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LITTLEWOOD'S MULTIPLE FORMULA FOR SPIN CHARACTERS OF SYMMETRIC GROUPS

HIROSHI MIZUKAWA AND HIRO-FUMI YAMADA

Introduction

This paper deals with some character values of the symmetric group S_n as well as its double cover \tilde{S}_n .

Let $\chi^\lambda(\rho)$ be the irreducible character of S_n , indexed by the partition λ and evaluated at the conjugacy class ρ . Comparing the character tables of S_2 and S_4 , one observes that

$$\begin{aligned}\chi^{(4)}(2\rho) &= \chi^{(2)}(\rho) \\ \chi^{(2^2)}(2\rho) &= \chi^{(2)}(\rho) + \chi^{(1^2)}(\rho)\end{aligned}$$

for $\rho = (2)$, $2\rho = (4)$ and $\rho = (1^2)$, $2\rho = (2^2)$. A number of such observations lead to what we call *Littlewood's multiple formula* (Theorem 1.1). This formula appears in Littlewood's book [2]. We include a proof that is based on an 'inflation' of the variables in a Schur function. This is different from one given in [2], and we claim that it is more complete than the one given there.

Our main objective is to obtain the spin character version of Littlewood's multiple formula (Theorem 2.3). Let $\zeta^\lambda(\rho)$ be the irreducible *negative* character of \tilde{S}_n (cf. [1]), indexed by the strict partition λ and evaluated at the conjugacy class ρ . One finds character tables ($\zeta^\lambda(\rho)$) in [1] for $n \leq 14$. This time we evidently see that

$$\zeta^{3\lambda}(3\rho) = \zeta^\lambda(\rho)$$

for $\lambda = (4), (3, 1)$ and $\rho = (3, 1), (1^4)$. The proof of Theorem 2.3 is achieved in a way that is similar to the case of ordinary characters. Instead of a Schur function, we deal with Schur's P-function, which is defined as a ratio of Pfaffians.

1. Littlewood's multiple formula

We first recall the multiple formula for ordinary characters of the symmetric groups that is due to Littlewood [2, Chapter 8].

Throughout this section, we fix a positive integer r . Let $\lambda = (\lambda_1, \dots, \lambda_N)$ be a partition. We always assume that N is at least the length $l(\lambda)$ of λ . An $(r+1)$ -tuple of partitions $(\lambda^c, \lambda[0], \dots, \lambda[r-1])$ is attached to λ ; λ^c is the r -core of λ and the collection $\lambda^* = (\lambda[0], \dots, \lambda[r-1])$ is the r -quotient of λ (cf. [4, p. 12]). The effect of changing N is to permute the $\lambda[k]$ cyclically. Since we will only consider the Littlewood–Richardson coefficients $\text{LR}_{\lambda[0], \dots, \lambda[r-1]}^\mu$, the ambiguity of choosing N can be ignored. Put $r\lambda = (r\lambda_1, \dots, r\lambda_N)$. We can easily verify that $(r\lambda)^c = \emptyset$ and

$(r\lambda)[0] = (\lambda_r, \lambda_{2r}, \lambda_{3r}, \dots)$, $(r\lambda)[k] = (\lambda_{r-k}, \lambda_{2r-k}, \lambda_{3r-k}, \dots)$ ($1 \leq k \leq r-1$). As is well known, the ordinary irreducible characters of the symmetric group S_n are parametrized by $P(n)$, the set of all partitions of n . Let $\chi^\lambda(\rho)$ denote the irreducible character of S_n indexed by λ , evaluated at the conjugacy class determined by the partition ρ .

THEOREM 1.1 (Littlewood's multiple formula).

$$\chi^{r\lambda}(r\rho) = \sum_{v \in P(n)} \text{LR}_{(r\lambda)[0], (r\lambda)[1], \dots, (r\lambda)[r-1]}^v \chi^v(\rho)$$

for any partitions λ and ρ of n .

Littlewood proved a more general formula in [3, pp. 340–342]. However, in order to contrast the formula with the spin character case, we here deal only with this form. In the rest of this section, we give a simple proof of the theorem by using Schur functions.

Put $\mathbf{x}_N = (x_1, x_2, \dots, x_N)$, $\mathbf{x}_N^r = (x_1^r, x_2^r, \dots, x_N^r)$ and $\mathbf{x}_{N,r} = (x_1, x_2, \dots, x_{rN})$ with $x_{kN+i} = \omega^k x_i$ ($0 \leq k \leq r-1$, $1 \leq i \leq N$), where $\omega = \exp(2\pi\sqrt{-1}/r)$. We call $\mathbf{x}_{N,r}$ the r -inflation of \mathbf{x}_N . Let $\delta_N = (N-1, N-2, \dots, 1, 0)$. Following [4, p. 40], we define the Schur function of variables \mathbf{x}_N , corresponding to the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$, by

$$s_\lambda(\mathbf{x}_N) = \frac{\det A_{\lambda+\delta_N}(\mathbf{x}_N)}{\det A_{\delta_N}(\mathbf{x}_N)},$$

where

$$A_\alpha(\mathbf{x}_N) = (x_i^{\alpha_j})_{1 \leq i, j \leq N}$$

for a sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ of non-negative integers.

THEOREM 1.2.

$$s_{r\lambda}(\mathbf{x}_{N,r}) = \prod_{k=0}^{r-1} s_{(r\lambda)[k]}(\mathbf{x}_N^r)$$

for any partition $\lambda = (\lambda_1, \dots, \lambda_{rN})$.

Proof. For a sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ of non-negative integers, put

$$X_N^{\alpha_j+N-j} = \begin{pmatrix} x_1^{\alpha_j+N-j} \\ x_2^{\alpha_j+N-j} \\ \vdots \\ x_N^{\alpha_j+N-j} \end{pmatrix},$$

the j th column of the matrix $A_{\alpha+\delta_N}(\mathbf{x}_N)$. We first compute the numerator of $s_{r\lambda}(\mathbf{x}_{N,r})$:

$$A_{r\lambda+\delta_{rN}}(\mathbf{x}_{N,r}) = (\omega^{(i-1)(r-j)} X_N^{r\lambda_j+rN-j})_{1 \leq i \leq r, 1 \leq j \leq rN}.$$

Let $\tau \in S_{rN}$ be defined by

$$\tau(ir+j) = N(j-1) + (i+1) \quad 0 \leq i \leq r-1, 1 \leq j \leq r.$$

Permuting columns of the above matrix according to τ , and setting $p =$

$\text{diag}(x_1, x_2, \dots, x_N)$, we obtain

$$\begin{aligned} \det A_{r\lambda+\delta_{rN}}(\mathbf{x}_{N,r}) &= \text{sgn}(\tau) \det(\omega^{(i-1)(r-j)} p^{r-j} A_{(r\lambda)[r-j]+\delta_N}(\mathbf{x}_N^r))_{1 \leq i, j \leq r} \\ &= \text{sgn}(\tau) \det(\omega^{(i-1)(r-j)} 1_N)_{1 \leq i, j \leq r} \det(p^{r-j})_{1 \leq j \leq r} \det(A_{(r\lambda)[r-j]+\delta_N}(\mathbf{x}_N^r))_{1 \leq j \leq r}. \end{aligned}$$

Thus we have

$$\det A_{r\lambda+\delta_{rN}}(\mathbf{x}_{N,r}) = \text{sgn}(\tau) (\det p)^\alpha c \prod_{k=0}^{r-1} \det(A_{(r\lambda)[r-k]+\delta_N}(\mathbf{x}_N^r)), \quad (1)$$

where

$$\begin{cases} \alpha = \sum_{j=1}^r (r-j) = \binom{r}{2} \\ c = \det(\omega^{(i-1)(r-j)} E_N). \end{cases}$$

Putting $\lambda = (0, \dots, 0)$, we get

$$\det A_{\delta_{rN}}(\mathbf{x}_{N,r}) = \text{sgn}(\tau) p^\alpha c (\det A_{\delta_N}(\mathbf{x}_N^r))^r. \quad (2)$$

From (1) and (2), we see that

$$\begin{aligned} s_{r\lambda}(\mathbf{x}_{N,r}) &= \frac{\det A_{r\lambda+\delta_{rN}}(\mathbf{x}_{N,r})}{\det A_{\delta_{rN}}(\mathbf{x}_{N,r})} \\ &= \prod_{k=0}^{r-1} \frac{\det A_{(r\lambda)[k]+\delta_{rN}}(\mathbf{x}_N^r)}{\det A_{\delta_N}(\mathbf{x}_N^r)} \\ &= \prod_{k=0}^{r-1} s_{(r\lambda)[k]}(\mathbf{x}_N^r). \end{aligned} \quad \square$$

We remark here that the right-hand side of the expression in Theorem 1.2 can also be expressed as

$$\prod_{k=0}^{r-1} s_{\lambda[k]}(\mathbf{x}_N^r) = \sum_{v \in P(n)} \text{LR}_{(r\lambda)[0], (r\lambda)[1], \dots, (r\lambda)[r-1]}^v s_v(\mathbf{x}_N^r).$$

In order to translate the above identity of Schur functions into that of irreducible characters of S_n , we need the power sum symmetric functions

$$p_m(\mathbf{x}_N) = \sum_{i=1}^N x_i^m \quad m = 1, 2, \dots$$

For a partition $\rho = (\rho_1, \rho_2, \dots)$, define

$$p_\rho = p_{\rho_1} p_{\rho_2} \dots$$

The well known Frobenius formula is

$$s_\lambda(\mathbf{x}_N) = \sum_{\rho \in P(n)} z_\rho^{-1} \chi^\lambda(\rho) p_\rho(\mathbf{x}_N),$$

where

$$z_\rho = \prod_{i=1}^n m_i! i^{m_i}$$

for the partition $\rho = (1^{m_1} 2^{m_2} \dots n^{m_n})$ of n . As for the r -inflation of \mathbf{x}_N , we have the following simple fact.

LEMMA 1.3.

$$p_m(\mathbf{x}_{N,r}) = \begin{cases} rp_m(\mathbf{x}_N) & m \equiv 0 \pmod{r} \\ 0 & m \not\equiv 0 \pmod{r}. \end{cases}$$

Proof. We easily see that

$$\begin{aligned} p_m(\mathbf{x}_{N,r}) &= \sum_{i=1}^{rN} x_i^m \\ &= \sum_{i=1}^N \left(\sum_{k=0}^{r-1} (\omega^k x_i)^m \right) = \begin{cases} r \sum_{i=1}^N x_i^m & m \equiv 0 \pmod{r} \\ \sum_{i=1}^N \frac{1 - (\omega^m)^r}{1 - \omega^m} x_i^m = 0 & m \not\equiv 0 \pmod{r}. \end{cases} \quad \square \end{aligned}$$

Combining this lemma with the Frobenius formula, we see that

$$\begin{aligned} s_{r\lambda}(\mathbf{x}_{N,r}) &= \sum_{\rho \in P(rn)} z_\rho^{-1} \chi^{r\lambda}(\rho) p_\rho(\mathbf{x}_{N,r}) \\ &= \sum_{\rho \in P(n)} z_{r\rho}^{-1} \chi^{r\lambda}(r\rho) r^{l(\rho)} p_{r\rho}(\mathbf{x}_N) \\ &= \sum_{\rho \in P(n)} z_\rho^{-1} \chi^{r\lambda}(r\rho) p_\rho(\mathbf{x}_N^r). \end{aligned}$$

Therefore we have

$$\begin{aligned} &\sum_{\rho \in P(n)} z_\rho^{-1} \chi^{r\lambda}(r\rho) p_\rho(\mathbf{x}_N^r) \\ &= \sum_{v \in P(n)} \text{LR}_{(r\lambda)[0], (r\lambda)[1], \dots, (r\lambda)[r-1]}^v \left[\sum_{\rho \in P(n)} z_\rho^{-1} \chi^v(\rho) p_\rho(\mathbf{x}_N^r) \right] \\ &= \sum_{\rho \in P(n)} \left(\sum_{v \in P(n)} \text{LR}_{(r\lambda)[0], (r\lambda)[1], \dots, (r\lambda)[r-1]}^v \chi^v(\rho) \right) z_\rho^{-1} p_\rho(\mathbf{x}_N^r). \end{aligned}$$

Since the p_ρ are linearly independent, we obtain

$$\chi^{r\lambda}(r\rho) = \sum_{v \in P(n)} \text{LR}_{(r\lambda)[0], (r\lambda)[1], \dots, (r\lambda)[r-1]}^v \chi^v(\rho),$$

as desired.

The proof of Lemma 1.3 given here can immediately be extended to the case of partition $\mu \in P(rn)$ whose r -core is empty. Littlewood's multiple formula is

$$s_\mu(\mathbf{x}_{N,r}) = \varepsilon \prod_{k=0}^{r-1} s_{\mu[k]}(\mathbf{x}_N^r),$$

and

$$\chi^\mu(r\rho) = \varepsilon \sum_{v \in P(n)} \text{LR}_{\mu[0], \mu[1], \dots, \mu[r-1]}^v \chi^v(\rho),$$

where ε is the sign depending only on μ .

2. Spin characters

We consider Littlewood's multiple formula for spin characters of symmetric groups. Since the theory of spin characters is Pfaffian by nature, Schur's P-functions play an important role. Here we adopt the definition of P-functions due to Nimmo [6, Appendix] (see also [4, p. 267]).

A partition $\lambda = (\lambda_1, \dots, \lambda_l)$ is said to be strict if $\lambda_1 > \dots > \lambda_l > 0$. The set of strict partitions of n is denoted by $\text{SP}(n)$.

Let

$$A(\mathbf{x}_N) = \left(\frac{x_i - x_j}{x_i + x_j} \right)_{1 \leq i, j \leq N},$$

and

$$D_\lambda(\mathbf{x}_N) = (x_i^{\lambda_j})_{1 \leq i \leq N, 1 \leq j \leq l}$$

for a strict partition $\lambda = (\lambda_1, \dots, \lambda_l)$. Put

$$A_\lambda(\mathbf{x}_N) = \begin{pmatrix} A(\mathbf{x}_N) & D_\lambda(\mathbf{x}_N) \\ -{}^t D_\lambda(\mathbf{x}_N) & 0 \end{pmatrix},$$

which is a skew-symmetric matrix of $N + l$ rows and columns. Define $\text{Pf}_\lambda(\mathbf{x}_N)$ to be the Pfaffian of $A_\lambda(\mathbf{x}_N)$ if $N + l$ is even, and to be the Pfaffian of $A_\lambda(\mathbf{x}_{N+1})$ if $N + l$ is odd, agreeing $x_{N+1} = 0$. When $\lambda = \emptyset$, we have

$$\text{Pf}_{\emptyset}(\mathbf{x}_N) = \begin{cases} \text{Pf} A(\mathbf{x}_N) & N \text{ is even} \\ \text{Pf} A(\mathbf{x}_{N+1}) & N \text{ is odd.} \end{cases}$$

It is a good exercise to verify that

$$\text{Pf}_{\emptyset}(\mathbf{x}_N) = \prod_{1 \leq i < j \leq N} \frac{x_i - x_j}{x_i + x_j}. \quad (3)$$

Now define Schur's P-function corresponding to the strict partition λ by

$$P_\lambda(\mathbf{x}_N) = \frac{\text{Pf}_\lambda(\mathbf{x}_N)}{\text{Pf}_{\emptyset}(\mathbf{x}_N)}.$$

Throughout this section we fix a positive odd integer r . The r -inflation of \mathbf{x}_N is defined, as in Section 1, by $\mathbf{x}_{N,r} = (x_1, x_2, \dots, x_{rN})$ with $x_{kN+i} = \omega^k x_i$ ($0 \leq k \leq r-1$, $1 \leq i \leq N$), where $\omega = \exp(2\pi\sqrt{-1}/r)$.

LEMMA 2.1.

$$\text{Pf}_{\emptyset}(\mathbf{x}_{N,r}) = c_r \{\text{Pf}_{\emptyset}(\mathbf{x}_N^r)\}^r,$$

where c_r is a non-zero complex number.

Proof. Put

$$B_k = \left(\frac{x_i - \omega^k x_j}{x_i + \omega^k x_j} \right)_{1 \leq i, j \leq N} \quad 0 \leq k \leq r-1,$$

$$\mathcal{B} = (B_{j-i})_{0 \leq i, j \leq r-1}.$$

Then we see that \mathcal{B} is a skew-symmetric matrix. We also see that

$$\text{Pf}_{\emptyset}(\mathbf{x}_{N,r}) = \text{Pf}(\mathcal{B})$$

if N is even, and

$$\text{Pf}_{\emptyset}(\mathbf{x}_{N,r}) = \text{Pf} \begin{pmatrix} \mathcal{B} & {}^t\theta \\ -\theta & 0 \end{pmatrix}$$

if N is odd. Here $\theta = (1, \dots, 1)$ is a row vector of length r . Put

$$b_k^+ = \prod_{1 \leq i < j \leq N} \frac{x_i - \omega^k x_j}{x_i + \omega^k x_j}, \quad b_k^- = \prod_{1 \leq j < i \leq N} \frac{x_i - \omega^k x_j}{x_i + \omega^k x_j},$$

and

$$d_k = \left(\frac{1 - \omega^k}{1 + \omega^k} \right)^N$$

for $0 \leq k \leq r-1$. Then we check that

$$b_k^- = (-1)^{N(N-1)/2} b_k^+,$$

and

$$\begin{aligned} \prod_{k=0}^{r-1} b_k^+ &= \prod_{1 \leq i < j \leq N} \left(\prod_{k=0}^{r-1} \frac{x_i - \omega^k x_j}{x_i + \omega^k x_j} \right) \\ &= \prod_{1 \leq i < j \leq N} \frac{x_i^r - x_j^r}{x_i^r + x_j^r} \\ &= \text{Pf}_{\emptyset}(\mathbf{x}_N^r). \end{aligned}$$

By (3), we have

$$\text{Pf}_{\emptyset}(\mathbf{x}_{N,r}) = c_r \left(\prod_{k=0}^{r-1} b_k^+ \right)^r = c_r \{ \text{Pf}_{\emptyset}(\mathbf{x}_N^r) \}^r,$$

where

$$c_r = \prod_{k=1}^{r-1} d_k^{r-k},$$

which is a non-zero constant. □

THEOREM 2.2.

$$P_{r\lambda}(\mathbf{x}_{N,r}) = P_{\lambda}(\mathbf{x}_N^r)$$

for any strict partition $\lambda = (\lambda_1, \dots, \lambda_l)$.

Proof. We give a proof for the case where $N+l$ is even, since the other case can be proved with only a slight modification. Set

$$D = D_{\lambda}(\mathbf{x}_N^r).$$

Since $rN+l$ is even, we have

$$\text{Pf}_{r\lambda}(\mathbf{x}_{N,r}) = \text{Pf} \begin{pmatrix} \mathcal{B} & {}^t\mathcal{D} \\ -\mathcal{D} & 0 \end{pmatrix}.$$

Here $\mathcal{D} = ({}^tD, \dots, {}^tD)$ is an $N \times rN$ matrix. Apply the following elementary row-block and column-block operations successively to the above matrix:

- (1) Subtract the r th row-block from the first, second, \dots , $(r-1)$ th row-blocks.
- (2) Subtract the r th column-block from the first, second, \dots , $(r-1)$ th column-blocks.
- (3) Add the r th row-block, multiplied by $1/r$, to the first, second, \dots , $(r-1)$ th row-blocks.
- (4) Add the r th column-block, multiplied by $1/r$, to the first, second, \dots , $(r-1)$ th column-blocks.

This series of operations preserves the skew-symmetry of the matrix and does not change its Pfaffian. Therefore we have, setting $B = 1/r \sum_{k=0}^{r-1} B_k$,

$$\begin{aligned} \text{Pf}_{r\lambda}(\mathbf{x}_{N,r}) &= \text{Pf} \begin{pmatrix} A' & 0 & 0 \\ 0 & B & D \\ 0 & -{}^t D & 0 \end{pmatrix} \\ &= \text{Pf}(A') \text{Pf} \begin{pmatrix} B & D \\ -{}^t D & 0 \end{pmatrix} \end{aligned}$$

for some skew-symmetric matrix A' of $r(N-1)$ rows and columns. By a simple calculation, we can show that the (i, j) -entry of B is equal to

$$\frac{1}{r} \sum_{k=0}^{r-1} \frac{x_i - \omega^k x_j}{x_i + \omega^k x_j} = \frac{x_i^r - x_j^r}{x_i^r + x_j^r},$$

and thus we see that $\text{Pf}(B) = \text{Pf}_{\emptyset}(\mathbf{x}_N^r)$. Hence we have

$$\text{Pf} \begin{pmatrix} B & D \\ -{}^t D & 0 \end{pmatrix} = \text{Pf}_{\lambda}(\mathbf{x}_N^r).$$

By the same elementary operations as above, we see that

$$\begin{aligned} \text{Pf}_{\emptyset}(\mathbf{x}_{N,r}) &= \text{Pf} \begin{pmatrix} A' & 0 \\ 0 & B \end{pmatrix} \\ &= \text{Pf}(A') \text{Pf}_{\emptyset}(\mathbf{x}_N^r). \end{aligned}$$

Since the left-hand side is equal to $c_r \{\text{Pf}_{\emptyset}(\mathbf{x}_N^r)\}^r$, we have

$$\text{Pf}(A') = c_r \{\text{Pf}_{\emptyset}(\mathbf{x}_N^r)\}^{r-1}.$$

Consequently we see that

$$\begin{aligned} P_{r\lambda}(\mathbf{x}_{N,r}) &= \frac{\text{Pf}_{r\lambda}(\mathbf{x}_{N,r})}{\text{Pf}_{\emptyset}(\mathbf{x}_{N,r})} \\ &= \frac{c_r \{\text{Pf}_{\emptyset}(\mathbf{x}_N^r)\}^{r-1} \text{Pf}_{\lambda}(\mathbf{x}_N^r)}{c_r \{\text{Pf}_{\emptyset}(\mathbf{x}_N^r)\}^r} \\ &= P_{\lambda}(\mathbf{x}_N^r). \end{aligned} \quad \square$$

Let \tilde{S}_n ($n \geq 4$) be the double cover of the symmetric group S_n , which is generated by t_1, \dots, t_{n-1} and z subject to the relations

$$\begin{cases} z^2 = 1 \\ zt_i = t_i z, \quad t_i^2 = z & 1 \leq i \leq n-1 \\ (t_i t_{i+1})^3 = z & 1 \leq i \leq n-2 \\ t_i t_j = z t_j t_i & |i-j| \geq 2. \end{cases}$$

The group \tilde{S}_n can be regarded as a central extension

$$1 \longrightarrow Q \longrightarrow \tilde{S}_n \xrightarrow{\theta} S_n \longrightarrow 1,$$

where $Q = \{1, z\} \cong S_2$. The central element z acts on an irreducible representation by ± 1 . An irreducible representation of \tilde{S}_n is said to be negative if $z = -1$.

The irreducible negative representations of \tilde{S}_n are parametrized up to associativity by $\text{SP}(n)$ [1, Chapter 8]. We denote by ζ^λ the character of the irreducible negative representation corresponding to $\lambda \in \text{SP}(n)$. These spin characters are related to the P-functions by the following Schur formula [5, p. 64]:

$$P_\lambda(\mathbf{x}_N) = \sum_{\rho \in \text{OP}(n)} 2^{(l(\rho) - l(\lambda) + \epsilon(\lambda))/2} z_\rho^{-1} \zeta^\lambda(\rho) p_\rho(\mathbf{x}_N),$$

where $\text{OP}(n)$ denotes the set of partitions of n consisting of odd parts, $\zeta^\lambda(\rho)$ is the value of ζ^λ on the conjugacy class determined by $\rho \in \text{OP}(n)$, and

$$\epsilon(\lambda) = \begin{cases} 0 & n - l(\lambda) \text{ is even} \\ 1 & n - l(\lambda) \text{ is odd.} \end{cases}$$

Using the Schur formula, we can translate Theorem 2.2 into the following multiple formula for spin characters.

THEOREM 2.3. *If r is an odd positive integer, we have*

$$\zeta^{r\lambda}(r\rho) = \zeta^\lambda(\rho)$$

for any $\lambda \in \text{SP}(n)$ and $\rho \in \text{OP}(n)$.

Proof. We have

$$\begin{aligned} P_{r\lambda}(\mathbf{x}_{N,r}) &= \sum_{\rho \in \text{OP}(rn)} 2^{(l(\rho) - l(r\lambda) + \epsilon(r\lambda))/2} z_\rho^{-1} \zeta^{r\lambda}(\rho) p_\rho(\mathbf{x}_{N,r}) \\ &= \sum_{\rho \in \text{OP}(n)} 2^{(l(r\rho) - l(r\lambda) + \epsilon(r\lambda))/2} z_{r\rho}^{-1} \zeta^{r\lambda}(r\rho) r^{l(\rho)} p_\rho(\mathbf{x}_N^r) \\ &= \sum_{\rho \in \text{OP}(n)} 2^{(l(r\rho) - l(r\lambda) + \epsilon(r\lambda))/2} z_\rho^{-1} \zeta^{r\lambda}(r\rho) p_\rho(\mathbf{x}_N^r). \end{aligned}$$

The left-hand side is equal to

$$P_\lambda(\mathbf{x}_N^r) = \sum_{\rho \in \text{OP}(n)} 2^{(l(\rho) - l(\lambda) + \epsilon(\lambda))/2} z_\rho^{-1} \zeta^\lambda(\rho) p_\rho(\mathbf{x}_N^r).$$

Since $l(r\rho) = l(\rho)$, $l(r\lambda) = l(\lambda)$ and $\epsilon(r\lambda) = \epsilon(\lambda)$, we can conclude, comparing the coefficients of $p_\rho(\mathbf{x}_N^r)$, that $\zeta^{r\lambda}(r\rho) = \zeta^\lambda(\rho)$. \square

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