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SEMI-CONVERGENCE OF FILTERS AND NETS

RAJA MOHAMMAD LATIF

ABSTRACT. In 1963, N. Levine introduced the concept of semi-open set and semi-continuity. Semi-convergence and semi-compactness were first introduced, investigated and characterized by C. Dorsett in 1978 and 1981 respectively. In this paper semi-convergence and semi-clusterence of filters are introduced, investigated and characterized.

Throughout, for a subset A of a topological space X, $Cl(A)$ denotes the closure of A in X; no map is assumed to be continuous or surjective unless mentioned explicitly. Moreover X and Y denote topological spaces. For more details on nets and filters we refer the reader to [Willard; 1970].

Definition 1. Let (X, τ) be a topological space and let $A \subseteq X$. Then A is semi-open if and only if there exists an open set U in X such that $U \subseteq A \subseteq Cl(U)$. Let $SO(X)$ denote the class of all semi-open sets in a topological space X.

Remark 2. N. Levine proved that a set A in a topological space X is semi-open if and only if A is contained in the closure of the interior of A in X. We note that every open set in a topological space X is a semi-open set but clearly a semi-open set may not be an open set in X. He also proved that the union of a collection of semi-open sets in a topological space is always semi-open. It is clear that a nowhere dense set in a space X is never semi-open in X and the complement of a nowhere dense set in X is always semi-open in X. In particular for any semi-open set S in a space X, the difference of the closure of S and S is not semi-open in X. The intersection of any family of semi-closed sets in a space X is always semi-closed in X. We observe that the intersection of two semi-open sets in a space X may not be a semi-open set in X. The semi-interior of a set A in a topological space X, denoted by $sInt(A)$, is the union of all semi-open sets contained

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in A. We note that a set A of a space X is semi-open in X if and only if $A = sInt(A)$.

Definition 3. If (X, τ) is a topological space, $A \subseteq X$ and $x \in X$, then x is a semi-limit point of A if and only if every semi-open set containing x contains a point of A different from x. The union of A and the set of all semi-limit points of A is called the semi-closure of A and is denoted by $sCl(A)$.

Definition 4. Let X be a topological space. We say that a set $M_x \subseteq X$ is a semi-neighborhood of a point $x \in X$ if and only if there exists a semi-open set S such that $x \in S \subseteq M_x$.

Definition 5. Let (X, τ) be a topological space. For each $x \in X$, let $S(x) = \{A \in SO(X) : x \in A\}$. Then $S(x)$ has the finite intersection property. Thus $S(x)$ *is* a filter subbasis on X. Let S_x be the filter generated by $S(x)$, i.e., $S_x = \{A \subseteq X$: there exists $\mu \subseteq S(x)$ such that μ is finite and $\cap\mu \subseteq A\}$. We will call S_x the semi-neighborhood filter at x.

Definition 6. Let (X, τ) be a topological space. Let $\mathcal{F}$ be a filter on X. Let $x \in X$. We say that $\mathcal{F}$ semi-converges to x if and only if $\mathcal{F}$ contains S_x, that is, if and only if $\mathcal{F}$ is finer than the semi-neighborhood filter at x.

Definition 7. Let (X, τ) be a topological space. Let $\mathcal{F}$ be a filter on X, and let $x \in X$. We say that $\mathcal{F}$ has x as a semi-cluster point (or, $\mathcal{F}$ semi-clusters at x) if and only if every $F \in \mathcal{F}$ meets each $S \in S(x)$.

In the following we consider an example for elaboration.

Example 8. Consider R with the usual metric. Let $A = \{\frac{1}{n}|n \geq 1\} \cup \{-\frac{1}{n}|n \geq 1\} \subset R$, $\mathcal{F} = \{F \subseteq R | A \subseteq F\}$, $U = \overset{\infty}{\underset{n=1}{\cup}} \left(\frac{1}{n+1}, \frac{1}{n}\right)$ and $S = U \cup \{0\}$. Then we have $U \subset S \subset Cl(U)$. So $S \in S_{(0)}$, but $S \cap A = \phi$. This implies that $\mathcal{F}$ does not semi-cluster at 0.

Proposition 9. *Let (X, τ) be a topological space. Let $\mathcal{F}$ be a filter on X, and let $x \in X$ such that $\mathcal{F}$ has x as a semi-cluster point. Then $x \in \cap\{sCl(F) : F \in \mathcal{F}\}$.*

Proof. Easy. □

Proposition 10. *If (X, τ) is a topological space and $\mathcal{F}$ is a filter on X such that $\mathcal{F}$ semi-converges to x in X, then $\mathcal{F}$ converges to x.*

Proof. The straightforward proof is omitted. □

The following example shows that the converse of proposition 10 may not hold in general.

Example 11. Let $X=\{1,2,3,4\}$. Let $\tau=\{\phi,\{1\},\{1,2\},\{1,2,3\},X\}$ be a topology on X. Consider the filter $\mathcal{F}=\{\{1,2\},\{1,2,3\},\{1,2,4\},X\}$ on X. The neighborhood filter at 3 is $N_3=\{\{1,2,3\},X\}$. Clearly $N_3\subseteq\mathcal{F}$ implies $\mathcal{F}$ converges to 3. Now $Cl(\{1\})=X$ implies that $\{1,3\}\in S_3$. But $\{1,3\}\notin\mathcal{F}$. Hence $\mathcal{F}$ does not semi-converge to 3.

Definition 12. Let (X,τ) be a topological space. Let $\mathcal{F}$ be a filter on X, and let $x\in X$. We say that $\mathcal{F}$ has x as a strong semi-cluster point (or $\mathcal{F}$ strongly semi-clusters at x) if and only if every $F\in\mathcal{F}$ meets each $S\in S_x$.

Proposition 13. *If (X,τ) is a topological space and $\mathcal{F}$ is a filter on X such that $\mathcal{F}$ strongly semi-clusters at x in X, then $\mathcal{F}$ semi-clusters at x.*

Proof. Obvious. □

The following example shows that the converse of proposition 13 is not true in general.

Example 14. Consider R with the usual metric. Let $A=(-1,0)\cup(0,1)\subset R$, and $\mathcal{F}=\{F\subseteq R|A\subseteq F\}$. Then $\mathcal{F}$ is a filter on R. Clearly $\mathcal{F}$ semi-clusters at 0. Note that $\{0\}=(-1,0]\cap[0,1)$ being the intersection of two semi-open sets is in S_0. But $A\cap\{0\}=\phi$. Hence $\mathcal{F}$ does not have 0 as a strongly semi-cluster point.

Definition 15. If $\mathcal{F}$ is a filter on X and $f:X\longrightarrow Y$ is a single-valued function where X and Y are topological spaces, then $f(\mathcal{F})$ is the filter on Y having for a base the sets $f(F)$, $F\in\mathcal{F}$.

Definition 16. Let $f:X\longrightarrow Y$ be a single-valued function where X and Y are topological spaces. Then $f:X\longrightarrow Y$ is called semi-continuous if and only if, for any open set V in Y, $f^{-1}(V)\in SO(X)$.

Theorem 17 ([Latif; 1993]). *Let $f:X\longrightarrow Y$ be a single-valued function where X and Y are topological spaces. Then $f:X\longrightarrow Y$ is semi-continuous if and only if, for each x in X and each neighborhood U of $f(x)$, there is a semi-neighborhood V of x such that $f(V)\subseteq U$.*

Theorem 18. *Let $f:X\longrightarrow Y$ be a single-valued function where X and Y are topological spaces. Then f is semi-continuous at $x^*\in X$ if and only if whenever $\mathcal{F}$ semi-converges to x^* in X then $f(\mathcal{F})$ converges to $f(x^*)$ in Y.*

Proof. Suppose f is semi-continuous at x^* and $\mathcal{F}$ semi-converges to x^*. Let V be any neighborhood of $f(x^*)$ in Y. Then for some semi-neighborhood U of x^* in X, $f(U) \subseteq V$. Then since $U \in \mathcal{F}$, $V \in f(\mathcal{F})$. Hence $f(\mathcal{F})$ converges to $f(x^*)$ in Y.

Conversely, suppose whenever $\mathcal{F}$ semi-converges to x^* in X then $f(\mathcal{F})$ converges to $f(x^*)$ in Y. Let $\mathcal{F}$ be the filter of all semi-neighborhoods of x^* in X. Then each neighborhood V of $f(x^*)$ belongs to $f(\mathcal{F})$. It follows that for some semi-neighborhood U of x^*, $f(U) \subseteq V$. Thus f is semi-continuous at x^*. □

Definition 19. Let X and Y be topological spaces. We say that a function $f : X \longrightarrow Y$ is irresolute at a point $x \in X$ if and only if for each semi-open subset T of Y containing $f(x)$, there exists a semi-open subset S of X such that $x \in S$ and $f(S) \subseteq T$. A function $f : X \longrightarrow Y$ will be called an irresolute if it is irresolute at each point $x \in X$.

In the following we give an equivalent definition of an irresolute function.

Definition 20. Let X and Y be topological spaces. Then a function $f : X \longrightarrow Y$ is said to be an irresolute if and only if for any semi-open subset S of Y, $f^{-1}(S)$ is semi-open in X.

Theorem 21 ([Latif; 1993]). *Let X and Y be topological spaces. Then a function $f : X \longrightarrow Y$ is irresolute if and only if for each x in X and each semi-neighborhood U of $f(x)$, there is a semi-neighborhood V of x such that $f(V) \subseteq U$.*

Theorem 22. *Let $f : X \longrightarrow Y$ be a single-valued function where X and Y are topological spaces. Then f is an irresolute at $x^* \in X$ if and only if whenever a filter $\mathcal{F}$ on X semi-converges to x^* in X then $f(\mathcal{F})$ semi-converges to $f(x^*)$ in Y.*

Proof. Suppose f is an irresolute at x^* and $\mathcal{F}$ semi-converges to x^*. Let V be any semi-neighborhood of $f(x^*)$ in Y. Then for some semi-neighborhood U of x^* in X, $f(U) \subseteq V$. Then since $U \in \mathcal{F}$, so $V \in f(\mathcal{F})$. Thus $f(\mathcal{F})$ semi-converges to $f(x^*)$ in Y.

Conversely, suppose whenever $\mathcal{F}$ semi-converges to x^* in X then $f(\mathcal{F})$ semi-converges to $f(x^*)$ in Y. Let $\mathcal{F}$ be the filter of all semi-neighborhoods of x^* in X. Then each semi-neighborhood V of $f(x^*)$ belongs to $f(\mathcal{F})$, so for some semi-neighborhood U of x^*, $f(U) \subseteq V$. Thus f is an irresolute at x^*. □

Definition 23. Let (X, τ) be a topological space. Let $(x_i : i \in I)$ be a net in X, and let $x \in X$. Then $(x_i : i \in I)$ semi-converges to x if and only if $(x_i : i \in I)$ is eventually in every semi-open set containing x.

Definition 24. If $\mathcal{F}$ is a filter on X, and $\Lambda_{\mathcal{F}} = \{(x, F) : x \in F \in \mathcal{F}\}$. Then $\Lambda_{\mathcal{F}}$ is directed by the relation $(x_1, F_1) \leq (x_2, F_2)$ if and if $F_2 \subseteq F_1$, so the map $P : \Lambda_{\mathcal{F}} \longrightarrow X$ defined by $P(x, F) = x$ is a net in X. It is called the net based on $\mathcal{F}$.

Theorem 25. *Let X be a topological space. Then a filter $\mathcal{F}$ semi-converges to x in X if and only if the net based on $\mathcal{F}$ semi-converges to x.*

Proof. Suppose $\mathcal{F}$ semi-converges to x. If S is a semi-neighborhood of x, then $S \in \mathcal{F}$. Pick $p \in S$. Then $(p, S) \in \Lambda_{\mathcal{F}}$ and if $(q, T) \geq (p, S)$, then $q \in T \subseteq S$. Thus the net based on $\mathcal{F}$ semi-converges to x.

Conversely, suppose the net based on $\mathcal{F}$ semi-converges to x. Let S be a semi-neighborhood of x. Then for some $(p^*, F^*) \in \Lambda_{\mathcal{F}}$, we have $(p, F) \geq (p^*, F^*)$ implies $p \in S$. But then $F^* \subseteq S$; otherwise, there is some $q \in F^* - S$, and then $(q, F^*) \geq (p^*, F^*)$, but $q \notin S$. Hence $S \in \mathcal{F}$, so $\mathcal{F}$ semi-converges to x. □

Definition 26. If $(x_i : i \in I)$ is a net in X, the fillter generated by the filter base $\mathfrak{C}$ consisting of the sets $B_{i_0} = \{x_i | i \geq i_0\}$, $i_0 \in I$, is called the filter generated by $(x_i : i \in I)$.

Theorem 27. *A net $(x_i : i \in I)$ semi-converges to x in X if and only if the filter generated by $(x_i : i \in I)$ semi-converges to x.*

Proof. The net $(x_i : i \in I)$ semi-converges to x if and only if each semi-neighborhood of x contains a tail of $(x_i : i \in I)$. Since the tails of $(x_i : i \in I)$ form a base for the filter generated by $(x_i : i \in I)$, the result follows. □

Definition 28. Let (X, τ) be a topological space. Let $(x_i : i \in I)$ be a net in X, and let $x \in X$. Then x is a semi-cluster point of $(x_i : i \in I)$ if and only if $(x_i : i \in I)$ is frequently in every semi-open set containing x.

Definition 29. A topological space (X, τ) is called semi-compact if and only if every semi-open cover of X, i.e., a cover of X by semi-open sets in X has a finite subcover.

Theorem 30. *The following conditions are equivalent for a topological space X.*

(a) X is semi-compact.

(b) Every filter in X has a semi-cluster point.

(c) Every net in X has a semi-cluster point.

Proof. $(a) \Longrightarrow (b)$. If $\mathcal{F}$ is a filter, then $\mathcal{F}^* = \{sCl(S) : S \in \mathcal{F}\}$ is a collection of semi-closed sets with the finite intersection property. Hence it is fixed by theorem 3.3 of [Dorsett; 1981] and each point in its intersection is a semi-cluster point.

$(b) \Longrightarrow (c)$. Given a net, its associated filter has a semi-cluster point; this is a semi-cluster point of the net, by definition.

$(c) \Longrightarrow (b)$. Let $\mathcal{F}$ be a filter on X. For any $F \in \mathcal{F}$, we fix a point $p_F \in F$. We give an order to $\mathcal{F}$, $E \leq F \Longleftrightarrow E \supseteq F$. Then $(\mathcal{F}, \leq)$ is a directed set. So, $(p_F : F \in \mathcal{F})$ is a net. Hence there exists a semi-cluster point p of $(p_F : F \in \mathcal{F})$. Then, p is a semi-cluster point of $\mathcal{F}$.

$(b) \Longrightarrow (a)$. Let $\mathcal{C}$ be a collection of semi-closed sets with finite intersection property. Let $\mathcal{B}$ be the set of all finite intersections of members of $\mathcal{C}$. Then clearly $\mathcal{B}$ is a filterbase for a filter $\mathcal{F}$ and $\mathcal{C}$ is included in $\mathcal{F}$. Let x be a semi-cluster point of $\mathcal{F}$. Then $x \in \cap\{sCl(S) : S \in \mathcal{F}\} \subseteq \cap\{sCl(S) : S \in \mathcal{C}\}$. Thus $\mathcal{C}$ is fixed, and X is semi-compact by theorem 3.3 of [Dorsett; 1981]. □

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Raja Mohammad Latif
Department of mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia.
e-mail address: raja@kfupm.edu.sa

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