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# Consensus Formation between Two Experts : More Theorems and a Discrete Case

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## 1. Introduction

De Groot(1974) obtained a convergence theorem among $n$ experts using a linear homogeneous model. And this result on averaging processes is the starting point of the following researches on consensus formation. Based on a linear inhomogeneous model, Chatterjee(1975) and Chatterjee and Seneta(1977) generalized the result due to De Groot. Recently Krause (1999) employed nonlinear inhomogeneous models, and established some convergence theorems. In his proofs, a sort of contraction mapping theorem is used. In a sense, Krause(1999) showed that the condition, which assures less discrepancy in each step of negotiation, can guarantee a path converging to consensus.

Fujimoto(1999) took up a nonlinear inhomogeneous model, which is similar to a nonlinear Leontief model discussed in Fujimoto(1986). A path of bargaining is either monotonically increasing or decreasing, and it is easy to show the convergence given a certain limit. Then Ekuni and Fujimoto(2000) considered a model of consensus formation between two experts. It is shown that under a set of very weak conditions, two experts can reach a quasi-consensus state, thanks to the Poincaré–Bendixson theorem.

In this note, we give some more theorems using the same model as in Ekuni and Fujimoto(2000), and also discuss the discrete adjustment case for a model of two experts. In Section 2, the continuous version of our model is presented, then some new propositions are presented. Section 3 explains a model in which adjustment is made in a discrete way. A main result for the discrete case is stated in Section 3. In the final Section 4, some remarks are given.

## 2. The Continuous Adjustment

We repeat the definitions and assumptions in Ekuni and Fujimoto (2000). The symbol $R^2$ means the Euclidean space of dimension two, and $R_+^2$ the nonnegative orthant of $R^2$. The evaluations of two experts are expressed by a vector in $R_+^2$. The adjustment of their opinion is made as time goes on following the differential equation :

$$\dot{x} = \frac{dx}{dt} = f(x) \quad \text{for } x \in R_+^2,$$

where $t$ shows time, and a given function $f(x) \equiv (f_1(x), f_2(x))'$ maps $x \in R_+^2$ to a vector in $R^2$, and is possibly nonlinear. (A prime indicates the transposition of a vector concerned.) The function $f(x)$ is continuously differentiable on $R_+^2$ with respect to $x_1$ and $x_2$ of $x \equiv (x_1, x_2)'$. Besides we make the following assumptions.

**Assumption 1.** If $x_1 \neq x_2$ for $x \in R_+^2$, then $f(x) \neq 0$.

**Assumption 2.** If $x_i = 0$, $i = 1$ or 2, then $f_i(x) \geq 0$.

**Assumption 3.** There exists a positive scalar $N$ such that if $0 \leq x_1 = x_2 < N$ for $x \in R_+^2$, then $f(x) = 0$.

**Assumption 4.** There exists a positive scalar $M$ such that if $x_i > M$, $i = 1$ or 2 then $f_i(x) < 0$.

Here are made two more definitions.

**Definition 1.** The set of equilibrium points, $E$, is $\{x \mid f(x) = 0, x \in R_+^2\}$.

**Definition 2.** Let $\varepsilon$ be an arbitrary positive scalar. The set of $\varepsilon$–equilibrium, $QE$, is defined as $\{x \mid \exists y \in E$ such that $| x - y | < \varepsilon, x \in R_+^2\}$.

In this section, we use one more symbol :

$$S \equiv \{x \mid x_1 < M, x_2 < M, x \in R_+^2\}.$$

Our Assumption 3 guarantees the nonemptiness of $E$, while Assumption 1 requires that at least one expert modifies his / her opinion when the two have different estimates, prohibiting the existence of equilibrium on the off–diagonal points. In Assumption 1, no compromise is postulated, but simply a revision.

What is shown in Ekuni and Fujimoto(2000) is

**Proposition 1.** Given Assumptions 1–4, the adjustment process with its initial vector anywhere in $R_+^2$ reaches a point of the $\varepsilon$–equilibrium set.

This tells us that after a certain time the adjustment process enters the set $S$, and $QE$ as well, but may leave the latter later. We can, however, make a supplementary rule that the adjustment process shall be ended when the discrepancy of the two is less than $\varepsilon$.

$$\dot{x} = \frac{dx}{dt} = \begin{cases} f(x) & \text{for } x \in (R_+^2 - QE), \\ 0 & \text{for } x \in QE \end{cases}$$

The problem associated with the above convergence to the set $QE$ is that we do not know how long it takes to reach there. Otherwise this Proposition is rather interesting because two experts do not have to make concessions in each step, i. e., they need not try to make the difference smaller.

Now we can state other results by adding more assumptions. The Jacobian matrix at each point of $E$ has two eigenvalues associated with two eigenvectors. One of them is zero with its associated eigenvector $(1,1)'$. The second eigenvalue may be positive, zero, or negative. From Proposition 1, it is clear that not all the points in $E$ have the second eigenvalue positive. Otherwise Proposition 1 is impossible. Then we have

**Proposition 2**. Given Assumptions 1–4, if all the equilibrium points have the second eigenvalue negative, then the adjustment process converges to an equilibrium point.

**Proof**. Since the set $E$ is compact, the scalar $\varepsilon$ can be chosen so that in each $\varepsilon$–neighbourhood, the second eigenvalue with its eigenvector is forcing the adjustment process to the diagonal. The distance with the diagonal is 'strictly decreasing'. By Proposition 1, after a certain time, the adjustment process enters the set $QE$. Hence, the process converges to an equilibrium point. (We need a more detailed discussion at the end points of $E$.) QED.

Proposition 2 amounts to saying that if each equilibrium point is locally stable, the equilibrium set $E$ as a whole is globally stable.

As a conjecture, we state

**Proposition 3**. Given Assumptions 1–4, the adjustment process converges to an equilibrium point.

As a subset of $QE$, we consider the set $SQE$ defined as

$$SQE \equiv \{x \mid x \in QE, \text{the second eigenvalue is negative or zero}\}.$$

Using an argument similar to that used in the proof of Poincaré–

Bendixson, we find that the direction of movement is the same near the diagonal in the set $E$. The direction along the diagonal becomes impossible as the orbit approaches the diagonal in $E$. The remaining possibility is a saddle point at an equilibrium where the second eigenvalue is also zero. This is, however, also impossible because the first eigenvalue is zero.

## 3. The Discrete Adjustment

In this section we consider a discrete adjustment process :

$$x(t+1) = \max(x(t) + f(x(t)), 0).$$

The function $f(x)$ is continuous on $R_+^2$. We make the same Assumptions 1, 2, and 3 as in Section 2. Without losing generality we suppose the point $(1/2, 1/2)'$ is in the equilibrium set $E$. Two more assumptions are made together with two symbols.

**Assumption 4 D**. The function $f(x)$ is homogeneous, i. e.,

$$f(kx) = s(k) \cdot f(x),$$

where $k$ is a positive scalar, and $s(k)$ is a continuous function of $k$ for which there exists a positive scalar $M$ such that

$$s(k) > 0 \text{ for } k < M, \text{ and } s(k) < 0 \text{ for } k > M.$$

**Definition 3**. The map $P$ is the projection from $R_+^2 - \{0\}$ to the unit simplex

$$U \equiv \{x \mid x \in R_+^2, \textstyle\sum_{i=1}^2 x_i = 1\}$$

defined as

$$Px \equiv (x_1/(x_1+x_2), x_2/(x_1+x_2)).$$

**Assumption 5**. The function $f(x)$ satisfies the condition that if the process jumps over the diagonal, then the distance between the diagonal and $Px(t)$ becomes smaller.

**Proposition 4**. Given Assumptions 1–5 with Assumption 4 replaced by 4 D, the adjustment process converges to an equilibrium point.

**Proof**. When we project any movement to the unit simplex, that movement should be directed toward the diagonal. Otherwise, there would be a point on $U$ such that the movement along the simplex vanishes because of Assumption 2. By Assumption 4 D, this implies the existence of equilibrium on the off–diagonal area, contradicting Assumption 1. Thus thanks to Assumption 5, the distance between the diagonal and the point projected $Px(t)$ monotonically decrease irrespective of jumps over the diagonal. QED.

## 4. Discussion

What is shown in Ekuni and Fujimoto(2000), here presented as Proposition 1, is that when at least one of the two experts is willing to change his / her opinion if two have different views, the continuous adjustment process will bring the two to a quasi–equilibrium point, i.e., a point where the discrepancy between two estimates can be smaller than any preassigned magnitude. The trouble with this result is that later in this process the discrepancy gets larger again, and we do not know how long it takes for the process to go into the quasi-equilibrium set.

Proposition 2 is of interest when we examine the adjustment process in the previous results for $n$ experts. When the situation is limited to two experts, we can calculate the second eigenvalue of the Jacobian matrix on the equilibrium set.

Proposition 3 seems to be valid as it stands. A counter-example, if any, should include an unnatural adjustment process as a negotiation behaviour, and can be excluded by an additional 'plausible' assumption.

The discrete cases present many challenging problems, which I will discuss in the future. The one that is dealt with in this note seems uninteresting because a kind of homogeneity is assumed, and Assumption 5 can be restrictive. This homogeneity can, however, be weakened to a considerable degree. Moreover, Assumption 5 may be natural in many situations. The relative evaluation between two experts can be reversed only when their relative difference gets decreased.

One more case we should take up is the one where the space itself is discrete : for example, only the lattice points in $R_+^2$ are the points of state vectors of which the estimates of the two experts can take the value.

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