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Univalence of Nonlinear Mappings : A Qualitative Approach

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1. Introduction

In a previous issue of this journal (Fujimoto and Ranade (1998)), I produced jointly with Professor Ravindra R. Ranade an article on the univalence of nonlinear mappings employing qualitative information about the sign patterns of the Jacobian matrix of a given differentiable transformation. The proof therein is based on one of the mean value theorems for multivariable functions. In the final section of the above paper, we suggested possible generalizations, among which is the one concerned with the domain of a mapping. A given space need not be the Euclidean space over the real field. All we need is an ordered vector space which accommodates the construction of matrix as well as determinant theory where $Mx = 0$ implies the vector $x = 0$ provided the determinant of M is not zero. Since we do not use inverse matrices, the above requirement is satisfied by any integral domains, more precisely *ordered integral domains*. By this extension we can deal with the univalence problem for models with indivisibility of commodities and / or processes. Certainly this can be useful only when the existence of a solution is guaranteed.

In this note, I will explain in more detail how the above extension proceeds. The underlying idea is the same as in the previous note, and we establish a sort of mean value theorem for a mapping from a space consisting of lattice points into itself.

2. Assumptions and Notation

Let us take up an ordered integral domain Q. This Q is an integral domain with a total order which has the following three properties :

O 1 : if $a \leq b$ and $c \leq d$, $a + c \leq b + d$.

O 2 : if $a \leq b$, $-a \geq -b$.

O 3 : if. $a > 0$ and $b > 0$, $ab > 0$.

A linear space over Q we consider is a set of n-tuples of Q, denoted as Q^n, each of which elements belongs to Q. Addition, subtraction, inner–product, and scalar multiplication are to be defined as in the normal Euclidean space over the real field.

Now consider a mapping $f \equiv (x_1, x_2, \ldots, x_n)$ from a subset D in Q^n to Q. For this mapping, we define the 'gradient' $\nabla f \equiv (g_1, g_2, \ldots, g_n)$, something like $(+, -, 0, *, \ldots)$. The four signs have the following meanings :

(1) $g_j = +$: when f is always increasing with respect to x_j.

(2) $g_j = -$: when f is always decreasing with respect to x_j.

(3) $g_j = 0$: when f is not dependent upon x_j.

(4) $g_j = *$: the remaining cases complementary to the above (1) to (3).

3. Main Proposition

Now we are ready to prove this

Lemma 1 : If $f(x) = f(y)$ for $x, y \in Q^n$ such that $x \neq y$, then there exists a vector p such that $p \cdot z = 0$, where and $z = x - y$, and p has the same sign pattern as the gradient. (The sign * can be any value. And $p \cdot z$ stands for the inner-product of two vectors.)

Proof. Let us prove this lemma by mathematical induction on the dimension. When $n = 1$, the sign in the gradient must be either 0 or *. So, the lemma is trivially true by setting $p = 0$. Consider the case where $n = 2$. If one of the elements in z is 0, this case reduces to the case $n = 1$. Thus, we assume both the elements in z is non–zero. Without loss of generality we suppose these two elements are 'positive' (> 0) because we can reverse the direction of any coordinate. Then the two gradients cannot be both positive or negative. Again without losing generality we can regard one of them as + and the other as –. We have to solve the equation : $p_1 z_1 + p_2 z_2 = 0$, where $z_1, z_2 > 0$, and a pair of solutions must satisfy $p_1 > 0$ and $p_2 < 0$. Just choose $p_1 = z_2$ and $p_2 = -z_1$, and indeed they are in Q .

Well now suppose the lemma is valid when $n = m - 2$ and $m - 1$ with $m > 2$, and prove it when $n = m$. Again we can safely assume all the elements of z is positive. If one of the gradients is either 0 or *, the case reduces to the case $n = m - 1$. So, every entry in the gradient is either + or –. Not all the entries can be positive nor negative, and so there should be at least one pair of entries (i, j) with the sign pattern (+, –). As in the case $n = 2$, we can find out a pair of values (p_i, p_j) such that they are in Q , and $p_i z_i + p_j z_j = 0$. Thus, the case can be handled as the case $n = m - 2$.

Therefore by mathematical induction, the desired result follows. □

Using Q^n, we can construct the theory of matrices and determinants as in the case of R^n except for inverse matrices and things related to inverse elements. Consider a system of simultaneous 'linear' equations with coefficients all in Q : $Az = 0$, or more precisely

$$\begin{cases} a_{11}z_1 + a_{12}z_2 + \dots + a_{1n}z_n = 0 \\ a_{21}z_1 + a_{22}z_2 + \dots + a_{2n}z_n = 0 \\ \quad \dots \\ a_{n1}z_1 + a_{n2}z_2 + \dots + a_{mn}z_n = 0 \end{cases}$$

Lemma 2 : If $Az = 0$ when $|A| \neq 0$, z must be 0. (The symbol $|A|$ means the determinant of A as in the usual theory.)

Proof. This can be proved by the elimination method. Note that the value $|A|$ is also in Q, and since Q is an integral domain, z must be 0. □

At last we come to consider a 'nonlinear' mapping $f \equiv (f_1, f_2, \dots, f_n)'$ from a domain D into a subset of Q^n. The 'Jacobian' matrix of f is defined to be a matrix whose i–th row is the gradient of f_i.

Proposition : If the Jacobian matrix of f keeps the same sign pattern on the domain D, and it is known to be 'regular' by the sign pattern only, then f is univalent on D.

Proof. Suppose there exist two solutions x and y. Defining $z \equiv x - y$, we have a set of equations by the above Lemma 1 : $Az = 0$. If x and y are distinct, z is not zero, while by assumption $|A| \neq 0$. This is a contradiction to Lemma 2. □

As in the previous article we raise four examples of sign–regular pattern :

$$\begin{pmatrix} + & * \\ 0 & - \end{pmatrix}, \quad \begin{pmatrix} + & + & + \\ - & + & + \\ 0 & + & - \end{pmatrix}, \text{ and } \begin{pmatrix} - & + & - & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{pmatrix}.$$

The second example comes from Morishima et al. (1973, p. 40), while the last matrix is again from Quirk (1997, p. 133) with the (1, 3)–element changed from 0 to –.

4. Conclusions and Remarks

The method based on the mean value theorem has at least one merit. This can be semi–*qualitative* : one can use semi–quantitative information to establish the univalence. For example, when we know some or all entries of the Jacobian matrix satisfy a certain set of quantitative constraints and these constraints are sufficient to show the regularity of the Jacobian together with the given qualitative information, the injectiveness follows. This is because by the very nature of the mean value theorem (Apostol (1974), p. 355), the value of each element in the Jacobian matrix satisfies the presumed quantitative constraints. On the other hand, the proposition presented here, as it stands now, cannot be strengthened by quantitative information. And yet it gives a benefit. The domain D need not be convex, and can be even discrete. When D is formed of a set of lattice points, the existence of a solution becomes a problem. One exception is the case in which $f(x) = x - T(x) - d$, where each member function of $T(x)$ is increasing with x. (See Fujimoto (1986).)

The reader may wonder how our result can be applied to models with indivisibility because every commodity is characterized by the same integral domain. This is not a problem as we can choose a suitable unit of measurement for each commodity. Or, we can allow for a different integral domain for each commodity, and proceed in almost the same manner. In the latter case, scalar multiplication is defined for the ring of integers.

A further research topic, if interesting enough so far, is how to restrict our target spaces so that some quantitative information can be incorporated, and by so doing, the first theorem in Bandyopadhyay and Biswas (1994) may be included in that version of proposition as a special case.

A final remark. The reader can be familiar with qualitative economics by first reading the books by Quirk and Saposnik (1968) and by Morishima et al. (1973), then a classical paper by Lancaster (1962), and a recent paper by Quirk (1997) which puts more emphasis on stability problem.

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Abstract

This note is a sequel to the previous one published in this journal (Vol. 30, No. 1). In that article, we used one of mean value theorems to prove the univalence of a nonlinear mapping based on the qualitative regularity of the Jacobian matrix. The qualitative regularity is a property of a matrix whose regularity is shown to be valid by using only the sign patterns of mappings involved.

In this note, we extend the result into a vector space over an integral domain. The vectors themselves are of n-tuples of elements in the integral domain. This integral domain is totally ordered, and some natural properties are assumed concerning this order.

First two lemmata are given, and the first one is in fact a sort of mean value theorem for mappings from a direct product of discrete spaces into a discrete space, and utilizes mathematical induction. The second lemma depends on the fact that theory of matrices and determinants can be constructed also on a ring except for inverse matrix. Finally, our main proposition derives from the very integrity of a given domain.

Another merit of the result is that the domain of a mapping need not be convex, and can be even a set of lattice points.