \times \tilde{I} \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. Although a subspace I of $\mathbf{R}$ and $\tilde{I}$ are not diffeomorphic, they are homotopy equivalent. Let x_0 be a based point of diffeological space X. Here, the homotopy group $\pi_n(X, x_0)$ is defined to be the set of smooth homotopy classes of smooth maps $(\tilde{I}^n, \partial\tilde{I}^n) \to (X, x_0)$. In subsection 4.2 we present the notion of diffeological CW complexes. The properties of diffeological CW complexes are similar to the properties of topological CW complexes. Characteristic maps of cell complex $(X, \{e_\lambda\})$ is defined by for any λ in Λ, subduction maps $\tilde{I}^{n_\lambda}$ to $\bar{e_\lambda}$. Moreover the definitions of C and W are similar to the definitions of C and W for topological CW complexes. Then if X and Y are diffeological CW complexes, $X \times Y$ is a diffeological CW complex.

We define a deformation retract. Although a subspace $\partial I^n \times I \cup I^n \times \{0\}$ of $\mathbf{R}^{n+1}$ is not a deformation retract of $I^n \times I$, a subspace $\partial\tilde{I}^n \times \tilde{I} \cup \tilde{I}^n \times \{0\}$ of $\tilde{I}^n \times \tilde{I}$ is a deformation retract of $\tilde{I}^n \times \tilde{I}$. We have the following by this result. If A is a subcomplex of diffeological CW complex X, the inclusion $i\colon A \to X$ is a cofibration(Definition 4.2.8). Now we define a Serre fibration as follows. A smooth map $p\colon X \to Y$ between diffeological spaces is called a Serre fibration if p has the right lifting property with respect to the inclusion map $\partial\tilde{I}^n \times \tilde{I} \cup \tilde{I}^n \times \{0\} \to \tilde{I}^n \times \tilde{I}$. Then a smooth map $p\colon X \to Y$ is a Serre fibration if and only if p has the homotopy extension lifting property with respect to a diffeological CW pair (A, B). In subsection 4.3 we introduce a homotopy theory of diffeological spaces. Let $I_{\mathbf{Diff}}$ be the set of boundary inclusions $\partial\tilde{I}^n \to \tilde{I}^n,\ n \geq 0$ and $J_{\mathbf{Diff}}$ be the set of inclusions $\partial\tilde{I}^n \times \tilde{I} \cup \tilde{I}^n \times \{0\} \to \tilde{I}^n \times \tilde{I},\ n \geq 0$. Let $W_{\mathbf{Diff}}$ be the class of weak homotopy equivalences. Then we shall show that there is a finitely generated model structure on **Diff** with $I_{\mathbf{Diff}}$ as the set of generating cofibrations, $J_{\mathbf{Diff}}$ as the set of generating trivial cofibrations, and $W_{\mathbf{Diff}}$ as the class of weak equivalences. In subsection 4.4 we shall show that the categories **Diff** and **Top** are a Quillen adjunction. For any $n \geq 0$, a D-topological space $T\tilde{I}^n$ of smashing space and a subspace I^n of Euclidean space $\mathbf{R}^n$ are homeomorphic. By using this result, we prove that the functor $D\colon \mathbf{Top} \to \mathbf{Diff}$ preserves all fibrations and trivial fibrations.

In section 5 we introduce the notion of numerically generated spaces. This notion is intoroduced by Simakawa in [13]. In subsection 5.1 we define numerically generated spaces by using two functors $D\colon \mathbf{Top} \to \mathbf{Diff}$ and $T\colon \mathbf{Diff} \to \mathbf{Top}$. Let $\nu = TD$. We say that a topological space X is a numerically generated space if X and νX are homeomorphic. Then A is an open set of numerically generated space X if and only if for any continuous

$P\colon U \to X$ from an open set of Euclidean spaces to X, $P^{-1}(A)$ is an open set of U. Let **NG** be the full subcategory of **Top** consisting of numerically generated spaces. Let X be a topological space. Since $\nu(\nu X) = \nu X$ holds, we have a functor $\nu\colon \mathbf{Top} \to \mathbf{Diff}$. The inclusion functor $i\colon \mathbf{NG} \to \mathbf{Top}$ is a left adjoint to $\nu\colon \mathbf{Top} \to \mathbf{NG}$. Now, the coproduct space $X \coprod Y$ of numerically generated spaces X and Y is a numerically generated space. Let $p\colon X \to Y$ be a surjective map from numerically generated space X to a set Y. Then a quotient space $(Y, p_*(O_X))$ is a numerically generated space where O_X is a topology of X. Moreover we define product space of numerically generated spaces X and Y by $X \times_\nu Y = \nu(X \times Y)$ and subspace A_ν of X by $A_\nu = \nu A$, where A is a subset of X. By the definition, it is clear that $X \times_\nu Y$ and A_ν are numerically generated spaces. Therefore the category **NG** is complete and cocomplete. In subsection 5.2 we prove that the category **NG** is a cartesian closed category. We introduce the notion of a numerically continuous. By using this notion, we show that there is a natural homeomorphism

$$\mathrm{Hom}_{\mathbf{NG}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathbf{NG}}(X, \mathrm{Hom}_{\mathbf{NG}}(Y, Z)).$$

In subsection 5.3 we prove that there exists a cofibrantly generated model structure on **NG** as an application of Theorem 2.4.1. Then since the identity map $1_X\colon \nu X \to X$ is a weak homotopy equivalence for any $X \in \mathbf{Top}$, the categories **NG** and **Top** are a Quillen equivalent by Proposition 2.4.3.

2 Model categories

2.1 Categories

In this subsection we review some basic ideas and constructions from category theory.

Definition 2.1.1 (categories). A category $\mathbf{C}$ consists of a family $\mathrm{Ob}(\mathbf{C})$ of objects and for each pair of objects X and Y of $\mathbf{C}$, a set $\mathrm{Hom}_C(X,Y)$ of morphisms from X to Y, together with a way of composing morphisms that match

$$\circ\colon \mathrm{Hom}_{\mathbf{C}}(X,Y)\times \mathrm{Hom}_{\mathbf{C}}(Y,Z)\to \mathrm{Hom}_{\mathbf{C}}(X,Z).$$

This data satisfies the following conditions;

1. Composition is associative, that is,

$$h\circ(g\circ f)=(h\circ g)\circ f.$$

2. For any object X in $\mathbf{C}$, there exists the identity morphism 1_X in $\mathrm{Hom}_{\mathbf{C}}(X,X)$ such that for any $f\in \mathrm{Hom}_{\mathbf{C}}(X,Y)$,

$$1_Y\circ f=f=f\circ 1_X.$$

Moreover we say that $\mathbf{C}$ is a small category if the collection $\mathrm{Ob}(\mathbf{C})$ of objects of $\mathbf{C}$ forms a set.

Definition 2.1.2 (functors). Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. We say that $F\colon \mathbf{C}\to\mathbf{D}$ is a functor if the following conditions;

1. For any object X of $\mathbf{C}$, $F(X)$ is a object of $\mathbf{D}$.
2. for any morphism $f\colon X\to Y$ of $\mathbf{C}$, $F(f)\colon F(X)\to F(Y)$ is a morphism of $\mathbf{D}$ and $F(g\circ f)=F(g)\circ F(f)$ holds.
3. For any object X of $\mathbf{C}$, $F(1_X)=1_{F(X)}$ holds, where 1_X is the identity morphism.

Moreover we say that $F\colon \mathbf{C}\to\mathbf{D}$ is a contravariant functor if for any morphism $f\colon X\to Y$ of $\mathbf{C}$, $F(f)\colon F(Y)\to F(X)$ is a morphism of $\mathbf{D}$ and $F(g\circ f)=F(f)\circ F(g)$ in the statement 2.

Definition 2.1.3 (constant functors). Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. For any object X of $\mathbf{C}$, we define a functor $\Delta(X)\colon \mathbf{D}\to\mathbf{C}$ by

1. for any object Y of $\mathbf{D}$, $\Delta(X)Y = X$, and
2. for any morphism $f\colon A \to B$ of $\mathbf{D}$, $\Delta(X)(f) = 1_X$.

This functor $\Delta(X)$ is called the constant functor.

Definition 2.1.4 (natural transformations). Let F and G be two functors between categories $\mathbf{C}$ and $\mathbf{D}$. We say that $\alpha\colon F \to G$ is a natural transformation if it consists of a morphism $\alpha_X\colon F(X) \to G(X)$ for each object X of $\mathbf{C}$ such that the following diagram commutes for each morphism $f\colon X \to Y$ of $\mathbf{C}$;

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \alpha_X}\big\downarrow & & {\scriptstyle \alpha_Y}\big\downarrow \\ G(X) & \xrightarrow{G(f)} & G(Y). \end{array}$$

Moreover the natural transformation α is called a natural equivalence if the morphism $\alpha_X\colon F(X) \to G(X)$ is an isomorphism of $\mathbf{D}$ for each object X of $\mathbf{C}$.

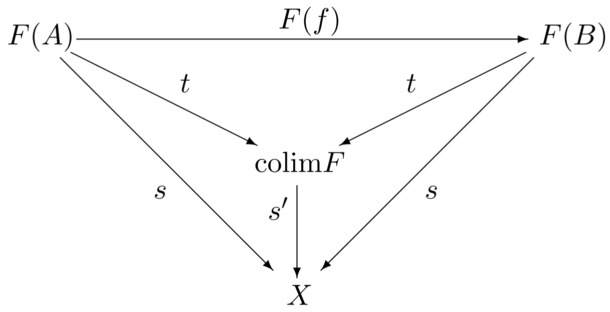
Definition 2.1.5. Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. We say that $\mathbf{C}$ and $\mathbf{D}$ are equivalent if there are functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ such that the composites FG and GF are related to the appropriate identity functors by natural equivalences.

Definition 2.1.6 (adjoint functors). Let $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ be a pair of functors. We say that F is a left adjoint to G and G is a right adjoint to F if for any objects X of $\mathbf{C}$ and Y of $\mathbf{D}$, there exists a natural map

$$\varphi\colon \mathrm{Hom}_{\mathbf{D}}(FX, Y) \to \mathrm{Hom}_{\mathbf{C}}(X, GY)$$

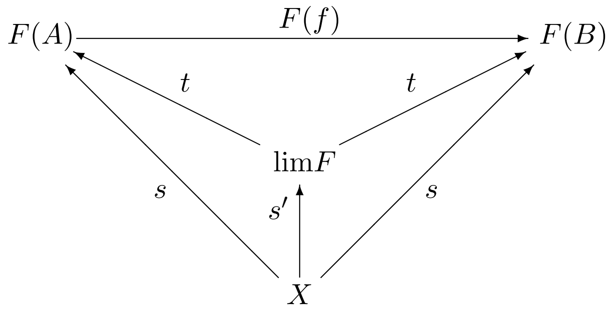
such that φ is a isomorphism. A pair $(F, G, \varphi)\colon \mathbf{C} \to \mathbf{D}$ is called an adjunction from $\mathbf{C}$ to $\mathbf{D}$.

Definition 2.1.7 (colimits). Let $\mathbf{C}$ and $\mathbf{D}$ be a category and a small category, respectively. Let $F\colon \mathbf{D} \to \mathbf{C}$ be a functor. A colimit for F is an object $\mathrm{colim}F$ of $\mathbf{C}$ together with a natural transformation $t\colon F \to \Delta(\mathrm{colim}F)$ such that for every object X of $\mathbf{C}$ and every natural transformation $s\colon F \to \Delta(X)$, there exists a unique morphism $s'\colon \mathrm{colim}F \to X$ of $\mathbf{C}$ such that $\Delta(s')t = s$.



Especially, A category **C** is said to have all small colimits if there exists colim(F) for any functor F from a small category **D** to **C**.

Definition 2.1.8 (limits). Let **C** and **D** be a category and a small category, respectively. Let $F\colon \mathbf{D} \to \mathbf{C}$ be a functor. A limit for F is an object lim$F$ of **C** together with a natural transformation $t\colon \Delta(\mathrm{lim}F) \to F$ such that for every object X of **C** and every natural transformation $s\colon \Delta(X) \to F$, there exists a unique morphism $s'\colon X \to \mathrm{lim}F$ of **C** such that $t\Delta(s') = s$.



Especially, a category **C** is said to have all small colimits if there exists lim$F$ for any functor F from a small category **D** to **C**.

2.2 Model structure

In this subsection we present our definition of a model category. The author learned the basic properties of a model category from [7] and [3].

Definition 2.2.1 (retracts). Let **C** be a category. Let $f\colon X \to Y$ and $g\colon X' \to Y'$ be two morphisms of **C**. We say that f is a retract of g if there exists the following commutative diagram;

$$\begin{array}{ccccc} X & \longrightarrow & X' & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$

such that the horizontal composites are identities.

Definition 2.2.2 (lifting property). Let $\mathbf{C}$ be a category. Let $i\colon A \to B$ and $p\colon X \to Y$ be two morphisms of $\mathbf{C}$. We say that i has the left lifting property with respect to p and that p has the right lifting property with respect to i if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & {\scriptstyle p}\downarrow \\ B & \xrightarrow{g} & Y \end{array}$$

there exists a lift $h\colon B \to Y$ such that $hi = f$ and $ph = g$.

Definition 2.2.3 (model structure). Let $\mathbf{C}$ be a category. A model structure on $\mathbf{C}$ is three subcategories of $\mathbf{C}$ called weak equivalences, cofibrations, and fibrations satisfying the following properties;

1.2-out-of-3 Let f and g be two morphisms of $\mathbf{C}$ such that gf is defined. If two of f, g and gf are weak equivalences, then so is the third.

2.Retracts Let f and g be two morphisms of $\mathbf{C}$. If f is a retract of g and g is a weak equivalence, cofibration, or fibration, then so is the third.

3.Lifting Define a map to be a trivial cofibration if it is both a cofibration and a weak equivalence. Similarly, define a map to be a trivial fibration if it is both a fibration and a weak equivalence. Then trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations.

4.Factorization For any morphism f of $\mathbf{C}$, we have the followings;

(i). There exist a cofibration i and a trivial fibration p such that $f = pi$ holds.

(ii). There exist a trivial cofibration i and a fibration p such that $f = pi$ holds.

Definition 2.2.4 (Model categories). A model category is a category $\mathbf{C}$ with all small limits and colimits together with a model structure on $\mathbf{C}$.

Definition 2.2.5 (Quillen adjunction). Let $\mathbf{C}$ and $\mathbf{D}$ be two model categories.

1. We call functor $F\colon \mathbf{C} \to \mathbf{D}$ a left Quillen functor if F is a left adjoint and preserves cofibrations and trivial cofibrations.

2. We call a functor $G\colon \mathbf{D} \to \mathbf{C}$ a right Quillen functor if G is a right adjoint and preserves fibrations and trivial fibrations.

3. Let (F, G, φ) be an adjunction from $\mathbf{C}$ to $\mathbf{D}$. We call (F, G, φ) a Quillen adjuntion if F is a left Quillen functor.

Lemma 2.2.6 ([7, 1.3.4])**.** *Let $\mathbf{C}$ and $\mathbf{D}$ be two model categories. Let (F, G, φ) be an adjunction from $\mathbf{C}$ to $\mathbf{D}$. Then (F, G, φ) is a Quillen adjunction if and only if G is a right Quillen functor.*

Definition 2.2.7 (**Quillen equivalence**)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be two model categories.

1. We call an object of $\mathbf{C}$ cofibrant if a morphism from initial object to it is a cofibration.

2. We call an object of $\mathbf{C}$ fibrant if a morphism from it to terminal object is a fibration.

3. A Quillen adjunction $(F, G, \varphi)\colon \mathbf{C} \to \mathbf{D}$ is called a Quillen equivalence if for all clfibrant X in $\mathbf{C}$ and fibrant Y in $\mathbf{D}$, a morphism $f\colon FX \to Y$ is a weak equivalence in $\mathbf{D}$ if and only if $\varphi(f)\colon X \to GY$ is a weak equivalence in $\mathbf{C}$.

2.3 Cofibrantly generated model structure

In this subsection we introduce the notion of cofibrantly generated model structure. Moreover we gives the example of cofibrantly generated model categories. These details have taken Chapter 2 of [7].

Definition 2.3.1. Let I be a class of morphisms in a category $\mathbf{C}$.

1. A morphism is I-injective if it has the right lifting property with respect to every morphism in I. The class of I-injective morphisms is denoted by I-inj.

2. A morphism is I-projective if it has the left lifting property with respect to every morphism in I. The class of I-projective morphisms is denoted by I-proj.

3. A morphism is an I-cofibration if it has the left lifting property with respect to every I-injective morphism. The class of I-cofibrations is class $(I - \text{inj}) - \text{proj}$ and is denoted by I-cof.

4. A morphism is an I-fibration if it has the right lifting property with respect to I-projective map. The class of I-fibrations is the class (I-proj)-inj and is denoted by I-fib.

Definition 2.3.2 (cofibrantly generated). Let $\mathbf{C}$ be a model category. We say that $\mathbf{C}$ is cofibrantly generated if there are sets I and J of morphisms of $\mathbf{C}$ such that the following conditions;

1. The domains of the morphisms of I are small relative to I-cell in the sense of [7, 2.1.3] and [7, 2.1.9].

2. The domains of the morphisms of J are small relative to J-cell.

3. The class of fibrations is J-inj.

4. The class of trivial fibrations is I-inj.

We refer to I as the set of generating cofibrations, and to J as the set of generateing trivial cofibrations.

Definition 2.3.3 (finitely generated). Let $\mathbf{C}$ be a model category. We say that $\mathbf{C}$ is finitely generated if the following conditions are satisfied;

1. $\mathbf{C}$ is a cofibrantly generated model category, and

2. domains and codomains of I and J are finite relative to I-cell.

Theorem 2.3.4 ([7, 2.1.19])**.** *Let $\mathbf{C}$ be a category with small colimits and limits. Suppose W is a subcategory of $\mathbf{C}$, and I and J are sets of morphisms of $\mathbf{C}$. Then there is a cofibrantly generated model structure on $\mathbf{C}$ with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and W as the subcategory of weak equivalences if and only if the following conditions are satisfied;*

1. *the subcategory W has the two out of three property and is closed under retracts.*

2. *The domains of I are small relative to I-cell.*

3. *The domains of J are small relative to J-cell.*

4. *J-cell $\subset W \cap I$-cof.*

5. *I-inj $\subset W \cap J$-inj.*

6. *Either $W \cap I$-cof $\subset J$-cof or $W \cap J$-inj $\subset I$-inj.*

Now we introduce the example of finitely generated model categories. For the category **Top** of topological spaces, We takes I' the set of inclusions $S^{n-1} \to D^n,\ n \geq 0$, and as J the set of inclusions $D^n \times \{0\} \to D^n \times I,\ n \geq 0$. Let $W_{\mathbf{Top}}$ be the class of weak homotopy equivalences in **Top**. Then we have the following.

Theorem 2.3.5 ([7, 2.4.19])**.** *There is a finitely generated model structure on* **Top** *with I' as the set of generating cofibrations, J as the set generating trivial cofibrations, and $W_{\mathbf{Top}}$ as the class of weak equivalences.*

2.4 On model structure for coreflective subcategories of a model category

Let $\mathbf{D}$ be a cofibrantly generated model category with generating cofibrations I, generating trivial cofibrations J and the class of weak equivalences $W_{\mathbf{D}}$. If the domains and codomains of I and J are finite relative to I-cell [7, 2.1.4], then $\mathbf{D}$ is said to be finitely generated.

Recall that a subcategory $\mathbf{C}$ of $\mathbf{D}$ is said to be coreflective if the inclusion functor $i\colon \mathbf{C} \to \mathbf{D}$ has a right adjoint $G\colon \mathbf{D} \to \mathbf{C}$, so that there is a natural isomorphism $\varphi\colon \mathrm{Hom}_{\mathbf{D}}(X, Y) \to \mathrm{Hom}_{\mathbf{C}}(X, GY)$. The counit of this adjunction $\epsilon\colon GY \to Y\ (Y \in \mathbf{D})$ is called the coreflection arrow.

Theorem 2.4.1. *Let $\mathbf{C}$ be a coreflective subcategory of a cofibrantly generated model category $\mathbf{D}$ which is complete and cocomplete. Suppose that the unit of the adjunction $\eta\colon X \to GX$ is a natural isomorphism, and that the classes I and J of cofibrations and trivial cofibrations in $\mathbf{D}$ are contained in $\mathbf{C}$. Then $\mathbf{C}$ has a cofibrantly generated model structure with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and $W_{\mathbf{C}}$ as the class of weak equivalences, where $W_{\mathbf{C}}$ is the class of all weak equivalences contained in $\mathbf{C}$. If $\mathbf{D}$ is finitely generated, then so is $\mathbf{C}$. Moreover, the adjunction $(i, G, \varphi)\colon \mathbf{C} \to \mathbf{D}$ is a Quillen adjunction in the sense of [7, 1.3.1].*

Proof. It suffices to show that $\mathbf{C}$ satisfies the six conditions of Theorem 2.3.4 with respect to I, J and $W_{\mathbf{C}}$. Clearly, the first condition holds because $W_{\mathbf{C}}$ satisfies the two out of three property and is closed under retracts. To see that the second and the third conditions hold, let $I_{\mathbf{C}}$-cell and $J_{\mathbf{C}}$-cell be the collections of relative I-cell and J-cell complexes contained in $\mathbf{C}$, respectively. Since $I_{\mathbf{C}}$-cell and $J_{\mathbf{C}}$-cell are subcollections of the collections of relative I-cell and J-cell complexes in $\mathbf{D}$, respectively, the domains of I and J are small relative to $I_{\mathbf{C}}$-cell and $J_{\mathbf{C}}$-cell, respectively. The rest of the conditions are verified as follows. Let $f\colon X \to Y$ be a map in $\mathbf{C}$. Since $\eta\colon X \to GX$ is isomorphic for $X \in \mathbf{D}$, f is I-injective in $\mathbf{C}$ if and only if it is I-injective in $\mathbf{D}$. Similarly, f is J-injective in $\mathbf{C}$ if and only if it is J-injective in $\mathbf{D}$. Let f be an I-cofibration in $\mathbf{D}$. Then it has the left lifting property with respect to all I-injective maps in $\mathbf{C}$. Hence f is an I-cofibration in $\mathbf{C}$. Conversely, let f be an I-cofibration in $\mathbf{C}$. Suppose we are given a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & A \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ Y & \longrightarrow & B \end{array}$$

where p is I-injective in $\mathbf{D}$. Then there is a relative I-cell complex $g\colon X \to Z$ [7, 2.1.9] such that f is a retract of g by [7, 2.1.15]. Since g is an I-cofibration in $\mathbf{D}$, there is a lift $Z \to A$ of g with respect to p. Then the composite $Y \to Z \to A$ is a lift of f with respect to p. Therefore f is an I-cofibration in $\mathbf{D}$. Similarly, f is a J-cofibration in $\mathbf{C}$ if and only if it is a J-cofibration in $\mathbf{D}$. Thus we have the desired inclusions

- $J_{\mathbf{C}}$-cell $\subseteq W_{\mathbf{C}} \cap I_{\mathbf{C}}$-cof,
- $I_{\mathbf{C}}$-inj $\subseteq W_{\mathbf{C}} \cap J_{\mathbf{C}}$-inj, and
- either $W_{\mathbf{C}} \cap I_{\mathbf{C}}$-cof $\subseteq J_{\mathbf{C}}$-cof or $W_{\mathbf{C}} \cap J_{\mathbf{C}}$-inj $\subseteq I_{\mathbf{C}}$-inj.

Here $I_{\mathbf{C}}$-inj and $I_{\mathbf{C}}$-cof denote the classes of I-injective maps and I-cofibrations in $\mathbf{C}$, and similarly for $J_{\mathbf{C}}$-inj and $J_{\mathbf{C}}$-cof. Therefore $\mathbf{C}$ is a cofibrantly generated model category by Theorem 2.3.4.

It is clear, by the definition, that $\mathbf{C}$ is finitely generated if so is $\mathbf{C}$.

Finally, to prove that (i, G, φ) is a Quillen adjunction, it suffices to show that $G\colon \mathbf{D} \to \mathbf{C}$ is a right Quillen functor, or equivalently, G preserves J-injective maps in $\mathbf{D}$ by [7, 1.3.4] and [7, 2.1.17]. Let $p\colon X \to Y$ be a J-injective map in $\mathbf{D}$. Suppose there is a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & GX \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle Gp}\big\downarrow \\ B & \longrightarrow & GY \end{array}$$

where $f \in J$. Then we have a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & GX & \xrightarrow{\epsilon} & X \\ {\scriptstyle f}\big\downarrow & & & & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & GY & \xrightarrow{\epsilon} & Y. \end{array}$$

Since p is J-injective in $\mathbf{D}$, there is a lift $h\colon B \to X$ of f. Thus we have a lift $Gh \circ \eta\colon B \cong GB \to GX$ of f with respect to Gp. Therefore $Gp\colon GX \to GY$ is J-injective in $\mathbf{C}$. Similarly, we can show that G preserves I-injective maps in $\mathbf{C}$, and so G preserves trivial fibrations in $\mathbf{C}$. Hence (i, G, φ) is a Quillen adjunction. □

We turn to the case of pointed categories [7, p.4]. Let $\mathbf{D}_*$ be the pointed category associated with $\mathbf{D}$, and let $U\colon \mathbf{D}_* \to \mathbf{D}$ be the forgetful functor. We denote by I_+ and J_+ the classes of those maps $f\colon X \to Y$ in $\mathbf{D}_*$ such that $Uf\colon UX \to UY$ belongs to I and J, respectively. Then we have the following. (Compare [7, 1.1.8], [7, 1.3.5], and [7, 2.1.21].)

Theorem 2.4.2. *Let $\mathbf{D}$ be a cofibrantly (resp. finitely) generated model category, and let $\mathbf{C}$ be a coreflective subcategory satisfying the conditions of Theorem 2.4.1. Then the pointed category $\mathbf{C}_*$ has a cofibrantly (resp. finitely) generated model structure, with generating cofibrations I_+ and generating trivial cofibrations J_+, such that the induced adjunction $(i_*, G_*, \varphi_*)\colon \mathbf{C}_* \to \mathbf{D}_*$ is a Quillen adjunction.*

We also have the following Proposition.

Proposition 2.4.3. *Suppose $\mathbf{C}$ and $\mathbf{D}$ satisfy the conditions of Theorem 2.4.1. Suppose, further, that the coreflection arrow $\epsilon\colon GY \to Y$ is a weak equivalence for any fibrant object Y in $\mathbf{D}$. Then the adjunctions $(i, G, \varphi)\colon \mathbf{C} \to \mathbf{D}$ and $(i_*, G_*, \varphi_*)\colon \mathbf{C}_* \to \mathbf{D}_*$ are Quillen equivalences.*

Proof. Let X be a cofibrant object in $\mathbf{C}$ and Y a fibrant object in $\mathbf{D}$. Let $f\colon X \to Y$ be a map in $\mathbf{D}$. Then we have $\varphi f = Gf \circ \eta\colon X \cong GX \to GY$. Since f coincides with the composite $X \xrightarrow{\varphi f} GY \xrightarrow{\epsilon} Y$ and ϵ is a weak equivalence in $\mathbf{D}$, φf is a weak equivalence in $\mathbf{C}$ if and only if f is a weak equivalence in $\mathbf{D}$. It follows by [7, 1.3.17] that that the induced adjunction (i_*, G_*, φ_*) is a Quillen equivalence. □

Now we give some examples of Theorem 2.4.1, Theorem 2.4.2, and Proposition 2.4.3.

In section 5, we introduce the notion of numerically generated spaces. Let **NG** be the full subcategory of **Top** consisting of numerically generated spaces. Let I' and J be the sets of inclusions $S^{n-1} \to D^n$ and $D^n \times \{0\} \to D^n \times I,\ n \geq 0$, respectively. Let $W_{\mathbf{NG}}$ be the class of maps $f\colon X \to Y$ in **NG** which is a weak equivalence in **Top**. Then we have the followings.

Theorem 2.4.4 (see §5)**.** *The category* **NG** *has a finitely generated model structure with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and $W_{\mathbf{NG}}$ as the class of weak equivalences. Moreover the adjunction $(i, \nu, \varphi)\colon \mathbf{NG} \to \mathbf{Top}$ is a Quillen equivalence.*

Corollary 2.4.5 (see §5)**.** *There is a finitely generated model structure on the category $\mathbf{NG}_*$ of pointed numerically generated spaces, with generating cofibrations I_+ and generating trivial cofibrations J_+. Moreover, the inclusion functor $i_*\colon \mathbf{NG}_* \to \mathbf{Top}_*$ is a Quilen equivalence.*

Moreover we introduce the definitive source for k-spaces and compactly generated spaces in [9]. We denote the full subcategory of **Top** consisting of k-spaces by **K**, and the full subcategory of **K** consisting compactly generated spaces by **T**. Let $W_{\mathbf{K}}$ and $W_{\mathbf{T}}$ be the classes of maps $f\colon X \to Y$ in **K** and **T** which is a weak equivalence in **Top**, respectively. Then we have the followings.

Theorem 2.4.6 ([7, 2.4.23] and [7, 2.4.25])**.** *The category* **K** *(resp.* **T***) has a finitely generated model structure with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and $W_{\mathbf{K}}$ (resp. $W_{\mathbf{T}}$) as the class of weak equivalences. Moreover* **K** *(resp.* **T***) and* **Top** *are a Quillen equivalent.*

Corollary 2.4.7 ([7, 2.4.24] and [7, 2.4.26])**.** *There is a finitely generated model structure on the category $\mathbf{K}_*$ (resp. $\mathbf{T}_*$) of pointed numerically generated spaces, with generating cofibrations I_+ and generating trivial cofibrations J_+. Moreover $\mathbf{K}_*$ (resp. $\mathbf{T}_*$) and $\mathbf{Top}_*$ are a Quillen equivalent.*

3 Diffeological spaces

3.1 Diffeological spaces

In this subsection we introduce the notion of diffeological spaces; for more details see [13], [8].

Definition 3.1.1 (diffeological spaces). Let X be a non empty set. Let D be a set of maps from open subsets of Euclidean spaces into X. A set D is a diffeology of X if and only if the following axioms are fulfilled;

D1.Covering Any constant parametrization $\mathbf{R}^n \to X$ belongs to D.

D2.Locality A parametrization $P\colon U \to X$ belongs to D if every point u of U has a neighborhood W such that $P|W\colon W \to X$ belongs to D.

D3.Smooth compatibility If $P\colon U \to X$ belongs to D, then so does the composite $P \circ Q\colon V \to X$ for any smooth map $Q\colon V \to U$ between open subsets of Euclidean spaces.

Formally, a diffeological space is a pair (X, D) where X is the underlying set and D its diffeology. Moreover we call each element of D a plot of X.

Example 3.1.2. Let $\mathbf{R}^n$ be an Euclidean space. Let $D_{\mathbf{R}^n}$ be a set of all smooth parametrizations from an open set Euclidean space to $\mathbf{R}^n$. Then it is clear that $D_{\mathbf{R}^n}$ satisfies the axioms of diffeology. We call a pair $(\mathbf{R}^n, D_{\mathbf{R}^n})$ the standard diffeological space and $D_{\mathbf{R}^n}$ the standard diffeology.

Proposition 3.1.3. *The first axiom* **D1** *of diffeology can be replaced by the following.* **D1′**. *The values of the elements of D cover X, that is,* $\cup_{p\in D}\mathrm{val}(P) = X$.

Proof. Let (X, D) be a diffeological space. It is clear that **D1** implies **D1′**. Conversely, for any $x \in X$, for any $n \in \mathbf{N}$, let $C_x\colon \mathbf{R}^n \to X$ be the constant map to x. Then there exists $P\colon U \to X \in D$ and $r \in U$ such that $P(r) = x$ by $\cup_{P\in D}\mathrm{val}(P) = X$. Let $F\colon \mathbf{R}^n \to U$ be the constant map to r. Since F is a smooth parametrization, the composite $F \circ P = C_x$ belongs to D by the third axiom. Therefore C_x belongs to D. $\square$

Lemma 3.1.4. *Let X be a set. Let D be a set of parametrizations. The second axiom* **D2** *of diffeology can be replaced by the following.* **D2′**. *For all integers n, for all families $\{P_\lambda\colon U_\lambda \to X\}_{\lambda\in\Lambda}$ of n-parametrizations such that $r \in U_\lambda \cap U_{\lambda'}$ implies $P_\lambda(r) = P_{\lambda'}(r)$, if all the P_λ belong to D, then the following parametrization P belong to D,*

$$P\colon U = \cup_{\lambda\in\Lambda} U_\lambda \to X, \text{ with } P(r) = P_\lambda(r) \text{ if } r \in U_\lambda.$$

Definition 3.1.5 (comparing diffeology). Let X be a set. Let D and D' be two diffeologies of X. If $D \subset D'$, we say that D is finer than D' and D' is coarser than D.

Definition 3.1.6 (discrete diffeology). Let X be a set. We say that a parametrization $P\colon U \to X$ is the locally constant if for any $r \in U$, there exists an open neighborhood V of r such that $P|V$ is constant. Then a set $D_\circ(X)$ consisting of all the locally constant parametrizations in X forms a diffeology called discrete diffeology. Moreover the discrete diffeology is the finest diffeology of X. The set X equipped with the discrete diffeology $D_\circ(X)$ will be denoted by $X_\circ$.

Definition 3.1.7 (coarse diffeology). Let X be a set. Then a set $D_\bullet(X)$ consisting of all the parametrizations in X forms a diffeology called the coarse diffeology. Moreover the coarse diffeology is the coarsest diffeology of X. The set X equipped with the discrete diffeology $D_\bullet(X)$ will be denoted by $X_\bullet$.

Definition 3.1.8 (smooth map). Let X and Y be two diffeological spaces. A map $f\colon X \to Y$ is smooth if and only if for any plot $P\colon U \to X$, the composite $f \circ P\colon U \to Y$ is a plot of Y.

Definition 3.1.9 (the category Diff). Since the composition of two smooth maps is a smooth, the class of diffeological spaces and smooth maps form a category **Diff**.

Proof. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two smooth maps between diffeological spaces. Since f is smooth, for any plot $P\colon U \to X$ of X, the composite $f \circ P\colon U \to Y$ is a plot of Y. Moreover $g \circ (f \circ P)\colon U \to Z$ is a plot of Z. Therefore the composite $g \circ f$ is smooth. □

Definition 3.1.10 (diffeomorphism). Let X and Y be two diffeological spaces. A map $f\colon X \to Y$ is a diffeomorphism if and only if the following conditions are satisfies.

1. A map f is bijective.
2. Both f and f^{-1} are smooth.

If there exists a diffeomorphism between X and Y, we say that X and Y are diffeomorphic. We will be dnoted by $X \cong Y$.

Definition 3.1.11 (subset diffeology). Let X be a diffeological space and D be its diffeology. Let A be a subset of X and let $j_A\colon A \to X$ be the inclusion. The subset A carries naturally the diffeology $j_A^*(D)$, pull back of the diffeology D by the inclusion j_A. This diffeology $j_A^*(D)$ is called the subset diffeology of A. The plots of the subset diffeology are just the plots of X with values in A, that is,

$$j_A^*(D) = \{P \in D | \mathrm{val}(P) \subset A\}.$$

Proposition 3.1.12. *Let X be a diffeological space. Let A and B be two subspaces of X such that $A \subset B \subset X$. It is equivalent to consider A as a subspace of B, regarded as a subspace of X, or to consider A as a subspace of X.*

Proof. Let $j_A\colon A \to X$, $i_B\colon B \to X$, and $i_{AB}\colon A \to B$ be the inclusions. Clearly, $j_A = i_B \circ i_{AB}$. Thus $j_A^*(D) = (i_B \circ i_{AB})^*(D) = i_{AB}^*(i_B^*(D))$. □

Definition 3.1.13 (sum diffeology). Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of diffeological spaces. Then there exists, on the direct sum

$$X = \coprod_{\lambda\in\Lambda} X_\lambda,$$

a finest diffeology such that the natural injecions $i_\lambda\colon X_\lambda \to \coprod_{\lambda\in\Lambda} X_\lambda$ are smooth, for each $\lambda \in \Lambda$. Then we have the following condition. A parametrization $P\colon U \to \coprod_{\lambda\in\Lambda} X_\lambda$ is a plot of $\coprod_{\lambda\in\Lambda} X_\lambda$ if and only if for any $r \in U$, there exists an open neighborhood V of r and a plot $Q_\lambda\colon V \to X_\lambda$ of X_λ such that $P|V = i_\lambda \circ Q_\lambda$.

Definition 3.1.14 (pushforward of diffeology). Let X be a diffeological space, and Y be a set. Let $f\colon X \to Y$ be a map. Then there exists a finest diffeology of Y such that the map of f is smooth. This diffeology is called the pushforward by f of the diffeology D of X, and will be denoted by $f_*(D)$. Now we have the following;

A parametrization $P\colon U \to Y$ is a plot for $f_*(D)$ if and only if for any $r \in U$, there exists an open neighborhood V of r such that, either $P|V$ is constant parametrization, or there exists a plot $Q\colon V \to X$ of X such that $P|V = f \circ Q$.

Proposition 3.1.15. *Let X be a diffeological space and D be its diffeology. Let Y and Z be two sets. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two maps. The pushforward by g of the pushforward by f of the diffeology D of X is equal to the pushforward of D by $g \circ f$, that is, $(g \circ f)_*(D) = g_*(f_*(D))$.*

Proof. We shall show that $(g \circ f)_*(D) \subset g_*(f_*(D))$. Let $P\colon U \to Z$ be a plot for $(g \circ f)_*(D)$. If P is the locally constant, $P \in g_*(f_*(D))$. If for any $r \in U$ there exists an open neighborhood V of r and a plot $Q\colon V \to X$ of X such that $(g \circ f) \circ Q = P|V$, the parametrization $P|V = g \circ (f \circ Q)$ is a plot for $g_*(f_*(D))$ since the parametrization $f \circ Q$ is a plot for $f_*(D)$. Thus P is a plot for $g_*(f_*(D))$ by the second axiom **D2**. Similarly, we can prove that $g_*(f_*(D)) \subset (g \circ f)_*(D)$. Therefore $(g \circ f)_*(D) = g_*(f_*(D))$. □

Definition 3.1.16 (Quotient diffeology). Let X and Y be two diffeological spaces. We say that a map $f\colon X \to Y$ is a subduction if the following conditions are satisfied.

1. The map f is surjective.
2. The diffeology D_Y of Y is the pushforward of the diffeology D of X, that is, $f_*(D) = D_Y$.

Now we call this diffeolofy D_Y the quotient diffeology.

Corollary 3.1.17. *Let X and Y be two diffeological spaces. Then $f\colon X \to Y$ is a subduction if and only if the following conditions are satisfied.*

1. *A map f is smooth surjective.*
2. *For any plot $P\colon U \to Y$ of Y, for any $r \in U$, there exists an open neighborhood V of r and a plot $Q\colon V \to Y$ of Y such that $P|V = f \circ Q$.*

Proposition 3.1.18. *Let X, Y and Z be three diffeological spaces, and let D_X, D_Y and D_Z be their diffeologies, respectively. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two subducions. Then $g \circ f\colon X \to Z$ is a subduction.*

Proof. By the definition of a subduction, we have

$$(g \circ f)_*(D_X) = g_*(f_*(D_X)) = g_*(D_Y) = D_Z.$$

Therefore $g \circ f$ is a subduction. □

Theorem 3.1.19. *Let X and Y be two diffeological spaces, and let $f\colon X \to Y$ be a smooth map. Then f is injective and subdction if and only if f is a diffeomorphism.*

Proof. Let f be injective. We shall show that $f^{-1}\colon Y \to X$ is smooth. Let $P\colon U \to Y$ be a plot of Y. By the definition of a subduction, for any $r \in U$, there exists an open neighborhood V of r and a plot $Q\colon V \to X$ of X such that $P|V = f \circ Q$. Then the composite

$$f^{-1} \circ P|V = f^{-1} \circ (f \circ Q) = Q$$

is a plot of X. Thus $f^{-1} \circ Q$ is a plot of X by the second axiom **D2**. Conversely, let f be a diffeomorphism. Let $P\colon U \to Y$ be a plot of Y. Then the composite $f^{-1} \circ Q$ is a plot of X. Moreover it is clear that $f \circ (f^{-1} \circ P) = P$. Therefore f is a subduction. It is clear that f is injective. □

Theorem 3.1.20. *Let X, Y and Z be three diffeological spaces. Let $\pi\colon X \to Y$ be a subduction. Then $f\colon Y \to Z$ is smooth if and only if $f \circ \pi\colon X \to Z$ is smooth.*

Proof. Let $f\colon Y \to Z$ be a smooth map. It is clear that the composite $f \circ \pi$ is smooth. Conversely, let $f \circ \pi\colon X \to Z$ be a smooth map. By the definition of a subduction, for any plot $P\colon U \to Y$ of Y, for any $r \in U$, there exists an open neighborhood V of r and a plot $Q\colon V \to X$ of X such that $\pi \circ Q = P|V$. Then the composite $f \circ (\pi \circ Q) = f \circ P|V$ is a plot of Z. Therefore $f \circ P$ is a plot of Z by the second axiom **D2**. □

Definition 3.1.21 (product diffeology). Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of diffeological spaces. Let us denoted by D_λ, $\lambda \in \Lambda$, their diffeologies. Then there exists on the product

$$X = \prod_{\lambda \in \Lambda} X_\lambda$$

a coarsest diffeology D such that, for each $\lambda \in \Lambda$, the projection

$$\pi_\lambda\colon \prod_{\lambda \in \Lambda} X_\lambda \to X_\lambda$$

is smooth. We call this diffeology $D_{\prod}$ the product diffeology of the family $\{D_\lambda\}_{\lambda \in \Lambda}$ and call a diffeological space $(X, D_{\prod})$ the diffeological product. Now we have the following;

A parametrization $P\colon U \to \prod_{\lambda \in \Lambda} X_\lambda$ is a plot of $\prod_{\lambda \in \Lambda} X_\lambda$ if and only if for any $\lambda \in \Lambda$, $\pi_\lambda \circ P\colon U \to X_\lambda$ is a plot of X_λ.

Proposition 3.1.22. *Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be the diffeological product of a family $\{X_\lambda\}_{\lambda \in \Lambda}$ of diffeological spaces. Then for any $\lambda \in \Lambda$, the projection $\pi_\lambda\colon X \to X_\lambda$ is a subduction.*

Proof. Let $P\colon U \to X_\lambda$ be a plot of X_λ. For every $\lambda' \neq \lambda \in \Lambda$, let x'_λ be a fixed point $X_{\lambda'}$. We define the inclusion $i_\lambda\colon X_\lambda \to X$ by

$$i_\lambda(x) = (x_1, \dots, x_{\lambda-1}, x, x_{\lambda+1}, \dots)$$

for any $x \in X$. Then $i_\lambda \circ P$ is a plot of X. It is clear that $P = \pi_\lambda \circ (i_\lambda \circ P)$. Thus π_λ is a subduction. □

Since the category **Diff** has equalizers, small products, coequalizers, and small coproducts, we have the following.

Theorem 3.1.23. *The category* **Diff** *is complete and cocomplete.*

3.2 Functional diffeological spaces

In this section we call show that the category **Diff** is a cartesian closed category.

Definition 3.2.1 (standard functional diffeology). Let $C^\infty(X, Y)$ be a set of all smooth maps between two diffeological spaces X and Y. Let ev be the evaluation map defined by

$$\mathrm{ev}\colon C^\infty(X, Y) \times X \to Y, \text{ and } \mathrm{ev}(f, x) = f(x).$$

We shall call functional diffeology any diffeology of $C^\infty(X, Y)$ such that the map ev is smooth. Note that the discrete diffeology is a functional diffeology. But there exists a coarsest functional diffeology D_{C^∞} on $C^\infty(X, Y)$, we shall call it the standard functional diffeology. Actually, the plots of this diffeology are explicitely given by the following condition;

A parametrization $P\colon U \to C^\infty(X, Y)$ is a plot for D_{C^∞} if and only if for any plot $Q\colon V \to X$ of X, the composite

$$U \times V \xrightarrow{P\times Q} C^\infty(X, Y) \times X \xrightarrow{\mathrm{ev}} Y$$

is a plot of Y.

The set $C^\infty(X, Y)$ equipped with the standard functional diffeology is called the standard functional diffeological space.

Lemma 3.2.2. *Let X, Y, and Z be three diffeological spaces and let $C^\infty(Y, Z)$ be the standard functional diffeological space. Let $f\colon X \times Y \to Z$ be a smooth map. We define $\tau(f)\colon X \to C^\infty(Y, Z)$ by for any $x \in X$ and $y \in y$, $\tau(f)(x)(y) = f(x, y)$. Then $\tau(f)$ is smooth.*

Proof. We shall show that $\tau(f)(x)\colon Y \to Z$ is smooth for any $x \in X$. Let $P\colon U \to Y$ be a plot of Y and let $C_x\colon U \to X$ be the constant map from the domain of P to x. Then $C_x \times P\colon U \to X \times Y$ is a plot of $X \times Y$. Thus $\tau(f)(x)$ is smooth since $f \circ (C_x \times P) = \tau(f)(x) \circ P$ is a plot of Z. Hence $\tau(f)(x) \in C^\infty(Y, Z)$. We shall show that $\tau(f)\colon X \to C^\infty(Y, Z)$ is smooth. Let $P\colon U \to X$ be a plot of X. For any plot $Q\colon V \to Y$ of Y, the composite

$$U \times V \xrightarrow{\tau(f)\circ P\times Q} C^\infty(Y, Z) \xrightarrow{\text{ev}} Z$$

should just be a plot of Z. Then we have

$$\text{ev} \circ ((\tau(f) \circ P) \times Q) = f \circ (P \times Q).$$

Therefore $\tau(f)$ is smooth since the composite $\text{ev} \circ ((\tau(f) \circ P) \times Q)$ is a plot of Z. $\square$

By lemma 3.2.2, we can define a map

$$\tau\colon C^\infty(X \times Y, Z) \to C^\infty(X, C^\infty(Y, Z)) \text{ with } \tau(f)(x)(y) = f(x, y).$$

Lemma 3.2.3. *Let X and Y, and Z be three diffeological spaces. Then a map*

$$\tau\colon C^\infty(X \times Y, Z) \to C^\infty(X, C^\infty(Y, Z))$$

is smooth.

Proof. Let $P\colon U \to C^\infty(X \times Y, Z)$ be a plot of $C^\infty(X \times Y, Z)$. By the definition of standard functional diffeology, the composite $\tau \circ P\colon U \to C^\infty(X, C^\infty(Y, Z))$ is a plot of $C^\infty(X, C^\infty(Y, Z))$ if and only if the following condition is satisfied. For any plot $Q\colon V \to X$, for any plot $L\colon W \to Y$, the composite

$$U \times V \times W \xrightarrow{(\text{ev}\circ(\tau(P))\times Q)\times L} C^\infty(Y, Z) \times Y \xrightarrow{\text{ev}} Z$$

is a plot of Z. Then we have

$$\text{ev} \circ ((\text{ev} \circ (\tau(P) \times Q)) \times L) = \text{ev} \circ (P \times Q \times L).$$

By the definition of standard functional diffeology, the composite $\text{ev} \circ (P \times Q \times L)$ is a plot of Z. Therefore τ is smooth. $\square$

Lemma 3.2.4. *Let X, Y, and Z be three diffeological spaces and let $C^\infty(Y, Z)$ be the standard functional diffeological space. Let $g\colon X \to C^\infty(Y, Z)$ be a smooth map. We define $\tau'(g)\colon X \times Y \to Z$ by for any $x \in X$ and $y \in Y$, $\tau'(g)(x, y) = g(x)(y)$. Then $\tau'(g)$ is smooth.*

Proof. For any plot $P\colon U \to X \times Y$ of $X \times Y$, $P(r) = (P_1(r), P_2(r))$. Then $P_1\colon U \to X$ and $P_2\colon U \to Y$ are plots of X and Y, respectively. Now we have

$$\tau'(g) \circ P = \mathrm{ev} \circ ((g \circ P_1) \times P_2).$$

By the definition of standard functional diffeology, the composite

$$U \xrightarrow{(g \circ P_1) \times P_2} C^\infty(Y, Z) \times Y \xrightarrow{\mathrm{ev}} Z$$

is a plot of Z. Therefore $\tau'(g)$ is smooth since $\tau'(g) \circ P$ is a plot of Z. □

By lemma 3.2.4, we can define a map

$$\tau'\colon C^\infty(X, C^\infty(Y, Z)) \to C^\infty(X \times Y, Z) \text{ with } \tau'(g)(x)(y) = g(x, y).$$

Lemma 3.2.5. *Let X, Y and Z be three diffeological spaces. Then a map*

$$\tau'\colon C^\infty(X, C^\infty(Y, Z)) \to C^\infty(X \times Y, Z)$$

is smooth.

Proof. Let $P\colon U \to C^\infty(X, C^\infty(Y, Z))$ be a plot of $C^\infty(X, C^\infty(Y, Z))$. The composite $\tau' \circ P\colon U \to C^\infty(X \times Y, Z)$ is a plot of $C^\infty(X \times Y, Z)$ if and only if the following condition is satisfied. For any plot $Q\colon V \to X \times Y$ of $X \times Y$, the composite

$$U \times V \xrightarrow{(\tau' \circ P) \times Q} C^\infty(X \times Y, Z) \times X \times Y \xrightarrow{\mathrm{ev}} Z$$

is a plot of Z. Now, let $Q(r) = (Q_1(r), Q_2(r))$ for any r in V. Then $Q_1\colon V \to X$ and $Q_2\colon V \to Y$ are plots of X and Y, respectively. By the definition of standard functional diffeology, the composite

$$\begin{aligned} U \times V \xrightarrow{P \times Q_1 \times Q_2} C^\infty(X, C^\infty(Y, Z)) \times X \times Y &\xrightarrow{\mathrm{ev} \times 1_Y} \\ C^\infty(Y, Z) \times Y &\xrightarrow{\mathrm{ev}} Z \end{aligned}$$

is a plot of Z. Now we have

$$\mathrm{ev} \circ ((\tau' \circ P) \times Q) = \mathrm{ev} \circ (\mathrm{ev} \times 1_Y) \circ (P \times Q_1 \times Q_2).$$

Therefore τ' is smooth since $\tau' \circ P$ is a plot of $C^\infty(X \times Y, Z)$. □

It is clear that the composite $\tau' \circ \tau$ and $\tau \circ \tau'$ are the identity map on $C^\infty(X \times Y, Z)$ and $C^\infty(X, C^\infty(X, C^\infty(Y, Z))$, respectively. Thus we have the following.

Theorem 3.2.6 (**Cartesian closed).** *Let X, Y and Z be three diffeological spaces. Then a smooth map*

$$\tau \colon C^\infty(X \times Y, Z) \to C^\infty(X, C^\infty(Y, Z))$$

is the diffeomorphism. Therefore the category **Diff** *is a cartesian closed category.*

3.3 D-Topology and T-Diffeology

In this subsection we introduce the notions of D-topology and T-diffeology. Let **Diff** and **Top** be the category diffeoloical and topological spaces, respectively.

Definition 3.3.1 (**D-Topology).** Let X be a diffeological space and let D be its diffeology. There exists, on X, a finest topology such that the plots of X are continuous. This topology is called the D-topology of X, and it will be denoted by $T(D)$. The open sets for the D-topology are called D-open sets, they are characterized by the following property;

A subset A of X is D-open if and only if for any plot $P \colon U \to X$ of X, $P^{-1}(A)$ is an open set of U.

A set X equipped with D-topology $T(D)$ is called D-topological space, and it will be denoted by TX. Similarly, a subset F of X is D-closed if and only if for any plot $P \colon U \to X$ of X, $P^{-1}(F)$ is a closed set of U. Let B be a subset of X. there is the smallest D-closed set containing B, this set will be called the D-closure set of B and will be denoted by $\overline{B}$.

Proof. Let X be a diffeological space and let D be its diffeology. We shall show that the D-topology $T(D)$ of X satisfies the axiom of topology. Let $P \colon U \to X$ be a plot of X. The sets X and $\emptyset$ are D-open since $P^{-1}(X)$ and $P^{-1}(\emptyset)$ are open in U. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be any finite family of D-open sets. A set $P^{-1}(\cap_{\lambda \in \Lambda} A_\lambda) = \cap_{\lambda \in \Lambda} P^{-1}(A_\lambda)$ is open in U since $P^{-1}(A_\lambda)$ is open in U every $\lambda \in \Lambda$. Thus $\cap_{\lambda \in \Lambda} A_\lambda$ is D-open. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be any family of D-open sets. A set $P^{-1}(\cup_{\lambda \in \Lambda} A_\lambda) = \cup_{\lambda \in \Lambda} P^{-1}(A_\lambda)$ is open in U since $P^{-1}(A_\lambda)$ is open in U every $\lambda \in \Lambda$. Thus $\cup_{\lambda \in \Lambda} A_\lambda$ is D-open.

Let O be another topology for which the plots of X are continuous. Let A be an element of $T(D)$. It is clear that A is an element of O since $P^{-1}(A)$ is open in U, for every plot $P \colon U \to X$ of X. Thus $T(D) \subset O$. $\square$

Proposition 3.3.2. *Let X and Y be two diffeological spaces. Let $f\colon X \to Y$ be a smooth map. Then a map $T(f)\colon TX \to TY$ is continuous. This map $T(f)$ is called D-continuous.*

Proof. Let A be an D-open set of Y. For any plot $P\colon U \to X$ of X, $f \circ P\colon U \to Y$ is a plot of Y. Since $(f \circ P)^{-1}(A) = P^{-1}f^{-1}(A)$ is open in U, $f^{-1}(A)$ is D-open. Therefore f is D-continuous. □

Hence there is a functor $T\colon \mathbf{Diff} \to \mathbf{Top}$ which maps a diffeological space X to the topological space TX.

Proposition 3.3.3. *Let X and Y be two diffeological spaces. Let A and B be two D-open sets of X such that $X = A \cup B$. Let $f\colon A \to Y$ and $g\colon B \to Y$ be two smooth maps such that $f(x) = g(x)$ for any x in $A \cap B$. If we define $h\colon X \to Y$ by for any x in X*

$$h(x) = \begin{cases} f(x) & x \in A \\ \\ g(x) & x \in B, \end{cases}$$

h is smooth.

Proof. It is clear that h is well-defined. For any plot $P\colon U \to X$ of X, $U_A = P^{-1}(A)$ and $U_B = P^{-1}(B)$ are open in U. Since $P|U_A\colon U_A \to A$ and $P|U_B\colon U_B \to B$ are plots of A and B, respectively, $h \circ P|U_A = f \circ P|U_A$ and $h \circ P|U_B = g \circ P|U_B$ are plots of Y. Therefore h is smooth by Lemma 3.1.4. □

Definition 3.3.4. Let X be a topological space and let O be its topology. Let $D(O)$ be a set of all continuous maps from open subsets of Euclidean spaces into X. It is clear that $D(O)$ satisfies the axioms of diffeology. The set $D(O)$ is called T-diffeology of X. A set X equipped with T-diffeology $D(O)$ is called T-diffeological space and it will be denoted by DX.

Proposition 3.3.5. *Let X and Y be two topological spaces. Let $f\colon X \to Y$ be a continuous map. Then $D(f)\colon DX \to DY$ is smooth. This map $D(f)$ is called T-smooth.*

Proof. Let $P\colon U \to X$ be a plot of DX. Since P and f are continuous, the composite $f \circ P$ is continuous. Thus $f \circ P$ is a plot of DY. Hence $D(f)$ is T-smooth. □

Therefore there is a functor $D\colon \mathbf{Top} \to \mathbf{Diff}$ which maps a topological space X to the diffeological space DX.

Lemma 3.3.6. *Let X be a diffeological space and let Y be a topological space. Then we have the following conditions;*

1. *The identity map $1_X: X \to DTX$ is smooth.*
2. *The identity map $1_Y: TDY \to Y$ is continuous.*

Proof. Let $P: U \to X$ be a plot of X. By the definition of D-topology, P is D-continuous. Thus $1_X \circ P$ is a plot of DTX. Hence 1_X is smooth. Let A be an open set of Y. For any continuous map $\sigma: U \to X$, $\sigma^{-1}(A)$ is open in U. Thus A is D-open in TDX. Therefore 1_Y is continuous. □

Theorem 3.3.7. *The functor T:* **Diff** $\to$ **Top** *is a left adjoint to D:* **Top** $\to$ **Diff**.

Proof. Let X and Y be a diffeological space and a topological space, respectively. By Lemma 3.3.6, the identity map $1_X: X \to DTX$ is smooth and the identity map $1_Y: TDY \to Y$ is continuous. We can define

$$\phi: \mathrm{Hom}_{\mathbf{Top}}(TX, Y) \to \mathrm{Hom}_{\mathbf{Diff}}(X, DY)$$

by letting $\phi(f) = D(f) \circ 1_X$ for every $f: TX \to Y \in \mathrm{Hom}_{\mathbf{Top}}(TX, Y)$. Similarly, we can define

$$\varphi: \mathrm{Hom}_{\mathbf{Diff}}(X, DY) \to \mathrm{Hom}_{\mathbf{Top}}(TX, Y)$$

by letting $\varphi(g) = 1_Y \circ g$ for every $g: X \to DY \in \mathrm{Hom}_{\mathbf{Diff}}(X, DY)$. Then we have $\varphi\phi = id$ and $\phi\varphi = id$. Therefore the functor T is a left adjoint to D. □

3.4 Weak diffeology

In this section we define the weak diffeology. Moreover we shall show the basic properties of the weak diffeology.

Definition 3.4.1 (weak diffeology). Let X be a diffeological space. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a cover of subspaces of X. Let $i_\lambda: X_\lambda \to X$ be the inclusion map for each $\lambda \in \Lambda$. We say that X has the weak diffeology with respect to $\{X_\lambda\}_{\lambda \in \Lambda}$ if the following condition is satisfied;
A parametrization $P: U \to X$ is a plot of X if and only if for any $r \in U$, there exists an open neighborhood V and a plot $Q_\lambda: V \to X_\lambda$ of X_λ such that $P|V = i_\lambda \circ Q_\lambda$.

Proposition 3.4.2. *Let X be a diffeological space. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a cover of D-open subspaces of X. Then X has the weak diffeology with respect to $\{X_\lambda\}_{\lambda\in\Lambda}$.*

Proof. Let $P\colon U \to X$ be a plot of X. For any $r \in U$ there exists $\lambda \in \Lambda$ such that $P(r) \in X_\lambda$. Since $p^{-1}(X_\lambda) = V_\lambda$ is open, $P|V_\lambda\colon V_\lambda \to X_\lambda$ is a plot of X_λ. It is clear that $i_\lambda \circ P|V_\lambda = P|V_\lambda$, where $i_\lambda\colon X_\lambda \to X$ is the inclusion. □

Proposition 3.4.3. *Let X be a diffeological space and let A be its subspace. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a cover of X. Let $A_\lambda = X_\lambda \cap A$ for each $\lambda \in \Lambda$. If X has the weak diffeology with respect to $\{X_\lambda\}_{\lambda\in\Lambda}$, then A has the weak diffeology with respect to $\{A_\lambda\}_{\lambda\in\Lambda}$.*

Proof. Let $P\colon U \to A$ be a plot of A. Let $j_A\colon A \to X$ be the inclusion map. By the definition of subset diffeology, $j_A \circ P$ is a plot of X. For each $\lambda \in \Lambda$, let $i_\lambda\colon X_\lambda \to X$ be the inclusion map. Since X has the weak diffeology with respect to $\{X_\lambda\}_{\lambda\in\Lambda}$, for every $r \in U$ there exists an open neighborhood V of r and a plot $Q\colon V \to X_\lambda$, for some $\lambda \in \Lambda$, such that $j_A \circ P|V = i_\lambda \circ Q$ holds. But we have $i_\lambda = j_A \circ j'_\lambda$, where $j'_\lambda\colon A_\lambda \to A$ is the inclusion map. Thus we have $P|V = j'_\lambda \circ Q$, implying that A has the weak diffeology with respect to $\{A_\lambda\}$. □

Theorem 3.4.4 (product of weak diffeology). *Let X and Y be two diffeological spaces. If X and Y have the weak diffeology with respect to $\{X_\lambda\}_{\lambda\in\Lambda}$ and $\{Y_{\lambda'}\}_{\lambda'\in\Lambda'}$, respectively, then $X \times Y$ has the weak diffeology with respect to $\{X_\lambda \times Y_{\lambda'}\}_{(\lambda,\lambda')\in\Lambda\times\Lambda'}$.*

Proof. Let $P\colon U \to X \times Y$ be a plot of $X \times Y$. For any $r \in U$, let us write $P(r) = (P_X(r), P_Y(r))$. By the definition of product diffeology, $P_X : U \to X$ and $P_Y : U \to Y$ are the plots of X and Y, respectively. Since X has the weak diffeology with respect to $\{X_\lambda\}_{\lambda\in\Lambda}$, there exist an open neighborhood V_X of r and a plot $Q_X : V_X \to X_\lambda$, for some $\lambda \in \Lambda$, such that $P_X|V_X = i_\lambda \circ Q_X$ holds, where $i_\lambda\colon X_\lambda \to X$ is the inclusion map. Similarly, there exist an open neighborhood V_Y of r and a plot $Q_Y : V_Y \to Y_{\lambda'}$ for some $\lambda' \in \Lambda'$, such that $P_Y|V_Y = j_{\lambda'} \circ Q_Y$ holds, where $j_{\lambda'} : Y_{\lambda'} \to Y$ is the inclusion map.

Let $W = V_X \cap V_Y$ and define $Q\colon W \to X_\lambda \times Y_{\lambda'}$ by $Q(r) = (Q_X(r), Q_Y(r))$. Then Q is a plot of $X_\lambda \times Y_{\lambda'}$, and we have $P|W = (i_\lambda \times j_{\lambda'}) \circ Q$. This shows that $X \times Y$ has the weak diffeology with respect to the covering $\{X_\lambda \times Y_{\lambda'}\}_{(\lambda,\lambda')\in\Lambda\times\Lambda'}$. □

Corollary 3.4.5. *Let X and Y be two diffeological spaces. If X has the weak diffeology with respect to $\{X_\lambda\}_{\lambda\in\Lambda}$, then $X \times Y$ has the weak diffeology with respect to $\{X_\lambda \times Y\}_{\lambda\in\Lambda}$.*

Theorem 3.4.6. *Let X, Y and A be three diffeological spaces. Suppose that X has the weak diffeology with respect to $\{X_\lambda\}_{\lambda\in\Lambda}$. Then we have the following conditions;*

1. *A map $f\colon X \to Y$ is smooth if and only if $f|X_\lambda\colon X_\lambda \to Y$ is smooth for every λ in Λ.*
2. *A map $f\colon X \times A \to Y$ is smooth if and only if $f|X_\lambda \times A\colon X_\lambda \times A \to Y$ is smooth for every λ in Λ.*

Proof. We show that *1* holds. It is obvious that if $f\colon X \to Y$ is smooth then so is $f|X_\lambda\colon X_\lambda \to Y$ for every $\lambda \in \Lambda$. Conversely, suppose that $f|X_\lambda\colon X_\lambda \to Y$ is smooth for every $\lambda \in \Lambda$, where X has the weak diffeology with respect to $\{X_\lambda\}_{\lambda\in\Lambda}$. Let $P\colon U \to X$ be a plot of X. Then for any $r \in U$ there exist an open neighborhood V of r and a plot $Q\colon V \to X_\lambda$ of X_λ such that $P|V = i_\lambda \circ Q$ holds, where $i_\lambda\colon X_\lambda \to X$ is the inclusion map. Thus $f \circ P|V = f \circ i_\lambda \circ Q$ is a plot of Y. By the second axiom **D2**, $f \circ P$ is the plot of Y. Thus $f\colon X \to Y$ is a smooth map.

The statement *2* follows from *1* and Corollary 3.4.5. □

4 Homotopy theory of diffeological spaces

4.1 Diffeological homotopy groups

In this subsection we introduce the notion of homotopy groups for diffeological spaces.

Let $I = [0, 1]$ be the unit interval. Following [9], let $\gamma\colon \mathbf{R} \to I$ be the smooth bump function given by

$$\gamma(t) = \begin{cases} 0, & t \leq 0 \\ \exp - (\frac{1}{t}), & t > 0 \end{cases}$$

and let $\lambda\colon \mathbf{R} \to I$ be the smashing function given by

$$\lambda(t) = \frac{\gamma(t)}{\gamma(t) + \gamma(1-t)}, \ t \in \mathbf{R}.$$

The map λ is smooth and satisfies $\lambda(t) = 0$ for $t \leq o,\ \lambda(t) = 1$ for $1 \leq t$.

We call the quotient diffeology $\lambda_*(D_{\mathbf{R}})$ of I the smashing diffeology. A diffeological space $(I, \lambda_*(D_{\mathbf{R}}))$ will be denoted by $\tilde{I}$. if i is the inclusion of I into $\mathbf{R}$ then the diffeological space $(I,\ i^*(D_{\mathbf{R}}))$, equipped with the subspace diffeology of $\mathbf{R}$, is simply denoted by I.

More generally, for every real number with $0 \leq \epsilon < \dfrac{1}{2}$, let $f_\epsilon\colon \mathbf{R} \to \mathbf{R}$ be a diffeomorphism given by

$$f_\epsilon(t) = \frac{1}{1-2\epsilon}(t - \epsilon),\ t \in \mathbf{R},$$

and let $\lambda_\epsilon\colon \mathbf{R} \to I$ be the composite $\lambda \circ f_\epsilon$. Then $\lambda_\epsilon(t) = 0$ for $t \leq \epsilon$, $\lambda_\epsilon(t) = 1$ for $1 - \epsilon \leq t$, and is called the ϵ-smashing function. Clearly, the quotient diffeological space $\tilde{I}_\epsilon = (I, \lambda_{\epsilon *}(D_{\mathbf{R}}))$ is diffeomorphic to $\tilde{I}$.

Definition 4.1.1. Let X and Y be diffeoloical spaces. For given smooth maps $f_0,\ f_1\colon X \to Y$, a homotopy between f_0 and f_1 is a smooth map $F\colon X \times \tilde{I} \to Y$ satisfying $F(x, 0) = f_0(x),\ F(x, 1) = f_1(x)$. We say that f_0 and f_1 are homotopic and write

$$f_0 \simeq f_1\colon X \to Y$$

if there is a homotopy F between f_0 and f_1. A map $f\colon X \to Y$ is called a homotopy equivalence if there exists a smooth map $g\colon Y \to X$ satisfying

$$g \circ f \simeq 1_X\colon X \to X,\ f \circ g \simeq 1_Y\colon Y \to Y.$$

We say that X and Y are homotopy equivalent, and write $X \simeq Y$, if there exists a homotopy equivalence $f\colon X \to Y$.

Although I and $\tilde{I}$ are not diffeomorphic, we have the following proposition.

Proposition 4.1.2. *The diffeological spaces* $\mathbf{R}$, I *and* $\tilde{I}$ *are homotopy equivalent with each other.*

Proof. Let $i\colon I \to \mathbf{R}$ be the inclusion of I into $\mathbf{R}$, and $j\colon \tilde{I} \to I$ the identity regarded as a smooth map from $\tilde{I}$ to I. It suffices to show that the composites $j \circ \lambda\colon \mathbf{R} \to I$ and $\lambda \circ i\colon I \to \tilde{I}$ are homotopy inverses to i and j, respectively.

To see this, let $F\colon \mathbf{R} \times \tilde{I} \to \mathbf{R}$ be the smooth map defined by

$$F(t, s) = (1 - s)\lambda(t) + st.$$

Then we have $F(t, 0) = \lambda(t) = i \circ j \circ \lambda(t)$ and $F(t, 1) = t = 1_I(t)$. Hence F determines a homotopy $i \circ j \circ \lambda \simeq 1_{\mathbf{R}}$. On the other hand, let $G\colon I \times \tilde{I} \to I$ be the restriction of F to $I \times \tilde{I}$. Then G determines a homotopy $j \circ \lambda \circ i \simeq 1_I$. Therefore, $j \circ \lambda$ is a homotopy inverse to $i\colon I \to \mathbf{R}$.

Similarly, let $\tilde{F}\colon \tilde{I} \times \tilde{I} \to \tilde{I}$ be the unique map such that the diagram

$$\begin{array}{ccc} \mathbf{R} \times \tilde{I} & \xrightarrow{F} & \mathbf{R} \\ {\scriptstyle \lambda \times 1_{\tilde{I}}} \downarrow & & \downarrow {\scriptstyle \lambda} \\ \tilde{I} \times \tilde{I} & \xrightarrow[\tilde{F}]{} & \tilde{I} \end{array}$$

commutes. Since $\lambda \times 1_{\tilde{I}}\colon \mathbf{R} \times \tilde{I} \to \tilde{I} \times \tilde{I}$ is a subduction, the map $\tilde{F}$ is smooth and so determines a homotopy $\lambda \circ i \circ j \simeq 1_{\tilde{I}}$. Thus $\lambda \circ i$ is a homotopy inverse to $j\colon \tilde{I} \to I$, proving the proposition. □

A smooth map $\gamma\colon \tilde{I} \to X$ and $\delta\colon \tilde{I} \to X$ are paths satisfying $\gamma(1) = \delta(0)$, then for any ϵ, with $0 < \epsilon < \dfrac{1}{2}$, we denote by $\gamma *_\epsilon \delta\colon \tilde{I} \to X$ the map given by the formula

$$\gamma *_\epsilon \delta(t) = \begin{cases} \gamma \circ \lambda_\epsilon(2t), & 0 \leq t < \frac{1+\epsilon}{2} \\ \delta \circ \lambda_\epsilon(2t - 1), & \frac{1-\epsilon}{2} < t \leq 1, \end{cases}$$

where λ_ϵ is the ϵ-smashing function. One easily checks that $f *_\epsilon g$ is smooth at every $t \in \tilde{I}$ and hence determines a path in X. Moreover $f *_\epsilon g$ and $f *_{\epsilon'} g$ are homotopic for any ϵ and ϵ'.

For given $x,\ y \in X$, let us write $x \sim y$ if there exists a path $\gamma\colon \tilde{I} \to X$ from x to y. Then the relation $x \sim y$ is an equivalence relation, because if γ and γ' are paths joining x to y and y to z, respectively, then $\gamma *_\epsilon \gamma'$ is a path joining x to z. The equivalence class of a point $x \in X$ is called a path component of X and will be denoted by $[x]$. We denote by $\pi_0(X)$ the set of the components of X, that is ,

$$\pi_0(X) = \{[x] | x \in X\}.$$

A path $\gamma\colon \tilde{I} \to X$ such that $\gamma(0) = \gamma(1) = x$ is called a loop based at $x \in X$. The set of all loops based at $x \in X$ is denoted by $\mathrm{Loops}(X, x)$, and is regarded as a diffeological subspace of $C^\infty(\tilde{I}, X)$ based at the constant path C_x. More generally, for every $n \geq 0$ we define a diffeological space $\mathrm{Loops}_n(X, x)$, with basepoint x_n, by the following recursive formula:

$$\mathrm{Loops}_n(X, x) = \mathrm{Loops}(\mathrm{Loops}_{n-1}(X, x), x_{n-1}),$$

where $\mathrm{Loops}_0(X, x) = X$ and $x_0 = x$. In particular, we have $\mathrm{Loops}_1(X, x) = \mathrm{Loops}(X, x)$. By the cartesian closedness of **Diff** (cf.Theorem 2.1), we may identity $\mathrm{Loops}_n(X, x)$ with the subspace of $C^\infty(\tilde{I}^n, X)$ consisting of those loops $\gamma\colon \tilde{I}^n \to X$ satisfying $\gamma(\partial\tilde{I}^n) = x$.

Given a pointed diffeological space (X, x), we denote by $\pi_0(X, x)$ the pointed set $\pi_0(X)$ based at the component $[x]$ of x, i.e.

$$\pi_0(X, x) = (\pi_0(X), [x]).$$

For every $n \geq 0$, the n-th homotopy set of (X, x) is defined to be the pointed set

$$\pi_n(X, x) = \pi_0(\mathrm{Loops}_n(X, x), C_{x_n}).$$

Clearly, every smooth map $f\colon X \to Y$ induces a basepoint preserving map $f_*\colon \pi_n(X, x) \to \pi_n(Y, f(x))$ for all $n \geq 0$.

Definition 4.1.3. A smooth map $p\colon X \to Y$ is called a weak homotopy equivalence if the induced map $p_*\colon \pi_n(X, x) \to \pi_n(Y, p(x))$ is a bijection for every $x \in X$ and $n \geq 0$.

4.2 Diffeological CW complexes

In this subsection we introduce the notion of diffeological CW complex. Let X and Y be two diffeological space, and let A be a subspace of X. Let $f\colon A \to Y$ be a smooth map. Then we define a diffeological space $X \cup_f Y$ to be the quotient $X \coprod Y/_\sim$, where $\sim$ is the equivalence relation generated by the relation $x \sim f(x),\ x \in A$

Definition 4.2.1. Let X be a diffeological space, and let $\{e_\lambda\}_{\lambda\in\Lambda}$ be a family of subsets of X. Then $(X, \{e_\lambda\}_{\lambda\in\Lambda})$ is called a diffeological cell complex if the following conditions hold:

1. We have $X = \coprod_{\lambda\in\Lambda} e_\lambda$.

2. For each e_λ, there exists a unique integer $n_\lambda \geq 0$, called the dimension of e_λ, and a subduction map $\Phi_\lambda \tilde{I}^{n_\lambda} \to \bar{e}_\lambda$, called a characteristic map of e_λ, which restricts to a diffeomorphism $\Phi_\lambda|\mathrm{Int}\tilde{I} \to e_\lambda$ and takes $\partial\tilde{I}^{n_\lambda}$ into $\bar{e}_\lambda \setminus e_\lambda$.

3. Let $X^q = \cup_{n_\mu \leq q} e_\mu$. Then we have $\bar{e}_\lambda \setminus e_\lambda$.

Let us call e_λ an n_λ-cell of X and $n_\lambda = \mathrm{dim} e_\lambda$. Thus X^q is the union of all n_λ-cells with $n_\lambda \geq q$ and is called the q-skeleton of X. A diifeological cell complex $(A, \{e_{\lambda'}\}_{\lambda'\in\Lambda'})$ is called a subcomplex of $(X, \{e_\lambda\}_{\lambda\in\Lambda})$ if A is a subspace of X and $\{e_{\lambda'}\}_{\lambda'\in\Lambda'}$ is a subfamily of $\{e_\lambda\}_{\lambda\in\Lambda}$. We simply denote $(X, \{e_\lambda\}_{\lambda\in\Lambda})$ by X when there is no fear of confusion.

Definition 4.2.2. A diffeological cell complex $(X, \{e_\lambda\}_{\lambda\in\Lambda})$ is called a CW complex if the following conditions (C) and (W) are satisfied:

(C) For any x in X, there is a finite subcomplex A such that x in A.

(W) X has the weak diffeology with respect to $\{\bar{e}_\lambda\}_{\lambda\in\Lambda}$.

If Λ is finite then $(X, \{e_\lambda\}_{\lambda\in\Lambda})$ is called a finite CW complex.

The next proposition is a consequence of Lemma 3.4.3.

Proposition 4.2.3. *Let X be a diffeological cell complex. If X is a CW complex, then so is its every subcomplex A.*

Proposition 4.2.4. *If $(X, \{e_\lambda\}_{\lambda\in\Lambda})$ and $(Y, \{e'\}_{\lambda'\in\Lambda'})$ are CW complexes then $(X \times Y, \{e_\lambda \times e'_{\lambda'}\}_{(\lambda,\lambda')\in\Lambda\times\Lambda'})$ is a CW complex.*

Proof. Let $\Phi_\lambda \colon \tilde{I}^{n_\lambda} \to \bar{e}_\lambda$ and $\Phi'_{\lambda'} \colon \tilde{I}^{n_{\lambda'}} \to \bar{e'}_{\lambda'}$ be characteristic maps of e_λ and $e'_{\lambda'}$, respectively. Then $(X \times Y, \{e_\lambda \times e'_{\lambda'}\}_{(\lambda,\lambda')\in\Lambda\times\Lambda'})$ is a diffeological cell complex with respect to characteristic maps $\Phi_\lambda \times \Phi'_{\lambda'} \colon \tilde{I}^{n_\lambda} \times \tilde{I}^{n_{\lambda'}} \to \bar{e}_\lambda \times \bar{e}'_{\lambda'}$. Clearly, the condition (C) is satisfied. On the other hand, the condition (W) follows from Theorem 3.4.4, because $X \times Y$ has the weak diffeology with respect to the family of closed cells $\overline{e_\lambda \times e'_{\lambda'}} = \bar{e}_\lambda \times \bar{e}'_{\lambda'}$, $(\lambda, \lambda') \in \Lambda \times \Lambda'$. Thus $X \times Y$ is a CW complex. □

Proposition 4.2.5. *Let $(X, \{e_\lambda\}_{\lambda\in\Lambda})$ be a diffeological CW complex, and A a subcomplex of X. Let $\bar{X}^q = X^q \cup A$ for $q \geq 0$, and let $\bar{X}^{-1} = A$. Then there is a diffeomorphism*

$$\bar{X}^q \cong \bar{X}^{q-1} \bigcup_{\coprod_\mu \Phi_\mu|\partial\tilde{I}^q_\mu} (\coprod_{\mu\in\Lambda_q} \tilde{I}^q_\mu)$$

where $\Lambda_q = \{\mu \in \Lambda | \dim e_\mu = q,\ e_\mu \not\subset A\}$.

Proof. Let $\Phi_\mu \colon \tilde{I}^q_\mu \to \bar{e}_\mu$ be a characteristic map for e_μ with $\mu \in \Lambda_q$, and define a surjection $f \colon \bar{X}^{q-1} \coprod(\coprod_{\mu\in\Lambda_q} \tilde{I}^q_\mu) \to \bar{X}^q$ by

$$f(x) = \begin{cases} x, & x \in \bar{X}^{q-1} \\ \coprod_{\mu\in\Lambda_q} \Phi_\mu(x), & x \in \coprod_{\mu\in\Lambda_q} \tilde{I}^q_\mu \end{cases}$$

Then f factors through a bijection $\tilde{f} \colon \bar{X}^{q-1} \bigcup_{\coprod_\mu \Phi_\mu|\partial\tilde{q}_\mu} (\coprod_{\mu\in\Lambda_q} \tilde{I}^q_\mu) \to \bar{X}^q$. To see that $\tilde{f}$ is a diffeomorphism we need only show that f is a subduction (cf Theorem 3.1.19 and 3.1.20). Let $P \colon U \to \bar{X}^q$ be a plot of $\bar{X}^q$ and let r be a point of U. By the property of weak diffeology, there exist an open neighborhood V of r and a plot $Q_\lambda \colon V \to \bar{e}_\lambda$, for some $\lambda \in \Lambda_q$, such that $P|V = j_\lambda \circ Q_\lambda$ holds. Here j_λ denotes the inclusion of $\bar{e}_\lambda$ into $\bar{X}^q$. Thus it suffices to show that $P|V$ lifts locally along f.

Suppose either $\dim e_\lambda \leq q-1$ or $e_\lambda \subset A$ holds. Let i_λ be the inclusion of $\bar{e}_\lambda$ into $\bar{X}^{q-1} \coprod(\coprod_{\mu\in\Lambda_q} \tilde{I}^q_\mu)$. Then, by the definition of sum diffeology, $i_\lambda \circ Q_\lambda$ is a plot of $\bar{X}^{q-1} \coprod(\coprod_{\mu\in\Lambda_q} \tilde{I}^q_\mu)$. Since we have $f \circ i_\lambda \circ Q_\lambda = j_\lambda \circ Q_\lambda = P|V$, $P|V$ surely lifts along f. Suppose, on the other hand, that $\dim e_\lambda = q$ and $e_\lambda \not\subset A$. Since $\Phi_\lambda \colon \tilde{I}^{n_\lambda} \to \bar{e}_\lambda$ is a subduction, there exist an open neighborhood W of r and a plot $Q'_\lambda \colon W \to \tilde{I}^q_\lambda$ such that $Q_\lambda|W = \Phi_\lambda \circ Q'_\lambda$ holds. Let $i'_\lambda \colon \tilde{I}^q_\lambda \to \bar{X}^{q-1} \coprod(\coprod_{\mu\in\Lambda_q} \tilde{I}^q_\mu)$ be the inclusion map. Then $i'_\lambda \circ Q'_\lambda \colon W \to \bar{X}^{q-1} \coprod(\coprod_{\mu\in\Lambda_q} \tilde{I}^q_\mu)$ is a plot of $\bar{X}^{q-1} \coprod(\coprod_{\mu\in\Lambda_q} \tilde{I}^q_\mu)$, and we have

$$f \circ i'_\lambda \circ Q'_\lambda(r) = \Phi_\lambda \circ Q'_\lambda = Q_\lambda(r) = P(r), r \in W.$$

This shows that $P|W$ lifts along f. Hence f is a subduction. □

Definition 4.2.6. Let A be a subset of a diffeological space X. We say that A is a retract of X if there exists a smooth map $\gamma \colon X \to A$ such that $\gamma \circ i = 1_A$, where $i \colon A \to X$ is the inclusion map. The smooth map $\gamma \colon X \to A$ is called a retraction from X to A. A retract A of X is called a deformation retract of X if there exists a homotopy $F \colon X \times \tilde{I} \to X$ satisfying

$$F(x, 0) = 1_X(x), F(x, 1) = i \circ \gamma(x).$$

If, moreover, F satisfies the conditions

$$F(x,0) = 1_X(x), F(x,1) = i \circ \gamma(x), F(a,t) = a \ (a \in A),$$

Then A is a called a strong deformation retract of X.

For $n \geq 0$, let J^n denote the subspace $\partial \tilde{I}^n \times \tilde{I} \cup \tilde{I}^n \times \{0\}$ of $\tilde{I}^n \times \tilde{I}$.

Theorem 4.2.7. *J^n is a strong deformation retract of $\tilde{I}^n \times \tilde{I}$.*

Proof. we need to show that there is a map $r \colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ such that the composite $\lambda^{n+1} \circ r \colon \mathbf{R}^{n+1} \to \tilde{I}^{n+1}$ is smooth, and that the map $\tilde{I}^{n+1} \to \tilde{I}^{n+1}$, characterized by the commutative diagram

$$\begin{array}{ccc} \mathbf{R}^{n+1} & \xrightarrow{r} & \mathbf{R}^{n+1} \\ {\scriptstyle \lambda^{n+1}}\big\downarrow & & \big\downarrow{\scriptstyle \lambda^{n+1}} \\ \tilde{I}^{n+1} & \longrightarrow & \tilde{I}^{n+1}, \end{array}$$

induces a smooth retraction $\tilde{r} \colon \tilde{I}^{n+1} \to J^n$. For $n \geq 1$ let $\mu_n \colon \mathbf{R} \to \mathbf{R}$ be the odd function satisfying

$$\mu_n(x) = \sqrt[n]{2\lambda((x+1)/2) - 1} \quad (x \geq 0).$$

For $0 < t < 1/2$ let

$$L(x,t) = (x-t)/(1-2t), \quad T_n(x,t) = \mu_n(x/t)\mu_n((1-x)/t).$$

Define $h_1, \ \cdots, \ h_n, \ v \colon \mathbf{R}^n \times (-\infty, 2) \to \mathbf{R}$ by the formulae:

$$h_j(x_1, \ldots, x_n, z) = \begin{cases} L\left(x_j, (z/4) \prod_{1 \leq k \leq n, \, k \neq j} T_{n-1}(x_k, z/4)\right), & z > 0 \\ x_j, & z \leq 0, \end{cases}$$

$$v(x_1, \ldots, x_n, z) = \begin{cases} z\left(1 - \prod_{1 \leq k \leq n} T_1(x_k, z/4)\right), & z > 0 \\ 0, & z \leq 0. \end{cases}$$

Let $\theta \colon \mathbf{R} \to (-\infty, 2)$ be a smooth map such that $\theta(x) = x$ holds for $x \leq 1$, and put

$$r = (h_1, \ldots, h_n, v) \circ (1, \ldots, 1, \theta) \colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}.$$

For every $(x_1, \ldots, x_n, z) \in \mathbf{R}^n \times \mathbf{R}$, the condition $v(x_1, \ldots, x_n, z) > 0$ implies $h_j(x_1, \ldots, x_n, z) \in \mathbf{R} - (0,1)$. In fact, suppose there is an α such that

$T_1(x_\alpha, z/4) < 1$ and that $T_1(x_\alpha, z/4) \leq T_1(x_j, z/4)$ holds for every j. By symmetricity, we may assume $0 \leq x_\alpha < z/4$. Then we have

$$\begin{aligned}(z/4)\prod_{k\neq\alpha} T_{n-1}(x_k, z/4) &= \prod_{k\neq\alpha} \sqrt[n-1]{z/4}\,T_{n-1}(x_k, z/4) \\ &\geq \prod_{k\neq\alpha} \sqrt[n-1]{\min\{x_k, z/4\}} \geq \prod_{k\neq\alpha} \sqrt[n-1]{x_\alpha} = x_\alpha\end{aligned}$$

Hence $h_\alpha(x_1, \ldots, x_n, z) \leq L(x_\alpha, x_\alpha) = 0$. Since r restricts to $\lambda^n \times \lambda$ on J^n, the correspondence

$$(x_1, \ldots, x_n, z) \mapsto r(\lambda^{-1}(x_1), \ldots, \lambda^{-1}(x_n), \lambda^{-1}(z))$$

induces a retraction $\tilde{r}$ of $\tilde{I}^{n+1}$ onto J^n.

Finally the linear homotopy $h\colon \mathbf{R}^{n+1} \times \mathbf{R} \to \mathbf{R}^{n+1}$ given by the formula

$$h(x,t) = (1-t)r(x) + tx$$

induces a desired deformation $i \circ \tilde{r} \simeq 1\colon \tilde{I}^{n+1} \to \tilde{I}^{n+1}$.

□

Definition 4.2.8. Let X, Y and A be three diffeological space. A map $i\colon A \to X$ is a cofibration if it satisfies the homotopy extension property (HEP). This means that for any smooth map $f\colon X \to Y$, for any smooth map $H\colon A \times \tilde{I} \to Y$ such that $H \circ i_0 = f \circ i$, there is a smooth map $\tilde{H}\colon X \times \tilde{I} \to Y$ such that $\tilde{H} \circ (i \times \mathrm{id}) = H$ and $\tilde{H} \circ i_0 = f$.

Theorem 4.2.9. *Let X be a diffeological CW complex. Let A be a subcomplex of X. Then the inclusion map $i\colon A \to X$ is a cofibration.*

Proof. For any smooth map $f\colon X \to Y$, for any smooth map $H\colon A \times \tilde{I} \to Y$ such that $H(a, 0) = f(a)$, let $G^q\colon \bar{X}^q \times \tilde{I} \to Y$ be a smooth map defined by recursion:

$$G^q(x, 0) = f(x), G^{-1} = H, G^q|\bar{X}^{q-1} \times \tilde{I} = G^{q-1}.$$

Let us assume that $\{e_\mu\}$ is a family of $q+1$-cells in X such that it is not included in A. Let $\Phi_\mu\colon \tilde{I}_\mu^{q+1} \to \bar{e_\mu}$ be a characteristic map for each e_μ. Define $L^{q+1}\colon \partial\tilde{I}_\mu^{q+1} \times \tilde{I} \bigcup \tilde{I}_\mu^{q+1} \times \{0\} \to Y$ by

$$L^{q+1}(x,t) = \begin{cases} G^q \circ (\Phi_\mu \times 1_{\tilde{I}}(x,t), & (x,t) \in \partial\tilde{I}_\mu^{q+1} \times \tilde{I} \\ f \circ \Phi_\mu(x), & (x,0) \in \tilde{I}_\mu^{q+1} \times \{0\}. \end{cases}$$

A map L^{q+1} is well defined and smooth by the property of diffeology of $\tilde{I}$. By Theorem 4.2.7, there is a retraction

$$\gamma\colon \tilde{I}^{q+1}_{\mu} \times \tilde{I} \to \partial\tilde{I}^{q+1}_{\mu} \times \bigcup \tilde{I}^{q+1}_{\mu} \times \{0\}.$$

Let $L'^{q+1} = L^{q+1} \circ \gamma$. Then

$$L'^{q+1} \coprod G^{q}\colon (\tilde{I}^{q+1}_{\mu} \times \tilde{I}) \coprod (\bar{X}^{q} \times \tilde{I} \to Y$$

is a smooth map. By Proposition 4.2.5,

$$((\tilde{I}^{q+1}_{\mu} \times \tilde{I}) \coprod (\bar{X}^{q} \times))/_{\sim} = (\bar{X}^{q} \bigcup_{\Phi_{\mu}|\partial\tilde{I}^{q+1}_{\mu}} \tilde{I}^{q+1}_{\mu}) \times \tilde{I} \cong (\bar{X}^{q} \cup e_{\mu}) \times \tilde{I}.$$

Thus, the map

$$(L'^{q+1} \coprod G^{q}/_{\sim} = F^{q+1}_{\mu}\colon (\bar{X}^{q} \cup e_{\mu}) \times \tilde{I} \to Y$$

induced by $L'^{q+1} \coprod G^{q}$ satisfies the following conditions:

$$F^{q+1}(x,0) = f(x), F^{-1} = H, F^{q+1}|\bar{X}^{q} \times \tilde{I} = G^{q}.$$

If we define $G^{q+1}\colon \bar{X}^{q+1} \times \tilde{I} \to Y$ by $G^{q+1}|\bar{X}^{q} \cup e_{\mu}) \times \tilde{I} = F^{q+1}_{\mu}$ for each μ, the map G^{q+1} is smooth by Theorem 3.4.6. Moreover, if we define $G\colon X \times \tilde{I} \to Y$ by $G|\bar{X}^{q} \times \tilde{I} = G^{q}$, the map G is a smooth map. Therefore, $i\colon A \to X$ is a cofibration. □

Definition 4.2.10. A smooth map $i\colon A \to B$ is said to have the left lifting property (LLP) with respect to a smooth map $p\colon X \to Y$ and p is said to have the right lifting property (RLP) with respect to i if a lift exists in any commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y. \end{array}$$

Definition 4.2.11. A smooth map $p\colon X \to Y$ is said to have the homotopy extension lifting property (HELP) with respect to a pair (A, B) if p has the RLP with respect to the inclusion map $i\colon B \times \tilde{I} \cup A \times \{0\} \to A \times \tilde{I}$.

Definition 4.2.12. A smooth map $p\colon X \to Y$ is called a Serre fibration if p has the HELP with respect a pair $(\tilde{I}^{n}, \partial\tilde{I}^{n})$ for all $n \geq 0$.

Theorem 4.2.13. *Let X and Y be two diffeological spaces. A smooth map $p\colon X \to Y$ is a Serre fibration if and only if a smooth map p has the HELP with respect to a CW pair (A, B).*

Proof. It is clear that p is a Serre fibration if p has the HELP with respect to a CW pair (A, B). Conversely, suppose p is a Serre fibration. Consider the following commutative diagram:

$$\begin{array}{ccc} B \times \tilde{I} \cup A \times \{0\} & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ A \times \tilde{I} & \xrightarrow[g]{} & Y. \end{array}$$

Let e_μ be an n_μ-cell of A such that $e_\mu \cap B = \emptyset$ and $\bar{e}_\mu \cap B \neq \emptyset$. Let $\Phi_\mu\colon \tilde{I}^{n_\mu} \to \bar{e}_\mu$ be a characteristic map of e_μ. We define $L\colon \partial\tilde{I}^{n_\mu} \times \tilde{I} \cup \tilde{I}^{n_\mu} \times \{0\} \to B \times \tilde{I} \cup A \times \{0\}$ by

$$L(t, s) = \begin{cases} (\Phi_\mu(t), s), & (t, s) \in \partial\tilde{I}^{n_\mu} \times \tilde{I} \\ (\Phi_\mu(t), 0), & (t, 0) \in \tilde{I}^{n_\mu} \times \{0\}. \end{cases}$$

By the property of diffeology of $\tilde{I}$, L is smooth. The following diagram commutes:

$$\begin{array}{ccccc} J^n & \xrightarrow{L} & B \times \tilde{I} \cup A \times \{0\} & \xrightarrow{f} & X \\ {\scriptstyle j}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{I}^{n_\mu} \times \tilde{I} & \xrightarrow[\Phi_\mu \times 1_{\tilde{I}}]{} & A \times \tilde{I} & \xrightarrow[g]{} & Y. \end{array}$$

There is a lift $K\colon \tilde{I}^{n_\mu} \times \tilde{I} \to X$ with respect to p since p is a Serre fibration. Let

$$(K \coprod f)/_\sim \colon (\tilde{I}^{n_\mu} \times \tilde{I}) \coprod (B \times \tilde{I} \cup A \times \{0\} \to X$$

be an induced map from

$$K \coprod f\colon (\tilde{I}^{n_\mu} \times \tilde{I}) \coprod (B \times \tilde{I} \cup A \times \{0\}) \to X.$$

Therefore there exists a lift

$$h_\mu\colon (B \cup e_\mu) \times \tilde{I} \cup A \times \{0\} \to X$$

with respect to p since

$$\begin{aligned} (\tilde{I}^{n_\mu} \times \tilde{I}) \coprod (B \times \tilde{I} \cup A \times \{0\})/_\sim &= (B \cup_{\Phi|\partial\tilde{I}^{n_\mu}} \tilde{I}^{n_\mu}) \times \tilde{I} \cup A \times \{0\} \\ &\cong (B \cup e_\mu) \times \tilde{I} \cup A \times \{0\} \end{aligned}$$

Therefore p has the HELP with respect to a CW pair (A, B). □

Corollary 4.2.14. *Let X and Y be two diffeological spaces. Let A be a CW complex. If a smooth map $p\colon X \to Y$ is a Serre fibration, then p has the RLP with respect to the inclusion map $i\colon A \times \{0\} \to A \times \tilde{I}$.*

Proof. Let $\emptyset$ be the empty set. Since p is a Serre fibration, p has the HELP with respect to a CW pair $(A, \emptyset)$. Therefore p has the RLP with respect to the inclusion map $i\colon A \times \{0\} \to A \times \tilde{I}$. □

4.3 Model structure of diffeological spaces

In this subsection we shall show that the category **Diff** has a finitely generated model structure, referring to [7].

Let $I_{\mathbf{Diff}}$ be the set of boundary inclusions $\partial\tilde{I}^n \to \tilde{I}^n,\ n \geq 0$ and $J_{\mathbf{Diff}}$ be the set of inclusions $J^n = \partial\tilde{I}^n \times \tilde{I} \cup \tilde{I}^n \times \{0\} \to \tilde{I}^n \times \tilde{I},\ n \geq 0$. Then a smooth map $p\colon X \to Y$ is a Serre fibration if and only if p is a J-injective. Let $W_{\mathbf{Diff}}$ be the classes of weak homotopy equivalences in **Diff**. Then we have the following.

Theorem 4.3.1. *There is a finitely generated model structure on* **Diff** *with $I_{\mathbf{Diff}}$ as the set of generating cofibrations, $J_{\mathbf{Diff}}$ as the set of generating trivial cofibrations, and $W_{\mathbf{Diff}}$ as the class of weak equivalences.*

The proof of this theorem should just fulfill conditions of Theorem 2.3.4. We shall show it in the following procedures. It is clear that we have the following.

Lemma 4.3.2. *The class $W_{\mathbf{Diff}}$ has the two out of three property and is closed under retracts.*

Lemma 4.3.3. *Let λ be an ordinal. Let a functor $X : \lambda \longrightarrow$* **Diff** *be a λ-sequence of inclusions. Then colim$X$ has the weak diffeology with respect to $\{X_\alpha\}_{\alpha<\lambda}$.*

Proof. Since the category **Diff** is closed under cocomplete, there exists a colimit $\mathrm{colim}X$ of $X\colon \lambda \to \mathbf{Diff}$. Then $\mathrm{colim}X = \coprod_{\alpha<\lambda} X_\alpha/_\sim$ holds. Let $P\colon U \to \mathrm{colim}X$ be a plot of $\mathrm{colim}X$. By the definition of quotient diffeology, for any r in U, there exist an open neighborhood V of r and a plot $Q\colon V \to \coprod_{\alpha<\lambda} X_\alpha$ of $\coprod_{\alpha<\lambda} X_\alpha$ such that $P|V = \pi \circ Q$, where $\pi\colon \coprod_{\alpha<\lambda} X_\alpha \to \mathrm{colim}X$ is the projection. By the definition of sum diffeology, there exist an open neighborhood W of r and a plot $Q'\colon W \to X_\alpha$ of X_α such that $Q|W = i_\alpha \circ Q'$, where $i_\alpha\colon X_\alpha \to \coprod_{\alpha<\lambda} X_\alpha$ is the inclusion. Then we have

$$P|W = \pi \circ Q|W = \pi \circ i_\alpha \circ Q'.$$

Therefore colim$X$ has the weak diffeology with respect to $\{X_\alpha\}_{\alpha<\lambda}$, since $\pi \circ i_\alpha\colon X_\alpha \to \mathrm{colim}X$ is the inclusion, $\square$

Proposition 4.3.4. *For any $n \geq 0$, the smashing spaces $\tilde{I}^n$, $\partial\tilde{I}^n$ and J^n are finitely relative to inclusions.*

Proof. Let λ be a limit ordinal, and let $X\colon \lambda \to \mathbf{Diff}$ be a λ-sequence of inclusions. We shall show that $\tilde{I}^n$ is finitely relative to inclusions. It suffices to show that, for any smooth map $f\colon \tilde{I}^n \to \mathrm{colim}X$ the image $f(\tilde{I}^n) \subset X_\alpha$ for some α. Let $\lambda\colon \mathbf{R} \to \tilde{I}$ be the smashing function. We define $\lambda^n\colon \mathbf{R}^n \to \tilde{I}^n$ by for any $(x_1, \ldots, x_n)$ in $\mathbf{R}^n$,

$$\lambda^n(x_1, \ldots, x_n) = (\lambda(x_1), \ldots, \lambda(x_n)).$$

Then $f \circ \lambda^n\colon \mathbf{R}^n \to \mathrm{colim}X$ is a plot of colim$X$. By the definition of λ, We have $\mathrm{Im}(f \circ \lambda^n|I^n) = \mathrm{Im}f$. By Lemma 4.3.3, colim$X$ has the weak diffeology with respect to $\{X_\alpha\}_{\alpha<\lambda}$. Thus for any r in $I^n \subset \mathbf{R}^n$, there exist an open neighborhood V_x of x and a plot $Q_{m_x}\colon V_x \to X_{m_x}$ of X_{m_x} such that $f \circ \lambda^n|V_x = i_{m_x} \circ Q_{m_x}$, where $i_{m_x}\colon X_{m_x} \to \mathrm{colim}X$ is the inclusion. Since $I^n \subset \cup_{x\in I^n} V_x$ holds and I^n is a compact set, there exists $1 \leq k \leq l$ such that

$$f(\tilde{I}^n) = f \circ \lambda^n(I^n) \subset \cup_{1\leq k\leq l} X_{mk}.$$

Let $M = \max\{m_{xk} | 1 \leq k \leq l\}$. Then we have $f(\tilde{I}^n) \subset X_M$. $\square$

By the definition of relative cell complexes, all morphisms of $I_{\mathbf{Diff}}$-cell and $J_{\mathbf{Diff}}$-cell are inclusions. Thus we have the following Corollary.

Corollary 4.3.5. *The domains of $I_{\mathbf{Diff}}$-cell and $J_{\mathbf{Diff}}$-cell are small relative to $I_{\mathbf{Diff}}$-cell and $J_{\mathbf{Diff}}$-cell, respectively, and the domains and codomains of $I_{\mathbf{Diff}}$ and $J_{\mathbf{Diff}}$ are finitely relative to $I_{\mathbf{Diff}}$-cell.*

Lemma 4.3.6. *Let λ be an ordinal and $X\colon \lambda \to \mathbf{Diff}$ be a λ-sequence of inclusions that are also weak homotopy equivalences. Then the inclusion $i_0\colon X_0 \to \mathrm{colim}X$ is a weak homotopy equivalence.*

Proof. Let x be a basepoint of X_0. We shall show that $i_{0*}\colon \pi_n(X_0, x) \to \pi_n(\mathrm{colim}X, x)$ is bijective. Let $[f]$ be an element of $\pi_n(\mathrm{colim}X, x)$. Then $f\colon \tilde{I}^n \to \mathrm{colim}X$ is a smooth map such that $f(\partial\tilde{I}^n) = \{x\}$. By Proposition 4.3.4, there exits $\alpha < \lambda$ such that $f(\tilde{I}^n) \subset X_\alpha$. Thus $[f] \in \pi_n(X_\alpha, x)$. Since the inclusion $j_\alpha\colon X_0 \to X_\alpha$ is a weak homotopy equivalence, there exists an element $[\tilde{f}] \in \pi_n(X_0, x)$ such that $j_{\alpha *}[\tilde{f}] = [f]$ holds. Then we have

$$\begin{array}{rcl} i_{\alpha *} j_{\alpha *}[\tilde{f}] & = & i_{\alpha *}[f] \\ (i_\alpha \circ j_\alpha)_*[\tilde{f}] & = & [f] \\ i_{0*}[\tilde{f}] & = & [f], \end{array}$$

where $i_\alpha \colon X_\alpha \to \mathrm{colim} X$ is the natural map.

We shall show that i_{0*} is injective. For any elements $[f]$ and $[g]$ in $\pi_n(X_0, x)$ such that $i_{0*}[f] = i_{0*}[g]$, there exists a homotopy $H\colon \tilde{I}^n \times \tilde{I} \to \mathrm{colim} X$ such that

$$H(x, 0) = i_0 \circ f = f \text{ and } H(x, 1) = i_0 \circ g = g.$$

By Proposition 4.3.4, there exists a $\alpha < \lambda$ such that $H(\tilde{I}^n \times \tilde{I}) \subset X_\alpha$ holds. Thus $[f] = [g]$ holds on $\pi_n(X_\alpha, x)$. Since the inclusion $j_\alpha \colon X_0 \to X_\alpha$ is a weak homotopy equivalence, $[f] = [g]$ holds on $\pi_n(X_0, x)$. Hence i_{0*} is bijective. □

Lemma 4.3.7. *Every morphism in $J_{\mathbf{Diff}}$-cell is a weak homotopy equivalence and in $I_{\mathbf{Diff}}$-cof.*

Proof. Let $i\colon Z \to Y$ be a relative $J_{\mathbf{Diff}}$-cell complex. We shall show that i is an $I_{\mathbf{Diff}}$-cofibration. It suffices to show that, for any $p\colon A \to B$ in $I_{\mathbf{Diff}}$-inj, i has the left lifting property with respect to p. Suppose the following diagram commutes:

$$\begin{array}{ccc} Z & \xrightarrow{f} & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[g]{} & B. \end{array} \qquad (1)$$

By the definition of relative $J_{\mathbf{Diff}}$-cell complex, there exist an ordinal λ and λ-sequence $X\colon \lambda \to \mathbf{Diff}$ such that i is the composition of X and such that for each β such that $\beta + 1 < \lambda$, there is a pushout diagram:

$$\begin{array}{ccc} J^{n_\beta} & \longrightarrow & X_\beta \\ {\scriptstyle j_{n_\beta}}\downarrow & & \downarrow{\scriptstyle k_\beta} \\ \tilde{I}^{n_\beta} \times \tilde{I} & \longrightarrow & X_{\beta+1} \end{array}$$

such that $j_{n_\beta} \in J$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc}
J^{n_0} & \longrightarrow & X_0 = Z & \xrightarrow{=} & X_0 = Z & \xrightarrow{f} & A \\
{\scriptstyle j_{n_0}}\big\downarrow & & {\scriptstyle k_0}\big\downarrow & & {\scriptstyle i=i_0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
\tilde{I}^{n_0} \times \tilde{I} & \longrightarrow & X_1 & \xrightarrow[i_1]{} & \mathrm{colim} X = Y & \xrightarrow[g]{} & B,
\end{array}$$

where k_0 and i_1 are inclusions. Since p is an $I_{\mathbf{Diff}}$-injective, p has the right lifting property with respect to j_{n_0} by Theorem 4.3.8. Thus there exists a lift $h_1 \colon \tilde{I}^{n_0} \times \tilde{I} \to A$. By the property of pushout, h_1 induces a lift $\tilde{h}_1 \colon X_1 \to A$. Hence k_0 has the left lifting property with respect to p. Hereinafter, if we constitute inductively, for each $\alpha \geq 1$, we have the following commutative diagram:

$$\begin{array}{ccccccc}
J^{n_\alpha} & \longrightarrow & X_\alpha & \xrightarrow{=} & X_\alpha & \xrightarrow{f} & A \\
{\scriptstyle j_{n_\alpha}}\big\downarrow & & {\scriptstyle k_\alpha}\big\downarrow & & {\scriptstyle i_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
\tilde{I}^{n_\alpha} \times \tilde{I} & \longrightarrow & X_{\alpha+1} & \xrightarrow[i_{\alpha+1}]{} & \mathrm{colim} X = Y & \xrightarrow[g]{} & B,
\end{array}$$

where i_α, $i_{\alpha+1}$ and k_α are inclusions. Then there exists a lift $\tilde{h}_{\alpha+1} \colon X_{\alpha+1} \to A$. Thus k_α has the left lifting property with respect to p. Since we have the following diagram:

$$\begin{array}{ccccccccc}
Z = X_0 & \xrightarrow{j_0} & X_1 & \xrightarrow{j_1} & \cdots & \longrightarrow & X_\alpha & \xrightarrow{j_\alpha} & \cdots \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle \tilde{h}_1}\big\downarrow & & & & {\scriptstyle \tilde{h}_\alpha}\big\downarrow & & \\
A & \xrightarrow[=]{} & A & \xrightarrow[=]{} & \cdots & \longrightarrow & A & \xrightarrow[=]{} & \cdots,
\end{array}$$

there exists a natural map $\tilde{h} \colon \mathrm{colim} X = Y \to A$ by the property of colimit. Therefore i has the left lifting property with respect to p since there exists a lift $\tilde{h}$.

We shall show that $i_0 : Z \longrightarrow Y$ is a weak homotopy equivalence. We consider the diagram (1). Then $X_{\beta+1}$ is the space which stuck $\tilde{I}^{n_\beta} \times \tilde{I}$ on X_β along with J^{n_β}. Hence it is clear that the inclusion $i_\beta \colon X_\beta \to X_{\beta+1}$ is a weak homotopy equivalence by the Theorem 4.2.7. Therefore $i_0 \colon Z \to Y$ is a weak homotopy equivalence by Lemma 4.3.6. $\square$

Theorem 4.3.8. *Let X and Y be two diffeological spaces. Let $p \colon X \to Y$ be a smooth map. Then the following conditions are equivalent:*

1. $p\colon X \to Y \in W_{\mathbf{Diff}} \cap$ *J-inj,*

2. $p\colon X \to Y \in I_{\mathbf{Diff}}$*-inj, and*

3. $p\colon X \to Y$ *has the right lifting property with respect to inclusion* $i\colon B \to A$ *such that* (A, B) *is a diffeological CW-pair.*

Proof. It is clear that *3* implies *2*.

We shall show that *2* implies *3*. Let (A, B) be diffeological CW-pair. Consider the following commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[g]{} & Y. \end{array}$$

Let us assume that $\{e_\lambda\}$ is a family of cells in A such that it is not induced in B. Let $\Phi_\lambda\colon \tilde{I}^{n_\lambda} \to \bar{e}_\lambda$ be a characteristic map for each e_λ. There exists $\lambda \in \Lambda$ such that $\Phi_\lambda(\partial \tilde{I}^{n_\lambda}) \subset B$. Then the following diagram commutes:

$$\begin{array}{ccccc} \partial \tilde{I}^{n_\lambda} & \xrightarrow{\Phi_\lambda|\partial \tilde{I}^{n_\lambda}} & B & \xrightarrow{f} & X \\ {\scriptstyle j_{n_\lambda}}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{I}^{n_\lambda} & \xrightarrow[\Phi_\lambda]{} & B \cup e_\lambda & \xrightarrow[g]{} & Y. \end{array}$$

By the condition *2*, there exists a smooth map $h\colon \tilde{I}^{n_\lambda} \to X$ such that $h \circ j_{n_\lambda} = f \circ \Phi_\lambda|\partial \tilde{I}^{n_\lambda}$ and $p \circ h = g \circ \Phi_\lambda$. Let

$$f \coprod h/\sim\colon B \bigcup_{\varphi|\partial \tilde{I}^q} \tilde{I}^q \to X$$

be the map induced from $f \coprod h\colon B \coprod \tilde{I}^q \to X$. By Proposition 5.5, we have $B \cup_{\Phi|\partial I^{n_\lambda}} \tilde{I}^{n_\lambda} \cong B \cup e_\lambda$. Thus there exists a smooth map $H_\lambda\colon B \cup e_\lambda \to X$ be defined by $H|B \cup e_\lambda = H_\lambda$. Then H is a smooth map by Theorem 4.6. Therefore $p\colon X \to Y$ has the RLP with respect to every inclusion $i\colon B \to A$ such that (A, B) is a diffeological CW-pair. Thus the conditions of *2* and *3* are equivalent.

We shall show that *2* implies *1*. It is clear that p is in J-inj since J^n is a subcomplex of diffeological CW-complex $\tilde{I}^n \times \tilde{I}$. Let x_0 be a basepoint of X. For each $n \geq 0$, let us prove first that $p_*\colon \pi_n(X, x_0) \to \pi_n(Y, p(x_0))$ is surjective. For any $[\gamma] \in \pi_n(Y, p(x_0))$, let $\gamma\colon \tilde{I}^n \to X$ be a smooth map such that $\gamma(\partial \tilde{I}^n) = \{p(x_0)\}$. Let $C_{x_0}\colon \partial \tilde{I}^n \to X$ be the constant map from $\partial \tilde{I}^n$ to $x_0 \in X$. Then the following diagram commutes:

$$\begin{array}{ccc} \partial \tilde{I}^n & \xrightarrow{C_{x_0}} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{I}^n & \xrightarrow{\gamma} & Y. \end{array}$$

Thus there exists a smooth map $\tilde{\gamma}\colon \tilde{I}^n \to X$ such that $\tilde{\gamma}(\partial \tilde{I}^n) = \{x_0\}$ and $p \circ \tilde{\gamma} = \gamma$. Therefore p_* is surjective from $p_*[\tilde{\gamma}] = [\gamma]$.

Next, let us prove that $p_*\colon \pi_n(X, x_0) \to \pi_n(Y, p(x_0))$ is injective. Let $[f]$ and $[f']$ be elements of $\pi_n(X, x_0)$ such that $p_*[f] = p_*[f']$. Then there exists a homotopy $F\colon \tilde{I}^n \times \tilde{I} \to Y$ such that $F(t,0) = p \circ f(t)$, $F|\partial \tilde{I}^n \times \tilde{I}(t) = p(x_0)$, and $F(t,1) = p \circ f'(t)$. Let

$$F'\colon \partial \tilde{I}^{n+1} \cong \tilde{I}^n \times \{0\} \cup \partial \tilde{I}^n \times \tilde{I} \cup \tilde{I}^n \times \{1\} \to X$$

be defined by $F'(t,0) = f(t)$, $F'|\partial \tilde{I}^n \times \tilde{I}(t) = x_0$ and $F'(t,1) = f'(t)$. By property of the smashing diffeology of $\tilde{I}$, F' is smooth. The following diagram commutes:

$$\begin{array}{ccc} \partial \tilde{I}^{n+1} & \xrightarrow{F'} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{I}^n \times \tilde{I} & \xrightarrow[F]{} & Y. \end{array}$$

By the condition of *2*, there exists a smooth map $\tilde{F}\colon \tilde{I}^n \times \tilde{I} \to X$ such that

$$\tilde{F}(t,0) = f(t),\ \tilde{F}|\partial \tilde{I}^n \times \tilde{I} = \{x_0\}, \tilde{F}(t,1) = f'(t) \text{ and } p \circ \tilde{F} = F.$$

Thus p_* is injective from $[f] = [f']$. Therefore $p\colon X \to Y$ is both a weak homotopy equivalence and in $J_{\mathbf{Diff}}$-inj.

We shall show that *1* implies *2*. Let $p\colon X \to Y$ be a weak homotopy equivalence and in $J_{\mathbf{Diff}}$-inj. Consider the following commutative diagram:

$$\begin{array}{ccc} \partial \tilde{I}^n & \xrightarrow{f} & X \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{I}^n & \xrightarrow[g]{} & Y. \end{array} \tag{2}$$

Let t_0 be a basepoint of $\partial \tilde{I}^n$ and $f(t_0) = x_0$. Since $\partial \tilde{I}^n$ and $\tilde{I}^{n-1}/\partial \tilde{I}^{n-1}$ are homotopy equivalent, $[f] \in \pi_{n-1}(X, x_0)$. Since $p_*[f] = 0$ and p is a weak homotopy equivalence, there exists a smooth map $F\colon \partial\tilde{I}^n \times \tilde{I} \to X$ such that $F(t,0) = f(t)$ and $F(t,1) = x_0$. Let

$$G\colon \partial\tilde{I}^{n+1} \cong \tilde{I}^n \times \{0\} \cup \partial\tilde{I}^n \times \tilde{I} \cup \tilde{I}^n \times \{1\} \to Y$$

be defined by

$$G|\tilde{I}^n \times \{0\} = g,\ G|\partial\tilde{I}^n \times \tilde{I} = p \circ F \text{ and } G|\tilde{I}^n \times \{1\} = p(x_0).$$

By property of the smashing diffeology of $\tilde{I}$, G is smooth. Since p is a weak homotopy equivalence, there exists a smooth map $\tilde{F}\colon \partial\tilde{I}^{n+1} \to X$ such that $p_*[\tilde{F}] = [G]$. Thus there exists a smooth map $\tilde{G}\colon \partial\tilde{I}^{n+1} \times \tilde{I} \to Y$ such that $\tilde{G}(t,0) = G(t)$ and $\tilde{G}(t,1) = p \circ \tilde{F}$. Let $K\colon \tilde{I}^n \to X$ and $L\colon \tilde{I}^n \times \tilde{I} \to Y$ be defined by $K = \tilde{F}|\tilde{I}^n \times \{0\}$ and $L = \tilde{G}|\tilde{I}^n \times \{0\} \times \tilde{I}$, respectively. Then the following diagram commutates:

$$\begin{array}{ccc} \tilde{I}^n & \xrightarrow{K} & X \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{I}^n \times \tilde{I} & \xrightarrow[L]{} & Y. \end{array}$$

If we use properties of a $J_{\mathbf{Diff}}$-injective and Corollary 5.14, we can construct a smooth map $K'\colon \partial\tilde{I}^n \times \tilde{I} \cup \tilde{I}^n \times \{1\} \to X$ such that

$$K'|\tilde{I}^n \times \{1\} = K \text{ and } p \circ K' = L|\partial\tilde{I}^n \times \tilde{I} \cup \tilde{I}^n \times \{1\}.$$

Then the following diagram:

$$\begin{array}{ccc} J^n & \xrightarrow{K'} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{I}^n \times \tilde{I} & \xrightarrow[L]{} & Y. \end{array}$$

Since p is in $J_{\mathbf{Diff}}$-inj, there exists a smoth map $H\colon \tilde{I}^n \times \tilde{I} \to X$ such that

$$H|\partial\tilde{I}^n \times \tilde{I} \cup \tilde{I}^n \times \{1\} = K' \text{ and } p \circ H = L.$$

Then we have

$$p \circ H|\tilde{I}^n \times \{0\} = L|\tilde{I}^n \times \{0\} = \tilde{G}|\tilde{I}^n \times \{0\} \times \{0\} = \tilde{G}|\tilde{I}^n \times \{0\} = g, \text{and}$$
$$H|\partial\tilde{I}^n \times \{0\} = K'|\partial\tilde{I}^n \times \{0\} = f.$$

Let $\tilde{H}\colon \tilde{I}^n \to X$ be defined by $\tilde{H} = H|\tilde{I}^n \times \{0\}$. It is now easy to see that $\tilde{H}$ is the desired lifting in the diagram (2). Therefore $p\colon X \to Y$ is in $I_{\mathbf{Diff}}$-inj. □

Therefore Theorem 4.3.1 is proved by Theorem 2.3.4.

4.4 Quillen adjunction between Diff and Top

In this subsection we shall show that an adjunction $(T, D, \varphi) : \mathbf{Diff} \to \mathbf{Top}$ is a Quillen adjunction with respect to the model structure of **Top**.

For the category **Top**, we takes as I' the set of boundary inclusions $S^{n-1} \to D^n, n \geq 0$, and as J the set of inclusions $D^n \times \{0\} \to D^n \times I, n \geq 0$. Let $W_{\mathbf{Top}}$ be the classes of weak homotopy equivalences in **Top**. The standard model structure on **Top** can be described as follows.

Theorem 4.4.1 ([7, 2.4.19])**.** *There is a finitely generated model structure on* **Top** *with I' as the set of generating cofibraitons, J as the set of generating trivial cofibrations, and $W_{\mathbf{Top}}$ as the class of weak equivalences.*

Proposition 4.4.2. *Let I be a subspace of the standard diffeological space* $\mathbf{R}$ *and let $\tilde{I}$ be the smashing space. Then TI and $T\tilde{I}$ are homeomorphic to subspace I of the Euclidean space* $\mathbf{R}$*.*

Proof. We shall show that TI and the subspace I of the Euclidean space $\mathbf{R}$ are homeomorphic. It is clear that the identity map $1_I : TI \longrightarrow I$ is continuous. Let A be an open set of TI. Since the smashing function $\lambda : \mathbf{R} \longrightarrow I$ is a plot of I, $\lambda^{-1}(A)$ is an open set of $\mathbf{R}$. Hence $\lambda^{-1}(A) \cap I$ is an open set of I. Since $\lambda|I : I \longrightarrow I$ is a homeomorphism,

$$\lambda(\lambda^{-1}(A) \cap I) = A \cap \lambda(I) = A \cap I = A$$

is an open set of I. Therefore TI and I are homeomorphic since 1_I is an open map. We shall show that $T\tilde{I}$ and I are homeomorphic. Obviously, the identity map $1_{\tilde{I}} : T\tilde{I} \longrightarrow I$ is continuous, and the smashing function $\lambda : \mathbf{R} \longrightarrow \tilde{I}$ is a plot of $\tilde{I}$. Similarly, the identity map $1_{\tilde{I}}$ is an open map. Therefor $T\tilde{I}$ and I are homeomorphic. □

For every $X \in \mathbf{Diff},\ Y \in \mathbf{Top}$, let φ denote the isomorphism

$$\mathrm{Hom}_{\mathbf{Top}}(TX, Y) \approx \mathrm{Hom}_{\mathbf{Diff}}(X, DY).$$

Theorem 4.4.3. *The adjunction (T, D, φ):* **Diff** $\to$ **Top** *is a Quillen adjunction.*

Proof. To see that (T, D, φ) is a Quillen adjunction, it suffices to show that D: **Top** $\to$ **Diff** is a right Quillen functor by Lemma 2.2.6. We first show that the functor D: **Top** $\to$ **Diff** preserves trivial fibrations. Let $p : X \longrightarrow$

Y be a trivial fibration in **Top**. Suppose there is a commutative diagram:

$$\begin{array}{ccc} \partial\tilde{I}^n & \longrightarrow & DX \\ {\scriptstyle i_n}\big\downarrow & & \big\downarrow{\scriptstyle D(p)} \\ \tilde{I}^n & \longrightarrow & DY. \end{array} \tag{3}$$

By Proposition 4.4.2, $T(\partial\tilde{I}^n) \cong \partial I^n$ and $T\tilde{I}^n \cong I^n$ hold, where ∂I^n and I^n are subspaces of the Euclidean space $\mathbf{R}^n$. Thus we have the following commutative diagram:

$$\begin{array}{ccccc} \partial I^n & \longrightarrow & TDX & \xrightarrow{1_X} & X \\ {\scriptstyle T(i_n)}\big\downarrow & & & & \big\downarrow{\scriptstyle p} \\ I^n & \longrightarrow & TDY & \xrightarrow{1_Y} & Y, \end{array}$$

where 1_X and 1_Y are two identity maps. Since p has the right lifting property with respect to $T(i_n)\colon \partial I^n \cong S^{n-1} \to I^n \cong D^n$, there exists a lift $h\colon I^n \to X$. It is clear that two identity maps $1_{\partial\tilde{I}^n}\colon \partial\tilde{I}^n \to D(\partial I^n)$ and $1_{\tilde{I}^n}\colon \tilde{I}^n \to DI^n$ are smooth. Thus there is the following commutative diagram:

$$\begin{array}{ccccc} \partial\tilde{I}^n & \xrightarrow{1_{\partial\tilde{I}^n}} & D(\partial I^n) & \longrightarrow & DX \\ {\scriptstyle i_n}\big\downarrow & & & & \big\downarrow{\scriptstyle D(p)} \\ \tilde{I}^n & \xrightarrow[1_{\tilde{I}^n}]{} & D(I^n) & \longrightarrow & DY. \end{array}$$

It is now easy to see that the composite $D(h) \circ 1_{\tilde{I}^n}\colon \tilde{I}^n \to DX$ is the desired lifting in the diagram (3). Hence the functor $D\colon$ **Top** $\to$ **Diff** preserves trivial fibrations. since $D(p)\colon DX \to DY$ has the riht lifting property with respect to i_n. Similarly, it is easy to prove that the functor D preserves fibrations. Therefore the adjunction $(T, D, \varphi)\colon$ **Diff** $\to$ **Top** is a Quillen adjunction. □

5 Numerically generated spaces

5.1 Numerically generated spaces

In this subsection we define numerically generated spaces. Moreover we present the basic properties of numerically generated spaces; for more detail see [13]. There are two functors $T\colon \mathbf{Diff} \to \mathbf{Top}$ and $D\colon \mathbf{Top} \to \mathbf{Diff}$. By using two functors T and D, we define a numerically generated space.

Definition 5.1.1 (numerically generated space). Let us write $\nu = TD$. A topological space X is a numerically generated space if and only if $\nu X = X$ holds. Clearly, we have the following condition. A subset A is an open set of numerically generated space X if and only if for any continuous map $\sigma\colon U \to X$, $\sigma^{-1}(A)$ is open in U. Let $\mathbf{NG}$ denote the full subcategory of $\mathbf{Top}$ consisting of numerically generated spaces.

Lemma 5.1.2. *For any open subset U of Euclidean space, we have $\nu U = U$.*

Proof. Suppose that νU and U are not homeomorphic. Then there exists an open set V of νU such that it is not open in U. Let 1_U be the identity map on U. Then $1_U^{-1}(V) = V$ is open in U. This is contradictory. □

Lemma 5.1.3. *Let X be a topological space. Then we have $\nu X = \nu(\nu X)$.*

Proof. Let A be an open set of $\nu(\nu X)$. Let $P\colon U \to X$ be continuous map. We shall show that $P^{-1}(A)$ is open in U. By Lemma 3.3.6, the identity map $1_X\colon \nu X \to X$ is continuous. By Lemma 5.1.2, we have a continuous $\nu(P)\colon U = \nu U \to \nu X$. Since $1_X \circ \nu(P) = P$ holds, $P^{-1}(A) = \nu(P)^{-1} 1_X^{-1}(A)$ is open in U. Therefore A is an open set of νX. □

By Lemma 5.1.3, we have a functor $\nu\colon \mathbf{Top} \to \mathbf{NG}$ which maps a topological space X to the numerically generated space νX.

Proposition 5.1.4. *The inclusion functor $i\colon \mathbf{NG} \to \mathbf{Top}$ is a left adjoint to $\nu\colon \mathbf{Top} \to \mathbf{NG}$.*

Proof. Let X and Y be a numerically generated space and a topological space, respectively. Since $\nu X = X$ holds, we can define

$$\varphi : \mathrm{Hom}_{\mathbf{Top}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathbf{NG}}(X, \nu Y)$$

by letting $\varphi(f) = \nu f$ for every $f\colon X \to Y \in \mathrm{Hom}_{\mathbf{Top}}(X, Y)$. On the other hand, let 1_Y be the identity map from νY to Y, and define

$$\psi : \mathrm{Hom}_{\mathbf{NG}}(X, \nu Y) \longrightarrow \mathrm{Hom}_{\mathbf{Top}}(X, Y)$$

by letting $\psi(g) = 1_Y \circ g$ for every $g\colon X \to \nu Y \in \mathrm{Hom}_{\mathbf{NG}}(X, \nu Y)$. Then we have $\psi\varphi = id$ and $\varphi\psi = id$. Therefore the functor i is a left adjoint to ν. □

Let X and Y be two topological spaces. Let us denoted by $X \times Y$ the product of X and Y.

Definition 5.1.5 (products). Let X and Y be two numerically generated spaces. We define the product of X and Y by $\nu(X \times Y)$, it will be denoted by $X \times_\nu Y$, that is,

$$X \times_\nu Y = \nu(X \times Y).$$

It is clear that $X \times_\nu Y$ is a numerically generated.

Let X and Y be two topological spaces. Let us write $X \coprod Y$ for the coproduct of X and Y with its usual topology. That is, a set A is open in $X \coprod Y$ if and only if $X \cap A$ and $Y \cap A$ are open sets of X and Y, respectively.

Lemma 5.1.6 (coproducts). *Let X and Y be two numerically generated spaces. Then a coproduct space $X \coprod Y$ of X and Y is also a numeically generated space.*

Proof. Let X and Y be two numerically generated spaces. We shall show that $X \coprod Y$ is a numerically generated space. Let A be an open set of $\nu(X \coprod Y)$. Let $P\colon U \to X$ and $Q\colon V \to Y$ be two continuous maps, where U and V are open sets of Euclidean spaces. Let $i\colon X \to X \coprod Y$ and $j\colon Y \to X \coprod Y$ be two the inclusions. Then $(i \circ P)^{-1}(A) = P^{-1}(X \cap A)$ and $(j \circ Q)^{-1}(A) = Q^{-1}(Y \cap A)$ are open sets of U and V, respectively. Thus A is open in $X \coprod Y$ since $X \cap A$ and $Y \cap A$ are open in X and Y, respectively. Therefore $X \coprod Y$ is a numerically generated space since $\nu(X \coprod Y) = X \coprod Y$ holds. □

Let $p\colon X \to Y$ be a surjection from a topologial space X to set Y. Let O_X be the topology of X. Let $p_*(O_X)$ be the quotient topology of Y. Then A is open for $p_*(O_X)$ if and only if $p^{-1}(A)$ is open in X.

Lemma 5.1.7 (quotients). *Let $p\colon X \to Y$ be a surjection from a numerically generated space X to a set Y. Then a quotient space $(Y, p_*(O_X))$ is a numerically generated space, where O_X is a topology of X.*

Proof. We shall show that the quotient space Y is a numerically generated space. Let A be an open set of νY. For any continuous map $Q\colon U \to X$ from an open set U of Euclidean spaces to X, the composite $p \circ Q\colon U \to Y$

is continuous. Thus $(p \circ Q)^{-1}(A) = Q^{-1}(p^{-1}(A))$ is an open set of U. Since $p^{-1}(A)$ is open in U, A is open in Y. Therefore Y is a numerically generated since $Y = \nu Y$ holds. □

Definition 5.1.8 (subspaces). Let X be a numerically generated space. Let A be a subset of X. We define the subspace of X by νA.

Since the category **NG** has equalizers, small products, coequalizers, and small coproducts, we have the following.

Theorem 5.1.9. *The category* **NG** *is complete and cocomplete.*

5.2 Exponentials in NG

In this subsection we prove that the category **NG** is a cartesian closed category by using an adjunction between **Top** and **Diff**.

Definition 5.2.1. Let X and Y be two topological spaces. A map $f\colon X \to Y$ is said to be numerically continuous if the composite $f \circ \sigma\colon \Delta^n \to Y$ is continuous for every singular simplex $\sigma\colon \Delta^n \to X$.

Proposition 5.2.2. *Let $f\colon X \to Y$ be a map between topological spaces. Then the following conditions are equivalent;*

1. *$f\colon X \to Y$ is numerically continuous.*
2. *$f \circ \sigma\colon U \to Y$ is continuous for any continuous map $\sigma\colon U \to X$ from an open subset U of an Euclidean space X.*
3. *$f\colon \nu X \to Y$ is continuous.*
4. *$f\colon DX \to DY$ is smooth.*

Proof. We shall show that *1* implies *2*. Let $f\colon X \to Y$ be a numerically continuous. Let $\sigma\colon U \to X$ be a continuous map from an open subset U of an Euclidean space to X. For any r in U, there exists an open neighborhood V such that V and $\mathrm{Int}\Delta^n$ are diffeomorphic, where the dimension of V is n. Then $f \circ \sigma|V$ is continuous. Therefore $f \circ \sigma$ is continuous. It is clear that *2* implies *3*. We shall show that *3* implies *4*. Let $f\colon \nu X \to Y$ be a continuous map. Let $1_{DX}\colon DX \to DTDX$ be the inclusion map. Then 1_{DX} is smooth. It is clear that the composite $D(f) \circ 1_{DX} = f\colon DX \to DY$ is smooth. We shall show that *4* implies *1*. Let $f\colon DX \to DY$ be a smooth map. Since the identity map $1_Y\colon TDY \to Y$ is continuous, $1_Y \circ T(f) = f\colon TDX \to Y$ is continuous. Let $\sigma\colon \Delta^n \to X$ be a continuous map. There exists an open ball V such that Δ^n is a retract of V. Let $\gamma\colon V \to \Delta^n$ be a retraction. Then the composite $f \circ (\sigma \circ \gamma)$ is continuous. Therefore $f \circ \sigma$ is continuous, since $f \circ (\sigma \circ \gamma)|\Delta^n = f \circ \sigma$ holds. □

Let us denote by $\mathrm{map}(X, Y)$ the set of continuous maps from X to Y equipped with the compact-open topology, and let $\mathrm{smap}(X, Y)$ be the set of numerically continuous maps equipped with the initial topology with respect to the maps

$$\sigma^*\colon \mathrm{smap}(X, Y) \to \mathrm{map}(\Delta^n, Y), \quad \sigma^*(f) = f \circ \sigma,$$

where $\sigma\colon \Delta^n \to X$ runs through singular simplexes of X. More explicitly, the space $\mathrm{map}(X, Y)$ has a subbase consisting of those subsets

$$W(K, U) = \{f \mid f(K) \subset U\},$$

where K is a compact subset of X and U is an open subset of Y. On the other hand, $\mathrm{smap}(X, Y)$ has a subbase consisting of those subsets

$$W(\sigma, L, U) = \{f \mid f(\sigma(L)) \subset U\},$$

where $\sigma\colon \Delta^n \to X$ is a singular simplex, L a compact subset of Δ^n, and U an open subset of Y. Since we have

$$W(\sigma, L, U) \cap \mathrm{map}(X, Y) = W(\sigma(L), U),$$

the inclusion map $\mathrm{map}(X, Y) \to \mathrm{smap}(X, Y)$ is continuous.

Proposition 5.2.3. *The inclusion map* $\mathrm{map}(X, Y) \to \mathrm{smap}(X, Y)$ *is bijective for all Y if, and only if, X is numerically generated.*

Proof. Let the inclusion map $\mathrm{map}(X, Y) \to \mathrm{smap}(X, Y)$ be bijective, for all topological space Y. Since $\mathrm{map}(X, \nu X) \to \mathrm{smap}(X, \nu X)$ is surjective, the identity map $1_X\colon X \to \nu X$ is continuous. Therefore X is a numerically generated space since $X \cong \nu X$ holds. Conversely, let X be a numerically generated space. By Proposition 5.2.2, $f\colon X \to Y$ is numerically continuous if and only if f is continuous. Therefore the inclusion map $\mathrm{map}(X, Y) \to \mathrm{smap}(X, Y)$ is bijective. □

Lemma 5.2.4. *The category* **NG** *contains all CW-complexes. That is, If X is a CW-complex, X and νX are homeomorphic.*

Proof. Let A be an open set of νX. For each λ in Λ, let φ_λ is a characteristic map. Then $\varphi_\lambda^{-1}(A)$ is an open set of D^{n_λ}. Thus A is an open set of X since φ_λ is a quotient map. Therefore X and νX is homeomorphic. □

Proposition 5.2.5. *Suppose that X is a CW-complex. Then the inclusion map* $\mathrm{map}(X,Y) \to \mathrm{smap}(X,Y)$ *is a homeomorphism for any Y.*

Proof. By Proposition 5.2.3 and Lemma 5.2.4, the inclusion map

$$\mathrm{map}(X,Y) \to \mathrm{smap}(X,Y)$$

is bijective. To prove the continuity of its inverse, we have to show that every subset of the form $W(K,U)$ is open in $\mathrm{smap}(X,Y)$. Since X is closure-finite, there exists a finite subcomplex A such that $K \subset A$. Let $\{e_1, \dots e_k\}$ be the set of cells of A, and let $L_i = \varphi_\lambda^{-1}(\overline{e}_\lambda \cap K) \subset \Delta^{n_\lambda}$, where $\varphi_\lambda \colon \Delta^{n_\lambda} \to X$ is a characteristic map for each $\lambda \in \Lambda$. Then we have

$$W(K,U) = W(\varphi_1, L_1, U) \cap \cdots \cap W(\varphi_k, L_k, U)$$

Hence $W(K,U)$ is open in $\mathrm{smap}(X,Y)$. □

Proposition 5.2.6. *For any topological spaces X and Y, we have*

$$D\mathrm{smap}(X,Y) \cong C^\infty(DX, DY).$$

Proof. Let $\sigma \colon U \to \mathrm{smap}(X,Y)$ be a map from an open subset $U \subset \mathbf{R}^n$. Then σ is a plot of $D\mathrm{smap}(X,Y)$ if and only if the composite

$$U \xrightarrow{\sigma} \mathrm{smap}(X,Y) \xrightarrow{\tau^*} \mathrm{map}(\Delta^m, Y)$$

is continuous for every singular simplex $\tau \colon \Delta^m \to X$. But $\tau^*\sigma$ corresponds to the the composite

$$U \times \Delta^m \xrightarrow{\sigma \times \tau} \mathrm{smap}(X,Y) \times X \xrightarrow{\mathrm{ev}} Y,$$

under the homeomorphism

$$\mathrm{map}(U \times \Delta^m, Y) \cong \mathrm{map}(U, \mathrm{map}(\Delta^m, Y)).$$

Thus σ is a plot of $D\mathrm{smap}(X,Y)$ if and only if $\mathrm{ev}(\sigma,\tau)$ is continuous for every τ. which is equivalent to say that σ is a plot of $C^\infty(DX, DY)$. □

For any numerically generated spaces X and Y, let us denote

$$Y^X = \nu\mathrm{smap}(X,Y).$$

The following result says that exponential law holds for Y^X.

Theorem 5.2.7. *There is a natural homeomorphism $Z^{X\times Y} \cong (Z^Y)^X$ induced by the correspondence $f \mapsto \alpha(f)$, where $\alpha(f)(x)(y) = f(x,y)$, $x \in X$, $y \in Y$.*

Proof. In fact, the map

$$\alpha\colon Z^{X\times Y} \to (Z^Y)^X$$

is identical with the composite of homeomorphism below:

$$\begin{aligned} Z^{X\times Y} &= \nu\mathrm{smap}(X\times Y, Z) \\ &= TC^\infty(D(X\times Y), DZ) \\ &= TC^\infty(DX\times DY, DZ) \\ &\cong TC^\infty(DX, C^\infty(DY, DZ)) && (1) \\ &= TC^\infty(DX, D\mathrm{smap}(Y,Z)) \\ &= \nu\mathrm{smap}(X, \mathrm{smap}(Y,Z)) \\ &\cong \nu\mathrm{smap}(X, \nu\mathrm{smap}(Y,Z)) = (Z^Y)^X && (2) \end{aligned}$$

where (1) is induced by the smooth isomorphism given by Theorem 3.2.6, and (2) is induced by the numerical isomorphism $\mathrm{smap}(Y,Z) \to \nu\mathrm{smap}(Y,Z)$. □

ince the homeomorphism $Z^{X\times Y} \cong (Z^Y)^X$ implies a bijection

$$\mathrm{Hom}_{\mathbf{NG}}(X\times Y, Z) \cong \mathrm{Hom}_{\mathbf{NG}}(X, Z^Y),$$

we have the following corollary.

Corollary 5.2.8. *The category* **NG** *is a cartesian closed category.*

5.3 Model structure of NG

In this subsection we shall show that the category **NG** has a finitely generated model structure and also that the inclusion functor $i\colon \mathbf{NG} \to \mathbf{Top}$ is a Quillen equivalence with respect to the model structure of **Top** described in Theorem 4.4.1.

By Lemma 5.2.4, a CW complex is a numerically generated space, we can define a homotopy group $\pi_n(X, x_0)$ by the set of homotopy classes of continuous maps $(I^n, \partial I^n) \to (X, x_0)$.

Definition 5.3.1. Let X and Y be two numerically generated spaces. A continuous map $f\colon X \to Y$ is called a weak homotopy equivalence in **NG** if it induces an isomorphism of homotopy groups

$$f_* : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$$

for all $n \geq 0$ and $x \in X$. That is, $f\colon X \to Y$ between numerically generated spaces is a weak homotopy equivalence in **NG** if and only if it is a weak homotopy equivalence in **Top**.

Let I' be the set of boundary inclusions $S^{n-1} \to D^n,\ n \geq 0$ and J be the set of inclusions $D^n \times \{0\} \to D^n \times I,\ n \geq 0$. Then **Top** has a finitely generated model structure with I' as the set of generating cofibrations, with J as the set of generating trivial cofibrations, and with $W_{\mathbf{Top}}$ as the class of weak equivalences.

The full subcategory **NG** of **Top** holds the following conditions:

1. **NG** is complete and cocomplete by Theorem 5.1.9.

2. By Proposition 5.1.4, there exists an adjunction (i, ν, φ) from **NG** to **Top**, where φ is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Top}}(X, Y) \to \mathrm{Hom}_{\mathbf{NG}}(X, \nu Y).$$

3. The unit of the adjunction $\eta\colon X \to \nu X$ is an isomorphism for every $X \in \mathbf{NG}$ by the definition of numerically generated spaces.

4. The classes I' and J are contained in **NG** by Lemma 5.2.4.

Let $W_{\mathbf{NG}}$ be the set of weak homotopy equivalences in **NG**. Then we have the following by Theorem 2.4.1.

Theorem 5.3.2. *The category* **NG** *has a finitely generated model structure with I' as the set of generating cofibrations, J as the set of generating trivial cofibrations, and $W_{\mathbf{NG}}$ as the class of weak equivalences. Moreover the adjunction $(i, \nu, \varphi)\colon \mathbf{NG} \to \mathbf{Top}$ is a Quillen adjunction.*

Lemma 5.3.3. *Let X be a topological space. Let 1_X be the identity map from νX to X. Then 1_X is a weak homotopy equivalence.*

Proof. We shall show that $1_{X*}\colon \pi_n(\nu X, x) \to \pi_n(X, x)$ is a bijection. Let $[f]$ be a element of $\pi_n(X, x)$. Then we have $[\nu f] \in \pi_n(\nu X, x)$, since νf is a map from $S^n = \nu S^n$ to νX. Now, $1_{X*}([\nu f]) = [1_X \circ \nu f] = [f]$. Thus 1_{X*} is a surjection. On the other hand, for any $[g]$ and $[g']$ in $\pi_n(\nu X, x)$ such that $1_{X*}([g]) = 1_{X*}([g'])$, there is a homotopy $H\colon S^{n-1} \times I \to X$ from $1_X \circ g$ to $1_X \circ g'$, which induces a homotopy $\nu H\colon S^{n-1} \times I \to \nu X$ from g to g'. Thus 1_{X*} is a bijection. □

Hence we have the following by Proposition 2.4.3 and Lemma 5.3.3.

Theorem 5.3.4. *The Quillen adjunction* $(i, \nu, \varphi)\colon \mathbf{NG} \to \mathbf{Top}$ *is a Quillen equivalence.*

We turn to the case of pointed spaces. Let $\mathbf{Top}_*$ be the category of pointed topological spaces. By [7, 2.4.20], there is a finitely generated model structure on the category $\mathbf{Top}_*$, with generating cofibrations I'_+ and generating trivial cofibrations J_+. Then we have the following by Theorem 2.4.2 and Proposition 2.4.3.

Corollary 5.3.5. *There is a finitely generated model structure on the category* $\mathbf{NG}_*$ *of pointed numerically generated spaces, with generating cofibrations* I'_+ *and generating trivial cofibrations* J_+*. Moreover, the inclusion functor* $i_*\colon \mathbf{NG}_* \to \mathbf{Top}_*$ *is a Quilen equivalence.*

References

[1] C. Berger and I. Moerdijk. Axiomatic homotopy theory for operads. Comment. Math. Helv. , 78(4):805-831, 2003.

[2] S. E. Crans. Quillen closed model structures for sheaves. J. Pure Appl. Algebra , 101(1):35-57, 1995.

[3] W. G. Dwyer and J. Spalinski. Homotopy theories and model categories, Handbook of Algebraic Topology, Elsevier, 1995, 73-126.

[4] T. Haraguchi and K. Simakawa, A model structure on the category of diffeological spaces (in preparation).

[5] T. Haraguchi, On model structure for coreflective subcategories of a model category, Mathematical Journal of Okayama University, 2013.

[6] P. S. Hirschhorn, Model categories and their localizations, Mathematical Surveys and Monographs, 99, American Mathematical Society, 2003.

[7] M. Hovey, Model categories, Mathematical Surveys and Monographs, 63, American Mathematical Society, Providence, RI, 1999.

[8] P. Iglesias-Zemmour, Diffeology, CNRS, Marseille, France, and The Hebrew University of Jerusalem, Israel.

[9] L. G. Lewis, Jr., The stable category and generalized Thom spectra, Ph.D. thesis, University of Chicago, 1978.

[10] J. W. Milnor. Topology from the differentiable viewpoint. Princeton Landmarks.

[11] D. G. Quillen, Homotopical Algebra, SLNM 43, Springer, Berlin (1967).

[12] D. G. Quillen, Rational homotopy theory, Ann. of Math. 90 (1969), 205-295.

[13] K. Shimakawa, K. Yoshida, and T. Haraguchi, Homology and cohomology via bifunctors, arXiv:1010.3336v1.

[14] Souriau, J. M. Groupes differentiels. In Differential geometrical methods in mathematical

[15] E. H. Spanier, Algebraic Topology, McGraw-Hill, New York (1966).

[16] O. Wyler, Convenient categories for topology, General Topology and Appl. 3(1973), 225-242.