The Theory of Market and The Theory of Political Decision

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I Prologue to A Theory of The State

It seems indisputable that one of the most serious problems with which public economics and welfare economics are now confronted is "theory of the state". Economics, for a long time, has paid attention to the market, and tried to analyse the rational behavior of consumers and firms that come onstage.(1) It is well-known that one of the most important contributions made in this area of "market theory" were the studies about existence and stability of competitive equilibrium, and about the relation between competitive equilibrium and Pareto optimality. And it seems that these studies are now going ahead toward the construction of extensive theories containing some initially excluded assumptions —— such as externalities, uncertainties, and transaction costs.

But it should be noticed that the market itself, however it may be treated abstractly, could not exist independently. The market, indeed,

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(1) See, for example, Arrow and Hahn (1) chapter 1.

exists as it is determined to be by many other factors. So the structure of the market also cannot help changing, if the frameworks restricting the market happen to change. In the area of the market theory, it is the (dis) externalities and the public goods that make us realize this fact completely. Suppose, for instance, some economic unit J, a producer, by air pollution, has a disadvantageous effect on another economic unit, some consumers, K. Then, both should have, besides their present transactions on the market, some direct negotiations (bargaining, merger etc.) or ask for the arbitration of a third party about this activity (governmental legislation, tax-subsidy policy, decisions of justice etc.). And, as we have found in case studies, the most knotty problem in this discussion is that the cost of clearing up the pollution would have to be charged to J or K, according to whether we define the environmental property rights as belonging to K or J. In such a case the validity of pareto optimality looks weak. Suppose we define them as belonging to J. What on earth does the Pareto optimum mean if we say that the present position is a Pareto optimum, so K must compensate J in order to make J stop its activities of air pollution, while K is suffering from a serious disease due to the pollutant ? Here do we not need the "theory of political decision-making", assisted by the human hand or based upon explicit value judgement, instead of the harmonious "theory of market" led by an invisible hand ?

Next, let us consider public goods. It is well-known, by the suggestions of Samuelson (7) , Musgrave (5) etc., that in order to determine the optimum supply of public goods and the optimum tax-sharing upon each consumer, we have to resort to a social welfare function based on some specific economic unit's value judgement or some political methods under the voting system, in so far as we reject the ability-to-pay approach to taxation resting on interpersonal comparisons of utility.

The need for political decision, however, should not be limited to the above cases. When we wish to determine the "equitable" distribution of incomes among all members of society; when we wish to define the "merit goods" of Musgrave (5); or, more generally, when we wish to determine the appropriate frameworks to put limitations upon the market, such as the permissible extent of oligopoly, level of minimum wage rate, "proper" social security level for the weak, and the selection on trade-off relations …… in all these cases, the theory of political decision, in addition to the market theory, should be called for.

Now let us compare the theory of political decision with the market theory in order to make more distinct our objects of study. We have already pointed out that attention is paid to the "market", and consumers and firms are the economic units that enter there in the market theory. Each of these units wants to maximize its own objective function — utility or profit — with the given resources (incomes) and production facilities which it has already procured. On the other hand, it is the "assembly"(2) that is the center of attention in the political decision theory, and the political units which occupy the stage are citizens and political parties. What the citizen possesses is "civil, political rights"(3) and what a party holds is "freedom of political activi-

(2) It should be noticed that the term of "assembly" used here is extremely abstract, corresponding exactly to the abstractness of "market" in market theory.

(3) In some cases one ballot for voting, but not restricted to this.

ties". Now what are the objective functions of a citizen and a party? They do not seem necessarily clear in comparison with "utilities and profits". Is the object of a citizen also utility? (Buchanan and Tullock (2)) Is that of a party, winning elections? (Downs)

Next, in the market theory, the parameter is price, and according to what theories state, the market will be cleared up if the price mechanism is sufficiently flexible, and furthermore "a competitive allocation is in the *core*" (4) (Debreu and Scarf (3)). On the other hand, the parameter in the political decision theory is probably "power", and power negotiation among citizens (parties) will make some (not necessarily consistent) political decisions. It is worth noticing that in the "assembly", the coalition of various types of units would be all but indespensable to be effective. It is such coalition that make possible all kinds of citizens' movements, party activities, and the action of pressure groups, and that, at the same time, make more complicated and difficult the political decision theory than the market theory.

I shall take up, in extremely restricted forms, these various aspects of political theory —— which I should like to call the "theory of the state" —— in the sequential papers. This paper, as a first step, will take up —— using the most simple models —— one of the differences between the two theories.

II The Most Simplified Competitive Equilibrium

(4) "An allocation is in the core if it cannot be recontracted out by any set of consumers S. i. e. if no set of consumers S can redistribute their own initial supply among themselves so as to improve the position of any one member of S without deterioration of that of any other." (Debreu and Scarf (3) p. 238)

The most simplified example of a competitive equilibrium may be given by Edgeworth's boxdiagram. Suppose there are two goods X and Y, and two persons I and J whose initial endowments of both goods are expressed as (X_i, Y_i) and (X_j, Y_j). We assume that both persons have "well-defined" preferences which are shown by two indifference maps I_1, I_2, ……, and J_1, J_2, ……. (See diagram 1 which locates I in the bottom left-hand corner and J in the upper right-hand corner. We measure X on the abscissa from the left for I and from the right for J, and Y on the ordinate from below for I and from above for J.) The initial situation when I and J enter the market is indicated by point A. Since each individual is assumed to behave as a *pricetaker* in the "perfect competitive market", we have only to persue the responses of both persons when the relative price of two goods changes. It is obvious that neither person will trade unless the relative price falls in between p_1 and p_4. Furthermore it may easily be understood that since both want to occupy the best position (maximizing their utilities), individual I will select the points on the curve extending from A to R through C, D and E, while J will select the points on the curve from A to S, as the relative price changes. (Notice that the indifference curves are tangent to the price lines at each point of AR and AS.) We shall call them I' s offer-curve and J' s offer-curve. Now, it is a proof of the existence of competitive equilibrium that these two offer-curves indeed have at least one point of intersection in this case.(5) At the same time, the proposition that "a competitive equilibrium is a Pareto optimum" will be proved also, since at the equilibrium point D, both persons' indifference curves (I_3, J_2) are

(5) See Nikaidô (6) for the rigorous proof in a general case.

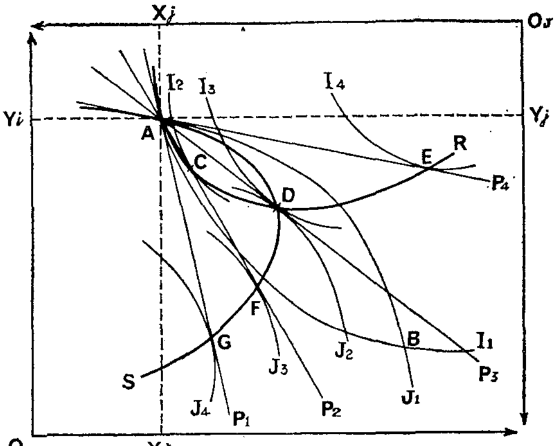

Fig.1

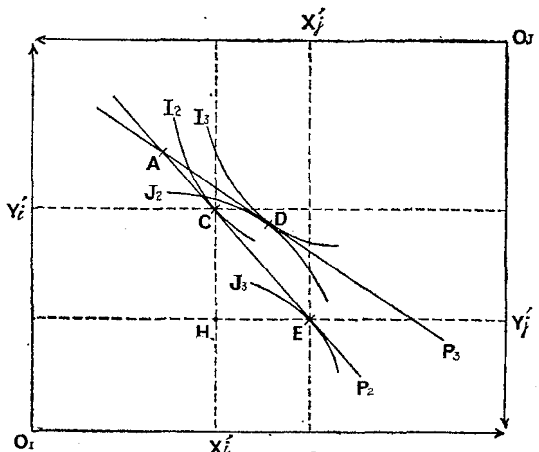

Fig.2

tangential to each other, or in other words, the equilibrium point is on the "contract curve".(6)

(6) By the way, since the contract curve is equivalent to the core in Edgeworth's boxdiagram, another proposition, namely that "a competitive equilibrium is in the core", can be proved also.

Next, we may be able to prove the stability of this equilibrium by the same boxdiagram. See figure 2 with the same notations as figure 1. Suppose that the relative price shifts to p_2 from the equilibrium price p_3. Then individual I will select point C, according to the same logic as before, while J will select the point E. In this situation these two persons' demands are expressed by (X'_i, Y'_i) and (X'_j, Y'_j), and as is easily shown from the figure, $X'_i+X'_j<X_i+X_j$ and $Y'_i+Y'_j>Y_i+Y_j$. Then there will be an excess supply for X of EH, and an excess demand for Y of CH in the market. Thus the relative price will necessarily change again so that the market may clear these excess supplies and excess demands i. e. toward the direction in which the relative price of X will fall. Consequently p_2 will begin to draw near p_3 again. Thus we see a proof of stability in this simple example.

Well, one of the most conspicuous characteristics of the "market theory" is that both I and J will expect to make themselves better off than their initial position by their market negotiations, so long as the market equilibrium will fall in the area surrounded by I_1 and J_1 curves. The market is the place of voluntary participation and nobody makes himself worse off by his negotiations in the market.

III The Most Simplified Political Allocation

The most simplified example of allocation and distribution based on the political process will be illustrated by a society composed of three citizens. Since some stimulating examples have already been given in Buchanan and Tullock (2), we shall make some extentions of their study.

Let us suppose a society composed of three farmers has some political projects concerning cultivation. All three are assumed to be equally situated, but the productivities of their farms differ. They are urged to decide whether they will or will not take part in some project of joint production. Let us look at these assumptions in more details.

I. Three farmers —— 1, 2 and 3 —— are equally situated except the productivities of their farms.

II. As to the fertilizer investment, each farm has the following differential production function with decreasing returns to scale:

$y_1 = f_1(x_1),\ \ y_2 = f_2(x_2),\ \ y_3 = f_3(x_3):$

$f_i(x_i) > 0,\ \ f'_i(x_i) > 0,\ \ f''_i(x_i) < 0 \qquad \text{for all } x_i > 0$

$f_i(0) = 0,\ \ f_i(\infty) = \infty,\ \ f'_i(0) = \infty,\ \ f'_i(\infty) = 0,\ \ (i = 1,\ 2,\ 3)$

$f_1(x) > f_2(x) > f_3(x) \qquad \text{for all } x > 0.$

(Here x_i shows input of fertilizer, y_i output for the i person, each measured in dollar terms.)

Each farm's production has no external or spillover effects on the others.

III. In taking part in any project, each farmer is required to pay k dollars. Since this project is assumed to be a single, isolated decision, there is no possibility of log-rolling.

Under these assumptions, we shall ask in what situations each citizen would want to take part in such a project, and what political situation each would desire. To that end we have to compare several cases: a) one in which each invests privately k dollars for fertilizer; b) one in which all three wish to "collectivize" their production with only unanimity rule; c) one in which they wish to adopt the majority rule in the allocation or distribution process; and d) the case where one of the farmers possesses dictatorial power. Let us take these cases up one by one.

a) The private investment case is the most simple. Each person's benefits are;

$$y_1=f_1(k),\quad y_2=f_2(k),\quad y_3=f_3(k) \text{ and } y_1>y_2>y_3.$$

If $y_i<k$, for some person i, then person i may not invest at all, since he will make himself worse off by investment.

b) Production collectivization is more interesting. In this case, the optimal resources allocation will be realized by the following procedure;

$$\max\ \Sigma f_i(x_i) \qquad \text{sub. to. } \Sigma x_i=3\,k.$$

In Lagrangian,

$$L\equiv\Sigma f_i(x_i)+\lambda(\Sigma x_i-3k).$$

Then the first order condition will be

$$f'_1(x_1)=f'_2(x_2)=f'_3(x_3)$$

$$\Sigma x_i=3k.$$

On the other hand, the second order conditions are met as shown below:

$$\begin{vmatrix} f''_1(x_1) & 0 & 1 \\ 0 & f''_2(x_2) & 1 \\ 1 & 1 & 0 \end{vmatrix} = -[f''_1(x_1)+f''_2(x_2)]>0$$

$$\begin{vmatrix} f''_1(x_1) & 0 & 0 & 1 \\ 0 & f''_2(x_2) & 0 & 1 \\ 0 & 0 & f''_3(x_3) & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -[f''_1(x_1)\cdot f''_2(x_2)+f''_2(x_2)\cdot f''_3(x_3)+f''_3(x_3)\cdot f''_1(x_1)]<0$$

since $f''_i(x_i)<0$, $(i=1, 2, 3)$ and $\frac{\partial}{\partial x_j}f'_i(x_i)=0\ (j\neq i)$ from assumption II. In denoting each optimal imputs as $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$, the maximum benefits will be $\hat{y}=\Sigma f_i\ (\hat{x}_i)$ subject to $f'_1(\hat{x}_1)=f'_2(\hat{x}_2)=f'_3$

$(\hat{x}_3)$ and $\Sigma\hat{x}_i = 3k$.

In comparison with case a, case b is obviously feasible, since $\Sigma f_i(\hat{x}_i) > \Sigma f_i(k)$.[7] But in this case there could occur some serious troubles about the distribution of outcome, because actually some funds of those farmers with farms of low productivity have been invested in a farm (or farms) with high productivity. This aspect is of much interest and worth more examination.

If the three are quite gentle, and look for unanimous agreement, they will perform the following distribution.

$$D = \{z_1,\ z_2,\ z_3) \mid z_1 \geqq f_1(k),\ z_2 \geqq f_2(k),\ z_3 \geqq f_3\ (k),\ \Sigma z_i = \hat{y}\}$$

(z_i is the i person's benefit, $i=1,\ 2,\ 3$)

Note: Even in this case there remain some room, though little,[8] for bargaining about how to distribute $\hat{y} - \Sigma f_i(k)$.

c) The majority decision rule is the most interesting. There are two types of majority decision in this case: one is only about the distribution after collective production, and the other is about fund allocation. Both cases can be interpreted as three-person cooperative games of Von Neumann and Morgenstern.

c-1) In a majority decision with collective allocation, the three farmers play this game about the distribution after they perform the optimal collective production. Then the characteristic function of the game is:

i) $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$

ii) $v(\{1,\ 2\}) = v(\{1,\ 3\}) = v(\{2,\ 3\}) = \hat{y}$

(7) See Appendix A for proof.

(8) In our specification that $y_1 = \sqrt{ax_1}$, $y_2 = \sqrt{bx_2}$, and $y_3 = \sqrt{cx_3}$ ($a > b > c$), the difference is $\hat{y} - \Sigma f_i\ (k) = \sqrt{\frac{3k\,(a_2 + b_2 + c_2)}{a+b+c}} - \sqrt{(a+b+c)\,k}$.

iii) $v(\{1, 2, 3\})=\hat{y}$

Since the effective sets in this game are obviously $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$, if they agree to the symmetrical sharing of gains, the solution will be:

$$E=\{(\frac{\hat{y}}{2}, \frac{\hat{y}}{2}, 0), (\frac{\hat{y}}{2}, 0, \frac{\hat{y}}{2}), 0, \frac{\hat{y}}{2}, \frac{\hat{y}}{2})\}^{(9)}$$

But the situation will be more complicated on account of the differential productivities which go into the collective production. So the solution may be generally given as the following discriminatory one:

$$F=\{(z_1, z_2, c_3), (z_1, c_2, z_3), (c_1, z_2, z_3)\}$$
$$(0\leqq c_i, z_j\geqq f_j (k), \Sigma z_j+c_i=\hat{y}; i, j=1, 2, 3: i\neq j)$$

Note: First of all, person i who is subject to discrimination, is assigned c_i by the majority vote of the group, and after that the negotia-

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(9) Let us briefly describe the precise definition of the solution in Von Neumann and Morgenstern's n-person game.

Denoting vector $x=(x_1, x_2, \cdots\cdots, x_n)$ as the n players' distribution in the game, any n-tuple x of a real number satisfying the following two conditons (1) individual rationality and (2) Pareto optimality, is called an *imputation* of the game with characteristic function v:

(1) $v(\{i\})\leqq x_i$ for every $i\in I_n=\{1, 2, \cdots\cdots, n\}$

(2) $\Sigma x_i =v(I_n)$

Next one imputation $y=(y_1, y_2, \cdots\cdots, y_n)$ is said to *dominate* another imputation $x=(x_1, x_2, \cdots\cdots, x_n)$ with respect to a coalition S $(S\subset I_n)$, provided that S is not an empty set, and the following two conditions are met:

(3) $v(S)\leqq \sum_{i\in S} y_i$

(4) $y_i > x_i$ for every $i\in S$.

The set in such a domination is called an *effective set.* Finally the set T of imputations is the *solution* of the game, if it possesses the following properties:

(5) No imputation y in T is dominated by another x in T

(6) Every imputation z not in T is dominated by at least one imputation in T.

See also Von Neumann and Morgenstern (8) pp. 263—264.

tion about how to distribute $\hat{y}-c_i$ between the other two will be carried out. Further the total benefits are $\hat{y}$ in this case, too.[10]

c-2) If the majority game is performed firstly about the fund allocation, the situation will be a bit different from c-1. The characteristic function is:

i) $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$

ii) $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 3k$

iii) $v(\{1, 2, 3\}) = 3k$

And the solution is:

$$G=\{(\frac{3}{2}k,\ \frac{3}{2}k,\ 0),\ (\frac{3}{2}k,\ 0,\ \frac{3}{2}k),\ (0,\ \frac{3}{2}k,\ \frac{3}{2}k)\}$$

If in each imputation, the losing person would not be asked to participate in production, the maximum output will be achieved by the following procedure:

$$\max\ f_i(x_i)+f_j(x_j) \qquad \text{sub. to. } x_i+x_j=3k.$$
$$(i,\ j=1,\ 2,\ 3:\ i\neq j)$$

Thus, by the same caluculation as before, the following outcomes will be shown:

$$H=\{(z_1,\ z_2,\ 0),\ (z'_1,\ 0,\ z'_3),\ (0,\ z''_2,\ z''_3)\}$$

$$z_1+z_2=f_1(\tilde{x}_1)+f_2(\tilde{x}_2) \quad \text{such that} \quad f'_1(\tilde{x}_1)=f'_2(\tilde{x}_2),\ \tilde{x}_1+\tilde{x}_2=3k.$$

$$z'_1+z'_3=f_1(\bar{x}_1)+f_3(\bar{x}_3) \quad \text{such that} \quad f'_1(\bar{x}_1)=f'_3(\bar{x}_3),\ \bar{x}_1+\bar{x}_3=3k.$$

$$z''_2+z''_3=f_2(\overset{*}{x}_2)+f_3(\overset{*}{x}_3) \quad \text{such that} \quad f'_2(\overset{*}{x}_2)=f'_3(\overset{*}{x}_3),\ \overset{*}{x}_2+\overset{*}{x}_3=3k.$$

Obviously, under assumption II, $z_1+z_2>z'_1+z'_3>z''_2+z''_3$,[11] i. e. the [1, 2] coalition would bring the largest benefits, but, of course, there is no absolute guarantee for this.[12]

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(10) See Von Neumann and Morgensern (8) pp. 288—290, for a discriminatory solution.

(11) See Appendix A for proof.

(12) This is the "without sidepayments case" in Buchanan and Tullock (2).

c-3) On the other hand, if the losing person takes part in production in exchange for compensation, there will be a negotiation between the winning coalition and the losing person, and the situation will be quite similar to case b. Then the solution will be given by the set D.(13)
Note: If compensation is not offered by the winning coalition, even the largest benefits among the above three will be less than that of collective production $(z_1+z_2<\hat{y})$.(14)

d) Finally, if this society has the characteristics of dictatorial control, the outcome will be shown by either of the following:

d-1) collective allocation and dictatorial distribution

$$K=\{(\hat{y},\ 0,\ 0),\ (0,\ \hat{y},\ 0),\ (0,\ 0,\ \hat{y})\}$$

d-2) dictatorial fund allocation without compensation

$$L=\{(f_1(3k),\ 0,\ 0),\ (0,\ f_2(3k),\ 0),\ (0,\ 0,\ f_3(3k))\}$$

d-3) dictatorial fund allocation with compensation, where there will be a negotiation between the dictator and the others, and the similar situation to case b will occur.

Note: In cases d-1 and d-3, the social benefits are equal, and, generally, more than in d-2.

Now what implications can we draw from our model ? But before considering that, let us compare the results of all the above cases.

a) $Y=\{(f_1(k),\ f_2(k),\ f_3(k))\}$

b) $D=\{(z_1,\ z_2,\ z_3)\mid z_i\geqq f_i(k),\ \Sigma z_i=\hat{y}\}$

c-1) $F=\{(z_1,\ z_2,\ c_3),\ (z_1,\ c_2,\ z_3),\ (c_1,\ z_2,\ z_3)\}$

$(c_i\geqq 0,\ z_j\geqq f_j(k),\ \Sigma z_j+c_i=\hat{y}:\ i,\ j=1,\ 2,\ 3:\ i\neq j)$

c-2) $H=\{(z_1,\ z_2,\ 0),\ (z'_1,\ 0,\ z'_3),\ (0,\ z''_2,\ z''_3)\}$

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(13) This is the "with full sidepayments case" in Buchanan and Tullock (2).
(14) See Appendix A for proof.

$(\hat{y} > z_1 + z_2 > z'_1 + z'_3 > z''_2 + z''_3)$

c-3) D

d-1) $K = \{(\hat{y},\ 0,\ 0),\ (0,\ \hat{y},\ 0),\ (0,\ 0,\ \hat{y})\}$

d-2) $L = \{(f_1(3k),\ 0,\ 0),\ (0,\ f_2(3k),\ 0),\ (0,\ 0,\ f_3(3k))\}$

d-3) D

We can now draw some interesting conclusions from the above comparison.

In the first place, the largest social benefits $\hat{y}$ can be realized in any of the following systems: a unanimous decision system (b); majority distribution with collective production (c-1); majority allocation with compensation (c-3); dictatorial distribution with collective production (d-1); or dictatorial allocation with compensation (d-3). These cases have the common characteristics of collective production whether or not there is compensation. It is worth noticing that even though there are no externalities in production, the largest social benefits are achieved only in collective production. This may provide a motive for cooperation or joint management. But at the same time, it should be noticed that if the optimal collective production is organized, the largest social benefits will be realized regardless of the different political system (e. g. unanimous agreement, majority decision rule or dictatorship). In other words, this means that every system can produce the same largest social benefits. This may be a bit shocking.

Secondly, in the political decision process, even if the social benefits exceed the sum of the outcomes of private investments $(\hat{y} > \Sigma f_i(k))$, there could occur a situation in which some person finds himself worse off than in his private investment …… for instance, in majority allocation (c-1) (c-2) or dictatorial allocation (d-2). This seems to be an inevitable accompaniment of political process, in contrast to market process where nobody makes himself worse off by participation in market trade.

Finally, if each person has one ballot based on "political equality", the dictatorial political system (d) cannot be realized, and one of the other "games" (b), (c-1), (c-2) or (c-3) will be played. Since the distribution z_i in F or H could exceed z_i in D, the unanimous distribution project (b) will not necessarily be realized. But in connection with this, it should be noticed that person 1 will be extremely unwilling to participate in this political project, because his private investment would carry more benefits except in cases of unanimous distribution, of sufficient compensation or of his dictatorship. On the other hand, person 3 will be eager to organize this project for the opposite reason. Then in the distribution according to majority decision rule F or H, a symmetrical distribution among the members in the winning coalition cannot be achieved because of the differential productiviies of three farm. Here we find a conflict between political equality and the economically unequal state. And, as observed in our own actual society, formal political equality is often infringed upon by the differential economic conditions of the members in society.

IV Some Generalizations

In the preceding section, we dealt with a society composed of only three persons. Some implications drawn there, however, can be applied to a society of n persons. Let us denote n differential production functions as $f_1(x_1)$, $f_2(x_2)$, ……, $f_n(x_n)$. Without losing generality, we can put the same assumptions as before and assume

$f_1(x) > f_2(x) > \cdots\cdots > f_n(x)$ for all $x > 0$. Then the largest social benefits are achieved by the same procedure as before:

$$\max \sum_{i=1}^{n} f_i\ (x_i) \qquad \text{sub. to.} \sum_{i=1}^{n} x_i = nk.$$

The first condition is

$$f'_i(x_i) = f'_j(x_j) \qquad (i,\ j=1,\ 2,\ \cdots\cdots,\ n:\ i \neq j)$$

$$\sum_{i=1}^{n} x_i = nk.$$

The second condition is also met as will be shown in Appendix B.

Furthermore, for $N = \{1,\ 2,\ \cdots\cdots,\ n\}$ and $M \subset N$, the following inequalities will be proved by the same calculations as in the preceding section:

1) $\sum_{i \in N} f_i(\hat{x}_i) > \sum_{i \in N} f_i(k)$

2) $\sum_{i \in N} f_i(\hat{x}_i) > f_1(nk)$

3) $\sum_{i \in N} f_i(\hat{x}_i) > \sum_{j \in M} f_j(\tilde{x}_j)$

(Here $\hat{x}_i$ $(\tilde{x}_j)$ expresses, similarly to before, the optimal imput into the i (j) farm in production organized by N (M) members.)

Thus it can be shown that the more collectively people are organized in production, the more social benefits can be realized in contrast with the private investment. But at the same time, the cumulatively increasing negotiation costs must be taken into consideration in appraising. the distribution of this result. And we can not simply describe what kind of coalition will be made or what final imputation will result.

Appendix A

Under the assumption II:

$f_i(x_i) > 0$, $f'_i(x_i) > 0$, $f''_i(x_i) > 0$, for all $x_i > 0$;

$f_i(0) = 0$, $f_i(\infty) = \infty$, $f'_i(0) = \infty$, $f_i(\infty) = 0$ $\qquad (i=1,\ 2,\ 3)$

$f_1(x) > f_2(x) > f_3(x)$, for all $x > 0$, the following four formulations are proved together.

1) $\sum_{i=1}^{3} f_i(\hat{x}_i) > \sum_{i=1}^{3} f_i(k)$

2) $\sum_{i=1}^{3} f_i(\hat{x}_i) > f_1(3k)$

3) $\sum_{i=1}^{3} f_i(\hat{x}_i) > f_1(\tilde{x}_1) + f_2(\tilde{x}_2)$

4) $f_1(\tilde{x}_1) + f_2(\tilde{x}_2) > f_1(\bar{x}_1) + f_3(\bar{x}_3) > f_2(\overset{*}{x}_2) + f_3(\overset{*}{x}_3)$

such that

$$\sum_{i=1}^{3} \hat{x}_i = \tilde{x}_1 + \tilde{x}_2 = \bar{x}_1 + \bar{x}_3 = \overset{*}{x}_2 + \overset{*}{x}_3 = 3k.$$

$$f'_1(\hat{x}_1) = f'_2(\hat{x}_2) = f'_3(\hat{x}_3), \quad f'_1(\tilde{x}_1) = f'_2(\tilde{x}_2),$$

$$f'_1(\bar{x}_1) = f'_3(\bar{x}_3), \qquad f'_2(\overset{*}{x}_2) = f'_3(\overset{*}{x}_3).$$

Inequalities 1, 2 and 3 can be simply proved by a reduction to absurdity. Since $k+k+k=3k$, $\tilde{x}_1+\tilde{x}_2+0=3k$, and $f_2(0)=f_3(0)=0$, if either $\Sigma f_i(\hat{x}_i) < \Sigma f_i(k)$, $\Sigma f_i(\hat{x}_i) < f_1(3k)$, or $\Sigma f_i(\hat{x}_i) < f_1(\tilde{x}_1) + f_2(\tilde{x}_2)$ holds, then $\Sigma f_i(\hat{x}_i)$ is not the maximum under $\Sigma x_i = 3k$. This leads to a contradiction of the property of $\Sigma f_i(\hat{x}_i)$.

But let us try another proof.

Lemma $f_i(x_i) > f_i'(x_i) \cdot x_i$ for all i and all $x_i > 0$.

Proof. Since from Taylor's theorem, under the above assumptions, there exist $0 < \theta_i < x_i$ such that $f_i(x_i) = f'_i(\theta_i) \cdot x_i$. From the property of f, $f'_i(x_i) < f'_i(\theta_i)$. These prove the lemma.

Proof of 1)

In considering $f'_i(x_i) > 0$, $f''_i(x_i) < 0$, for all $x_i > 0$ $(i=1, 2, 3)$, from Taylor's theorem, we can find some $\mu_i = \theta_i^1 k + (1-\theta_i^1)\hat{x}_i$ $(0<\theta<1)$ for all $i=1$, 2 and 3, such that

$$f_i(\hat{x}_i) - f_i(k) = f'_i(\mu_i)(\hat{x}_i - k).$$

From the assumption, if $\hat{x}_i \gtrless k$, then $f_i'(\mu_i) \gtrless f_i'(\hat{x}_i)$. Hence,

$$f_i(\hat{x}_i)-f_i(k)=f'_i(\mu_i)(\hat{x}_i-k)>f'_i(\hat{x}_i)(\hat{x}_i-k).$$

Summing up,

$$\Sigma f_i(\hat{x}_i)-\Sigma f_i(k)>\Sigma f'_i(\hat{x}_i)(\hat{x}_i-k)=f'_1(\hat{x}_1)\Sigma(\hat{x}_i-k)=0.$$

(Q. E. D.)

Proof of 2)

From Taylor's theorem, we can find some $\eta=\theta^2 k+(1-\theta^2)\hat{x}_1$ $(0<\theta<1)$ such that $f_1(\hat{x}_1)-f_1(3k)=f'_1(\eta)(\hat{x}_1-3k)$. Thus, in connection with lemma,

$$\begin{aligned}\Sigma f_i(\hat{x}_i)-f_1(3k)&=f'_1(\eta)(\hat{x}_1-3k)+\sum_{i=2,\ 3} f_i(\hat{x}_i)\\&>f'_1(\hat{x}_1)(\hat{x}_1-3k)+\sum_{i=2,\ 3} f'_i(\hat{x}_i)\cdot\hat{x}_i\\&=f'_1(\hat{x}_1)\sum_{i=1}^{3}(\hat{x}_i-3k)=0.\end{aligned}$$

(Q. E. D.)

Proof of 3)

From Taylor's theorem, we can find some $\lambda_i=\theta_i^3\hat{x}_i+(1-\theta_i^3)\tilde{x}_i$ $(i=1,2)$ and $\lambda_3=\theta_3^3\hat{x}_3$ $(0<\theta_i^3<1$, for $i=1, 2, 3)$, such that

$$f_i(\hat{x}_i)-f_i(\tilde{x}_i)=f'_i(\lambda_i)(\hat{x}_i-\tilde{x}_1)\qquad(i=1,\ 2)$$

$$f_3(\hat{x}_3)\qquad=f'_3(\lambda_3)\cdot\hat{x}_3.$$

From the assumptions, if $\hat{x}_i\gtreqless\tilde{x}_i$, then $f'_i(\hat{x}_i)\lesseqgtr f'_i(\lambda_i)$ for $i=1$ and 2, and from the lemma, $f_3(\hat{x}_3)>f'_3(\hat{x}_3)\cdot\hat{x}_3$. Hence,

$$f_i(\hat{x}_i)-f_i(\tilde{x}_i)>f'_i(\hat{x}_i)(\hat{x}_i-\tilde{x}_i)\qquad(i=1,\ 2)$$

$$f_3(\hat{x}_3)\qquad>f'_3(\hat{x}_3)\cdot\hat{x}_3.$$

Summing up, we can prove 3). (Q. E. D.)

Proof of 4)

If we prove $f_1(\tilde{x}_1)+f_2(\tilde{x}_2)>f_1(\bar{x}_1)+f_3(\bar{x}_3)$, then the second part of this inequality will be proved similarly. From the assumptions and the continuity of function, it is shown that there is a $\bar{x}_2$ on $f_2(x_2)$ such that $f_2(\bar{x}_2)=f_3(\bar{x}_3)$. Furthermore from Taylor's theorem, we can find some $v_i=\theta_i^4\tilde{x}_i+(1-\theta_i^4)\bar{x}_i$ $(0<\theta_i^4<1)$ such that

$$f_i(\tilde{x}_i)-f_i(\bar{x}_i)=f'_i(v_i)(\tilde{x}_i-\bar{x}_i) \qquad (i=1,\ 2).$$

Then with the same procedure as before,

$$\sum_{i=1,\,2} f_i(\tilde{x}_i)-\sum_{i=1,\,3} f_i(\bar{x}_i)=\sum_{i=1,\,2}[f_i(\tilde{x}_i)-f_i(\bar{x}_i)]$$

$$=\sum_{i=1,\,2} f'_i(v_i)(\tilde{x}-\bar{x}_i)>\sum_{i=1,\,2} f'_i(\tilde{x}_i)(x_i-\bar{x}_i)=f'_i(\tilde{x}_i)\sum_{i=1,\,2}(\tilde{x}-\bar{x}_i)=0.$$

(Q. E. D.)

Appendix B

Let us denote $\begin{vmatrix} f''_1(x_1) & 0 & \cdots & 0 & 1 \\ 0 & f''_2(x_2) & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f''_m(x_m) & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix}$ as $|A_m|$

It was shown that $|A_2|>0$ and $|A_3|<0$ in section III. Furthermore,

$$A_{m+1}=\begin{vmatrix} f''_1(x_1) & 0 & \cdots & 0 & 1 \\ 0 & f''_2(x_2) & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f''_{m+1}(x_{m+1}) & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix}$$

$$=f''_{m+1}(x_{m+1})\begin{vmatrix} f''_1(x_1) & 0 & \cdots & 0 & 1 \\ 0 & f''_2(x_2) & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f''_m(x_m) & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix}$$

$$-\begin{vmatrix} f''_1(x_1) & 0 & \cdots & 0 & 0 \\ 0 & f''_2(x_2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f''_m(x_m) & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}.$$

$$= f''_{m+1}(x_{m+1})\,|A_m| - \prod_{i=1}^{m} f''_i(x_i)$$

Hence, in considering $f''_i(x_i)<0$, for all i, $|A_4|>0$, $|A_5|<0$, …….

(Q. E. D.)

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