Math. J. Okayama Univ. **54** (2012), 97–131

# HOMOGENIZATION OF NON-LINEAR VARIATIONAL PROBLEMS WITH THIN INCLUSIONS

ABDELAZIZ AÏT MOUSSA AND LOUBNA ZLAÏJI

ABSTRACT. We are concerned in this work with the asymptotic behavior of an assemblage whose components are a thin inclusion with higher rigidity modulus included into an elastic body. We aim at finding the approximating energy functional of the above structure in a Γ-convergence framework, and making use also of the subadditive theorem and the blow-up method.

## 1. INTRODUCTION

This work focuses on a junction problem of two different kind of materials. The first one is an elastic body divided into two parts and the other is a material intercalated between them and characterized by its small thickness equal to $2\varepsilon$ where $0 < \varepsilon << 1$ and its higher rigidity modulus $\mu >> 1$. The stored energy of the whole body is then dependent on the double parameter $\delta = (\varepsilon, \mu)$ and modelized by the family of functionals $F_\delta : W_0^{1,p}(\mathcal{O}; \mathbb{R}^N) \to [0, \infty)$ defined by

$$F_\delta(u) := \int_{\mathcal{O}} f_\delta(x, Du)\, dx, \tag{1.1}$$

where $\mathcal{O}$ is an open bounded subset of $\mathbb{R}^n$, $n \geq 3$, which represents the reference configuration of the assemblage and the stored energy density $f_\delta : \mathbb{R}^n \times \mathbb{R}^{N\times n} \to [0, +\infty)$, $N \geq 1$, is a Borel function satisfying an appropriate growth and p-Lipschitz conditions, with $1 < p < +\infty$ (we refer to (2.4)). For instance, we may occur that the structure is clamped on its boundary and subjected to external forces $g$, so that the total energy is given by

$$G_\delta(u) := F_\delta(u) - \int_{\mathcal{O}} g(x).u(x)dx.$$

Our purpose is to characterize the asymptotic behavior of the family of problems

$$m_\delta = \min\{G_\delta(u),\ u \in W_0^{1,p}(\mathcal{O}, \mathbb{R}^N)\},$$

whenever $\delta \to (0, +\infty)$ making use of the $\Gamma-$convergence approach.

---

*Mathematics Subject Classification.* 49J45; 60G10.

*Key words and phrases.* blow-up, Γ-convergence, subadditive theorem.

As stated by the well known *fundamental Theorem of* $\Gamma-$*convergence* (see Theorem 3.2), if the sequence of functionals $(G_\delta)$ is equi-coercive and admits a $\Gamma-$limit $G$, then we have the convergence of minima

$$\lim_\delta \inf\{G_\delta(u) \,:\, u \in W_0^{1,p}(\mathcal{O}, \mathbb{R}^N)\} = \min\{G(u) \,:\, u \in \mathrm{D}\},$$

where D is the domain of definition of G. Moreover, we have also convergence of minimizers, i.e. if $(u_\delta)$ is such that $\lim_\delta G_\delta(u_\delta) = \lim_\delta \inf G_\delta$ and if $u_\delta \to u_0$, then $u_0$ is a minimizer for G. The convergence type of the sequence $u_\delta$ must be chosen in order to provide the equi-coercivity of the sequence of functionals $(G_\delta)$, and it is often the strong $L^p$-metric. However, the stability of $\Gamma$-convergence with respect to continuous perturbations leads to looking for the limit of the sequence $(F_\delta)$ (in the sense of $\Gamma$-convergence) as the latest and the most important step in order to apply the fundamental Theorem of $\Gamma-$convergence, and it will be the subject of the present study.

Many works on dimension reduction of thin films have been investigated through a $\Gamma-$convergence analysis, and we can mention here for instance [5, 6, 8, 20, 12, 14] and the references therein. But, as a particular motivation of this work, Acerbi, Buttazzo and Percivale in [2] have considered a thin inclusion problem in elasticity by assuming the energy density to be convex. There is also an other work by Licht and Michaille in [28, 29] who have been interested in a junction problem of an elastic body constituted of adherent and adhesive materials with nonconvex bulk energy density both in the deterministic and the stochastic case; they assume the stiffness of the adhesive to be too small. The present study will be different of the first work by making no convexity assumption on the energy density and also by working with the strain tensor $Du$ instead of the linearized strain tensor $e(u)$ (the last is considerably more complicated), and of the second one by supposing the intermediate body to have a higher rigidity.

We aim at finding the $\Gamma$-limit $F$ of the sequence of functionals $(F_\delta)$, and the main tools will be, like in [29], the global subadditive theorem of Licht and Michaille in [28, 29] and the blow-up technique introduced by Fonseca and Müller in [26] which is especially exploited to prove the so-called liminf inequality (see Sec. 5.1).

We shall prove that in one hand the $\Gamma$-limit depends strongly on the product thickness-rigidity of the layer, and in the other hand, the critical case is whenever the effects of thickness and rigidity are equilibrate (i.e, whenever $0 < \lim \varepsilon\mu < \infty$). In this particular case, as in [2], a new homogenized density energy appears on the intermediate face between the two parts of the elastic body, and it possesses suitable properties. Namely, we expect on the quasiconvexity to get existence of the minimization limit-problem $\min F(u)$. Let us also notice that this energy density depends only on the first derivatives of the displacement u, which means that we have no plate or shell phenomena. We point out that an other interesting case appears in the convex case when replacing the strain tensor $Du$ with the linearized strain tensor $e(u)$, especially if $\lim \varepsilon\mu = +\infty$ and $\lim \varepsilon^{p+1}\mu < +\infty$ (see [2]). The limit energy density in this situation depends on the second derivatives of the displacement u which means that the inclusion behaves like a shell.

The paper is organized as follows: Sec. 2 is devoted to the problem statement and main result. In Sec. 3, we give some preliminaries which are necessary in the sequel. In Sec. 4, we establish some properties of the limit integrand $f_2^{hom}$ defined in (2.10), and finally, Sec. 5 treats proof of the main Theorem.

## 2. Problem statement and main result

2.1. **Notations.** In the sequel, $\mathcal{L}^d$ is the Lebesgue measure in $\mathbb{R}^d$, $\widetilde{Y}$ stands for the unit cube in $\mathbb{R}^{n-1}$, $x = (\widetilde{x}, x_n)$ is a generic point in $\mathbb{R}^n$, where $\widetilde{x} \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$, $Q_r(x)$ is the cube of center $x$ and side $r$, for a given real $1 < p < \infty$ we denote $p' = \frac{p}{p-1}$ his conjugate exponent, if $\mathcal{O}$ is an open bounded subset of $\mathbb{R}^n$ and $u \in W^{1,p}(\mathcal{O};\mathbb{R}^N)$, where $N \geq 1$, $Du$ denotes the Jacobian matrix in $\mathbb{R}^{N\times n}$ defined by

$$Du := \left[\frac{\partial u_i}{\partial x_j}\right]_{\substack{1\le i\le N\\ 1\le j\le n}},$$

and we shall need also the notation

$$D_{\tilde{x}}u := \left[\frac{\partial u_i}{\partial x_j}\right]_{\substack{1\le i\le N\\ 1\le j\le n-1}},$$

for a matrix $\xi = (\xi', \xi_N)$, where $\xi' \in \mathbb{R}^{N\times n-1}$ and $\xi_N \in \mathbb{R}^N$, we denote $\xi_T = (\xi', 0)$, and finally, C is a generic constant which is independent on the varying parameters $\varepsilon$ and $\mu$, and may be different from line to line.

2.2. **Problem statement.** Let $n \geq 2$ and $N \geq 1$, let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary denoted by $\partial\mathcal{O}$ and $S$ be an open bounded convex subset of $\mathbb{R}^{n-1}$ with Lipschitz boundary $\partial S$. For every $\varepsilon > 0$, we suppose that there exists a part $B_\varepsilon$ of $\mathcal{O}$ which takes the form:

$$B_\varepsilon = S\times ]-\varepsilon,\varepsilon[, \tag{2.1}$$

and we set

$$\mathcal{O}_\varepsilon = \mathcal{O}\setminus B_\varepsilon. \tag{2.2}$$

Let us define a function $f_\delta : \mathcal{O} \times \mathbb{R}^{N\times n} \to [0, +\infty)$ so that:

$$f_\delta(x, .) = \begin{cases} f_1(x, .) & \text{a.e. } x \in \mathcal{O}_\varepsilon, \\ \mu f_2(\frac{\tilde{x}}{\varepsilon}, .) & \text{a.e. } x \in B_\varepsilon, \end{cases} \tag{2.3}$$

where for each $i \in \{1, 2\}$, $f_i$ is a Carathéodory function and there exist constants $\alpha_i, \beta_i, c_i > 0$ such that:

$$\alpha_i|\xi|^p \leq f_i(z, \xi) \leq \beta_i(1 + |\xi|^p) \text{ (growth condition)}, \tag{2.4}$$

$$|f_i(z, \xi) - f_i(z, \xi')| \leq c_i|\xi - \xi'|(1 + |\xi|^{p-1} + |\xi'|^{p-1}) \text{ (p-Lipschitz condition)}$$

for every $z \in \mathbb{R}^n$ for i=1 and $z \in \mathbb{R}^{n-1}$ for i=2 and every $\xi, \xi'$ in $\mathbb{R}^{N\times n}$; in particular, we assume that

$$f_2 \text{ is } \widetilde{Y}\text{-periodic with respect to the variable } \tilde{x}. \tag{2.5}$$

2.3. **Main result.** Let us recall that the subject of our study is the asymptotic behavior of the family of functionals $F_\delta : W_0^{1,p}(\mathcal{O}; \mathbb{R}^N) \to [0, \infty)$ defined by:

$$F_\delta(u) = \int_{\mathcal{O}} f_\delta(x, Du)\, dx. \tag{2.6}$$

In particular, we look for the corresponding limit in the sense of Γ-convergence (definition in Sec. 3.1) with respect to the $L^p(\mathcal{O}; \mathbb{R}^N)$-metric, whenever $\varepsilon$ and $\mu$ tend simultaneously to zero and $+\infty$. To this aim, as usual we extend $F_\delta$ in the whole space $L^p(\mathcal{O}; \mathbb{R}^N)$ as follows:

$$F_\delta(u) = \begin{cases} \int_{\mathcal{O}} f_\delta(x, Du)\, dx & \text{if } u \in W_0^{1,p}(\mathcal{O}; \mathbb{R}^N), \\ +\infty & \text{if } u \in L^p(\mathcal{O}; \mathbb{R}^N) \setminus W_0^{1,p}(\mathcal{O}; \mathbb{R}^N). \end{cases} \tag{2.7}$$

Now, we are in measure to state the main result. To do so, assume that

$$\lim_{\delta \to (0,\infty)} \varepsilon\mu = \eta \in [0, +\infty], \tag{2.8}$$

and define the effective domain of Γ-limit as:

$$\operatorname{dom}(\Gamma\text{-}\liminf F_\delta) := \{u \in L^p(\mathcal{O}; \mathbb{R}^N) \,:\, (\Gamma\text{-}\liminf_\delta F_\delta)(u) < \infty\}. \tag{2.9}$$

Let ${f_2}^{hom}$ be the function defined for every $\xi \in \mathbb{R}^{N\times n}$ by:

$$\begin{aligned} {f_2}^{hom}(\xi) := \inf_k \inf \Bigg\{ & \frac{1}{k^{n-1}} \int_{k\widetilde{Y}\times]-1,1[} f_2(\tilde{y}, \xi_T + Du(y))\, dy \,: \\ & u \in W_0^{1,p}(k\widetilde{Y}\times]-1, 1[; \mathbb{R}^N) \Bigg\}. \end{aligned} \tag{2.10}$$

Then, the principal result is given through the following Γ-limit theorem:

**Theorem 2.1.** *Assume that the functions $f_i$ satisfy conditions (2.4) and (2.5). Then*

$$\operatorname{dom}(\Gamma - \liminf_{\delta} F_\delta) = D$$

$$:= \begin{cases} D_1 = W_0^{1,p}(\mathcal{O};\mathbb{R}^N), & \text{if } \eta = 0, \\ D_2 = \{u \in D_1 \,:\, u_{|S} \in W^{1,p}(S;\mathbb{R}^{N\times n})\}, & \text{if } 0 < \eta < \infty, \\ D_3 = \{u \in D_1 \,:\, u_{|S} = 0\}, & \text{if } \eta = \infty, \end{cases} \tag{2.11}$$

*where by notation, $u_{|S}(\tilde{x}) = u(\tilde{x}, 0)$, and the sequence of functionals $F_\delta$ defined in (2.7) $\Gamma$-converges for the $L^p(\mathcal{O};\mathbb{R}^N)$-metric to the functional $F$ defined in $L^p(\mathcal{O};\mathbb{R}^N)$ by:*

(1) *If $\eta = 0$ , then*

$$F(u) := \begin{cases} \int_{\mathcal{O}} \mathcal{Q}f_1(x, Du)\, dx & \text{if } u \in D_1, \\ +\infty & \text{if } u \in L^p(\mathcal{O};\mathbb{R}^N) \setminus D_1. \end{cases}$$

(2) *If $0 < \eta < +\infty$ , then*

$$F(u) := \begin{cases} \int_{\mathcal{O}} \mathcal{Q}f_1(x, Du)\, dx + \eta \int_S {f_2}^{hom}(D_{\tilde{x}}(u_{|S}))d\widetilde{x} & \text{if } u \in D_2, \\ +\infty & \text{if } u \in L^p(\mathcal{O};\mathbb{R}^N) \setminus D_2. \end{cases}$$

(3) *If $\eta = \infty$ and $f_2$ is p-homogeneous with respect to $\xi$, then*

$$F(u) := \begin{cases} \int_{\mathcal{O}} \mathcal{Q}f_1(x, Du)\, dx + & \text{if } u \in D_3, \\ +\infty & \text{if } u \in L^p(\mathcal{O};\mathbb{R}^N) \setminus D_3. \end{cases}$$

*In all cases, $\mathcal{Q}f_1$ is the quasiconvex envelope of $f_1$ (defined in (3.1)).*

**Lemma 2.1.** *$D_2$ endowed with the following norm*

$$\|u\|_2 = \|u\|_{W^{1,p}(\mathcal{O},\mathbb{R}^N)} + \|u_{|S}\|_{W^{1,p}(S,\mathbb{R}^N)}, \tag{2.12}$$

*is a reflexive Banach space. Moreover, $\mathcal{C}^1(\overline{\mathcal{O}})$ is dense in $D_2$.*

*Proof.* It's easy to check that $D_2$ is a reflexive Banach space. The proof of density relies on the density in $D_2$ of functions which are almost everywhere independent of $x_n$ in an interval $[-\alpha, \alpha]$ where $\alpha$ depends on the function (this density is the nonlinear version of the one in [27], p:299). To prove it, let $u \in D_2$ and $\alpha > 0$. Define the function $u_\alpha$ on $\mathcal{O}$ by:

$$u_\alpha(\widetilde{x}, x_n) = \begin{cases} u(\widetilde{x}, x_n + \alpha) & if \quad x_n \le -\alpha, \\ u(\widetilde{x}, 0) & if \quad -\alpha \le x_n \le \alpha, \\ u(\widetilde{x}, x_n - \alpha) & if \quad x_n \ge \alpha. \end{cases}$$

Then, $u_\alpha \in D_2$. Indeed, we have

$$u_{\alpha|\mathcal{O}\cap\{x_n \le -\alpha\}} = u_{|\mathcal{O}\cap\{x_n\le 0\}},$$
$$u_{\alpha|\mathcal{O}\cap\{x_n \ge \alpha\}} = u_{|\mathcal{O}\cap\{x_n\ge 0\}},$$
$$u_{\alpha|\mathcal{O}\cap\{-\alpha \le x_n \le \alpha\}} = u_{|S},$$

which implies that $u_\alpha \in W^{1,p}(\mathcal{O};\mathbb{R}^N)$. Since $u = 0$ on $\partial\mathcal{O}$, it is the same for $u_\alpha$. Moreover, $u_{\alpha|S} = u_{|S}$, so $u_\alpha \in D_2$. Now, we prove that $\|u_\alpha - u\|_{D_2} \to 0$. Since $u_{\alpha|S} = u_{|S}$, $\|u_\alpha - u\|_{D_2} = \|u_\alpha - u\|_{W^{1,p}(\mathcal{O};\mathbb{R}^N)}$. We have

$$\begin{aligned}\int_{\mathcal{O}} |u_\alpha - u|^p dx &= \int_{\mathcal{O}\cap\{x_n\leq -\alpha\}} |u(\widetilde{x}, x_n+\alpha) - u(\widetilde{x}, x_n)|^p dx \\ &\quad + \int_{\mathcal{O}\cap\{-\alpha\leq x_n\leq \alpha\}} |u(\widetilde{x}, 0) - u(\widetilde{x}, x_n)|^p dx \\ &\quad + \int_{\mathcal{O}\cap\{x_n\geq \alpha\}} |u(\widetilde{x}, x_n-\alpha) - u(\widetilde{x}, x_n)|^p dx.\end{aligned}$$

With the change of variable $y_n = x_n + \alpha$ in $\mathcal{O}\cap\{x_n \leq -\alpha\}$ and $y_n = x_n - \alpha$ in $\mathcal{O}\cap\{x_n \geq \alpha\}$ and by applying Theorem 1.1 in [32] (page:57) on the first and third integral terms in the right hand side, it follows that

$$\int_{\mathcal{O}} |u_\alpha - u|^p dx \to 0.$$

To prove that $\int_{\mathcal{O}} |Du_\alpha - Du|^p dx \to 0$, it suffices to apply the same argument now with $Du$ instead of $u$ noticing that we can apply theorem 1.1 in [32] since $u \in W^{1,p}(\mathcal{O};\mathbb{R}^N)$. To conclude, it remains to approximate $u_\alpha$ by convolutions. □

## 3. Preliminaries

Through this section, we recall some definitions and useful results. We begin by the concept of Γ-convergence.

3.1. **Γ-convergence.** Let $(X,\tau)$ be a topological space. The set of all open neighborhoods of x in X will be denoted by $\mathcal{N}(x)$. Let $(F_k)$ be a sequence of functions from X into $\overline{\mathbb{R}}$. For every $x \in X$, we call $\Gamma(\tau)$-lower limit and $\Gamma(\tau)$-upper limit of $(F_k)$ the functions $\Gamma(\tau)\text{-}\liminf\limits_{k\to\infty} F_k$ and $\Gamma(\tau)\text{-}\limsup\limits_{k\to\infty} F_k$ defined as

$$\Gamma(\tau)\text{-}\liminf_{k\to\infty} F_k(x) = \sup_{U\in\mathcal{N}(x)} \liminf_{k\to\infty} \inf_{y\in U} F_k(y),$$
$$\Gamma(\tau)\text{-}\limsup_{k\to\infty} F_k(x) = \sup_{U\in\mathcal{N}(x)} \limsup_{k\to\infty} \inf_{y\in U} F_k(y).$$

If the two functions are equal to a function $F : X \to \overline{\mathbb{R}}$, then we say that the sequence $(F_k)$ $\Gamma(\tau)$-converges to $F$ (in $X$) or that $F$ is the Γ-limit of $(F_k)$ and we write $F = \Gamma(\tau)\text{-}\lim\limits_{k\to\infty} F_k$. In particular (see e.g.Theorem 2.1 in Braides [16]).

**Theorem 3.1.** *Let $X$ be a metric space and $F_k, F : X \to \overline{\mathbb{R}}$. Then, the $\Gamma$-convergence of the sequence $(F_k)$ to $F$ at a point $x$ is equivalent to each of the following assertions:*

(a) *we have*

$$\begin{aligned} F(x) &= \Gamma(\tau)\text{-}\liminf_{k\to\infty} F_k(x) = \inf\{\liminf F_k(x_k) \,:\, x_k \to x\} \\ &= \Gamma(\tau)\text{-}\limsup_{k\to\infty} F_k(x) = \inf\{\limsup F_k(x_k) \,:\, x_k \to x\}; \end{aligned}$$

(b) *(*sequential $\Gamma$-convergence *) we have:*

(i) *(*inequality $\liminf$*) for every sequence $(x_k)$ converging to $x$*

$$\liminf_{k\to+\infty} F_k(x_k) \geq F(x).$$

(ii) *(*inequality $\limsup$*) there exists a sequence $(x_{0,k})$ converging to $x$ such that*

$$\limsup_{k\to+\infty} F_k(x_{0,k}) \leq F(x).$$

From now on, $(X, \tau)$ is a metric space. Hereafter are some known and useful properties of the concept of $\Gamma$-convergence.

*Proposition* 3.1. (1) (*Stability under continuous perturbations*). If $F_k$ $\Gamma$-converges to $F$ and if $G : X \to \overline{\mathbb{R}}$is a continuous function for the metric $\tau$, then $(F_k + G)$ $\Gamma$-converges to $F + G$.

(2) (*Lower semicontinuity of* $\Gamma$*-limits*). The functions $\Gamma$-liminf and $\Gamma$-limsup of $(F_k)$ are lower semicontinuous for the metric $\tau$.

We say that a sequence $F_k : X \to [0, +\infty]$ is $\tau-$*equicoercive* if there exists a $\tau-$compact set K (independent of k) such that

$$\inf\{F_k(x) \,:\, x \in X\} = \inf\{F_k(x) \,:\, x \in K\}.$$

**Theorem 3.2.** *(The Fundamental Theorem of $\Gamma-$convergence) Let $(F_k)$ be a $\tau-$equicoercive sequence $\Gamma(\tau)-$converging on $X$ to the function $F$. Then we have the convergence of minima*

$$\min\{F(x) \,:\, x \in X\} = \liminf_{k} \{F_k(x) \,:\, x \in X\},$$

*moreover we have also convergence of minimizers: if $x_k \to x$ and*

$$\lim_k F_k(x_k) = \liminf_k F_k,$$

*then $x$ is a minimizer for $F$.*

For more about $\Gamma$-convergence, we refer the reader to [7, 17, 22].

3.2. **Quasiconvex functions.** Let $f$ be a Borel measurable function defined on $\mathbb{R}^{N\times n}$, and $p \in [1, +\infty]$. We say that $f$ is $W^{1,p}$-quasiconvex at $\xi \in \mathbb{R}^{N\times n}$ if for every (or only for one) open bounded set $A \subset \mathbb{R}^n$ with $\mathcal{L}^n(\partial A) = 0$ (if f takes infinite values) and for every $\varphi \in W_0^{1,p}(A, \mathbb{R}^N)$,

$$f(\xi) \leq \frac{1}{\mathcal{L}^n(A)} \int_A f(\xi + \nabla\varphi)dx.$$

$f$ is $W^{1,p}$-quasiconvex if it is $W^{1,p}$-quasiconvex at every $\xi \in \mathbb{R}^{N\times n}$. We say that f is quasiconvex (in Morrey's sence) if it is $W^{1,\infty}$-quasiconvex.

We give here some relations between different notions of $W^{1,p}$-quasiconvexity (we refer to the original paper[10]).

*Remark* 3.1.
(1) If $1 \leq p, q \leq +\infty$, then $W^{1,p}$-quasiconvexity implies $W^{1,q}$-quasiconvexity.
(2) If $1 \leq p < +\infty$, f is continuous and
$$0 \leq f(\xi) \leq c(1 + |\xi|^p)$$
for every $\xi \in \mathbb{R}^{N\times n}$, then f is $W^{1,p}$-quasiconvex if and only if it is $W^{1,\infty}$-quasiconvex.
(3) If $1 \leq p < +\infty$ and f satisfies
$$|\xi|^p \leq f(\xi)$$
for every $\xi \in \mathbb{R}^{N\times n}$, then f is $W^{1,p}$-quasiconvex if and only if it is $W^{1,1}$-quasiconvex.

In general, a function f can be a nonquasiconvex function. However, it is always possible to work with his quasiconvex envelope given by

$$Qf := \sup\{g \leq f : \text{ g is quasiconvex }\}. \tag{3.1}$$

If f is locally bounded, then the definition of $Qf$ can be expressed for every $\xi \in \mathbb{R}^{N\times n}$ as [21, Page 201]

$$Qf(\xi) = inf\left\{\frac{1}{\mathcal{L}^n(A)} \int_A f(\xi + \nabla\varphi)dx : \varphi \in W_0^{1,\infty}(A, \mathbb{R}^N)\right\},$$

and this definition does not depend on the set D.

As it is well known, a quasiconvex function f (more generally convex in each variable) which verifies the following growth condition: there exist $C > 0$ such that

$$0 \leq f(\xi) \leq C(1 + |\xi|^r)$$

for all $\xi \in \mathbb{R}^{N\times n}$, where $1 \leq r < +\infty$, then the r-Lipschitz property

$$|f(\xi) - f(\xi')| \leq \beta(1 + |\xi|^{r-1} + |\xi'|^{r-1})|\xi - \xi'| \tag{3.2}$$

holds for all $\xi, \xi' \in \mathbb{R}^{N\times n}$, and some $\beta > 0$ (see [21] or [30]).

Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^n$. We say that a function $f : \mathcal{O} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is quasiconvex if for a.e. $x \in \mathcal{O}$ we have $\xi \mapsto f(\xi, A)$ is quasiconvex. The following proposition establishes sufficiency of quasiconvexity to obtain weak lower semicontinuity in $W^{1,p}$.

*Proposition* 3.2. Let $0 \leq p < +\infty$ and let $f : \mathcal{O} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Carathéodory quasiconvex function such that there exists a constant $C > 0$ with

$$0 \leq f(x,\xi) \leq C(1 + |\xi|^p), \quad \forall (x,\xi) \in \mathcal{O} \times \mathbb{R}^{N\times n}.$$

Then, the functional $F : u \to \int_{\mathcal{O}} f(x, \nabla u(x))\, dx$ is weakly lower semicontinuous on $W^{1,p}(\mathcal{O}, \mathbb{R}^N)$.

The proof of the above proposition can be found in [21, Theorem 2.4 and Remark iv].

3.3. **A global subadditive theorem.** For $m \geq 1$, let $\mathcal{B}_b(\mathbb{R}^m)$ be the family of Borel bounded subsets of $\mathbb{R}^m$ and $d$ the Euclidean distance in $\mathbb{R}^m$. For every $A \in \mathcal{B}_b(\mathbb{R}^m)$, set $\rho(A) = \sup\{r \geq 0 : \exists \overline{B}_r(x) \subset A\}$, where $\overline{B}_r(x) = \{y \in \mathbb{R}^m : d(x,y) \leq r\}$. A sequence $(B_k)_{k\in\mathbb{N}} \subset \mathcal{B}_b(\mathbb{R}^m)$ is called *regular* if there exist an increasing sequence of intervals $(I_k)_k \subset \mathbb{Z}^m$ and a constant C independent of k such that $B_k \subset I_k$ and $\mathcal{L}^n(I_k) \leq C\mathcal{L}^n(B_k)$, $\forall k$.

The global subadditive theorem is mainly based on *subadditive* $\mathbb{Z}^m$-*periodic* functions. A function $\Upsilon : A \in \mathcal{B}_b(\mathbb{R}^m) \to \Upsilon_A \in \mathbb{R}$ is called *subadditive* $\mathbb{Z}^m$-*periodic* if it satisfies the following conditions:

(i) For all $A, B \in \mathcal{B}_b(\mathbb{R}^m)$ such that $A \cap B = \emptyset$, $\Upsilon_{A\cup B} \leq \Upsilon_A + \Upsilon_B$.
(ii) For all $A \in \mathcal{B}_b(\mathbb{R}^m)$, all $z \in \mathbb{Z}^m$, $\Upsilon_{A+z} = \Upsilon_A$.

The following global subadditive theorem is contained in [29] (see Theorem 2.1).

**Theorem 3.3.** *Let $\Upsilon$ be a subadditive $\mathbb{Z}^m$-periodic function such that*

$$\varphi(\Upsilon) = \inf\left\{\frac{\Upsilon_I}{|I|} : I = [a,b[, a,b \in \mathbb{Z}^m \text{ and } a_i < b_i\ \forall 1 \leq i \leq m\right\} > -\infty.$$

*In addition, we suppose that $\Upsilon$ satisfies the dominant property: there exists $C(\Upsilon)$ such that for every Borel convex subset $A \subset [0,1[^m$, $|\Upsilon_A| \leq C(\Upsilon)$. Let $(A_k)_k$ be a regular sequence of convex subsets of $\mathcal{B}_b(\mathbb{R}^m)$ with $\lim_{k\to+\infty} \rho(A_k) = +\infty$. Then $\lim_{k\to+\infty} \dfrac{\Upsilon_{A_k}}{\mathcal{L}^n(A_k)}$ exists and is equal to*

$$\lim_{k\to+\infty} \frac{\Upsilon_{A_k}}{\mathcal{L}^n(A_k)} = \inf_{k\in\mathbb{N}^*}\left\{\frac{\Upsilon_{[0,k[^m}}{k^m}\right\} = \varphi(\Upsilon).$$

3.4. **Positive Radon measures.** To identify the $\Gamma$-limit inf of the functionals $F_\delta$, we use the blow-up method which rests on positive Radon measures. We give here some properties of such measures, which are needed precisely in Sec. 5.2.

**Definition 3.1.** (1) If $\lambda$ is a Radon measure with values in $\mathbb{R}^{N\times n}$, the total variation $|\lambda|$ is defined for every Borel subset $B \subset \mathcal{O}$ by:

$$|\lambda(B)| = \sup \sum_{i\in I} |\lambda(B_i)|$$

where the supremum is taking over all finite or quantable families $(B_i)_{i\in I}$ of subsets of B relatively compact in $\mathcal{O}$ and such that $B_i \cap B_j = \emptyset$ for $i \neq j$. It is evident to see that $|\lambda|$ is a positive $\sigma$-additive measure in $\mathcal{O}$. We denote by $\mathcal{M}(\mathcal{O}; \mathbb{R}^{N\times n})$ the space of Radon measures $\lambda$ with finite total variation, i.e. for which we have $|\lambda|(\mathcal{O}) < +\infty$. We say that a sequence $(\lambda_h)_h$ in $\mathcal{M}(\mathcal{O}; \mathbb{R}^{N\times n})$ *converges weakly* to $\lambda \in \mathcal{M}(\mathcal{O}; \mathbb{R}^{N\times n})$ if

$$\lim_{h\to+\infty} \int_{\mathcal{O}} \lambda_h \phi = \int_{\mathcal{O}} \lambda\phi, \quad \forall \phi \in \mathcal{D}(\mathcal{O}; \mathbb{R}^N).$$

(2) Let $\lambda : \mathbf{B}(\mathcal{O}) \to [0, +\infty[$ and $\mu : \mathbf{B}(\mathcal{O}) \to \mathbb{R}^N$ be two measures in $\mathcal{M}(\mathcal{O}; \mathbb{R}^{N\times n})$.

- We say that $\mu$ is *absolutely continuous with respect to* $\lambda$, and we write $\mu << \lambda$ if the following condition is satisfied :

$$\lambda(B) = 0 \Rightarrow \mu(B) = 0 \ \forall B \in \mathbb{B}(\mathcal{O}).$$

- We say that $\mu$ is *singular with respect to* $\lambda$, and we write $\mu \perp \lambda$ if there exist $B \in \mathbb{B}(\mathcal{O})$ such that

$$\lambda(B) = 0 \text{ and } |\mu|(\mathcal{O} \setminus B) = 0.$$

According to Radon-Nikodym, there exist an unique measure $\mu_\lambda$ such that $\mu_\lambda << \mu$ and $(\mu - \mu_\lambda) \perp \lambda$. The Besicovitch theorem of differentiation measures give the following relation:

$$\mu_\lambda = \frac{d\mu}{d\lambda}\lambda \tag{3.3}$$

where $\dfrac{d\mu}{d\lambda}$ is defined by the limit:

$$\frac{d\mu}{d\lambda}(x) = \lim_{r\to 0} \frac{\mu(B_r(x))}{\lambda(B_r(x))} \tag{3.4}$$

which exists $\lambda$-a.e. $x \in \mathcal{O}$. Here, $B_r(x)$ which is the usual ball of center $x$ and radius $r$ can be as well replaced by $Q_r(x)$. Such points, i.e. for which

(3.4) holds are called *Lebesgue points for $\mu$ with respect to $\lambda$.*

In literature, it is also common to work with Lebesgue points with respect to a function.

**Definition 3.2.** Let $u \in L^1(\mathcal{O}, \mathbb{R}^N)$. We say that $x \in \mathcal{O}$ is a *Lebesgue point with respect to $u$* if there exist $\tilde{u}(x) \in \mathbb{R}^N$ such that:

$$\lim_{r \to 0} \frac{1}{r^n} \int_{B_r(x)} |u(y) - \tilde{u}(x)| dy = 0.$$

By notation, $\mathcal{O}_u$ stands for the set of Lebesgue points for a given function $u$, and $S_u$ the discontinuity set of $u$, i.e. $S_u := \mathcal{O}_u \setminus \mathcal{O}$.

By Lebesgue differentiation theorem, $S_u$ is $\mathcal{L}^n-$negligible, and the function $\tilde{u}$ equal to $u$ a.e. in $\mathcal{O}_u$ is called the *Lebesgue representative of $u$.*

*Remark* 3.2. Later on in Sec. 5, Lebesgue points are needed with $Q_r(x)$ instead of $B_r(x)$, and it seems that it is a line of a number of works (e.g. [11, 19]).

3.5. **The Decomposition Lemma.** We recall that a sequence of functions $(u_m)_m \subset L^1(\mathcal{O})$ is said to be *equi-integrable* if for all $\varepsilon > 0$ there exist $\delta > 0$ such that

$$\sup_{m \in \mathbb{N}} \int_A |u_m| dx < \varepsilon$$

whenever $A \subset \mathcal{O}$ with $|A| < \delta$.

As a consequence of the next theorem (found e.g. in Baia and Fonseca [9] and the references therein), each sequence with bounded gradients in $L^p$, for $1 < p < \infty$, admits a subsequence that can be decomposed as a sum of a sequence with p-equi-integrable gradients and a remainder that converges to zero in measure.

**Theorem 3.4.** *(Decomposition Lemma) Let $1 < p < +\infty$ and assume that $\partial\mathcal{O}$ is Lipschitz and that $u_m \rightharpoonup v_0$ in $W^{1,p}(\mathcal{O}; \mathbb{R}^N)$. Then, there exists a subsequence $(u_{m_k})_k$ of $(u_m)_m$ and a sequence $(v_k)_k \subset W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^N)$ such that*
*i) $v_k \rightharpoonup v_0$ in $W^{1,p}(\mathcal{O}; \mathbb{R}^N)$,*
*ii) $v_k = v_0$ in a neighborhood of $\partial\mathcal{O}$,*
*iii)$(Dv_k)_k$ is p-equi-integrable,*
*iv) $\lim_{k \to +\infty} |\{x \in \mathcal{O} : v_k(x) \neq u_{m_k}(x)\}| = 0$.*

## 4. Properties of ${f_2}^{hom}$

The present section is devoted to check some properties of the function ${f_2}^{hom}$ defined by (2.10). Namely, that is a quasiconvex function which satisfies growth and p-Lipschitz conditions like in (2.4).

Firstly, by an argument analogous to that used in [9][Lemma 3.1], [13], [18][Proposition 14.4] and [19][Proposition 2.3], we show that ${f_2}^{hom}$ is well defined by the following lemma:

**Lemma 4.1.** *Let $f : \mathbb{R}^{n-1} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Carathéodory function $\widetilde{Y}$-periodic with respect to the variable $\tilde{x}$ and there exist $\beta > 0$ such that*

$$0 \leq f(\tilde{x}, \xi) \leq \beta(1 + |\xi|^p)$$

*for every $(\tilde{x}, \xi) \in \mathbb{R}^{n-1} \times \mathbb{R}^{N\times n}$. Let $k \in \mathbb{N}$, $\tilde{x}_k \in \mathbb{R}^{n-1}$ and denote by $Q_k(\tilde{x}_k)$ the open cube in $\mathbb{R}^{n-1}$ with center $\tilde{x}_k$ and side $L.k$ with $L > 0$. Then for all $\xi \in \mathbb{R}^{N\times(n-1)}$ the limit*

$$\lim_{k\to+\infty} \inf \left\{ \frac{1}{k^{n-1}} \int_{Q_k(\tilde{x}_k)\times]-1,1[} f(\tilde{x}, \xi_T + Du(x))\, dx \ : \right.$$
$$\left. u \in W_0^{1,p}(Q_k(\tilde{x}_k)\times]-1,1[; \mathbb{R}^N) \right\}$$

*exists and is equal to*

$$f^{hom}(\xi) := \inf_k \inf \left\{ \frac{1}{k^{n-1}} \int_{k\widetilde{Y}\times]-1,1[} f(\tilde{x}, \xi_T + Du)dx : \right.$$
$$\left. u \in W_0^{1,p}(k\widetilde{Y}\times]-1,1[; \mathbb{R}^N) \right\}.$$

*Proof.* Let $\xi \in \mathbb{R}^{N\times(n-1)}$, and define for every subset $A$ of $\mathcal{B}_b(\mathbb{R}^{n-1})$ the following map

$$\Upsilon_A(\xi) := \inf \left\{ \int_{\overset{o}{A}\times]-1,1[} f(\tilde{x}, \xi_T + Du(x))\, dx \ : \right.$$
$$\left. u \in W_0^{1,p}(\overset{o}{A} \times]-1,1[; \mathbb{R}^N) \right\}, \tag{4.1}$$

where $\overset{o}{A}$ stands for the interior of $A$. Then, the assumptions on the function $f$ imply that $\Upsilon_{(.)}(\xi) : A \in \mathcal{B}_b(\mathbb{R}^{n-1}) \to \Upsilon_A(\xi) \in \mathbb{R}$ satisfies all hypotheses of Theorem 3.3. Moreover,

$$\lim_{k\to+\infty} \rho(Q_k(\tilde{x}_k)) = \lim_{k\to+\infty} L.k = +\infty.$$

Hence $\lim\limits_{k\to+\infty} \dfrac{\Upsilon_{Q_k(\tilde{x}_k)}(\xi)}{k^{n-1}}$ exists and

$$\begin{aligned}\lim_{k\to+\infty} \frac{\Upsilon_{Q_k(\tilde{x}_k)}(\xi)}{k^{n-1}} &= \inf_{k\in\mathbb{N}^*} \frac{\Upsilon_{k\widetilde{Y}}(\xi)}{k^{n-1}} \\ &= f^{hom}(\xi),\end{aligned}$$

which proves lemma. $\square$

*Remark* 4.1. In the definition of $f^{hom}$ we may take as well any open convex subset of $\mathbb{R}^{n-1}$ as shown in Theorem 3.3.

In addition, ${f_2}^{hom}$ possess the following properties:

*Proposition* 4.1. ${f_2}^{hom}$ satisfies:
1.(Growth condition): there exist $\alpha_2'$ and $\beta_2'$ depending on $\alpha_2$, $\beta_2$ and p such that

$$\alpha_2'|\xi|^p \le {f_2}^{hom}(\xi) \le \beta_2'(1+|\xi|^p), \tag{4.2}$$

2.(p-Lipschitz condition): there exist a constant $c_2' > 0$ depending on $c_2$, $\alpha_2$, $\beta_2$, and p such that

$$|{f_2}^{hom}(\xi) - {f_2}^{hom}(\xi')| \le c_2'|\xi-\xi'|(1+|\xi|^{p-1}+|\xi'|^{p-1}), \tag{4.3}$$

for every $\xi, \xi' \in \mathbb{R}^{N\times(n-1)}$.

The p-Lipschitz condition is ensured as soon as we have the following:

**Lemma 4.2.** *Let $A$ be a subset of $\mathcal{B}_b(\mathbb{R}^{n-1})$ and $\Upsilon$ be the map defined as in (4.1). Then, there exist a positive constant $C$ depending on $c_2$, $\alpha_2$, $\beta_2$, and p such that for every $\xi$ and $\xi'$ in $\mathbb{R}^{N\times(n-1)}$*

$$\left|\frac{\Upsilon_A(\xi)}{\mathcal{L}^{n-1}(A)} - \frac{\Upsilon_A(\xi')}{\mathcal{L}^{n-1}(A)}\right| \le C(1+|\xi|^{p-1}+|\xi'|^{p-1})|\xi-\xi'|.$$

*Proof.* It is an adaptation of the proof of Proposition 2.1 in [31]. Let $A$ be a subset of $\mathcal{B}_b(\mathbb{R}^{n-1})$ and $\xi$, $\xi'$ in $\mathbb{R}^{N\times(n-1)}$. For a fixed $m>0$, let $u_m \in W_0^{1,p}(A\times]-1,1[;\mathbb{R}^N)$ so that

$$\Upsilon_A(\xi') \ge \int_{A\times]-1,1[} f_2(\tilde{y}, \xi_T' + Du_m)\,dy - \frac{1}{m}. \tag{4.4}$$

From the p-Lipchitz condition in (2.4) and by Holder inequality, it follows that

$$\begin{aligned}&\frac{1}{\mathcal{L}^{n-1}(A)}\left[\Upsilon_A(\xi) - \Upsilon_A(\xi') - \frac{1}{m}\right] \\ &\quad\le \frac{1}{\mathcal{L}^{n-1}(A)} \int_{A\times]-1,1[} [f_2(\tilde{y}, \xi_T + Du_m) - f_2(\tilde{y}, \xi_T' + Du_m)]\,dy\end{aligned} \tag{4.5}$$

$$\leq C|\xi-\xi'|\left(\frac{1}{\mathcal{L}^{n-1}(A)}\int_{A\times]-1,1[}(1+|\xi_T+Du_m(y)|^{p-1}\right.$$
$$\left.+|\xi'_T+Du_m(y)|^{p-1})^{\frac{p}{p-1}}dy\right)^{\frac{p-1}{p}}$$
$$\leq C|\xi-\xi'|\left(\frac{1}{\mathcal{L}^{n-1}(A)}\int_{A\times]-1,1[}1+|\xi|^p+|\xi'|^p\right.$$
$$\left.+|\xi'_T+Du_m(y)|^p dy\right)^{\frac{p-1}{p}},$$

where C is a constant depending on $c_2$ and p. On the other hand, by (2.4) and (4.4)

$$\begin{aligned}
&\frac{1}{\mathcal{L}^{n-1}(A)}\int_{A\times]-1,1[}|\xi'_T+Du_m(y)|^p)dy\\
&\leq \frac{1}{\alpha_2\mathcal{L}^{n-1}(A)}\int_{A\times]-1,1[}f_2(\tilde{y},\xi'_T+Du_m)\,dy\\
&\leq \frac{1}{\alpha_2\mathcal{L}^{n-1}(A)}\left(\int_{A\times]-1,1[}f_2(\tilde{y},\xi'_T)\,dy+\frac{1}{m}\right)\\
&\leq C(1+|\xi'|^p)+\frac{1}{\alpha_2 m\mathcal{L}^{n-1}(A)},
\end{aligned}$$

where C depends on $\alpha_2$ and $\beta_2$. Applying this in (4.5), taking the limit as $m\to+\infty$ and exchanging roles between $\xi$ and $\xi'$, we deduce that

$$\begin{aligned}
\left|\frac{\Upsilon_A(\xi)}{\mathcal{L}^{n-1}(A)}-\frac{\Upsilon_A(\xi')}{\mathcal{L}^{n-1}(A)}\right| &\leq C|\xi-\xi'|(1+|\xi|^p+|\xi'|^p)^{\frac{p-1}{p}},\\
&\leq C|\xi-\xi'|(1+|\xi|^{p-1}+|\xi'|^{p-1}).
\end{aligned}$$

□

*Proof of Proposition 4.1.* 1. The upper bound is easily deduced from (2.4). For the lower bound, let $\xi\in\mathbb{R}^{N\times(n-1)}$. Making use of (2.4), we get

$$\begin{aligned}
2\gamma_m|\xi|^p &\leq \frac{1}{k^{n-1}}\int_{k\widetilde{Y}\times]-1,1[}|\xi|^p dx=\frac{1}{k^{n-1}}\int_{k\widetilde{Y}\times]-1,1[}|\xi_T|^p dx\\
&\leq \frac{2^{p-1}}{k^{n-1}}\left\{\int_{k\widetilde{Y}\times]-1,1[}|\xi_T+Du|^p dx+\int_{k\widetilde{Y}\times]-1,1[}|Du|^p dx\right\},\\
&\leq \frac{2^{p-1}}{k^{n-1}}\left\{\frac{1}{\alpha_2}\int_{k\widetilde{Y}\times]-1,1[}f_2(\tilde{x},\xi_T+Du(x))dx+\int_{k\widetilde{Y}\times]-1,1[}|Du|^p dx\right\},
\end{aligned}$$

for any $u \in W_0^{1,p}(k\widetilde{Y}\times]-1,1[);\mathbb{R}^N)$. If we take the infimum over u, we obtain

$$\alpha_2'|\xi|^p \leq \frac{\Upsilon_{k\widetilde{Y}}}{k^{n-1}},$$

where $\alpha_2' = \dfrac{\alpha_2}{2^{p-2}}$. We have result after going to the limit in k.
2. It is immediately seen from Lemma 4.2. □

To end this section, it remains to establish the quasiconvexity property of ${f_2}^{hom}$. Arguing as in [9], we use the decomposition lemma to check that ${f_2}^{hom} = (\mathcal{Q}f_2)^{hom}$, where $\mathcal{Q}f_2$ is the quasiconvex envelope of $f_2$.

**Lemma 4.3.** *Assume $f_2$ to be a Carathdory function for which the growth condition in (2.4) holds. Then, we have that:*
*i). $\mathcal{Q}f_2$ is a Carathéodory function,*
*ii). ${f_2}^{hom}(\xi) = (\mathcal{Q}f_2)^{hom}(\xi)$ for all $\xi \in \mathbb{R}^{N\times(n-1)}$.*

*Proof.* We follow the same lines of lemma 3.2 in [9].
i). Let $(\tilde{x},\xi) \in \mathbb{R}^{n-1}\times\mathbb{R}^{N\times n}$. We can write

$$\mathcal{Q}f_2(\tilde{x},\xi) = \inf_{\phi\in\mathcal{D}_k} g_\phi(\tilde{x})$$

where

$$g_\phi(\tilde{x}) := \frac{1}{\mathcal{L}^n(k\widetilde{Y}\times]-1,1[)}\int_{k\widetilde{Y}\times]-1,1[} f_2(\tilde{x},\xi + D\phi(y))dy$$

and $\mathcal{D}_k$ is a countable subset of $\mathcal{D}(k\widetilde{Y}\times]-1,1[);\mathbb{R}^N)$ dense in $W_0^{1,p}(k\widetilde{Y}\times]-1,1[;\mathbb{R}^N)$. By Tonelli's Theorem the functions $g_\phi$ are measurable, and so is $\mathcal{Q}f_2(.,\xi)$ as the infimum of a countable family of measurable functions. For the continuity of $\mathcal{Q}f_2(\tilde{x},.)$, it is a consequence of its p-Lipschitz property (3.2), since it is a quasiconvex function which by (2.4) satisfies

$$0 \leq \mathcal{Q}f_2(\tilde{x},\xi) \leq f_2(\tilde{x},\xi) \leq C(1+|\xi|^p). \tag{4.6}$$

ii). Since by identity i) and (4.6) $\mathcal{Q}f_2$ satisfies all the conditions of Lemma 4.1, we can write for every $\xi \in \mathbb{R}^{N\times n}$

$$(\mathcal{Q}f_2)^{hom}(\xi) = \lim_k \inf\left\{\frac{1}{k^{n-1}}\int_{k\widetilde{Y}\times]-1,1[}\mathcal{Q}f_2(\tilde{y},\xi_T + Du)dy : \right.$$
$$\left. u \in W_0^{1,p}(k\widetilde{Y}\times]-1,1[;\mathbb{R}^N)\right\}.$$

Obviously $f_2^{hom}(\xi) \geq (\mathcal{Q}f_2)^{hom}(\xi)$. For the converse inequality, let $m \in \mathbb{N}$ and let $k_m \in \mathbb{N}$ and $u_m \in W_0^{1,p}(k_m\widetilde{Y}\times]-1,1[);\mathbb{R}^N)$ be such that

$$(\mathcal{Q}f_2)^{hom}(\xi) + \frac{1}{m} \geq \frac{1}{k_m^{n-1}}\int_{k_m\widetilde{Y}\times]-1,1[}\mathcal{Q}f_2(\tilde{y},\xi_T + Du_m(y))\,dy.$$

Thus

$$(\mathcal{Q}f_2)^{hom}(\xi) \geq \limsup_{m\to\infty} \frac{1}{k_m^{n-1}} \int_{k_m\widetilde{Y}\times]-1,1[} \mathcal{Q}f_2(\tilde{y}, \xi_T + Du_m(y))\, dy. \tag{4.7}$$

By Acerbi and Fusco Relaxation Theorem ([3, Statement III.7]) and as a consequence of assertion i) and (4.6), for every m fixed there exists a sequence $(u_{m,r})_r \subset W^{1,p}(k_m\widetilde{Y}\times]-1,1[;\mathbb{R}^N)$ such that $u_{m,r} \rightharpoonup u_m$ in $W^{1,p}(k_m\widetilde{Y}\times]-1,1[;\mathbb{R}^N)$ and

$$\begin{aligned} \frac{1}{k_m^{n-1}} \int_{k_m\widetilde{Y}\times]-1,1[} &\mathcal{Q}f_2(\tilde{y}, \xi_T + Du_m)dy \\ &= \lim_{r\to\infty} \frac{1}{k_m^{n-1}} \int_{k_m\widetilde{Y}\times]-1,1[} f_2(\tilde{y}, \xi_T + Du_{m,r})dy. \end{aligned} \tag{4.8}$$

By Theorem 3.4, there exists a subsequence (still denoted by $(u_{m,r})_r$) and a sequence $(v_{m,r})_r \subset W_0^{1,\infty}(\mathbb{R}^n;\mathbb{R}^N)$ such that $v_{m,r} \rightharpoonup u_m$ in $W^{1,p}(k_m\widetilde{Y}\times]-1,1[;\mathbb{R}^N)$ with

$$(|Dv_{m,r}|^p)_r \text{ equi-integrable} \tag{4.9}$$

and

$$\mathcal{L}^n(\{x \in k_m\widetilde{Y}\times]-1,1[ :\ v_{m,r}(x) \neq u_{m,r}(x)\}) \xrightarrow{r} 0. \tag{4.10}$$

As $f_2$ is nonnegative, by (2.4), (4.9) and (4.10)

$$\begin{aligned} &\lim_{r\to\infty} \frac{1}{k_m^{n-1}} \int_{k_m\widetilde{Y}\times]-1,1[} f_2(\tilde{y}, \xi_T + Du_{m,r}(y))\, dy \\ &\geq \limsup_{r\to\infty} \frac{1}{k_m^{n-1}} \int_{\{x\in k_m\widetilde{Y}\times]-1,1[\ v_{m,r}(x)=u_{m,r}(x)\}} f_2(\tilde{y}, \xi_T + Dv_{m,r}(y))\, dy \\ &= \limsup_{r\to\infty} \frac{1}{k_m^{n-1}} \int_{k_m\widetilde{Y}\times]-1,1[} f_2(\tilde{y}, \xi_T + Dv_{m,r}(y))\, dy. \end{aligned} \tag{4.11}$$

From (4.7), (4.8) and (4.11), we deduce that

$$\begin{aligned} (\mathcal{Q}f_2)^{hom}(\xi) &\geq \limsup_{m\to\infty} \limsup_{r\to\infty} \frac{1}{k_m^{n-1}} \int_{k_m\widetilde{Y}\times]-1,1[} f_2(\tilde{y}, \xi_T + Dv_{m,r}(y))\, dy \\ &\geq {f_2}^{hom}(\xi). \end{aligned}$$

□

Finally, we have the following proposition.

*Proposition* 4.2. Under the same assumptions of Lemma 4.3, the function $f_2^{hom}$ is quasiconvex.

*Proof.* In view of assertion ii). of Lemma 4.3, it is sufficient to prove it for $(\mathcal{Q}f_2)^{hom}$. So, let $\xi \in \mathbb{R}^{N\times(n-1)}$ and B be a bounded open set of $\mathbb{R}^{n-1}$. The quasiconvex envelope of $(\mathcal{Q}f_2)^{hom}$ reads as follows
(4.12)
$$\mathcal{Q}(\mathcal{Q}f_2)^{hom}(\xi) = \inf\left\{\frac{1}{\mathcal{L}^{n-1}(B)}\int_B (\mathcal{Q}f_2)^{hom}(\xi + D\varphi(\tilde{y}))d\tilde{y} \,:\, \varphi \in \mathrm{Aff}_0(B;\mathbb{R}^N)\right\},$$
where $\mathrm{Aff}_0(B;\mathbb{R}^N) := \{\varphi \in W_0^{1,\infty}(B;\mathbb{R}^N); \varphi \text{ piecewise affine}\}$ (see [21], page 207). We have to show that
$$\mathcal{Q}(\mathcal{Q}f_2)^{hom}(\xi) = (\mathcal{Q}f_2)^{hom}(\xi).$$
Firstly, it is clear that $\mathcal{Q}(\mathcal{Q}f_2)^{hom}(\xi) \leq (\mathcal{Q}f_2)^{hom}(\xi)$. Let us now prove the converse inequality. Let $\varphi \in \mathrm{Aff}_0(B;\mathbb{R}^N)$ and $m > 0$. By definition of $(\mathcal{Q}f_2)^{hom}$, there exists $k_m > 0$ and $u_{m,x} \in W_0^{1,p}(k_m\widetilde{Y}\times]-1,1[)$ (depending on x) such that

$$\begin{aligned}(4.13)\quad &(\mathcal{Q}f_2)^{hom}(\xi + D\varphi(\tilde{x})) + \frac{1}{m}\\ &\geq \frac{1}{k_m^{n-1}}\int_{k_m\widetilde{Y}\times]-1,1[} \mathcal{Q}f_2(\tilde{y}, (\xi + D\varphi(\tilde{x}))_T + Du_{m,x}(y))\,dy\\ &\geq \inf\left\{\frac{1}{k_m^{n-1}}\int_{k_m\widetilde{Y}\times]-1,1[} \mathcal{Q}f_2(\tilde{y}, (\xi + D\varphi(\tilde{x}))_T + Du(y))\,dy \,:\right.\\ &\qquad\left. u \in W_0^{1,p}(k_m\widetilde{Y}\times]-1,1[)\right\}.\end{aligned}$$

Since $\varphi \in \mathrm{Aff}_0(B;\mathbb{R}^N)$, there exist $B_i \subset B$ with $\cup_{i\in I}B_i = B$ and $\xi_i \in \mathbb{R}^{N\times(n-1)}$, $i \in I$ finite subset of $\mathbb{N}$, such that
$$(4.14)\qquad \int_B (\mathcal{Q}f_2)^{hom}(\xi + D\varphi(\tilde{x}))d\tilde{x} = \sum_{i\in I}(\mathcal{Q}f_2)^{hom}(\xi + \xi_i)\mathcal{L}^{n-1}(B_i).$$
Averaging on B in (4.13) and using (4.14) together with the quasiconvexity of $\mathcal{Q}f_2$, it follows that
$$\begin{aligned}(4.15)\qquad &\frac{1}{\mathcal{L}^{n-1}(B)}\int_B (\mathcal{Q}f_2)^{hom}(\xi + D\varphi(\tilde{x}))d\tilde{x} + \frac{1}{m}\\ &\geq \frac{1}{\mathcal{L}^{n-1}(B)}\inf\left\{\frac{1}{k_m^{n-1}}\int_B\left(\int_{k_m\widetilde{Y}\times]-1,1[} \mathcal{Q}f_2(\tilde{y}, (\xi + D\varphi(\tilde{x}))_T +\right.\right.\\ &\qquad\left.\left. +Du(y))dy\right)d\tilde{x} : u \in W_0^{1,p}(k_m\widetilde{Y}\times]-1,1[)\right\}\\ &= \frac{1}{\mathcal{L}^{n-1}(B)}\inf\left\{\frac{1}{k_m^{n-1}}\int_{k_m\widetilde{Y}\times]-1,1[}\sum_{i\in I}\mathcal{Q}f_2(\tilde{y}, (\xi + \xi_i)_T +\right.\end{aligned}$$

$$+Du(y))\mathcal{L}^{n-1}(B_i)\,dy:\ u\in W_0^{1,p}(k_m\widetilde{Y}\times]-1,1[)\Big\}$$

$$=\frac{1}{\mathcal{L}^{n-1}(B)}\inf\Big\{\frac{1}{k_m^{n-1}}\int_{k_m\widetilde{Y}\times]-1,1[}\sum_{i\in I}\int_{B_i}\mathcal{Q}f_2(\tilde{y},(\xi+\xi_i)_T+$$

$$+Du(y))dx\,dy: u\in W_0^{1,p}(k_m\widetilde{Y}\times]-1,1[)\Big\}$$

$$=\inf\Big\{\frac{1}{k_m^{n-1}}\int_{k_m\widetilde{Y}\times]-1,1[}\frac{1}{\mathcal{L}^{n-1}(B)}\int_B\mathcal{Q}f_2(\tilde{y},\xi_T+(D\varphi(\tilde{x}))_T+$$

$$+Du(y))d\tilde{x}\,dy: u\in W_0^{1,p}(k_m\widetilde{Y}\times]-1,1[)\Big\}$$

$$\geq\inf\Big\{\frac{1}{k_m^{n-1}}\int_{k_m\widetilde{Y}\times]-1,1[}\mathcal{Q}f_2(\tilde{y},\xi_T+Du(y))\,dy:$$

$$u\in W_0^{1,p}(k_m\widetilde{Y}\times]-1,1[)\Big\}.$$

The desired result is deduced after letting m to $+\infty$. □

## 5. Proof of Theorem 2.1

5.1. **Effective domain.** This paragraph aims at justifying (2.11), and to do so we need the following proposition:

*Proposition* 5.1. Let $(u_\delta)$ be a sequence in $W_0^{1,p}(\mathcal{O};\mathbb{R}^N)$ so that $F_\delta(u_\delta)\leq C$. Then, for a subsequence not relabeled, $(u_\delta)$ converges weakly in $W^{1,p}(\mathcal{O};\mathbb{R}^N)$ to a function $u\in W_0^{1,p}(\mathcal{O};\mathbb{R}^N)$. Moreover, $u_{|S}\in W^{1,p}(S;\mathbb{R}^N)$ if $0<\eta<\infty$, and $u_{|S}=0$ if $\eta=\infty$.

*Proof.* Let $(u_\delta)$ be a sequence in $W_0^{1,p}(\mathcal{O};\mathbb{R}^N)$ so that $F_\delta(u_\delta)\leq C$. By the growth condition in (2.4) we have the estimations

$$\mu\int_{B_\varepsilon}|Du_\delta|^p dx\leq C, \tag{5.1}$$

$$\int_{\mathcal{O}_\varepsilon}|Du_\delta|^p dx\leq C, \tag{5.2}$$

which imply that the sequence $(u_\delta)$ is bounded in $W^{1,p}(\mathcal{O};\mathbb{R}^N)$, and so, there exists a function $u$ such that for a subsequence

$$u_\delta\rightharpoonup u \text{ weakly in } W^{1,p}(\mathcal{O};\mathbb{R}^N), \tag{5.3}$$

which entails by the compactness of the trace mapping that $u=0$ on $\partial\mathcal{O}$ and $u_{|S}\in L^p(S;\mathbb{R}^N)$. Now, we proceed with the two cases, and we begin by treating the one where $0 < \eta < +\infty$. To this purpose we need to rescale the function $u_\delta$ on the fixed domain $S\times]-1,1[$ by introducing the function $U_\delta$ defined on $S\times]-1,1[$ by

$$U_\delta(\widetilde{x}, x_n) := u_\delta(\widetilde{x}, \varepsilon x_n). \tag{5.4}$$

Then, we have

$$D_{\tilde{x}} U_\delta(\widetilde{x}, x_n) = D_{\tilde{x}} u_\delta(\widetilde{x}, \varepsilon x_n), \tag{5.5}$$

$$\frac{\partial U_\delta}{\partial x_n}(\widetilde{x}, x_n) = \varepsilon \frac{\partial u_\delta}{\partial x_n}(\widetilde{x}, \varepsilon x_n). \tag{5.6}$$

The computations above together with estimates (5.1) and (5.2) lead to

$$\int_{S\times]-1,1[} |D_{\tilde{x}} U_\delta|^p dx \leq \frac{C}{\varepsilon\mu}, \tag{5.7}$$

$$\int_{S\times]-1,1[} \left|\frac{\partial U_\delta}{\partial x_n}\right|^p dx \leq \frac{C\varepsilon^p}{\varepsilon\mu}. \tag{5.8}$$

Hence, making use of (5.7) and (5.8) and applying the Poincaré inequality, since $U_\delta = 0$ on $\partial S\times]-1,1[$ (because $u_\delta = 0$ on $\partial S\times]-\varepsilon,\varepsilon[$), we deduce that the sequence $\|U_\delta\|_{W^{1,p}(S\times]-1,1[,\mathbb{R}^N)}$ is bounded, which implies that for a subsequence, there exists a function $U \in W^{1,p}(S\times]-1,1[,\mathbb{R}^N)$ such that

$$U_\delta \rightharpoonup U \text{ weakly in } W^{1,p}(S\times]-1,1[,\mathbb{R}^N). \tag{5.9}$$

The estimate (5.8) provides in particular that

$$\frac{\partial U_\delta}{\partial x_n} \to 0 \text{ strongly in } L^p(S\times]-1,1[,\mathbb{R}^N), \tag{5.10}$$

and consequently, the function $U$ is independent of $x_n$ and we can write $U(\widetilde{x}, x_n) = U(\widetilde{x})$ for every $(\widetilde{x}, x_n) \in S\times]-1,1[$. By (5.3), (5.4), (5.9) and the compactness of the trace mapping, we deduce that

$$u_{|S}(\widetilde{x}) = U(\widetilde{x}, 0) = U(\widetilde{x}) \in W^{1,p}(S,\mathbb{R}^N). \tag{5.11}$$

For the case $\eta = +\infty$, thanks to the estimates (5.1) and (5.2) and the Poincaré inequality, we get that

$$U_\delta \to 0 \text{ strongly in } W^{1,p}(S\times]-1,1[,\mathbb{R}^N),$$

and so, by (5.11) it follows that $u_{|S}(\widetilde{x}) = 0$, and the proof is then accomplished. $\square$

This Proposition allows naturally to the following inclusion

$$\operatorname{dom}(\Gamma - \liminf_{\delta} F_\delta) \subset D. \tag{5.12}$$

For the converse one and according to (5.41) later in this work, for every $u \in D$ there exist a sequence $(u_\delta)$ such that $u_\delta \to u$ strongly in $L^p(\mathcal{O};\mathbb{R}^N)$ and

$$\Gamma - \liminf_{\delta} F_\delta(u) \leq \limsup F_\delta(u_\delta) \leq F(u) < +\infty,$$

and consequently $u \in \operatorname{dom}(\Gamma - \liminf_\delta F_\delta)$, which makes true the converse inclusion and so (2.11) holds.

5.2. **Proof of the Γ-limit inf.** We show that for any sequence $(u_\delta)$ and $u$ in $L^p(\mathcal{O};\mathbb{R}^N)$ such that $u_\delta \to u$ strongly in $L^p(\mathcal{O};\mathbb{R}^N)$, we have

$$F(u) \leq \liminf_{\delta\to(0,\infty)} F_\delta(u_\delta). \tag{5.13}$$

5.2.1. *Cases $\eta = 0$ and $\eta = +\infty$.* We refer the reader to assertion (1) of Lemma 5.1 below and the fact that $F^1_\delta(u) \leq F_\delta(u)$, since the function $f_2$ is non negative.

5.2.2. *Case $0 < \eta < \infty$.* Suppose that we have a sequence $(u_\delta)$ and a function u so that $u_\delta \to u$ strongly in $L^p(O;\mathbb{R}^N)$. We may suppose also that $\liminf_{\delta\to(0,\infty)} F_\delta(u_\delta) < +\infty$, which implies by Proposition 5.1 that $u \in D_2$ defined in (2.11). Set

$$F_\delta(u) = F^1_\delta(u) + F^2_\delta(u), \tag{5.14}$$

where

$$F^1_\delta(u) = \int_{\mathcal{O}_\varepsilon} f_1(x, Du)dx, \quad F^2_\delta(u) = \mu \int_{B_\varepsilon} f_2\left(\frac{\tilde{x}}{\varepsilon}, Du\right) dx.$$

Then, (5.13) is obtained from the following lemma:

**Lemma 5.1.** *For every $u \in D_2$ and every sequence $(u_\delta) \in L^p(O;\mathbb{R}^N)$ such that $u_\delta \to u$ strongly in $L^p(O;\mathbb{R}^N)$, we have*

1. $\liminf_{\delta\to(0,+\infty)} F^1_\delta(u_\delta) \geq \int_{\mathcal{O}} \mathcal{Q}f_1(x, Du(x))dx.$
2. $\liminf_{\delta\to(0,+\infty)} F^2_\delta(u_\delta) \geq \eta \int_S {f_2}^{hom}(D_{\tilde{x}}(u_{|S})(\tilde{x}))d\tilde{x}.$

*Proof.* We shall use the Blow-up method as described in [1, 4, 19, 29].
(1). The proof of this assertion will be performed through 4 steps. For every $\delta$, define the positive measure

$$\nu_\delta(A) := \int_{A\setminus B_\varepsilon} f_1(x, Du_\delta)dx, \quad \forall A \in \mathcal{B}(\mathbb{R}^n).$$

**Step 1: The limit measure.** Since $\sup_\delta |\nu_\delta|(\mathcal{O}) \leq C$, by the weak* compactness of measures, there exists a subsequence of $(\nu_\delta)$ (with the same notation), and a positive measure $\nu$ on $\mathcal{O}$ such that

$$\nu_\delta \rightharpoonup \nu \quad \text{in } \mathcal{M}(\mathcal{O}), \tag{5.15}$$

where $\mathcal{M}(\mathcal{O})$ is the space of Radon Measures in $\mathcal{O}$. The Radon-Nikodym decomposition of the limit measure $\nu$ with respect to the n-dimensional Lebesgue measure $\mathcal{L}^n$ takes the following form

$$\nu = \frac{d\nu}{dx}\mathcal{L}^n + \nu^s, \tag{5.16}$$

where $\nu^s \perp \mathcal{L}^n$.

**Step 2: Local analysis.** Let $x_0 \in \mathcal{O} \setminus S$ be a Lebesgue point for $\nu$ with respect to $\mathcal{L}^n$, i.e.

$$\frac{d\nu}{dx}(x_0) = \lim_{r\to 0} \frac{\nu(Q_r(x_0))}{\mathcal{L}^n(Q_r(x_0))} = \lim_{r\to 0} \frac{\nu(Q_r(x_0))}{r^n}, \tag{5.17}$$

where $Q_r(x_0)$ is the open cube centered at $x_0$ and of side $r$. Moreover, suppose that

$$\lim_{r\to 0} \frac{1}{r^n} \int_{Q_r(x_0)} |\mathcal{Q}f_1(x, Du) - \mathcal{Q}f_1(x_0, Du(x_0))|dx = 0, \tag{5.18}$$

(cf. Remark 3.2). By Alexandroff theorem, if $r$ is such that $\nu(\partial Q_r(x_0)) = 0$, we have

$$\nu(Q_r(x_0)) = \lim_{\delta\to(0,+\infty)} \nu_\delta(Q_r(x_0)). \tag{5.19}$$

Since $\nu(\mathcal{O}) < +\infty$, for all $r \in ]0, r_0[\setminus R$, where $R$ is a countable set, we have $\nu(\partial Q_r(x_0)) = 0$. So, we choose $r$ in $]0, r_0[\setminus R$, and in addition, we assume that $r$ is such that $r < dist(x_0, S)$ (thus, for a very small $\varepsilon$, $Q_r(x_0) \cap B_\varepsilon = \emptyset$). Hence, using (5.17) and (5.19) we get

$$\begin{aligned}\frac{d\nu}{dx}(x_0) &= \lim_{r\to 0} \lim_{\delta\to(0,+\infty)} \frac{\nu_\delta(Q_r(x_0))}{r^n} \\ &= \lim_{r\to 0} \lim_{\delta\to(0,+\infty)} \frac{1}{r^n} \int_{Q_r(x_0)\setminus B_\varepsilon} f_1\left(x,\, Du_\delta\right) dx.\end{aligned} \tag{5.20}$$

**Step 3: Local estimates**. At this step, we prove that

$$\frac{d\nu}{dx}(x_0) \geq \mathcal{Q}f_1(x_0\,,\, Du(x_0)). \tag{5.21}$$

By (5.20), the fact that $Q_r(x_0) \setminus B_\varepsilon = Q_r(x_0)$ for $\varepsilon$ small enough and that $\mathcal{Q}f_1(x,\xi) \leq f_1(x,\xi)$, we have

$$\begin{aligned}\frac{d\nu}{dx}(x_0) &= \lim_{r\to 0} \lim_{\delta\to(0,+\infty)} \frac{1}{r^n} \int_{Q_r(x_0)\setminus B_\varepsilon} f_1\left(x,\, Du_\delta(x)\right) dx \\ &\geq \liminf_{r\to 0} \liminf_{\delta\to(0,\infty)} \frac{1}{r^n} \int_{Q_r(x_0)} \mathcal{Q}f_1(x, Du_\delta(x))dx.\end{aligned} \tag{5.22}$$

By Proposition 3.2, the function $u \mapsto \int_{Q_r(x_0)} \mathcal{Q}f_1(x, Du(x))dx$ is weakly lower semicontinuous in $W^{1,p}(Q_r(x_0))$. So, making use of (5.18) the limit in the last line of (5.22) is superior or equal to

$$\liminf_{r\to 0} \frac{1}{r^n} \int_{Q_r(x_0)} \mathcal{Q}f_1(x, Du)dx = \mathcal{Q}f_1(x_0, Du(x_0)). \tag{5.23}$$

Hence, (5.21) is given in view of (5.22) and (5.23).
**Step 4: Global estimates.** (5.16) together with (5.21) allow to get

$$\nu(\mathcal{O}) \geq \int_{\mathcal{O}} \frac{d\nu}{dx}(x)dx \geq \int_{\mathcal{O}} \mathcal{Q}f_1(x, Du(x))dx.$$

Since by (5.15) $\nu_\delta \rightharpoonup \nu$, we have

$$\liminf_{\delta} F_\delta^1(u_\delta) = \liminf_{\delta} \nu_\delta(\mathcal{O}) \geq \nu(\mathcal{O}) \geq \int_{\mathcal{O}} \mathcal{Q}f_1(x, Du)dx,$$

which end the proof of assertion (1).
(2). The proof of this assertion will be accomplished after 5 steps. For every $\delta$, let us define the positive measure

$$\lambda_\delta(A) := \mu \int_{A\times]-\varepsilon,\varepsilon[} f_2\left(\frac{\tilde{x}}{\varepsilon}, Du_\delta\right) dx, \quad \forall A \in \mathcal{B}(\mathbb{R}^{n-1}).$$

**Step 1: The limit measure.** Since $\sup_\delta |\lambda_\delta|(S) \leq C$, by the weak* compactness of measures, there exists a subsequence of $(\lambda_\delta)$ (with the same notation), and a positive measure $\lambda$ on $S$ such that

$$\lambda_\delta \rightharpoonup \lambda \quad \text{in } \mathcal{M}(S),$$

where $\mathcal{M}(S)$ is the space of Radon Measures in $S$. According to the Radon-Nikodym decomposition of the limit measure $\lambda$ with respect to the (n-1)-dimensional Lebesgue measure $\mathcal{L}^{n-1}$, we have

$$\lambda = \frac{d\lambda}{d\tilde{x}}\mathcal{L}^{n-1} + \lambda^s, \tag{5.24}$$

where $\lambda^s \perp \mathcal{L}^{n-1}$.
**Step 2: Local analysis.** Let $\tilde{x}_0 \in S$ be a Lebesgue point for $\lambda$ with respect to $\mathcal{L}^{n-1}$, i.e.

$$\frac{d\lambda}{d\tilde{x}}(\tilde{x}_0) = \lim_{r\to 0} \frac{\lambda(Q_r(\tilde{x}_0))}{\mathcal{L}^{n-1}(Q_r(\tilde{x}_0))} = \lim_{r\to 0} \frac{\lambda(Q_r(\tilde{x}_0))}{r^{n-1}},$$

where $Q_r(\tilde{x}_0)$ is the open cube centered at $\tilde{x}_0$ and of side $r$. Since $u_{|S} \in W^{1,p}(S; \mathbb{R}^{N\times n})$, up to a set of zero Lebesgue measure, we can assume also that $\tilde{x}_0$ satisfies the following condition:

$$\lim_{r\to 0} \frac{1}{r^{p+n-1}} \int_{Q_r(\tilde{x}_0)} |u_{|S}(\tilde{x}) - u_{|S}(\tilde{x}_0) - \langle D_{\tilde{x}} u_{|S}(\tilde{x}_0), \tilde{x} - \tilde{x}_0\rangle|^p = 0 \tag{5.25}$$

(see e.g. [33][Theorem 3.4.2]). By Alexandroff theorem and since $\lambda(S) < +\infty$, as before we choose $r \in ]0, r_0[\backslash R$ where $R$ is a countable set. Thus

$$\begin{aligned} \frac{d\lambda}{d\tilde{x}}(\tilde{x_0}) &= \lim_{r\to 0} \lim_{\delta\to(0,+\infty)} \frac{\lambda_\delta(Q_r(\tilde{x}_0))}{r^{n-1}} \\ &= \lim_{r\to 0} \lim_{\delta\to(0,+\infty)} \frac{\mu}{r^{n-1}} \int_{Q_r(\tilde{x}_0)\times]-\varepsilon,\varepsilon[} f_2\left(\frac{\tilde{x}}{\varepsilon}, Du_\delta\right) dx. \end{aligned} \tag{5.26}$$

**Step 3: Cut-off and slicing method of De Giorgi.** We use an argument of De Giorgi [24] (see also Dal Maso and Modica [23]) in which we subdivide $Q_r(\tilde{x_0})$ in the following way:
Let $m \in \mathbb{N}^*$, $0 < \varsigma < 1$ and

$$Q_0 := Q_{\varsigma r}(\tilde{x_0});\ Q_i := Q_{\varsigma r + i\frac{r(1-\varsigma)}{m}}(\tilde{x_0}) \text{ for } i = 1, 2, ..., m. \tag{5.27}$$

so that

$$Q_0 = Q_{\varsigma r}(\tilde{x_0}) \subset Q_1 \subset .. \subset Q_i \subset Q_{i+1} \subset .. \subset Q_m = Q_r(\tilde{x_0}).$$

Before introducing the cut-off functions, let us recall firstly their definition.

**Definition 5.1.** Let $\Omega$ an open subset of $\mathbb{R}^d$ and $A$, $A'$ two open subsets of $\Omega$ such that $A \subset\subset A'$ (i.e., $\bar{A}$ is compact and included into $A'$). We say that a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a *cut-off* function between $A$ and $A'$ if $\varphi \in \mathcal{D}(A')$, $0 \le \varphi \le 1$ in $\mathbb{R}^d$, $\varphi = 1$ in a neighborhood of $\bar{A}$.

Now, let us consider for every $i = 1, 2, ..., m$ a cut-off function $\Phi_i(\tilde{x}) \in \mathcal{C}_0^\infty(Q_r(\tilde{x_0}))$ so that $0 \le \Phi_i \le 1$, and

$$\begin{cases} \Phi_i = 1 \ \text{ in } Q_{i-1} \\ \Phi_i = 0 \ \text{ outside of } Q_i \\ |D\Phi_i|_{L^\infty(Q_i)} \le C\frac{m}{r(1-\varsigma)}. \end{cases} \tag{5.28}$$

Let $u_{\delta,i} \in W^{1,p}(Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[;\mathbb{R}^N)$ be the function defined as follows

$$u_{\delta,i}(x) := u_0(\tilde{x}) + \Phi_i(\tilde{x})(u_\delta(x) - u_0(\tilde{x})),$$

where $u_0(\tilde{x}) := u_{|S}(\tilde{x}_0) + < D_{\tilde{x}}u_{|S}(\tilde{x}_0), \tilde{x} - \tilde{x}_0 >$. Let us now fixe the integer $i \in \{1, 2, ..., m\}$. By the growth condition in (2.4)

$$\begin{aligned} \inf\Bigg\{ &\frac{\mu}{r^{n-1}} \int_{Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, (D_{\tilde{x}}u_{|S}(\tilde{x}_0))_T + Dw(x))dx \\ &: w \in W_0^{1,p}(Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[;\mathbb{R}^N) \Bigg\} \end{aligned} \tag{5.29}$$

$$\le \frac{\mu}{r^{n-1}} \int_{Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, Du_{\delta,i}(x))dx$$

$$= \frac{\mu}{r^{n-1}} \int_{Q_{i-1}\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, Du_\delta)dx + \frac{\mu}{r^{n-1}} \int_{(Q_i\setminus Q_{i-1})\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, Du_{\delta,i})$$

$$+\frac{\mu}{r^{n-1}} \int_{(Q_r(\tilde{x_0})\setminus Q_i)\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, (D_{\tilde{x}} u_{|S}(\tilde{x}_0))_T)dx$$

$$\leq \frac{\mu}{r^{n-1}} \int_{Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, Du_\delta(x))dx$$

$$+\frac{\mu}{r^{n-1}} \int_{(Q_i\setminus Q_{i-1})\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, Du_{\delta,i}(x))dx + \frac{C\mathcal{L}^{n-1}(Q_r(\tilde{x_0})\setminus Q_0)(\varepsilon\mu)}{r^{n-1}}$$

$$= \frac{\mu}{r^{n-1}} \int_{Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, Du_\delta(x))dx +$$

$$\frac{\mu}{r^{n-1}} \int_{(Q_i\setminus Q_{i-1})\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, Du_{\delta,i}(x))dx + C(1-\varsigma)^{n-1}(\varepsilon\mu),$$

Let us estimate the first term of the last line of (5.29). The growth condition in (2.4), (5.28), the well known inequality $(a+b+c)^p \leq 3^{p-1}(a^p+b^p+c^p)$ for any positive numbers a,b,c and the fact that $\mathcal{L}^{n-1}(Q_i\setminus Q_{i-1}) \leq \mathcal{L}^{n-1}(Q_r(\tilde{x_0})\setminus Q_0)$ yield

$$\frac{\mu}{r^{n-1}} \int_{(Q_i\setminus Q_{i-1})\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, Du_{\delta,i}(x))dx \tag{5.30}$$

$$= \frac{\mu}{r^{n-1}} \int_{(Q_i\setminus Q_{i-1})\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, (D_{\tilde{x}} u_{|S}(\tilde{x}_0))_T + D\Phi_i\otimes(u_\delta-u_0) + +\Phi_i D(u_\delta-u_0))dx$$

$$\leq C_u(1-\varsigma)^{n-1}(\varepsilon\mu) + \frac{C\mu}{r^{n-1}} \int_{(Q_i\setminus Q_{i-1})\times]-\varepsilon,\varepsilon[} |D(u_\delta-u_0)|^p dx$$

$$+\frac{Cm^p\mu}{(1-\varsigma)^p r^{n+p-1}} \int_{(Q_i\setminus Q_{i-1})\times]-\varepsilon,\varepsilon[} |u_\delta-u_0|^p dx.$$

We take the sum over $i = 1, 2, ..., m$ and we divide by m in the two sides of (5.29), we use (5.1), (5.30), the fact that $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ and $\bigcup_{i=1}^{m}(Q_i\setminus Q_{i-1}) = Q_r(\tilde{x_0})\setminus Q_0$ to get

$$\inf\left\{\frac{\mu}{r^{n-1}} \int_{Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, (D_{\tilde{x}} u_{|S}(\tilde{x}_0))_T + Dw(x))dx \ : \ w\in W_0^{1,p}(Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[;\mathbb{R}^N)\right\} \tag{5.31}$$

$$\leq \frac{\mu}{r^{n-1}} \int_{Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, Du_\delta(x))dx + C(1-\varsigma)^{n-1}(\varepsilon\mu)$$

$$+\frac{C\mu}{mr^{n-1}}\int_{(Q_r(\tilde{x_0})\setminus Q_0)\times]-\varepsilon,\varepsilon[}|D(u_\delta-u_0)|^p dx+$$
$$\frac{Cm^{p-1}\mu}{(1-\varsigma)^p r^{n+p-1}}\int_{Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[}|u_\delta(x)-u(x_0)-< D_{\tilde{x}}u(x_0),\tilde{x}-\tilde{x}_0>|^p dx.$$

Let us estimate the third term in the right-hand side of inequality (5.31). Since $\tilde{x}_0$ is a Lebesgue point for $\lambda$ with respect to $\mathcal{L}^{n-1}$ and by (5.26) and the growth condition in (2.4), we have

$$\limsup_{r\to 0}\limsup_{\delta\to(0,+\infty)}\frac{C\mu}{mr^{n-1}}\int_{(Q_r(\tilde{x_0})\setminus Q_0)\times]-\varepsilon,\varepsilon[}|D(u_\delta-u_0)|^p dx \tag{5.32}$$
$$\le \limsup_{r\to 0}\limsup_{\delta\to(0,+\infty)}\frac{C\mu}{mr^{n-1}}\int_{(Q_r(\tilde{x_0})\setminus Q_0)\times]-\varepsilon,\varepsilon[}|Du_\delta|^p+|Du_0|^p dx$$
$$\le \limsup_{r\to 0}\limsup_{\delta\to(0,+\infty)}\left\{\frac{C\mu}{mr^{n-1}}\int_{Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[}f_2(\frac{x}{\varepsilon},Du_\delta(x))dx+\right.$$
$$\left.+\frac{C_u(1-\varsigma)^{n-1}\varepsilon\mu}{m}\right\}$$
$$=\frac{C}{m}\frac{d\lambda}{d\tilde{x}}(\tilde{x}_0)+\frac{C_u(1-\varsigma)^{n-1}\eta}{m}=O(m),$$

where $\lim_{m\to+\infty}O(m)=0$. In what concerns the last term in the right-hand side of (5.31), at first we claim that

$$\lim_{\delta\to(0,+\infty)}\mu\int_{B_\varepsilon}|u_\delta(x)-u_{|S}(\tilde{x})|^p dx=0. \tag{5.33}$$

Indeed, recalling the scaled function

$$U_\delta(\widetilde{x},x_n):=u_\delta(\widetilde{x},\varepsilon x_n),$$

by (5.9) and (5.11), we have

$$\lim_{\delta\to(0,+\infty)}(\varepsilon\mu)\int_{B_\varepsilon}|U_\delta(x)-u_{|S}(\tilde{x})|^p dx=0. \tag{5.34}$$

With the change of scale $y=(\tilde{y},y_n)=(\tilde{x},\frac{x_n}{\varepsilon})$ in the above equation, (5.33) then follows. Hence, thanks to (5.33) and (5.25), we get

$$\limsup_{r\to 0}\limsup_{\delta\to(0,+\infty)}\frac{\mu}{r^{n+p-1}}\int_{Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[}|u_\delta(x)-u_{|S}(\tilde{x})- \tag{5.35}$$
$$< D_{\tilde{x}}u_{|S}(\tilde{x}),\tilde{x}-\tilde{x}_0>|^p dx$$
$$\le C\limsup_{r\to 0}\limsup_{\delta\to(0,+\infty)}\left\{\frac{\mu}{r^{n+p-1}}\int_{Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[}|u_\delta(x)-u_{|S}(\tilde{x})|^p dx\right.$$

$$+\frac{C(\varepsilon\mu)}{r^{n+p-1}}\int_{Q_r(\tilde{x_0})}|u_{|S}(\tilde{x})-u_{|S}(\tilde{x}_0)-<D_{\tilde{x}}u_{|S}(\tilde{x}_0),\tilde{x}-\tilde{x}_0>|^p dx\Bigg\}$$
$$=0.$$

Therefore, if we tend successively $\delta\to(0,\infty)$, $r\to 0$, $m\to+\infty$ and $\varsigma\to 1$ in (5.31) and we take into account (5.32) and (5.35), we obtain

$$\limsup_{r\to 0}\limsup_{\delta\to(0,\infty)}\inf\Bigg\{\frac{\mu}{r^{n-1}}\int_{Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[}f_2(\frac{\tilde{x}}{\varepsilon},(D_{\tilde{x}}u_{|S}(\tilde{x}_0))_T+Dw)dx$$
$$:w\in W_0^{1,p}(Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[)\Bigg\}$$
$$\leq\limsup_{r\to 0}\limsup_{\delta\to(0,\infty)}\frac{\mu}{r^{n-1}}\int_{Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[}f_2(\frac{\tilde{x}}{\varepsilon},Du_\delta(x))dx. \tag{5.36}$$

Coming back to (5.26) and making us of (5.36) give us

$$\frac{d\lambda}{d\tilde{x}}(\tilde{x}_0)\geq\limsup_{r\to 0}\limsup_{\delta\to(0,\infty)}\inf\Bigg\{\frac{\mu}{r^{n-1}}\int_{Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[}f_2(\frac{\tilde{x}}{\varepsilon},$$
$$(D_{\tilde{x}}u_{|S}(\tilde{x}_0))_T+Dw(x))dx:w\in W_0^{1,p}(Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[)\Bigg\}. \tag{5.37}$$

Now, giving a function $w\in W_0^{1,p}(Q_r(\tilde{x_0})\times]-\varepsilon,\varepsilon[)$ and if we set

$$w'(y)=\frac{1}{\varepsilon}w(\varepsilon y),$$

then it is immediate that $w\in W_0^{1,p}(Q_{\frac{r}{\varepsilon}}(\frac{\tilde{x_0}}{\varepsilon})\times]-1,1[)$ and $Dw'(y)=Dw(\varepsilon y)$. So, by the change of scale $y=\frac{x}{\varepsilon}$ in inequality (5.37), it follows that his right member is superior or equal to

$$\limsup_{r\to 0}\limsup_{\delta\to(0,\infty)}\inf\Bigg\{\frac{\varepsilon\mu}{(\frac{r}{\varepsilon})^{n-1}}\int_{Q_{\frac{r}{\varepsilon}}(\frac{\tilde{x_0}}{\varepsilon})\times]-1,1[}f_2(\tilde{y},(D_{\tilde{x}}u_{|S}(\tilde{x}_0))_T+Dw(y))dy:$$
$$w\in W_0^{1,p}(Q_{\frac{r}{\varepsilon}}(\frac{\tilde{x_0}}{\varepsilon})\times]-1,1[)\Bigg\}. \tag{5.38}$$

Consequently, applying a diagonalization argument to (5.38) (see e.g. [7], page: 32) and taking into account (5.37) gives as result of this step that

$$\frac{d\lambda}{d\tilde{x}}(\tilde{x}_0)\geq\limsup_{\delta\to(0,\infty)}\inf\Bigg\{\frac{\varepsilon\mu}{(\frac{r_\delta}{\varepsilon})^{n-1}}\int_{Q_{\frac{r_\delta}{\varepsilon}}(\frac{\tilde{x_0}}{\varepsilon})\times]-1,1[}f_2(\tilde{y},$$
$$(D_{\tilde{x}}u_{|S}(\tilde{x}_0))_T+Dw(y))dy:w\in W_0^{1,p}(Q_{\frac{r_\delta}{\varepsilon}}(\frac{\tilde{x_0}}{\varepsilon})\times]-1,1[)\Bigg\}. \tag{5.39}$$

**Step 4: Local estimates**. By assuming that $\lim_\delta \frac{r_\delta}{\varepsilon} = +\infty$ and in view of Lemma 4.1, we have then

$$\eta {f_2}^{hom}(D_{\tilde{x}}(u_{|S})(\tilde{x}_0)) = \limsup_{\delta\to(0,\infty)} \inf \Bigg\{ \frac{\varepsilon\mu}{(\frac{r_\delta}{\varepsilon})^{n-1}} \int_{Q_{\frac{r_\delta}{\varepsilon}}(\frac{\tilde{x_0}}{\varepsilon})\times]-1,1[} f_2(\tilde{y},$$

$$(D_{\tilde{x}}u_{|S}(\tilde{x}_0))_T + Dw(y))dy : w \in W_0^{1,p}(Q_{\frac{r_\delta}{\varepsilon}}(\frac{\tilde{x_0}}{\varepsilon})\times]-1,1[) \Bigg\},$$

and so, according to (5.39) we deduce that

$$\frac{d\lambda}{d\tilde{x}}(\tilde{x}_0) \geq \eta {f_2}^{hom}(D_{\tilde{x}}(u_{|S})(\tilde{x}_0)). \tag{5.40}$$

**Step 5: Global estimates.** (5.24) together with (5.40) allow to get

$$\lambda(S) \geq \int_S \frac{d\lambda}{d\tilde{x}}(\tilde{x})d\tilde{x} \geq \eta \int_S {f_2}^{hom}(D_{\tilde{x}}(u_{|S})(\tilde{x}))d\tilde{x},$$

which gives assertion (2) and end the proof of this lemma. □

5.3. **Proof of the Γ-limit sup.** To close the proof of theorem 2.1, it remains to check the Γ-limit sup. In other terms we have to show that for every $u \in L^p(\mathcal{O};\mathbb{R}^N)$ there exist a sequence $(u_\delta)_\delta \subset L^p(\mathcal{O};\mathbb{R}^N)$ such that

$$\limsup_{\delta\to(0,\infty)} F_\delta(u_\delta) \leq F(u). \tag{5.41}$$

Let us begin with the most simple case.

5.3.1. *Case* $\eta = 0$. Let $u \in L^p(\mathcal{O};\mathbb{R}^N)$. If $u \notin D_1$, the result is obvious (we take for instance $u_\delta = u$). Otherwise, let us remark that under the growth condition in (2.4), for each $u \in D_1$ we have $\mu \int_{B_\varepsilon} f_2(x, Du) \overset{\delta}{\to} 0$. So, by taking $u_\delta = u$ for every $\delta > 0$, we get

$$\limsup_{\delta\to(0,\infty)} F_\delta(u_\delta) \leq \int_{\mathcal{O}} f_1(x, Du)dx,$$

and then, it is sufficient to take the lower semicontinuous envelope with respect to the weak topology in $W^{1,p}(\mathcal{O};\mathbb{R}^N)$ as for (5.59)-(5.61).

5.3.2. *Case* $0 < \eta < +\infty$. Consider a function $u \in L^p(\mathcal{O};\mathbb{R}^N)$. If $u \notin D_2$, the result is obvious (we take for instance $u_\delta = u$). Otherwise, we lead the proof in two steps:

**Step 1**. We begin by proving (5.41) whenever $u \in \mathcal{C}^1(\overline{\mathcal{O}})$. To this purpose, we follow in one hand Licht and Michaille in [28]: by a Riemann approach, we subdivide S into a finite family of subsets, we thus get result under a small error which disappears when we integrate in the whole S. In other hand, as in Acerbi and al [2], we construct a test function equal to u far from S. Consider a parameter $m > 0$, and $I(m)$ a finite subset of $\mathbb{N}$. Let $(R_i)_{i\in I(m)}$ be the family of open bounded disconnected cubes of $\mathbb{R}^{n-1}$ with side $\dfrac{1}{m}$ so that $\mathcal{L}^{n-1}(\mathbb{R}^{N-1} \setminus \cup_{i\in I(m)} R_i) = 0$. Let us take $S_i := R_i \cap S$, $\tilde{x}_i \in S_i$ ($\tilde{x}_i$ may be taken to be the center of the cube $S_i$). Let $\Phi_{\varepsilon,i}$ be an $\varepsilon$-minimizer of

$$\frac{\Upsilon_{\frac{1}{\varepsilon}S_i}(D_{\tilde{x}}u(x_i))}{\mathcal{L}^{n-1}(\frac{1}{\varepsilon}S_i)} = \frac{1}{\mathcal{L}^{n-1}(\frac{1}{\varepsilon}S_i)} \inf\left\{ \int_{\frac{1}{\varepsilon}S_i\times]-1,1[} f_2(\tilde{y}, D_T u(x_i) + D\Phi(y))dy : \right.$$
$$\left. \Phi \in W_0^{1,p}(\frac{1}{\varepsilon}S_i\times]-1,1[) \right\},$$

i.e.,

$$\int_{\frac{1}{\varepsilon}S_i\times]-1,1[} f_2(\tilde{y}, (D_{\tilde{x}}u_{|S}(\tilde{x}_i))_T + D\Phi_{\varepsilon,i}(y))dy \leq \Upsilon_{\frac{1}{\varepsilon}S_i}(D_{\tilde{x}}u_{|S}(\tilde{x}_i)) + $$
$$+h(\varepsilon)\mathcal{L}^{n-1}(\frac{1}{\varepsilon}S_i), \tag{5.42}$$

with $h(\varepsilon) \to 0$ whenever $\varepsilon \to 0$, and we extend $\Phi_{\varepsilon,i}$ by 0 in the whole $\mathbb{R}^n$. Define the following function

$$\Phi_{\varepsilon,m}(x) := \sum_{i\in I(m)} \varepsilon\, \Phi_{\varepsilon,i}(\frac{x}{\varepsilon}),$$

so that $\Phi_{\varepsilon,m} \in W^{1,p}(\mathbb{R}^n;\mathbb{R}^N)$ and vanishes outside of $B_\varepsilon$. Next, we consider as a test-function

$$u_{\delta,m}(x) := [u_{|S}(\widetilde{x}) + \Phi_{\varepsilon,m}(x)]\theta_\varepsilon(x) + u(x)[1-\theta_\varepsilon(x)], \tag{5.43}$$

where for a given $x \in \mathbb{R}^n$,

$$\theta_\varepsilon(x) := \theta(\frac{x_N}{\varepsilon}), \tag{5.44}$$

and $\theta$ being a smooth function which fulfils

$$\theta(t) = 1 \text{ if } |t| \leq 1, \quad \theta(t) = 0 \text{ if } |t| \geq 2, \quad |\theta'(t)| \leq 2.$$

Firstly, let us prove that

$$\int_{\mathcal{O}} |u_{\delta,m} - u|^p\, dx \to 0. \tag{5.45}$$

Since $u_{\delta,m} = u$ in $\mathcal{O} \setminus B_{2\varepsilon}$, we have

$$\int_{\mathcal{O}} |u_{\delta,m} - u|^p\, dx = \int_{\mathcal{O}\setminus B_{2\varepsilon}} |u_{\delta,m} - u|^p\, dx + $$
$$\int_{B_{2\varepsilon}\setminus B_\varepsilon} |u_{\delta,m} - u|^p\, dx + \int_{B_\varepsilon} |u_{\delta,m} - u|^p\, dx \tag{5.46}$$

$$= \int_{B_{2\varepsilon}\setminus B_\varepsilon} |u_{\delta,m} - u|^p\,dx + \int_{B_\varepsilon} |u_{\delta,m} - u|^p\,dx.$$

In one hand, the fact that $\Phi_{\varepsilon,m}$ vanishes outside of $B_\varepsilon$, that $|\theta_\varepsilon| \leq 1$ and the regularity of the function $u$ provide together that

$$\begin{aligned} \int_{B_{2\varepsilon}\setminus B_\varepsilon} |u_{\delta,m} - u|^p\,dx &= \int_{B_{2\varepsilon}\setminus B_\varepsilon} |[u(\widetilde{x}) - u(x)].\theta_\varepsilon(x)|^p\,dx \\ &\leq \int_{B_{2\varepsilon}\setminus B_\varepsilon} |u(\widetilde{x}) - u(x)|^p\,dx \\ &\leq C\mathcal{L}^n(B_{2\varepsilon}\setminus B_\varepsilon) \xrightarrow{\delta} 0. \end{aligned} \tag{5.47}$$

In other hand, $\theta_\varepsilon = 1$ on $B_\varepsilon$. Thus

$$\begin{aligned} \int_{B_\varepsilon} |u_{\delta,m} - u|^p\,dx &= \int_{B_\varepsilon} |u(\widetilde{x}) + \Phi_{\varepsilon,m}(x) - u(x)|^p\,dx \\ &\leq C\left\{\int_{B_\varepsilon} |u(\widetilde{x}) - u(x)|^p\,dx + \int_{B_\varepsilon} |\Phi_{\varepsilon,m}|^p\,dx\right\}. \end{aligned} \tag{5.48}$$

By a change of scale and use of Poincaré inequality, it is immediate that

$$\begin{aligned} \int_{B_\varepsilon} |\Phi_{\varepsilon,m}(x)|^p\,dx &= \sum_{i\in I(m)} \int_{S_i\times]-\varepsilon,\varepsilon[} |\Phi_{\varepsilon,m}(x)|^p\,dx \\ &= \varepsilon^p \sum_{i\in I(m)} \int_{S_i\times]-\varepsilon,\varepsilon[} |\Phi_{\varepsilon,i}(\frac{x}{\varepsilon})|^p\,dx \\ &= \varepsilon^{p+n} \sum_{i\in I(m)} \int_{\frac{1}{\varepsilon}S_i\times]-1,1[} |\Phi_{\varepsilon,i}(y)|^p\,dy \\ &\leq \varepsilon^{p+n} \sum_{i\in I(m)} \int_{\frac{1}{\varepsilon}S_i\times]-1,1[} |D\Phi_{\varepsilon,i}(y)|^p\,dy. \end{aligned} \tag{5.49}$$

By the growth condition (2.4) and (5.42), we see that

$$\begin{aligned} &\int_{\frac{1}{\varepsilon}S_i\times]-1,1[} |D\Phi_{\varepsilon,i}(y)|^p\,dy \\ &\quad = \int_{\frac{1}{\varepsilon}S_i\times]-1,1[} |(D_{\tilde{x}}u_{|S}(\tilde{x}_i))_T + D\Phi_{\varepsilon,i}(y) - (D_{\tilde{x}}u_{|S}(\tilde{x}_i))_T|^p\,dy \\ &\quad \leq C\int_{\frac{1}{\varepsilon}S_i\times]-1,1[} |(D_{\tilde{x}}u_{|S}(\tilde{x}_i))_T + D\Phi_{\varepsilon,i}(y)|^p\,dy + C\mathcal{L}^{n-1}(\frac{1}{\varepsilon}S_i) \\ &\quad \leq C\int_{\frac{1}{\varepsilon}S_i\times]-1,1[} f_2(y,(D_{\tilde{x}}u_{|S}(\tilde{x}_i))_T + D\Phi_{\varepsilon,i}(y))\,dy + C\mathcal{L}^{n-1}(\frac{1}{\varepsilon}S_i) \end{aligned}$$

$$\leq C\Upsilon_{\frac{1}{\varepsilon}S_i}(D_{\tilde{x}}u_{|S}(\tilde{x}_i)) + C\mathcal{L}^{n-1}(\frac{1}{\varepsilon}S_i)(1+h(\varepsilon))$$
$$\leq C\int_{\frac{1}{\varepsilon}S_i\times]-1,1[} f_2(y,(D_{\tilde{x}}u_{|S}(\tilde{x}_i))_T)\,dy + C\mathcal{L}^{n-1}(\frac{1}{\varepsilon}S_i)(1+h(\varepsilon))$$
$$\leq C\int_{\frac{1}{\varepsilon}S_i\times]-1,1[} \left(1+|(D_{\tilde{x}}u_{|S}(\tilde{x}_i))_T|^p\right)\,dy + C\mathcal{L}^{n-1}(\frac{1}{\varepsilon}S_i)(1+h(\varepsilon)),$$

and so

$$\int_{\frac{1}{\varepsilon}S_i\times]-1,1[} |D\Phi_{\varepsilon,i}(y)|^p\,dy \leq \frac{C\mathcal{L}^{n-1}(S_i)(1+h(\varepsilon))}{\varepsilon^{n-1}}. \tag{5.50}$$

(5.50) together with (5.49) provide that

$$\int_{B_\varepsilon} |\Phi_{\varepsilon,m}(x)|^p\,dx \xrightarrow{\delta} 0. \tag{5.51}$$

Hence, coming back to (5.48) and taking into account this last result, we find

$$\int_{B_\varepsilon} |u_{\delta,m} - u|^p dx \to 0, \tag{5.52}$$

consequently, (5.45) is easily deduced from (5.46), (5.47) and (5.52). Furthermore, we may write

$$\begin{aligned} F_\delta(u_{\delta,m}) = \int_{\mathcal{O}\setminus B_{2\varepsilon}} f_1(x,Du(x)dx + \int_{B_{2\varepsilon}\setminus B_\varepsilon} f_1(x,Du_{\delta,m})dx +& \\ \mu\int_{B_\varepsilon} f_2(\frac{\widetilde{x}}{\varepsilon}, Du_{|S}(\widetilde{x}) + D\Phi_{\varepsilon,m}(x))dx.& \end{aligned} \tag{5.53}$$

Let us now focus on the second term of the right-hand side of (5.53). By condition (2.4) and since $\Phi_{\varepsilon,m}$ vanishes outside of $B_\varepsilon$, we have

$$\begin{aligned} \int_{B_{2\varepsilon}\setminus B_\varepsilon} f_1(x,Du_{\delta,m})dx &\leq \beta_1\int_{B_{2\varepsilon}\setminus B_\varepsilon} 1+|Du_{\delta,m}|^p dx \\ &= \beta_1\int_{B_{2\varepsilon}\setminus B_\varepsilon} 1+|Du_{|S}(\widetilde{x})\theta_\varepsilon(x) + Du(x)(1-\theta_\varepsilon(x))+ \\ &\qquad + (u_{|S}(\widetilde{x}) - u(x))D\theta_\varepsilon(x)|^p dx \\ &\leq \beta_1 2^{p-1}\left\{\int_{B_{2\varepsilon}} 1+|Du_{|S}(\widetilde{x})|^p + |Du(x)|^p dx + \right. \\ &\qquad \left. + \frac{C}{\varepsilon^p}\int_{B_{2\varepsilon}} |u_{|S}(\widetilde{x}) - u(x)|^p dx\right\}. \end{aligned} \tag{5.54}$$

By means of Hölder inequality, it is not difficult to show that

$$\frac{1}{\varepsilon^p}\int_{B_{2\varepsilon}} |u(\widetilde{x})-u(x)|^p dx = \frac{1}{\varepsilon^p}\int_{B_{2\varepsilon}} \left|\int_0^{x_n} \frac{\partial u}{\partial t} dt\right|^p dx \leq C\int_{B_{2\varepsilon}} |Du(x)|^p dx. \tag{5.55}$$

Hence, giving the regularity of $u$ and as a direct consequence of (5.54) and (5.55), it follows that

$$\int_{B_{2\varepsilon}\setminus B_\varepsilon} f_1(x, Du_{\delta,m})dx \xrightarrow{\delta} 0. \tag{5.56}$$

Moreover, we claim that

$$\lim_{\delta\to(0,\infty)} \mu\int_{B_\varepsilon} f_2(\frac{\tilde{x}}{\varepsilon}, Du_{|S}(\widetilde{x}) + D\Phi_{\varepsilon,m}(x))dx \leq \sum_{i\in I(m)} \mathcal{L}^{n-1}(S_i)\, \eta {f_2}^{hom}(D_{\tilde{x}} u_{|S}(\tilde{x}_i)) + O(m), \tag{5.57}$$

where $\lim_{m\to\infty} O(m) = 0$. Indeed, Lemma 4.1 applied to ${f_2}^{hom}$ and a change of scale lead to

$$\begin{aligned} \eta\mathcal{L}^{n-1}(S_i){f_2}^{hom}(D_{\tilde{x}}u(x_i)) &= \lim_{\delta\to(0,\infty)} \varepsilon^n\mu\int_{\frac{1}{\varepsilon}S_i\times]-1,1[} f_2(\tilde{y}, (D_{\tilde{x}}u_{|S}(\tilde{x}_i))_T + D\Phi_{\varepsilon,i}(y))dy \\ &= \lim_{\delta\to(0,\infty)} \mu\int_{S_i\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, (D_{\tilde{x}}u_{|S}(\tilde{x}_i))_T + D\Phi_{\varepsilon,i}(\frac{x}{\varepsilon}))dx \\ &\geq \lim_{\delta\to(0,\infty)} \mu\int_{S_i\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, (D_{\tilde{x}}u_{|S}(\tilde{x}))_T + D\Phi_{\varepsilon,i}(\frac{x}{\varepsilon}))dx - O(m) \\ &\geq \lim_{\delta\to(0,\infty)} \mu\int_{S_i\times]-\varepsilon,\varepsilon[} f_2(\frac{\tilde{x}}{\varepsilon}, Du_{|S}(\widetilde{x}) + D\Phi_{\varepsilon,i}(\frac{x}{\varepsilon}))dx - O(m). \end{aligned} \tag{5.58}$$

The two last inequalities in (5.58) are deduced from the Lipschitz condition in (2.4) satisfied by $f_2$ and the following estimate

$$\mu\int_{S_i\times]-\varepsilon,\varepsilon[} |D\Phi_{\varepsilon,i}(\frac{x}{\varepsilon})|^p dx \leq C(\varepsilon\,\mu)\mathcal{L}^{n-1}(S_i)(1+h(\varepsilon)),$$

which is derived from (5.50) by means of a change of scale. Hence, (5.57) follows by Summing over $I(m)$ in (5.58). Thus, applying (5.56) and (5.57) in (5.53), it follows that

$$\limsup_{m\to+\infty}\limsup_{\delta} F_\delta(u_{\delta,m}) \leq \int_{\mathcal{O}} f_1(x, Du)dx + \eta\int_S {f_2}^{hom}(D_{\tilde{x}}(u_{|S})(\tilde{x}))d\widetilde{x}.$$

According to a diagonalization argument, there exist a map $\delta \mapsto m(\delta)$ so that the sequence $(u_\delta)_\delta$ defined by $u_\delta := u_{\delta,m(\delta)}$ converges strongly to u in $L^p(\mathcal{O};\mathbb{R}^{\mathbb{N}})$ and

$$\begin{aligned} \lim_{\delta\to(0,\infty)} F_\delta(u_\delta) &\leq \limsup_{m\to+\infty}\limsup_{\delta\to(0,\infty)} F_\delta(u_{\delta,m}) \\ &\leq \int_{\mathcal{O}} f_1(x,Du)dx + \eta\int_S {f_2}^{hom}(D_{\tilde{x}}(u_{|S}))d\widetilde{x}. \end{aligned} \tag{5.59}$$

Hence

$$\begin{aligned} G(u) &= \inf\{\limsup_\delta F_\delta(v_\delta) : v_\delta \to u \text{ in } L^p(\mathcal{O};\mathbb{R}^{\mathbb{N}})\} \\ &\leq \textstyle\int_{\mathcal{O}} f_1(x,Du)dx + \eta\int_S {f_2}^{hom}(D_{\tilde{x}}(u_{|S}))d\widetilde{x}. \end{aligned} \tag{5.60}$$

For a given topology $\tau$, let us denote by $\mathcal{L}_\tau$ his lower semicontinuous envelope, and denote by $\tau_2$ the weak topology of $(D_2, \|.\|_2)$ (see Lemma 2.1). Then we take the lower semicontinuous envelope with respect to $\tau_2$ in the above inequality. Accordingly

$$\mathcal{L}_{\tau_2}G(u) \leq \int_{\mathcal{O}} \mathcal{Q}f_1(x,Du)dx + \eta\int_S {f_2}^{hom}(D_{\tilde{x}}(u_{|S}))d\widetilde{x}. \tag{5.61}$$

Here,we use the integral representation of quasiconvex envelopes for the first integral term in the right hand side of the above inequality (see for instance [3, 21]) and the quasiconvexity of ${f_2}^{hom}$ by Proposition 4.2 for the second term. Since $G$ is the $\Gamma$-limsup of $F_\delta$ (Theorem 3.1), it will be lower semicontinuous in $L^p(\mathcal{O};\mathbb{R}^N)$ (Proposition 3.1). Hence

$$G(u) = \mathcal{L}_{L^p(\mathcal{O};\mathbb{R}^N)}G(u) \leq \mathcal{L}_{weak-W^{1,p}(\mathcal{O};\mathbb{R}^N)}G(u) \leq \mathcal{L}_{\tau_2}G(u).$$

We conclude noticing that the infimum in the definition of G is attained.
**Step 2**. We take now any $u \in D_2$. By density as in Lemma 2.1, there exist a sequence $(u_k)_k \in \mathcal{C}^1(\overline{\mathcal{O}})$ such that $\|u - u_k\|_{D_2} \to 0$ . By the first step, there exist a sequence $(u_{k,\delta})_\delta$ such that

$$\begin{cases} u_{k,\delta} \to u_k \text{ in } L^p(\mathcal{O};\mathbb{R}^N) \\ \limsup_{\delta\to(0,\infty)} F_\delta(u_{k,\delta}) \leq F(u_k). \end{cases} \tag{5.62}$$

On the other hand, since $\mathcal{Q}f_1$ is quasiconvex and satisfies the growth condition

$$0 \leq \mathcal{Q}f_1(x,\xi) \leq f_1(x,\xi) \leq \beta_1(1+|\xi|^p),$$

it also satisfies the p-Lipschitz property (3.2) with constant independent of $x$ (see proof of Lemma 2.2 [21], page:156). If combined with Proposition 4.1, we deduce that F is continuous on $D_2$ with respect to the norm defined in (2.12). Thus, taking the limit $k \to +\infty$ in (5.62), we get

$$\limsup_{k\to+\infty}\limsup_{\delta\to(0,\infty)} F_\delta(u_{k,\delta}) \leq F(u).$$

Using a diagonalization argument, there exists a map $\delta \mapsto k(\delta)$ such that the sequence $(u_\delta)_\delta$ defined by $u_\delta := u_{k(\delta),\delta}$ satisfies

$$\begin{cases} u_\delta \to u \text{ in } L^p(\mathcal{O};\mathbb{R}^N) \\ \limsup_\delta F_\delta(u_\delta) \leq F(u), \end{cases}$$

which end the proof of this case.

5.3.3. *Case* $\eta = +\infty$. As a particular case of the preceding one, it is sufficient to make a few remarks, otherwise the reminding follows the same lines. Firstly, given a regular function $u \in D_3$, $u_{\delta,m}$, the function defined as in (5.43) is reduced to

$$u_{\delta,m} = u(x)(1 - \theta_\varepsilon(x)),$$

because $u_{|S} = \Phi_{\delta,m} = 0$. However, the fact that $f_2$ is $p-$homogeneous provides $f_2(\tilde{x}, 0) = 0$, and so the third term in the right of (5.53) is simplified to zero. The proof is accomplished by a density argument.

## References


[1] Y. Abddaimi, C. Licht, and G. Michaille, Stochastic homogenization for an integral of a quasiconvex function with linear growth, *Asymptotic Analysis*, **15**, IOS press (1997), 183–202.

[2] E. Acerbi, G. Buttazzo and D. Percivale, Thin inclusions in linear elasticity: a variational approach, *J. reine angew. Math.*, **386** (1988), 99–115.

[3] E. Acerbi and N. Fusco, Semicontinuity problems in the calculus of variations, *Arch. Rational Mech. Anal.*, **86** (1984), 125–145.

[4] F. Alvarez and J-P. Mandallena, Multi-parameter homogenization by localization and blow-up. *Proc. Roy. Soc. Edin. A*, **134** (2004), 801–814.

[5] N. Ansini and A. Braides, Homogenization of oscillating boundaries and applications to thin films, Preprint SISSA, Trieste (1999).

[6] G. Anzellotti, S. Baldo and D. Percivale, Dimension reduction in variational problems, asymptotic development in Γ-convergence and thin structures in elasticity, *Asymptotic Anal.*, **9** (1994), 61–100.

[7] H. Attouch, Variational Convergence for Functions and Operators. Pitman Advance Publishing Program, (1984).

[8] J-F. Babadjian and M. Baia, 3D-2D analysis of a thin film with periodic microstructure, *Proc. Roy. Soc. Edinburgh Sect. A*, **136** (2006), 223–243.

[9] M. Baia and I. Fonseca, The limit behavior of a family of variational multiscale problems, *Indiana Univ. Math. J.*, **56**(1) (2007), 1–50.

[10] J. M. Ball and F. Murat, $W^{1,p}$-quasiconvexity and variational problems for multiple integrals, *J. Func. Anal.*, **58** (1984), 225–253.

[11] A. C. Barroso, I. Fonseca, R. Toader, A relaxation theorem in the space of functions of bounded deformation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **29**(4) (2000), 19–49.

[12] M. Bocea, Young measure minimizers in the asymptotic analysis of thin films, *Electronic Journal of Differential Equations*, **15** (2007), 41–50.

[13] G. Bouchitté, F. Fonseca and L. Mascarenhas, A global method for relaxation, *Arch. Rational Mech.Anal.*, **145** (1998), 51–98.

[14] G. Bouchitté and I. Fragalà, Optimal design of thin plates by a dimension reduction for linear constrained problems, *SIAM J. Control Optim.*, **46**(5) (2007), 1664–1682.
[15] A. Braides, An Introduction to homogenization and gamma-convergence, School on Homogenization ICTP, Trieste (1993), 6–17.
[16] A. Braides, A handbook of Γ-convergence, in Handbook of Differential Equations. Stationary Partial Differential Equations, Volume 3 (M. Chipot and P. Quittner, eds.) Elsevier (2006).
[17] A. Braides, Γ-convergence for Beginners, Oxford University Press, Oxford (2002).
[18] A. Braides and A. Defranceschi, Homogenization of multiple integrals. Oxford University Press, Oxford (1998).
[19] A. Braides, M. Maslennikov and L. Sigalotti, Homogenizaton by blow-up, *Applicable Anal.*, **87** (2008), 1341–1356.
[20] A. Braides, I. Fonseca, G. Francfort, 3D-2D asymptotic analysis for inhomogeneous thin films, *Indiana Univ. Math. J.*, **49** (2000), 1367–1404.
[21] B. Dacorognat, Direct methods in calculus of variations, Applied Mathematical sciences, $n^0$78, Springer-Verlag, Berlin (1989).
[22] G. Dal Maso, An introduction to Γ-convergence, Progress in Nonlinear Differential Equations and their Applications, Birkäuser, Boston (1993).
[23] G. Dal Maso and L. Modica, Non linear stochastic homogenization and ergodic theory, *J. Für die Reine angew. Math.*, **363** (1986), 27–42.
[24] E. De Giorgi, Sulla convergenza di alcune successioni di integrali del tipo dell'area, *Rend. Mat. Appl.*, **8**(6) (1975), 277-294.
[25] I. Ekeland and R. Temam, Convex analysis and variational problems, North-Holland (1978).
[26] I. Fonseca and S. Müller, Quasiconvex integrands and lower semicontinuity in L1, *SIAM J. Math. Anal.* **23** (1992), 1081–1098.
[27] H. P. Huy and E. S. Palencia, Phénomènes de transmission à travers des couches minces de conductivité élevée, *J. Math. Anal. Appl.*, **47** (1974), 284–309.
[28] C. Licht and G. Michaille, A modelling of elastic adhesive bonded joints, Preprint 1995/12, département des sciences mathématiques, Université Montpellier II, (1995).
[29] C. Licht and G. Michaille Global-local subadditive ergodic theorems and application to homogenization in elasticity, *Annales mathématiques Blaise Pascal*, **1** (2002), 21–62.
[30] P. Marcellini, Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals, *Manuscripta Math.*, **51** (1985), 1–28.
[31] K. Messaoudi and G. Michaille, Stochastic homogenization of nonconvex integral functionals, *Mathematical Modelling and Numerical Analysis*, **28**(3) (1994), 329–356.
[32] J. Nečas, Les méthodes directes en théorie des équations elliptiques , Masson et Cie (1967).
[33] W. P. Ziemer, Weakly differentiable functions, Springer, Berlin (1989).

Abdelaziz Aït Moussa
Department of Mathematics and Informatics,
Faculty of Science,
Mohammed Premier University,
Oujda, Morocco

*e-mail address*: moussa@sciences.univ-oujda.ac.ma

Loubna Zlaïji
Department of Mathematics and Informatics
Faculty of Science,
Mohammed Premier University,
Oujda, Morocco
*e-mail address*: l.zlaiji@yahoo.fr

(*Received September 27, 2009*)
(*Revised October 13, 2009*)