Exponential integrability of stochastic convolutions

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EXPONENTIAL INTEGRABILITY OF STOCHASTIC CONVOLUTIONS

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Abstract

Sufficient conditions are found for stochastic convolution integrals driven by a Wiener process in a Hilbert space to belong to the Orlicz space $\exp L^2$; standard exponential tail estimates follow from these results. Proofs are based on the extrapolation theory and are rather simple.

0. *Introduction*

Let H be a Hilbert space, (e^{tA}) a C_0-semigroup on H, W a Wiener process in H, and ψ a progressively measurable process taking values in a suitable space of operators on H. Stochastic convolution integrals like

$$W_A(t) = \int_0^t e^{(t-s)A}\psi(s)\,dW(s), \qquad t \geqslant 0,$$

appear (with the choice $\psi = \Sigma(\cdot, X)$) in the variation of constants formula for a solution to a stochastic partial differential equation

$$dX = \{AX + F(t,X)\}\,dt + \Sigma(t,X)\,dW;$$

see the monograph [**8**] for a thorough account of the semigroup theory of stochastic evolution equations. Unfortunately, the process W_A is not a martingale and even its basic properties like continuity of trajectories are rather difficult to verify. Nowadays standard proofs are based on the factorization method (originating in the papers [**7**, **9**] by G. Da Prato, S. Kwapień and J. Zabczyk) which yields also L^p-estimates of the form

$$\boldsymbol{E} \sup_{0\leqslant t\leqslant T} \|W_A(t)\|^p \leqslant K_p \boldsymbol{E}\int_0^T \|\psi\|^p\,ds. \tag{0.1}$$

Let us recall further a particular case of the classical Zygmund extrapolation theorem: if (X,μ) and (Y,ν) are finite measure spaces and $T : L^p(\mu) \longrightarrow L^p(\nu)$ is a sublinear operator satisfying

$$\|Tf\|_{L^p} \leqslant C\sqrt{p}\|f\|_{L^p}$$

for all $p \geqslant p_0$ and $f \in L^\infty(\mu)$, then

$$\int_Y \exp(\lambda|Tf|^2)\,d\nu \leqslant K$$

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for some constants $\lambda > 0$, $K < \infty$ and all $f \in L^\infty(\mu)$, $\|f\|_{L^\infty} \leqslant 1$. Tracing the proof of (0.1) we shall show that

$$\sqrt[p]{K_p} = O(\sqrt{p}), \qquad p \to +\infty. \tag{0.2}$$

Thus we may hope that an extension of the Zygmund theorem to vector-valued functions together with (0.1) and (0.2) imply that

$$\boldsymbol{E} \exp(\lambda \sup_{0 \leqslant t \leqslant T} \|W_A(t)\|^2) \leqslant K \tag{0.3}$$

for some constants $\lambda > 0$, $K < \infty$ and all processes ψ with $\operatorname{ess\,sup}\|\psi\| \leqslant 1$.

It is the purpose of this paper to show that exponential estimates of the type (0.3) do hold and owing to the extrapolation theory may be proved quite easily. Obviously, (0.3) yields that

$$\boldsymbol{P}\left\{\sup_{0 \leqslant t \leqslant T} \|W_A(t)\| \geqslant \delta\right\} \leqslant K e^{-\lambda\delta^2}, \quad \delta \geqslant 0 \tag{0.4}$$

holds for all processes ψ with $\operatorname{ess\,sup}\|\psi\| \leqslant 1$. Exponential tail estimates closely related to (0.4) were studied in the papers [**2**, **6**, **16**, **20**]. Moreover, various types of exponential estimates for stochastic convolutions appear in proofs of the large deviation principle for stochastic partial differential equations, see, for example, [**8**, Chapter 12] and [**3–5**, **17**, **18**] and the references therein. It may be shown that the estimates (0.3) and (0.4) are equivalent, however, our direct proof of (0.3) is based on ideas different from those employed in the cited papers to derive (0.4).

This paper is organized as follows. In the next sections, we state three theorems on the validity of the estimate (0.3) under various hypotheses on the process W and the semigroup (e^{tA}). We consider the non-autonomous case, replacing semigroups with 2-parameter evolution operators. Further, for exponentially stable semigroups it is shown that the estimate (0.3) holds uniformly in $T \geqslant 0$; the necessity to have exponential estimates uniform in T is faced when large deviations for invariant measures for stochastic partial differential equations are investigated, cf. [**19**] or [**11**]. Proofs are deferred to Section 2. As we have already indicated, they are based on finding the dependence of the constant K_p in the L^p-estimate (0.1) on p and are not difficult from the technical point of view. For completeness and to convince the reader that Zygmund's theorem holds also for Banach space valued functions, we present a full proof of the version of the theorem that we need in Appendix A.

We close this section by introducing some notation. Let U, V be Hilbert spaces, by $\mathscr{L}(U,V)$ we denote the space of all bounded linear operators from U to V (endowed with the uniform norm) and by $\mathscr{L}_s(U,V)$ the same space but equipped with the strong operator topology. Let $(S, \mathscr{S}, \mu)$ be a (complete) measure space, recall that if U, V are separable then a function $f: S \longrightarrow \mathscr{L}_s(U,V)$ is measurable (that is, $f^{-1}(B) \in \mathscr{S}$ for every Borel set B in $\mathscr{L}_s(U,V)$) if and only if $f(\cdot)u : S \longrightarrow V$ is Bochner measurable for all $u \in U$ (see [**13**, Theorem 2]). Due to separability, $\|f\|_{\mathscr{L}(U,V)}$ is a measurable real function whenever $f: S \longrightarrow \mathscr{L}_s(U,V)$ is measurable. Further, we shall denote by $\mathscr{J}_2(U,V)$ the Hilbert space of all Hilbert–Schmidt operators from U into V. Let E be a Banach space; we denote its norm by $\|\cdot\|_E$ the subscript being omitted if there is no danger of confusion. We denote by $L^p(S, \mathscr{S}, \mu; E)$ the standard Banach space of all Bochner measurable mappings $f: S \longrightarrow E$ such that either $\|f\|_E^p$ is μ-integrable ($p < \infty$), or $\|f\|_E$ is essentially bounded ($p = \infty$). If S is a compact space, then we use $\mathscr{C}(S; E)$ to denote the space of all continuous mappings from S to E endowed with the sup-norm. If $E = \mathbb{R}$ then we simplify the notation in the usual way.

1. *Main results*

Let H and Υ_0 be real separable Hilbert spaces, $Q \in \mathscr{L}(\Upsilon_0)$ a non-negative self-adjoint operator and $(\Omega, \mathscr{F}, (\mathscr{F}_t), \boldsymbol{P})$ a stochastic basis. Denote by Υ the range $\mathrm{Rng} Q^{1/2}$ endowed with the norm $\|x\|_\Upsilon = \|Q^{-1/2}x\|_H$, $Q^{-1/2}$ being the pseudo-inverse to the square root $Q^{1/2}$ of Q; note that $(\Upsilon, \|\cdot\|_\Upsilon)$ is again a Hilbert space. Let W be a possibly cylindrical $(\mathscr{F}_t)$-Wiener process in Υ_0 with the covariance operator Q. Denote by $\mathscr{M}$ the σ-algebra of $(\mathscr{F}_t)$-progressively measurable sets over $\mathbb{R}_+ \times \Omega$ and set, for a fixed $T > 0$, $\boldsymbol{L}^p = L^p([0, T] \times \Omega, \mathscr{M}, dt \otimes \boldsymbol{P}; \mathscr{J}_2(\Upsilon, H))$. For brevity, we shall denote the norm in $\boldsymbol{L}^p$ by $|||\cdot|||_p$, that is

$$|||f|||_p \equiv \begin{cases} \left(\boldsymbol{E} \displaystyle\int_0^T \|f(s)\|^p_{\mathscr{J}_2(\Upsilon,H)}\, ds \right)^{1/p} < \infty, & 1 \leqslant p < \infty, \\ \displaystyle\operatorname*{ess\,sup}_{(s,\omega) \in [0,T] \times \Omega} \|f(s,\omega)\|_{\mathscr{J}_2(\Upsilon,H)} < \infty, & p = \infty. \end{cases}$$

Let $\Delta = \{(s,t);\ 0 \leqslant s \leqslant t \leqslant T\}$ and suppose that $U = (U(t,s), (s,t) \in \Delta)$ is an evolution operator: $U \in \mathscr{C}(\Delta; \mathscr{L}_s(H))$, $U(s,s) = I$ for all $s \in [0,t]$ and $U(t,s)U(s,r) = U(t,r)$ whenever $0 \leqslant r \leqslant s \leqslant t \leqslant T$. Let us recall that the stochastic convolution integral

$$\int_0^t U(t,s)\psi(s)\, dW(s), \qquad 0 \leqslant t \leqslant T, \tag{1.1}$$

is well defined provided that ψ is an $\mathscr{M}$-measurable $\mathscr{L}_s(\Upsilon, H)$-valued process and

$$\int_0^t \|U(t,s)\psi(s)\|^2_{\mathscr{J}_2(\Upsilon,H)}\, ds < \infty \ \boldsymbol{P}\text{-almost surely}, \qquad 0 \leqslant t \leqslant T, \tag{1.2}$$

in particular, if $\psi \in \boldsymbol{L}^2 \supseteq \boldsymbol{L}^\infty$.

Now we are prepared to state our first result.

THEOREM 1.1. *There exist constants $K < \infty$ and $\lambda > 0$ such that*

$$\boldsymbol{E} \exp\left(\frac{\lambda}{|||\psi|||^2_\infty} \sup_{0 \leqslant t \leqslant T} \left\| \int_0^t U(t,s)\psi(s)\, dW(s) \right\|^2 \right) \leqslant K \tag{1.3}$$

holds for every $\psi \in \boldsymbol{L}^\infty$.

We may strengthen the estimate (1.3) if the evolution operator U obeys suitable 'parabolicity' assumptions. Namely, we shall adopt the following hypothesis.

(P) *Let Ξ_α, $\alpha \in [0,1]$, be Banach spaces such that $\Xi_0 = H$, Ξ_β is continuously embedded into Ξ_γ whenever $1 \geqslant \beta \geqslant \gamma \geqslant 0$, and for each $\varrho \in (0,1]$ there exists a constant L_ϱ such that*

$$U(t,s) \in \mathscr{L}(H, \Xi_\varrho) \quad \textit{and} \quad \|U(t,s)\|_{\mathscr{L}(H,\Xi_\varrho)} \leqslant \frac{L_\varrho}{(t-s)^\varrho}$$

for all $0 \leqslant s < t \leqslant T$.

We shall denote the norm in Ξ_α simply by $\|\cdot\|_\alpha$. The following example is well known. If (e^{tA}) is a holomorphic C_0-semigroup on H, then the evolution operator $U(t,s) = e^{(t-s)A}$ satisfies (P) if we set $\Xi_\alpha = [H, \mathrm{Dom}(A)]_\alpha$, $\Xi_\alpha = (H, \mathrm{Dom}(A))_{\alpha,q}$ (the complex and the real interpolation spaces, respectively), or $\Xi_\alpha = \mathrm{Dom}((\beta I - A)^\alpha)$ equipped with the graph norm, the constant β being chosen sufficiently large for the operator $\beta I - A$ to be positive.

THEOREM 1.2. *Let hypothesis* (P) *be satisfied. For every* $\delta \in (0, \frac{1}{2})$ *there exist constants* $K_\delta < \infty$ *and* $\lambda_\delta > 0$ *such that*

$$\boldsymbol{E} \exp\left(\frac{\lambda_\delta}{|||\psi|||_\infty^2} \sup_{0 \leqslant t \leqslant T} \left\| \int_0^t U(t,s)\psi(s)\, dW(s) \right\|_\delta^2 \right) \leqslant K_\delta$$

holds for every $\psi \in \boldsymbol{L}^\infty$.

REMARK 1.1. Let (e^{tA}) be a holomorphic C_0-semigroup on H, $\delta \in [0, \frac{1}{2})$, $\theta \in [0, \frac{1}{2} - \delta)$. Define the space Ξ_δ as $\mathrm{Dom}((\beta I - A)^\delta)$ for β sufficiently large and endow it with the graph norm, set

$$W_A(t) = \int_0^t e^{(t-s)A}\psi(s)\, dW(s), \qquad 0 \leqslant t \leqslant T,$$

and

$$\boldsymbol{H}(\psi) = \sup_{0 \leqslant t \leqslant T} \|W_A(t)\|_\delta + \sup_{\substack{0 \leqslant s,t \leqslant T \\ s \neq t}} \frac{\|W_A(t) - W_A(s)\|_\delta}{|t-s|^\theta}.$$

Modifying slightly the proof of Theorem 1.2 by taking into account [**9**], Lemma 2 we may show that

$$\boldsymbol{E} \exp\left(\frac{\lambda_{\delta,\theta} \boldsymbol{H}(\psi)^2}{|||\psi|||_\infty^2} \right) \leqslant K_{\delta,\theta}$$

for some constants $K_{\delta,\theta} < \infty$, $\lambda_{\delta,\theta} > 0$ and all $\psi \in \boldsymbol{L}^\infty$.

The preceding theorems are quite satisfactory if Q is a nuclear operator. In the opposite case, the processes ψ appearing in applications to stochastic evolution equations satisfy $|||\psi|||_2 < \infty$ only scarcely. However, reasonable sufficient conditions on differential operators generating an evolution operator U are known, implying that U consists of Hilbert–Schmidt operators and (1.2) may hold. If U is of this type then we may integrate processes ψ taking values in $\mathscr{L}(\Upsilon, H)$. However, the space $\mathscr{L}(\Upsilon, H)$, unlike $\mathscr{J}_2(\Upsilon, H)$, is not separable and it is restrictive to assume that the process ψ in (1.1) is Bochner measurable. In fact, the stochastic integral (1.1) may be defined for processes measurable as $\mathscr{L}_s(\Upsilon, H)$-valued mappings, hence let us define $\boldsymbol{SL}^p$ as the set of all measurable mappings $f : ([0,T] \times \Omega, \mathscr{M}) \longrightarrow \mathscr{L}_s(\Upsilon, H)$ such that $[\![f]\!]_p < \infty$, where

$$[\![f]\!]_p \equiv \begin{cases} \left(\boldsymbol{E} \displaystyle\int_0^T \|f(s)\|_{\mathscr{L}(\Upsilon,H)}^p\, ds \right)^{1/p} < \infty, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{(s,\omega) \in [0,T] \times \Omega} \|f(s,\omega)\|_{\mathscr{L}(\Upsilon,H)} < \infty, & p = \infty. \end{cases}$$

The corresponding modification of Theorem 1.1 reads as follows.

THEOREM 1.3. *Assume that there exist a measurable function* $k : [0, T] \longrightarrow \mathbb{R}_+$ *and a* $\gamma \in (0, 1]$ *such that*

$$\| U(t,s) \|_{\mathscr{J}_2(H)} \leqslant k(t-s), \qquad 0 \leqslant s < t \leqslant T, \tag{1.4}$$

and

$$\kappa \equiv \int_0^T s^{-\gamma} k^2(s)\, ds < \infty. \tag{1.5}$$

Then there exist constants $\tilde{K} < \infty$ *and* $\tilde{\lambda} > 0$ *such that*

$$\boldsymbol{E} \exp\left(\frac{\tilde{\lambda}}{|\psi|_\infty^2} \sup_{0 \leqslant t \leqslant T} \left\| \int_0^t U(t,s)\psi(s) dW(s) \right\|^2 \right) \leqslant \tilde{K}$$

holds for every $\psi \in \boldsymbol{SL}^\infty$.

A 'parabolic' version of Theorem 1.3 may be obtained in a straightforward way, hence we shall not dwell upon it.

In Theorems 1.1–1.3, the time interval $[0, T]$ was fixed and compact. However, sometimes it is desirable to have exponential estimates uniform in T. We shall show that it is possible to derive such estimates provided that the evolution operator U is exponentially stable. We shall assume the following.

(ES) $U = (U(t,s),\ t \geqslant s \geqslant 0)$ *is an evolution operator on* H *such that*

$$\| U(t,s) \|_{\mathscr{L}(H)} \leqslant \tilde{L} e^{-\mu(t-s)}$$

for some constants $\tilde{L} < \infty$ *and* $\mu > 0$ *and for all* $t, s \in \mathbb{R}_+,\ t \geqslant s$.

We shall use $\boldsymbol{L}^p$ and $|||\cdot|||_p$ to denote also the space $L^p(\mathbb{R}_+ \times \Omega, \mathscr{M}, dt \otimes \boldsymbol{P}; \mathscr{J}_2(\Upsilon, H))$ and its norm, respectively.

THEOREM 1.4. *Let hypothesis* (ES) *be satisfied. Then for any* $q \in (2, \infty)$ *there exist constants* $\widehat{K} < \infty$ *and* $\widehat{\lambda} > 0$ *such that*

$$\boldsymbol{E} \exp\left(\frac{\widehat{\lambda} \sup_{t \geqslant 0} \left\| \int_0^t U(t,s)\psi(s)\, dW(s) \right\|^2}{|||\psi|||_q^2 \vee |||\psi|||_\infty^2} \right) \leqslant \widehat{K}$$

holds for all $\psi \in \boldsymbol{L}^q \cap \boldsymbol{L}^\infty$.

Despite the fact that to suppose that $\|\psi\|^q_{\mathscr{J}_2(\Upsilon,H)}$ is integrable over $\mathbb{R}_+ \times \Omega$ may look rather restrictive the following example indicates that this assumption may be checked by standard Lyapunov functions techniques in many reasonable cases.

EXAMPLE 1.1. Let us consider a stochastic evolution equation

$$dX_t = (AX_t + f(X_t))\, dt + \sigma(X_t)\, dW_t, \qquad X_0 = x_0 \in H$$

in H, where A is an infinitesimal generator of a C_0-semigroup on H and $f : H \longrightarrow H$ and $\sigma : H \longrightarrow \mathscr{J}_2(\Upsilon, H)$ are Lipschitz continuous mappings, that is

$$\|f(x) - f(y)\| \leqslant \mathrm{Lip}(f)\|x-y\|, \quad \|\sigma(x) - \sigma(y)\|_{\mathscr{J}_2(\Upsilon,H)} \leqslant \mathrm{Lip}(\sigma)\|x-y\| \tag{1.6}$$

for some constants $\mathrm{Lip}(f)$, $\mathrm{Lip}(\sigma)$ and all $x, y \in H$. Suppose that there exists a constant $\beta > 0$ such that for some $q \in (2, \infty)$ we have

$$\sigma(0) = 0, \tag{1.7}$$

$$\langle Ax + f(x), x \rangle \leqslant -\beta \|x\|^2, \quad x \in \mathrm{Dom}(A), \tag{1.8}$$

$$\tfrac{1}{2}\mathrm{Lip}(\sigma)^2(q-1)\mathrm{Tr}Q < \beta.$$

Then by [**12**, Corollary 3.2], $\boldsymbol{E}\|X_t\|^q \leqslant e^{at}\|x_0\|^q$ holds for an $a < 0$ and all $t \geqslant 0$. (In particular, note that (1.8) is satisfied provided that $\|e^{At}\|_{\mathscr{L}(H)} \leqslant e^{-\gamma t}$ for some $\gamma > 0$ and every $t > 0$, $f(0) = 0$ and $-\gamma + \mathrm{Lip}(f) < 0$.) Since $\|\sigma(x)\|_{\mathscr{I}_2(\Upsilon, H)} \leqslant \mathrm{Lip}(\sigma)\|x\|$ by (1.6) and (1.7), we obtain $\sigma(X) \in \boldsymbol{L}^q$. If, moreover, σ is bounded then $\sigma(X) \in \boldsymbol{L}^q \cap \boldsymbol{L}^\infty$ and the process $\psi = \sigma(X)$ obeys the hypotheses of Theorem 1.4.

2. *Proofs*

As we have already indicated, our proofs are based on invoking the Zygmund theorem, therefore, we need to compute the constants in L^p-estimates of stochastic convolution integrals precisely to see their dependence on p. We derive these estimates by means of the factorization method, so to begin with we investigate the generalized Riemann–Liouville operator, defined by the formula

$$R_\alpha f(t) = \int_0^t (t-s)^{\alpha-1} U(t,s) f(s)\, ds, \qquad 0 \leqslant t \leqslant T, \ f \in L^p([0,T]; H)$$

for $p > 1$, $\alpha > 1/p$. Setting

$$L_0 = \sup_{0 \leqslant s \leqslant t \leqslant T} \|U(t,s)\|_{\mathscr{L}(H)}$$

we may prove the following estimate.

Lemma 2.1. *For all $p \in (1, \infty)$ and $\alpha \in (1/p, \infty)$ the mapping R_α is a bounded linear operator from $L^p([0,T]; H)$ into $\mathscr{C}([0,T]; H)$ whose norm satisfies*

$$\|R_\alpha\| \leqslant L_0 \max(1, T^\alpha) \left(\frac{p-1}{\alpha p - 1} \right)^{1-1/p}. \tag{2.1}$$

If, in addition, (P) *is satisfied, $\delta \in (0,1]$ and $\alpha > \delta + 1/p$ then R_α maps boundedly $L^p([0,T]; H)$ into $\mathscr{C}([0,T]; \Xi_\delta)$ and*

$$\|R_\alpha\| \leqslant L_\delta \max(1, T^\alpha) \left(\frac{p-1}{(\alpha - \delta)p - 1} \right)^{1-1/p}. \tag{2.2}$$

Proof. The boundedness of R_α is known; we repeat here the easy proof to obtain the constants explicitly. Take $f \in L^p([0,T]; H)$, then

$$\begin{aligned}
\|R_\alpha f(t)\| &= \left\| \int_0^t (t-s)^{\alpha-1} U(t,s) f(s)\, ds \right\| \\
&\leqslant L_0 \int_0^t (t-s)^{\alpha-1} \|f(s)\|\, ds \\
&\leqslant L_0 \|f\|_{L^p(0,T;H)} \left(\int_0^t (t-s)^{(\alpha-1)p/(p-1)}\, ds \right)^{(p-1)/p} \\
&= L_0 \|f\|_{L^p(0,T;H)} \left(\frac{p-1}{\alpha p - 1} \right)^{1-1/p} t^{\alpha - 1/p},
\end{aligned}$$

for all $t \in [0,T]$ by the Hölder inequality, and (2.1) follows.

Let (P) be satisfied. The same procedure yields

$$\|R_\alpha f(t)\|_\delta \leqslant L_\delta \int_0^t (t-s)^{\alpha-\delta-1} \|f(s)\|_H \, ds$$
$$\leqslant L_\delta \|f\|_{L^p(0,T;H)} \left(\frac{p-1}{(\alpha-\delta)p-1}\right)^{1-1/p} t^{\alpha-\delta-1/p},$$

which proves (2.2). □

As the next step, let us recall a particular case of the Burkholder–Davis–Gundy inequality.

Lemma 2.2. *For any $p \in [2,\infty)$ there exists a constant $C_p < \infty$ such that*

$$\boldsymbol{E} \max_{0\leqslant r\leqslant t} \left\| \int_0^r \varphi(s)\, dW(s) \right\|^p \leqslant C_p \boldsymbol{E} \left(\int_0^t \|\varphi(s)\|^2_{\mathscr{I}_2(\Upsilon,H)} \, ds \right)^{p/2} \tag{2.3}$$

holds for all $t \in [0,T]$ and $\varphi \in \boldsymbol{L}^p$. Moreover, we may take

$$C_p = \left(\frac{4p}{p-1}\right)^p \left(p + \frac{1}{2}\right)^{p/2}.$$

The standard stochastic calculus proof of Lemma 2.2 (see, for example, [**8**, Lemma 7.2]) leads to the estimate (2.3) with a constant

$$\left(\frac{p(p-1)}{2} \left(\frac{p}{p-1}\right)^p \right)^{p/2}$$

which, however, grows faster than C_p as $p \to \infty$, so we need an alternative argument taking into account that martingales with continuous paths are considered.

Proof of Lemma 2.2. Let us fix $p \in [2,\infty)$ and $\varphi \in \boldsymbol{L}^p$ arbitrarily. Set $\varphi(s) = 0$ for $s > T$ and define

$$M_r = \int_0^r \varphi(s)\, dW(s), \qquad r \geqslant 0.$$

Then M is an H-valued martingale with continuous trajectories whose quadratic variation process $\langle M \rangle$ is given by

$$\langle M \rangle_r = \int_0^r \|\varphi(s)\|^2_{\mathscr{I}_2(\Upsilon,H)} ds, \qquad r \geqslant 0.$$

By a standard random time change argument we may find an H-valued martingale N with continuous trajectories such that $N_0 = 0$, $M_r = N \circ \langle M \rangle_r$, $\langle N \rangle_r = r$ for all $r \geqslant 0$ almost surely. There exists a real-valued Wiener process B such that

$$\sup_{0\leqslant r\leqslant t} \|N_r\| \leqslant \sup_{0\leqslant r\leqslant t} (|B_r| + \sqrt{t-r}) \leqslant \sup_{0\leqslant r\leqslant t} |B_r| + \sqrt{t}$$

for all $t \geqslant 0$ almost surely by [**14**, Theorem 4.4]. Hence also

$$\sup_{0\leqslant r\leqslant \tau} \|N_r\| \leqslant \sup_{0\leqslant r\leqslant \tau} |B_r| + \sqrt{\tau}$$

almost surely for any finite stopping time τ for N. According to [**10**, Theorem 1.1], we have

$$\boldsymbol{E}|B_\sigma|^p \leqslant z_p^p \boldsymbol{E}\sigma^{p/2} \tag{2.4}$$

for any stopping time σ, where z_p is the largest positive zero of the parabolic cylinder function D_p of parameter p. Using (2.4) and the Doob maximal inequality we obtain

$$\begin{aligned}\boldsymbol{E}\sup_{0\leqslant r\leqslant\tau}\|N_r\|^p &\leqslant 2^{p-1}\boldsymbol{E}\sup_{0\leqslant r\leqslant\tau}|B_r|^p+2^{p-1}\boldsymbol{E}\tau^{p/2}\\ &\leqslant 2^{p-1}\lim_{n\to\infty}\boldsymbol{E}\sup_{0\leqslant r\leqslant\tau\wedge n}|B_r|^p+2^{p-1}\boldsymbol{E}\tau^{p/2}\\ &\leqslant 2^{p-1}\left(\frac{p}{p-1}\right)^p\lim_{n\to\infty}\boldsymbol{E}|B_{\tau\wedge n}|^p+2^{p-1}\boldsymbol{E}\tau^{p/2}\\ &\leqslant 2^{p-1}\left(\frac{p}{p-1}\right)^p z_p^p\lim_{n\to\infty}\boldsymbol{E}(\tau\wedge n)^{p/2}+2^{p-1}\boldsymbol{E}\tau^{p/2}\\ &\leqslant 2^{p-1}\left[\left(\frac{p}{p-1}\right)^p z_p^p+1\right]\boldsymbol{E}\tau^{p/2}.\end{aligned}$$

Therefore,

$$\boldsymbol{E}\sup_{0\leqslant r\leqslant t}\|M_r\|^p=\boldsymbol{E}\sup_{0\leqslant r\leqslant\langle M\rangle_t}\|N_r\|^p\leqslant 2^{p-1}\left[\left(\frac{p}{p-1}\right)^p z_p^p+1\right]\boldsymbol{E}\langle M\rangle_t^{p/2}.$$

From [**1**, § 19.26], we know that $z_p<2\sqrt{p+\frac{1}{2}}$, so

$$\boldsymbol{E}\sup_{0\leqslant r\leqslant t}\|M_r\|^p\leqslant 2^{2p-1}\left[\left(\frac{p}{p-1}\right)^p\left(p+\frac{1}{2}\right)^{p/2}+1\right]\boldsymbol{E}\langle M\rangle_t^{p/2}$$

and our claim follows. □

For $\delta\in[0,\frac{1}{2})$ define

$$\boldsymbol{I}_\delta(\psi)=\sup_{0\leqslant t\leqslant T}\left\|\int_0^t U(t,s)\psi(s)\,dW(s)\right\|_\delta,\qquad \psi\in\boldsymbol{L}^2,$$

setting $\|\cdot\|_0=\|\cdot\|_H$ in accord with (P). Da Prato–Zabczyk's maximal inequality says that $\boldsymbol{I}_0$ maps $\boldsymbol{L}_p$ into $L^p(\Omega)$ for $p>2$; an analogous statement holds for $\boldsymbol{I}_\delta$ under suitable assumptions on U. In fact, we have the following theorem.

Theorem 2.3. *There exists a constant $M<\infty$, depending only on T and L_0, such that for all $p\in(2,\infty)$ the estimate*

$$\|\boldsymbol{I}_0(\psi)\|_{L^p(\Omega)}\leqslant M\left(\frac{p-1}{p-2}\right)^{1-1/p}\left(\frac{p}{p-2}\right)^{1/2}\left(p+\frac{1}{2}\right)^{1/2}|||\psi|||_p$$

holds for all $\psi\in\boldsymbol{L}^p$.

If hypothesis (P) *is also satisfied, then for each $\delta\in(0,\frac{1}{2})$ there exists a constant $M_\delta<\infty$, dependent only on L_0, L_δ and T, such that*

$$\|\boldsymbol{I}_\delta(\psi)\|_{L^p(\Omega)}\leqslant M_\delta\left(\frac{p-1}{(1-2\delta)p-2}\right)^{1-1/p}\left(\frac{p}{(1-2\delta)p-2}\right)^{1/2}\left(p+\frac{1}{2}\right)^{1/2}|||\psi|||_p$$

holds whenever $p\in(2/(1-2\delta),\infty)$ and $\psi\in\boldsymbol{L}^p$.

Proof. Take $p>2$, $\alpha\in(1/p,1/2)$, $\psi\in\boldsymbol{L}^p$ and define

$$Y(t)=\int_0^t(t-s)^{-\alpha}U(t,s)\psi(s)\,dW(s).$$

By the factorization formula

$$\int_0^t U(t,s)\psi(s)\,dW(s) = \frac{\sin\pi\alpha}{\pi}(R_\alpha Y)(t) \quad \boldsymbol{P}\text{-almost surely}$$

for any $t \in [0,T]$ (cf., for example, [**8**, § 7.1]), whence

$$\begin{aligned}\|\boldsymbol{I}_0(\psi)\|_{L^p(\Omega)} &= \left(\boldsymbol{E}\sup_{0\leqslant t\leqslant T}\left\|\left(\frac{\sin\pi\alpha}{\pi}\right)R_\alpha Y(t)\right\|^p\right)^{1/p}\\ &\leqslant \frac{1}{\pi}\|R_\alpha\|(\boldsymbol{E}\|Y\|^p_{L^p(0,T;H)})^{1/p}.\end{aligned}$$

Invoking Lemma 2.2 and the Young inequality we get

$$\begin{aligned}\boldsymbol{E}\|Y\|^p_{L^p(0,T;H)} &= \boldsymbol{E}\int_0^T \|Y(t)\|^p dt\\ &= \int_0^T \boldsymbol{E}\left\|\int_0^t (t-s)^{-\alpha}U(t,s)\psi(s)\,dW(s)\right\|^p dt\\ &\leqslant C_p\boldsymbol{E}\int_0^T\left(\int_0^t (t-s)^{-2\alpha}\|U(t,s)\psi(s)\|^2_{\mathscr{I}_2(\Upsilon,H)}\,ds\right)^{p/2} dt\\ &\leqslant C_pL_0^p\boldsymbol{E}\int_0^T\left(\int_0^t (t-s)^{-2\alpha}\|\psi(s)\|^2_{\mathscr{I}_2(\Upsilon,H)}ds\right)^{p/2} dt\\ &\leqslant C_pL_0^p\left(\int_0^T s^{-2\alpha}ds\right)^{p/2}\boldsymbol{E}\int_0^T \|\psi(s)\|^p_{\mathscr{I}_2(\Upsilon,H)}ds\\ &\leqslant C_pL_0^pT^{(1-2\alpha)p/2}\left(\frac{1}{1-2\alpha}\right)^{p/2}|||\psi|||_p^p\\ &\leqslant C_pL_0^p\max\left(1,T^{p/2}\right)\left(\frac{1}{1-2\alpha}\right)^{p/2}|||\psi|||_p^p.\end{aligned}$$

Combining this estimate with (2.1) we obtain

$$\|\boldsymbol{I}_0(\psi)\|_{L^p(\Omega)} \leqslant \frac{1}{\pi}L_0^2\max(1,T)C_p^{1/p}\left(\frac{p-1}{\alpha p-1}\right)^{1-1/p}\left(\frac{1}{1-2\alpha}\right)^{1/2}|||\psi|||_p.$$

Now we have to choose $\alpha \in (1/p, 1/2)$. Set

$$\alpha = \frac{1}{p}+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right) = \frac{1}{4}+\frac{1}{2p}, \tag{2.5}$$

then

$$1-2\alpha = \frac{p-2}{2p} \quad\text{and}\quad \alpha p - 1 = \frac{p-2}{4}.$$

Therefore, with this choice of α we get

$$\|\boldsymbol{I}_0(\psi)\|_{L^p(\Omega)} \leqslant MC_p^{1/p}\left(\frac{p-1}{p-2}\right)^{1-1/p}\left(\frac{p}{p-2}\right)^{1/2}|||\psi|||_p,$$

with a constant M dependent only on T and L_0 as required.

The assertion about $\boldsymbol{I}_\delta$ may be established by the same argument. It is only necessary to replace (2.1) with (2.2) to obtain for $\alpha \in (\delta + 1/p, 1/2)$ the estimate

$$\|\boldsymbol{I}_\delta(\psi)\|_{L^p(\Omega)} \leqslant \frac{1}{\pi}L_0L_\delta\max(1,T)C_p^{1/p}\left(\frac{p-1}{(\alpha-\delta)p-1}\right)^{1-1/p}\left(\frac{1}{1-2\alpha}\right)^{1/2}|||\psi|||_p.$$

Setting

$$\alpha = \frac{1}{p} + \delta + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p} - \delta\right) = \frac{1}{4} + \frac{1}{2p} + \frac{\delta}{2},$$

we complete the proof. □

Proof of Theorem 1.1. This is now straightforward. $\boldsymbol{I}_0$ is a sublinear operator from $\boldsymbol{L}^p$ into $L^p(\Omega)$, in particular positive homogeneous, and by Theorem 2.3 constants $A < \infty$ and $q \in (2, \infty)$ may be found such that

$$\|\boldsymbol{I}_0(\psi)\|_{L^p(\Omega)} \leqslant A\sqrt{p}|||\psi|||_p, \qquad p \in (q, \infty), \ \psi \in \boldsymbol{L}^\infty.$$

Hence Theorem 1.1 follows from Theorem A.1. □

The proof of Theorem 1.2 is almost identical and may be omitted.

Proof of Theorem 1.3. The proof is again very similar to that of Theorem 1.1. However, since $\psi \notin \boldsymbol{L}^2$ the estimate of the process Y introduced above has to be modified. Take $p > 2/\gamma$ and $\alpha \in (1/p, \gamma/2)$. By (1.4) and (1.5) we have

$$\begin{aligned}
\boldsymbol{E}\|Y\|^p_{L^p(0,T;H)} &\leqslant C_p \boldsymbol{E}\int_0^T \left(\int_0^t (t-s)^{-2\alpha}\|U(t,s)\psi(s)\|^2_{\mathscr{I}_2(\Upsilon,H)}\,ds\right)^{p/2} dt \\
&\leqslant C_p \boldsymbol{E}\int_0^T \left(\int_0^t (t-s)^{-2\alpha}k^2(t-s)\|\psi(s)\|^2_{\mathscr{L}(\Upsilon,H)}\,ds\right)^{p/2} dt \\
&\leqslant C_p \left(\int_0^T s^{-2\alpha}k^2(s)\,ds\right)^{p/2} \boldsymbol{E}\int_0^T \|\psi(s)\|^p_{\mathscr{L}(\Upsilon,H)}\,ds \\
&\leqslant C_p \max(1, T^{\gamma p/2})\left(\int_0^T s^{-\gamma}k^2(s)\,ds\right)^{p/2} |\psi|^p_p \\
&= C_p \max(1, T^{\gamma p/2})\kappa^{p/2}|\psi|^p_p.
\end{aligned}$$

All other steps of the proof remain the same, therefore

$$\|\boldsymbol{I}_0(\psi)\|_{L^p(\Omega)} \leqslant \frac{4}{\pi}L_0 \max(1, T^\gamma)\kappa^{1/2}\left(\frac{p-1}{\gamma p - 2}\right)^{1-1/p} C_p^{1/p}|\psi|_p$$

for every $p > 2/\gamma$ and the proof may be completed in an obvious way. □

Proof of Theorem 1.4. Set

$$\boldsymbol{J}(\psi) = \sup_{t \geqslant 0}\left\|\int_0^t U(t,s)\psi(s)\,dW(s)\right\|, \qquad \psi \in \boldsymbol{L}^2.$$

We aim at finding, for any $q \in (2, \infty)$, a constant $A < \infty$ so that

$$\|\boldsymbol{J}(\psi)\|_{L^p(\Omega)} \leqslant A\sqrt{p}|||\psi|||_p \tag{2.6}$$

for all $p \in [q, \infty)$ and $\psi \in \boldsymbol{L}^q \cap \boldsymbol{L}^\infty$. Proceeding as above, we derive (2.6) by means of the factorization method. First, we have to estimate the norm of R_α as an operator from $L^p(\mathbb{R}_+; H)$ to the space $\mathscr{C}_b(\mathbb{R}_+; H)$ of bounded continuous H-valued functions on $\mathbb{R}_+$. Take $p > 1$, $\alpha > 1/p$ and $f \in L^p(\mathbb{R}_+; H)$. Using (ES) we get

$$\begin{aligned}
\|R_\alpha f(t)\| &\leqslant \tilde{L} \int_0^t (t-s)^{\alpha-1} e^{-\mu(t-s)} \|f(s)\|\, ds \\
&\leqslant \tilde{L} \|f\|_{L^p(\mathbb{R}_+;H)} \left(\int_0^t (t-s)^{(\alpha-1)p/(p-1)} e^{-\mu p(t-s)/(p-1)}\, ds \right)^{1-1/p} \\
&\leqslant \tilde{L} \|f\|_{L^p(\mathbb{R}_+;H)} \left(\int_0^\infty s^{(\alpha-1)p/(p-1)} e^{-\mu ps/(p-1)}\, ds \right)^{1-1/p} \\
&= \tilde{L} \|f\|_{L^p(\mathbb{R}_+;H)} \left(\frac{p-1}{\mu p} \right)^{\alpha-1/p} \Gamma \left(\frac{\alpha p - 1}{p-1} \right)^{1-1/p}
\end{aligned}$$

for each $t \geqslant 0$, where Γ denotes the Gamma-function. Further, for $p > 2$, $\alpha \in (1/p, 1/2)$ and $\psi \in \boldsymbol{L}^p$ we have

$$\begin{aligned}
\boldsymbol{E} \int_0^\infty \|Y(t)\|^p dt &= \int_0^\infty \boldsymbol{E} \left\| \int_0^t (t-s)^{-\alpha} U(t,s) \psi(s)\, dW(s) \right\|^p dt \\
&\leqslant \tilde{L}^p C_p \boldsymbol{E} \int_0^\infty \left(\int_0^t (t-s)^{-2\alpha} e^{-2\mu(t-s)} \|\psi(s)\|^2_{\mathscr{J}_2(\Upsilon,H)}\, ds \right)^{p/2} dt \\
&\leqslant \tilde{L}^p C_p \left(\int_0^\infty s^{-2\alpha} e^{-2\mu s}\, ds \right)^{p/2} \boldsymbol{E} \int_0^\infty \|\psi(s)\|^p_{\mathscr{J}_2(\Upsilon,H)}\, ds \\
&= \frac{1}{(2\mu)^{(1-2\alpha)p/2}} \tilde{L}^p C_p \Gamma(1-2\alpha)^{p/2} |||\psi|||_p^p;
\end{aligned}$$

hence

$$\|\boldsymbol{J}(\psi)\|_{L^p(\Omega)} \leqslant \frac{\tilde{L}^2}{(2\mu)^{(1-2\alpha)/2}} \left(\frac{p-1}{\mu p} \right)^{\alpha-1/p} \Gamma \left(\frac{\alpha p-1}{p-1} \right)^{1-1/p} \Gamma(1-2\alpha)^{1/2} C_p^{1/p} |||\psi|||_p.$$

Choosing α as in (2.5) we obtain

$$\begin{aligned}
\|\boldsymbol{J}(\psi)\|_{L^p(\Omega)} \leqslant{}& \frac{\tilde{L}^2}{(2\mu)^{(p-2)/4p}} \left(\frac{p-1}{\mu p} \right)^{1/4-1/(2p)} \\
&\times \Gamma \left(\frac{p-2}{4p-4} \right)^{1-1/p} \Gamma \left(\frac{p-2}{2p} \right)^{1/2} C_p^{1/p} |||\psi|||_p
\end{aligned}$$

and (2.6) follows. □

Appendix A. *Zygmund's extrapolation theorem*

We aim at proving Zygmund's extrapolation theorem (see [**21**, Theorem XII.4.41]) in a form we need it, that is, for vector-valued functions; our proof follows the one in [**21**] rather closely.

Assume that $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{G}, \nu)$ are measure spaces and $\mu(X) < \infty$. Let E be a Banach space. In this appendix, we denote by $\|\cdot\|_p$ the norm in both $L^p(\mu)$ and $L^p(\nu; E)$ since there is no danger of confusion.

First, recall the elementary fact that if $f \in L^p(\nu; E) \cap L^\infty(\nu; E)$ and $q \in (p, \infty)$ then $f \in L^q(\nu; E)$ and

$$\|f\|_q \leqslant \|f\|_p \vee \|f\|_\infty, \tag{A.1}$$

since

$$\int_Y \|f\|_E^q \, d\nu = \int_Y \|f\|_E^p \, \|f\|_E^{q-p} \, d\nu \leqslant \operatorname{ess\,sup} \|f\|_E^{q-p} \int_Y \|f\|_E^p \, d\nu$$
$$= \|f\|_\infty^{q-p} \|f\|_p^p \leqslant (\|f\|_\infty \vee \|f\|_p)^q.$$

Now we may state the theorem.

THEOREM A.1. *Let $p \in [1, \infty)$, suppose that $T : L^p(\nu; E) \cap L^\infty(\nu; E) \longrightarrow L^0(\mu)$ is a mapping satisfying the following*:

(i) *T is positive homogeneous, $T(\gamma f) = \gamma T(f)$ for all $f \in L^p(\nu; E) \cap L^\infty(\nu; E)$ and $\gamma \geqslant 0$;*

(ii) *there exist constants $A < \infty$ and $\alpha > 0$ such that*

$$\|Tf\|_q \leqslant A q^\alpha \|f\|_q \tag{A.2}$$

for all $q \in [p, \infty)$ and $f \in L^p(\nu; E) \cap L^\infty(\nu; E)$.

Then for any $\lambda \in (0, \alpha A^{-1/\alpha} e^{-1})$ there exists a constant $C_\lambda < \infty$ such that

$$\int_X \exp\left(\lambda \left[\frac{|Tf|}{\|f\|_p \vee \|f\|_\infty}\right]^{1/\alpha}\right) d\mu \leqslant C_\lambda \tag{A.3}$$

holds for every $f \in L^p(\nu; E) \cap L^\infty(\nu; E)$.

REMARK A.1. The constant C_λ depends on A, α, p, λ and $\mu(X)$. If $\nu(Y) < \infty$ then $L^p(\nu; E) \supseteq L^\infty(\nu; E)$ and it will be clear from the proof that (A.3) may be replaced with an estimate

$$\int_X \exp\left(\lambda \left[\frac{|Tf|}{\|f\|_\infty}\right]^{1/\alpha}\right) d\mu \leqslant \tilde{C}_\lambda$$

valid for the same λ as (A.3), for a constant $\tilde{C}_\lambda$ which depends also on $\nu(Y)$ and for all $f \in L^\infty(\nu; E)$.

REMARK A.2. In the proof of Theorem A.1 we do not use the Bochner measurability of functions in $L^p(\nu; E)$; only the fact that $\|f\|_E$ is measurable for each $f \in L^p(\nu; E)$ is relevant. Therefore, the theorem remains valid for mappings $T : \boldsymbol{SL}^p \cap \boldsymbol{SL}^\infty \longrightarrow L^0(\mu)$ satisfying hypotheses (i), (ii) above.

REMARK A.3. The estimate (A.3) may be naturally interpreted in terms of Orlicz spaces. Define a Young function Φ by

$$\Phi(t) = e^{t^{1/\alpha}} - \sum_{k=0}^{[\alpha]} \frac{t^{k/\alpha}}{k!}, \qquad t \geqslant 0.$$

(Hence, in particular, $\Phi(t) = e^{t^2} - 1$ for $\alpha = \frac{1}{2}$.) Let $L^\Phi(\mu)$ be the corresponding Orlicz space (denoted often by $\exp L^{1/\alpha}$), see, for example, [**15**, § 3.2 and § 3.6]. Theorem A.1 states that T maps $L^p(\nu; E) \cap L^\infty(\nu; E)$ into $L^\Phi(\mu)$ and

$$\|Tf\|_\Phi \leqslant \frac{C_\lambda^\alpha (\|f\|_p \vee \|f\|_\infty)}{\lambda^\alpha},$$

$\|\cdot\|_\Phi$ denoting the Luxemburg norm in $L^\Phi(\mu)$ (cf. [**15**, § 3.8]).

Proof of Theorem A.1. Denote by $\lceil \alpha p \rceil$ the least integer greater than or equal to αp. For every integer $k \geqslant \lceil \alpha p \rceil$ we have

$$\int_X |Tf|^{k/\alpha} d\mu \leqslant A^{k/\alpha} \left(\frac{k}{\alpha}\right)^k \int_Y \|f\|_E^{k/\alpha} d\nu$$
$$\leqslant \left(\frac{A^{1/\alpha}}{\alpha}\right)^k k^k (\|f\|_p \vee \|f\|_\infty)^{k/\alpha}$$

by (A.2) and (A.1), hence

$$\int_X \left[\frac{|Tf|}{\|f\|_p \vee \|f\|_\infty}\right]^{k/\alpha} d\mu \leqslant \left(\frac{A^{1/\alpha}}{\alpha}\right)^k k^k. \tag{A.4}$$

Choose $\lambda \in (0, \alpha A^{-1/\alpha} e^{-1})$ and multiply (A.4) by $\lambda^k/k!$ obtaining

$$\int_X \frac{\lambda^k}{k!} \left(\frac{|Tf|}{\|f\|_p \vee \|f\|_\infty}\right)^{k/\alpha} d\mu \leqslant \left(\frac{\lambda A^{1/\alpha}}{\alpha}\right)^k \frac{k^k}{k!} \leqslant \left(\frac{\lambda e A^{1/\alpha}}{\alpha}\right)^k \equiv \beta^k,$$

as obviously $k^k/k! \leqslant e^k$. Note that $\beta \in (0,1)$ due to the choice of λ. Summing up the terms in this estimate we arrive at

$$\int_X \sum_{k=\lceil \alpha p \rceil}^{\infty} \frac{\lambda^k}{k!} \left(\frac{|Tf|}{\|f\|_p \vee \|f\|_\infty}\right)^{k/\alpha} d\mu \leqslant \sum_{k=\lceil \alpha p \rceil}^{\infty} \beta^k \leqslant \frac{1}{1-\beta}. \tag{A.5}$$

Setting

$$P(t) = \sum_{k=0}^{\lceil \alpha p \rceil - 1} \frac{t^k}{k!} \quad \text{and} \quad Uf = \frac{|Tf|}{\|f\|_p \vee \|f\|_\infty}$$

we may rewrite (A.5) as

$$\int_X \{\exp(\lambda (Uf)^{1/\alpha}) - P(\lambda (Uf)^{1/\alpha})\} \, d\mu \leqslant \frac{1}{1-\beta}. \tag{A.6}$$

For $t \geqslant 1$ and $0 \leqslant k \leqslant \lceil \alpha p \rceil - 1$ we have $t^k \leqslant t^{\lceil \alpha p \rceil - 1}$, thus

$$P(\lambda t^{1/\alpha}) \leqslant \sum_{k=0}^{\lceil \alpha p \rceil - 1} \frac{\lambda^k}{k!} t^{(\lceil \alpha p \rceil - 1)/\alpha} \leqslant e^{\lambda} t^p, \qquad t \geqslant 1. \tag{A.7}$$

Using positive homogeneity of T and hypothesis (A.2) we get

$$\int_X |Uf|^p \, d\mu = \int_X \left| T\left(\frac{f}{\|f\|_p \vee \|f\|_\infty}\right)\right|^p d\mu \leqslant A^p p^{p\alpha} \left\| \frac{f}{\|f\|_p \vee \|f\|_\infty} \right\|_p^p$$
$$\leqslant A^p p^{p\alpha}.$$

Therefore,

$$\int_X \exp(\lambda (Uf)^{1/\alpha}) \, d\mu = \int_{\{Uf \leqslant 1\}} \exp(\lambda (Uf)^{1/\alpha}) \, d\mu$$
$$+ \int_{\{Uf > 1\}} \{\exp(\lambda (Uf)^{1/\alpha}) - P(\lambda (Uf)^{1/\alpha})\} \, d\mu$$
$$+ \int_{\{Uf > 1\}} P(\lambda (Uf)^{1/\alpha}) \, d\mu$$

$$\leqslant e^{\lambda}\mu(X) + \frac{1}{1-\beta} + e^{\lambda}\int_{\{Uf>1\}} |Uf|^p \, d\mu$$
$$\leqslant e^{\lambda}\mu(X) + \frac{1}{1-\beta} + e^{\lambda}A^p p^{p\alpha}$$

follows from (A.6) and (A.7), which completes the proof. □

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