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SURGERY OBSTRUCTION OF TWISTED PRODUCTS

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1. Introduction. Let (G,χ) be a pair of a finite group G and a homomorphism $\chi: G \to \{\pm 1\}$. Then we call an oriented closed PL (or smooth) G-manifold L^m a G-χ-manifold when the action of $g \in G$ preserves the orientation of L^m if and only if $\chi(g) = +1$, and we can define the cobordismgroup $\Omega_m^\chi(G)$ of G-χ-manifolds as in [1]. Moreover, let (π, w) be a pair of a finitely presentable group π and a homomorphism $w: \pi \to \{\pm 1\}$. Then the Wall group $L_n^s(\pi, w)$ is defined in [7]. Its element σ can be represented as the surgery obstruction $\sigma(f)$ of a normal map of degree one $f: M^n \to N^n$ between compact PL (or smooth) manifolds to deform to a simple homotopy equivalence, where $\pi_1(N^n) = \pi$ and $w: \pi_1(N^n) \to \{\pm 1\}$ is the characteristic map of the orientation bundle of N^n.

Now, assume that there is an epimorphism $\phi: \pi \to G$. Then we can define a homomorphism

$$\Omega_m^\chi(G) \otimes L_n^s(\pi, w) \longrightarrow L_{m+n}^s(\pi, w\chi)$$

($w\chi: \pi \to \{\pm 1\}$ is the homomorphism defined by $w\chi(h) = w(h)(\chi(\phi(h))))$ as follows: For $\sigma(f) \in L_n^s(\pi, w)$ of $f: M^n \to N^n$, consider the covering map $\tilde{f}: \tilde{M}^n \to \tilde{N}^n$, where $\tilde{N}^n$ is the universal covering of N^n and $\tilde{M}^n$ is the covering of M^n induced from $\tilde{N}^n$ by f. Further, let L^m be a G-χ-manifold. Then π acts on L^m through ϕ, and the product manifolds $\tilde{M}^n \times L^m$ and $\tilde{N}^n \times L^m$ have the diagonal π-actions. Thus we have a map of degree one, $\tilde{f} \times_\pi 1: \tilde{M}^n \times_\pi L^m \to \tilde{N}^n \times_\pi L^m (1 = 1_{L^m}: L^m \to L^m)$, between the orbit spaces of the diagonal π-actions. This map has a natural structure of a normal map of degre one, and the characteristic map of the orientation bundle of $\tilde{N}^n \times_\pi L^m$ is given by $(w\chi)p_*$, where $p_*: \pi_1(\tilde{N}^n \times_\pi L^m) \to \pi_1(N^n) = \pi$ is the map induced by the projection p. Thus $\sigma(\tilde{f} \times_\pi 1) \in L_{m+n}^s(\pi_1(\tilde{N}^n \times_\pi L^m), (w\chi)p_*)$, and we denote by the same letter $\sigma(\tilde{f} \times_\pi 1)$ $\in L_{m+n}^s(\pi, w\chi)$ its image under the homomorphism induced by p. We define a desired homomorphism by sending $(L_m, \sigma(f))$ to $\sigma(\tilde{f} \times_\pi 1)$.

On the other hand, we can define the G-χ-equivariant Witt group $W_m^\chi(G,Z)$ (cf. §3) and a homomorphism

$$\rho: \Omega_m^\chi(G) \longrightarrow W_m^\chi(G,\boldsymbol{Z}) \text{ by setting}$$
$$\rho([L^{2k}]) = \langle H^k(L^{2k}, Z)/\mathrm{Tor}, \text{ the intersection form}\rangle$$

$$\rho([L^{2k+1}]) = \langle \mathrm{Tor}\ H^{k+1}(L^{2k+1}, \boldsymbol{Z}), \text{ the linking form} \rangle,$$

where Tor denotes the $\boldsymbol{Z}$-torsion subgroup. An algebraic action of $W_m^\chi(G, \boldsymbol{Z})$ on the Wall group of π is defined by the tensor product, $W_m^\chi(G, \boldsymbol{Z}) \otimes L_n^s(\pi, w) \longrightarrow L_{m+n}^s(\pi, w\chi)$ (cf. §8).

Our main result is Theorem 2 of §9, which claims that the following diagram is commutative ;

$$\begin{array}{ccc} \Omega_m^\chi(G) \otimes L_n^s(\pi, w) & \longrightarrow & L_{m+n}^s(\pi, w\chi) \\ \rho\otimes 1 \downarrow & & \| \\ W_m^\chi(G, \boldsymbol{Z}) \otimes L_n^s(\pi, w) & \longrightarrow & L_{m+n}^s(\pi, w\chi). \end{array}$$

This can be considered to be a generalization of the product formula of J.Morgan [4] to the equivariant case. The proof is not analogous to Morgan. We use the algebraic surgery theory due to A.Ranicki [5, 6]. The construction of the equivariant analogue $L_{G,\chi}^*(\boldsymbol{Z})$ of Ranicki's symmetric Poincaré cobordism group $L^*(\boldsymbol{Z})$ and the isomorphism $L_{G,\chi}^*(\boldsymbol{Z}) \cong W_*^\chi(G,\boldsymbol{Z})$ (Theosem 1, §7) are the main steps to the proof of Theorem 2.

The paper is organized as follows. In §2, we discuss the normal structure of the map $\tilde{f} \times_\pi 1$. In §§ 3 and 4, we define the G-χ-equivariant Witt group $W_*^\chi(G,\boldsymbol{Z})$ and G-χ-equivariant symmetric Poincaré cobordism group $L_{G,\chi}^*(\boldsymbol{Z})$. In §§ 5 and 6, we define homomorphisms

$$\Phi : W_*^\chi(G,\boldsymbol{Z}) \longrightarrow L_{G,\chi}^*(\boldsymbol{Z}) \text{ and } \Psi : L_{G,\chi}^*(\boldsymbol{Z}) \longrightarrow W_*^\chi(G,\boldsymbol{Z}),$$

which will be shown to be the mutual inverses in §7. In §8, we define the algebraic pairing $W_*^\chi(G,\boldsymbol{Z}) \otimes L_*^s(\pi,w) \longrightarrow L_*^s(\pi,w\chi)$ which is mentioned above concerning the main theorem. In §§ 9 and 10, the main theorem is presented and proved.

2. Twisted product of a normal map with a G-χ-manifold. Let $f : M^n \longrightarrow N^n$ be a map of degree one between n-dimensional compact PL (or smooth) manifolds with $\pi_1(N^n) = \pi$. Let $F : \nu_M \to \xi$ be a bundle map covering f, where ν_M is the stable normal bundle of M^n and ξ is a bundle over N^n. The map $f : M^n \to N^n$ of degree one equipped with the bundle map data $F : \nu_M \to \xi$ is called a normal map of degree one. In case the boundaries ∂M^n and ∂N^n are not empty, we assume that the restriction of f to the boundaries, $f|\partial M^n : \partial M^n \to \partial N^n$, is a simple homotopy equivalence. The surgery obstruction $\sigma(f) \in L_n^s(\pi, w)$ of f to deforming f to a simple homotopy equivalence relative boundary is defined in [7], where $w : \pi \to \{\pm 1\}$ is the characteristic map of the orientation bundle of N^n.

Let L^m be an m-dimensional closed G-χ-manifold and $\tilde{f}\times_\pi 1 : \tilde{M}^n \times_\pi L^m \to \tilde{N}^n \times_\pi L^m$ the map of degree one defined in §1. We make $\tilde{f}\times_\pi 1$ a normal map of degree one as follows. For a manifold W, τW denotes its tangent bundle. Let $p_M : \tilde{M}^n \times_\pi L^m \to M^n$ and $p_N : \tilde{N}^n \times_\pi L^m \to N^n$ be tne projections to the first factors. Then $\tau(\tilde{M}^n \times_\pi L^m)$ is isomorphic to the Whitney sum $p_M^*(\tau M) \oplus \tilde{M} \times_\pi \tau L = p_M^*(\tau M) \oplus (\tilde{f}\times_\pi 1)^*(\tilde{N}\times_\pi \tau L)$. Let η be a bundle over $\tilde{N}^n \times_\pi L^m$ such that the Whitney sum $(\tilde{N}\times_\pi \tau L) \oplus \eta$ is trivial. Then the bundle $(\tilde{M}\times_\pi \tau L \oplus (\tilde{f}\times_\pi 1)^*\eta) = (\tilde{f}\times_\pi 1)^*(\tilde{N}\times_\pi \tau L \oplus \eta)$ is trivial. Hence the bundle $\tau(\tilde{M}\times_\pi L) \oplus (p_M^*(\nu_M) \oplus (\tilde{f}\times_\pi 1)^*\eta) = p_M^*(\tau M) \oplus \nu_M \times_\pi \tau L \oplus (\tilde{f}\times_\pi 1)^*\eta$ is trivial. Therefore we may take the bundle $p_M^*(\nu_M) \oplus (\tilde{f}\times_\pi 1)^*\eta$ as the stable normal bundle of $\tilde{M}^n \times_\pi L^m$. The bundle map F: $\nu_M \to \xi$ can be lifted to the bundle map $\tilde{F} : p_M^*(\nu_M) \to p_N^*(\xi)$ canonically, and we obtain the following bundle map,

$$\begin{array}{ccc} p_M^*(\nu_M) \oplus (\tilde{f}\times_\pi 1)^*\eta & \xrightarrow{\tilde{F}+(\tilde{f}\times_\pi 1)} & p_N^*(\xi) \oplus \eta \\ \uparrow & & \uparrow \\ \tilde{M}^n \times_\pi L^m & \xrightarrow{\tilde{f}\times_\pi 1} & \tilde{N}^n \times_\pi L^m \end{array}$$

where the vertical maps are the bundle projections. Since $p_M^*(\nu_M) \oplus (\tilde{f}\times_\pi 1)^*$ is the stable normal bundle of $\tilde{M}^n \times_\pi L^m$, the above diagram gives a structure of a normal map of degree one to the map $\tilde{f}\times_\pi 1$.

Now the bundle $p_N^*(\nu_N) \oplus \eta$ may be regarded as the stable normal bundle of $\tilde{N}^n \times_\pi L^m$. The difference bundle $((p_N^*(\nu_N) \oplus \eta) - (p_N^*(\xi) \oplus \eta))$ is the induced virtual bundle $p_N^*(\nu_N - \xi)$. This means that the normal invariant of $\tilde{f}\times_\pi 1$ endowed with the above bundle data is the image of the normal invariant of f endowed with F under the map induced by p_N, $p_N^* : [N^n, G/PL] \to [\tilde{N}^n \times_\pi L^m, G/PL]$(or $p_N^* : [N^n, G/O] \to [\tilde{N}^n \times_\pi L^m, G/O]$ in the smooth case).

3. G-χ-equivariant Witt group. We denote the ring of integers by $\boldsymbol{Z}$, the field of rational numbers by $\boldsymbol{Q}$ and the quotient map from $\boldsymbol{Q}$ to $\boldsymbol{Q}/\boldsymbol{Z}$ by ω. The dual module V^* of a finitely generated (abbreviated f.g.) free $\boldsymbol{Z}$-module V is defined by $V^* = \mathrm{Hom}_{\boldsymbol{Z}}(V, \boldsymbol{Z})$, and the dual module W^* of a f.g. torsion $\boldsymbol{Z}$-module W by $W^* = \mathrm{Hom}_{\boldsymbol{Z}}(W, \boldsymbol{Q}/\boldsymbol{Z})$. For a morphism $\beta : V_1 \to V_2$, $\beta^* : V_2^* \to V_1^*$ denotes the dual morphism of β, where V_1 and V_2 are both together either f.g. free $\boldsymbol{Z}$-modules or f.g. torsion $\boldsymbol{Z}$-modules.

Let G be a finite group with a homomorphism $\chi : G \to \{\pm 1\}$. Then the integral group ring $\boldsymbol{Z}[G]$ has the involution $-$ defined by $\overline{\sum n_g g} = \sum n_g \chi(g) g^{-1}$ for $n_g \in \boldsymbol{Z}$ and $g \in G$. A f.g. G-module V means a left

$\boldsymbol{Z}[G]$-module which is a f.g. $\boldsymbol{Z}$-module. For a f.g. $\boldsymbol{Z}$-free G-module V, the dual module V^* has a structure of a f.g. $\boldsymbol{Z}$-free G-module defined by $(xu)(v) = u(\bar{x}v)$ for $u \in V^*$, $v \in V$ and $x \in \boldsymbol{Z}[G]$. Similarly for a f.g. $\boldsymbol{Z}$-torsion G-module U, the dual module U^* is also a f.g. $\boldsymbol{Z}$-torsion G-module. For a f.g. $\boldsymbol{Z}$-free or $\boldsymbol{Z}$-torsion G-module V, the dual module of V^* is canonically identified with V as a G-module. A G-map between f.g. G-modules means a $\boldsymbol{Z}[G]$-map between them.

Now we define the G-χ-equivariant Witt group.

(1) Even dimensional case. The following defintioh is due to A. Dress [2] when $\chi = 1$. For $\varepsilon = \pm 1$, let us define an ε-symmetric G-χ-equivariant form (V, α) to be a f.g. $\boldsymbol{Z}$-free G-module V together with a G-isomorphism $\alpha: V \to V^*$ such that $\alpha = \varepsilon\alpha^*$. In other words, there is a non-singular bilinear pairing $\tilde{\alpha}: V \times V \to \boldsymbol{Z}$ such that $\tilde{\alpha}(gv, gv') = \chi(g)\tilde{\alpha}(v, v')$, $\tilde{\alpha}(v, v') = \varepsilon\tilde{\alpha}(v', v)$ and $ad\ \tilde{\alpha} = \alpha$, where $ad\ \tilde{\alpha}$ is the adjoint of $\tilde{\alpha}$, $ad\ \tilde{\alpha}(v)(v') = \tilde{\alpha}(v', v)$, for $v, v' \in V$ and $g \in G$. For any two such forms (V_1, α_1) and (V_2, α_2), one has an orthogonal sum $(V_1, \alpha_1) \oplus (V_2, \alpha_2)$ which is an ε-symmetric G-χ-equivariant form as well. A G-isomorphism $\beta: (V_1, \alpha_1) \to (V_2, \alpha_2)$ satisfying $\beta^*\alpha_2\beta = \alpha_1$ is an isomorphism in our setting. One may form the half-group of isomorphism classes of ε-symmetric G-χ-equivariant form with respect to orthogonal sum and its associated universal group $y_\varepsilon^\chi(G, \boldsymbol{Z})$. One now defines a G-lagrangean P of (V, α) to be a $\boldsymbol{Z}[G]$-submodule of V which coincides with its orthogonal complement $V^\perp = \ker(i^*\alpha: V \to P^*)$, where $i: P \to V$ is the inclusion. If (V, α) has a G-iagrangean, it is called a split form.

For each integer $k \geqq 0$, we define the G-χ-equivariant Witt group $W_{2k}^\chi(G, \boldsymbol{Z})$ to be the residue class group of $y_{(-1)^k}^\chi(G, \boldsymbol{Z})$ with respect to the subgroup generated by all split $(-1)^k$-symmetric G-χ-equivariant form in $y_{(-1)^k}^\chi(G, \boldsymbol{Z})$.

(2) Odd dimensional case. For $\varepsilon = \pm 1$, an ε-symmetric G-χ-equivariant linking form (S, λ) is a f.g. $\boldsymbol{Z}$-torsion G-module S together with a G-isomorphism $\lambda: S \to S^*$ such that $\lambda = \varepsilon\lambda^*$. This means that there is a non-singular bilinear pairing $\tilde{\lambda}: S \times S \to \boldsymbol{Q}/\boldsymbol{Z}$ such that $\tilde{\lambda}(gs, gs') = \chi(g)\tilde{\lambda}(s, s')$, $\tilde{\lambda}(s, s') = \varepsilon\tilde{\lambda}(s', s)$ and $ad\tilde{\lambda} = \lambda$ for $s, s' \in S$ and $g \in G$.

Definition. A *G-resolution of length* 1 of an ε-symmetric G-χ-equivariant linking form (S, λ) is a short exact sequence of G- modules

$$0 \longrightarrow U \xrightarrow{\beta} V \xrightarrow{\gamma} S \longrightarrow 0$$

together with a bilinear pairing $\Lambda: V \times V \longrightarrow \boldsymbol{Q}$ such that

(i) U and V are both f.g. $\boldsymbol{Z}$-free G-modules.

(ii) $\Lambda(gv, gv') = \chi(g)\Lambda(v, v')$, $\Lambda(\beta(u), v) \in \boldsymbol{Z}$ and $\Lambda(v, \beta(u)) \in \boldsymbol{Z}$, and

(iii) $\tilde{\lambda}(\gamma(v), \gamma(v')) = \omega(\Lambda(v, v')) \in \boldsymbol{Q}/\boldsymbol{Z}$,

for $v, v' \in V$, $u \in U$ and $g \in G$.

Lemma 3.1. *Let* (S, λ) *be an* ε*-symmetric* G-χ*-equivariant linking form. Then, there is a* G*-resolution of length* 1 *of* (S, λ).

Proof. There are a f.g. $\boldsymbol{Z}[G]$-free module V and a G-epimorphism $\gamma : V \to S$. Let U be the kernel of γ and $\beta : U \to V$ the inclusion. Since $V \otimes_Z V$ is a f.g. $\boldsymbol{Z}[G]$-free module by the diagonal G-action, there is a bilinear form $\Lambda : V \times V \to \boldsymbol{Q}$ such that $\omega(\Lambda(v, v')) = \tilde{\lambda}(\gamma(v), \gamma(v'))$ and $\Lambda(gv, gv') = \chi(g)\Lambda(v, v')$ for $v, v' \in V$. Since $\beta(U) = \ker \gamma$, $\Lambda(\beta(u), v) \in \boldsymbol{Z}$ and $\Lambda(v, \beta(u)) \in \boldsymbol{Z}$ for $u \in U$ and $v \in V$. q.e.d.

For any two ε-symmetric G-χ-equivariant linking forms (S_1, λ_1) and (S_2, λ_2), one has an orthogonal sum $(S_1, \lambda_1)+(S_2, \lambda_2)$ which is an ε-symmetric G-χ-equivariant linking form as well. Two ε-symmetric G-χ-equivariant linking forms (S_1, λ_1) and (S_2, λ_2) are called isomorphic if there is a G-isomorphism $\delta : S_1 \to S_2$ such that $\delta^*\lambda_2\delta = \lambda_1$. One may form the half group of isomorphism classes of ε-symmetric G-χ-equivariant linking forms with respect to orthogonal sums and its associated universal group $w_\varepsilon^\chi(G, \boldsymbol{Z})$.

Consider the following two conditions on an ε-symmetric G-χ-equivariant linking form (S, λ):

(a) There is a G-resolution of length 1, $0 \to U \xrightarrow{\beta} V \xrightarrow{\gamma} S \to 0$, such that the map $\Lambda_V : V \to U^*$, defined by $\Lambda_V(v)(u) = \Lambda(v, \beta(u))$ for $v \in V$ and $u \in U$, is an isomorphism.

(b) There is a $\boldsymbol{Z}[G]$-submodule Q of S which coincides with its orthogonal complement $S^\perp = \ker(i^*\lambda : S \to Q^*)$, where $i : Q \to S$ is the inclusion.

For each integer $k \geqq 0$, the G-χ-equivariant Witt group $W_{2k+1}^\chi(G, \boldsymbol{Z})$ is defined to be the residue class group of $w_{(-1)^{k+1}}^\chi(G, \boldsymbol{Z})$ with respect to the subgroup generated by those $(-1)^{k+1}$-symmetric G-χ-equivariant linking forms which satisfy either (a) or (b).

4. Equivariant symmetric algebraic Poincaré cobordism group. Let G be a finite group with a homomorphism $\chi : G \to \{\pm 1\}$. An n-dimensional G-chain complex $\{C, d_C\}$ is a chain complex

$$C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0$$

such that each C_r is a f.g. $\boldsymbol{Z}$-free G-module and each d_r is a G-homomorphism. A G-chain map $f: C \to D$ between two G-chain complexes is a chain map such that each $f_r: C_r \to D_r$ is a G-homomorphism. For a G-chain complex $\{C, d_C\}$, the cochain complex $\{C^*, d_{C^*}\}$ is a G-chain complex, where $C^r = (C_r)^*$ is a f.g. $\boldsymbol{Z}$-free G-module as in §3 for each r. The homology and cohomology groups of a G-chain complex are f.g. G-modules. Given two G-chain complexes $\{C, d_C\}$ and $\{D, d_D\}$, $\mathrm{Hom}_Z(C^*, D_*)$ is the G-chain complex such that $(\mathrm{Hom}_Z(C^*, D_*))_r = \sum_{p+q=r} \mathrm{Hom}_Z(C^p, D_q)$ with G-action defined by $(g\psi)(c) = g(\psi(\chi(g)g^{-1}c))$ for $g \in G$, $\psi \in \mathrm{Hom}_Z(C^p, D_q)$ and $c \in C^p$, and the differential is given by $d(\psi) = d_D\psi + (-1)^q \psi d_C^*$ for $\psi \in \mathrm{Hom}_Z(C^p, D_q)$. Let $\mathrm{Hom}_Z^G(C^*, D_*)$ be the subcomplex of $\mathrm{Hom}_Z(C^*, D_*)$ consisting of all the G-module maps, that is,

$$\mathrm{Hom}_Z^G(C^*, D_*) = \{\psi \in \mathrm{Hom}_Z(C^*, D_*) \mid g\psi = \psi \text{ for any } g \in G\}.$$

For a G-chain complex $\{C, d_C\}$, the generator $T \in \boldsymbol{Z}_2$ acts on $\mathrm{Hom}^G(C^*, C_*)$ by the transposition involution

$$T(\{\psi : C^p \to C_q\}_{p+q=r}) = \{(-1)^{pq}\psi^* : C^q \to C_p\}_{p+q=r}\}.$$

Let W be the free $\boldsymbol{Z}[\boldsymbol{Z}_2]$-module chain complex given by

$$W_r = \begin{cases} \boldsymbol{Z}[\boldsymbol{Z}_2] & (r \geqq 0) \\ 0 & (r < 0) \end{cases} \qquad d_r = \begin{cases} 1+(-1)^r T & (r > 0) \\ 0 & (r \leqq 0). \end{cases}$$

For a G-chain complex $\{C, d_C\}$, we define the equivariant $\boldsymbol{Z}_2$-hypercohomology group by $Q_G^n(C) = H_n(\mathrm{Hom}_{Z[Z_2]}(W, \mathrm{Hom}_Z^G(C^*, C_*))$. An element $\psi \in Q_G^n(C)$ is represented by a collection of G-chain maps $\{\psi_s \in \mathrm{Hom}_Z^G(C^{n-r+s}, C_r) \mid r, s \geqq 0\}$ such that

$$d_C\psi_s + (-1)^r\psi_s d_C^* + (-1)^{n+s-1}(\psi_{s-1} + (-1)^s T\psi_{s-1}) = 0 \quad (s \geqq 0, \ \psi_{-1} = 0).$$

An n-dimensional symmetric Poincaré G-complex (C, ψ) is an n-dimensional G-chain complex $\{C, d_C\}$ together with an element $\psi \in Q_G^n(C)$ such that the chain map $\psi_0 : C^{n-*} \to C_*$ is a chain equivalence (forgetting the G-actions) with $(C^{n-*})_r = C^{n-r}$, and $d_{C^{n-*}} = (-1)^r d_C^* : C^{n-r} \to C^{n-r+1}$.

A G-chain map $f: C \to D$ induces the chain map $\mathrm{Hom}^G(f) : \mathrm{Hom}_Z^G(C^*, C_*) \to \mathrm{Hom}_Z^G(D^*, D_*)$ defined by $\mathrm{Hom}^G(f)(\psi) = f\psi f^*$ for $\psi \in \mathrm{Hom}_Z^G(C^*, C_*)$. This is a $\boldsymbol{Z}[\boldsymbol{Z}_2]$-chain map since $T \in \boldsymbol{Z}_2$ acts as the transposition. Hence, it induces a homomorphism $f^{\%} : Q_G^n(C) \to Q_G^n(D)$. Let (C, ψ_C) and (D, ψ_D) be two n-dimensional symmetric Poincaré G-complexes. A G-isomorphism f from (C, ψ_C) to (D, ψ_D) is a G-chain isomorphism f such that $f^{\%}\psi_C = \psi_D$.

A G-chain map from (C, ψ_C) to (D, ψ_D) is called a G-quasi equivalence if $f^{\%}\psi_C = \psi_D$ and f induces an isomorphism of the homology groups in each dimension. Two n-dimensional symmetric Poincaré G-complexes (C, ψ_C) and (D, ψ_D) are called G-quasi equivalent if there is a sequence of n-dimensional svmmetric Poincaré G-complexes $(C_1, \psi_1), \cdots, (C_m, \psi_m)$ such that $(C_1, \psi_1) = (C, \psi_C), (C_m, \psi_m) = (D, \psi_D)$ and there is a G-quasi equivalence either $f_i : C_i \to C_{i+1}$ or $f_i : C_{i+1} \to C_i$ for each i $(i = 1, \cdots, m-1)$.

Let $f : C \to D$ be a G-chain map. Let $C(\mathrm{Hom}^G(f))$ be the algebraic mapping cone of the chain map $\mathrm{Hom}^G(f)$, that is, the chain complex defined by $(C(\mathrm{Hom}^G(f)))_r = \mathrm{Hom}_Z^G(D^*, D_*)_r \oplus \mathrm{Hom}_Z^G(C^*, C_*)_{r-1}$, and $d(\theta, \psi) = (d\theta + (-1)^{r-1}\mathrm{Hom}^G(f)(\psi), d\psi)$, where $\theta \in \mathrm{Hom}_Z^G(D^*, D_*)_r$ and $\psi \in \mathrm{Hom}_Z^G(C^*, C_*)_{r-1}$. This becomes a $Z[Z_2]$-chain complex in an obvious way. We define the $(n+1)$-dimensional relative Q_G-group by $Q_G^{n+1}(f) = H_{n+1}(\mathrm{Hom}_{Z[Z_2]}(W, C(\mathrm{Hom}^G(f)))$. An element $(\delta\psi, \psi) \in Q_G^{n+1}(f)$ is represented by a collection of G-chain map pairs

$$\{(\delta\psi, \psi)_s = (\delta\psi_s \in \mathrm{Hom}_Z^G(D^{n+1-r+s}, D_r) \oplus \mathrm{Hom}_Z^G(C^{n-r+s}, C_r)\},\ r, s \geqq 0,$$

satisfying the following two conditions with $s \geqq 0$, $\delta\psi_{-1} = 0$ and $\psi_{-1} = 0$:

(*) $d\delta\psi_s + (-1)^r\delta\psi_s d^* + (-1)^{n+s}(\delta\psi_{s-1} + (-1)^s T\delta\psi_{s-1}) + (-1)^n \mathrm{Hom}^G(f)(\psi_s) = 0$, and

(**) $d\psi_s + (-1)^r\psi_s d^* + (-1)^{n+s-1}(\psi_{s-1} + (-1)^s T\psi_{s-1}) = 0$.

An $(n+1)$-dimensional connected symmetric G-pair $(f : C \to D, (\delta\psi, \psi))$ is a G-chain map f from an n-dimensional Poincaré G-complex C to an $(n+1)$-dimensional G-chain complex D together with a class $(\delta\psi, \psi) \in Q_G^{n+1}(f)$ which satisfies the condition:

(***) $H_0(C(\Delta)) = 0, \quad \Delta : D^{n+1-*} \longrightarrow C(f)_*$,

where $C(f)$ is the algebraic mapping cone of the chain map f and $C(\Delta)$ is the algebraic mapping cone of the chain map $\Delta : D^{n+1-*} \to C(f)_*$ defined by $\Delta(c) = (\delta\psi_0(c), \psi_0 f^*(c)) \in D_* \oplus C_{*-1} = C(f)_*$ for $c \in D^{n+1-*}$. Remark that the condition $H_0(C(f)) = 0$ implies (***), since we have the exact sequence $\longrightarrow H_0(D^{n+1-*}) \longrightarrow H_0(C(f)) \longrightarrow H_0(C(\Delta)) \longrightarrow 0$.

For an $(n+1)$-dimensional connected symmetric G-pair $(f : C \to D, (\delta\psi, \psi))$, define the n-dimensional symmetric Poincaré G-complex (C', ψ') as follows:

$$d_{C'} = \begin{bmatrix} d_C & 0 & (-1)^{n+1}\psi_0 f^* \\ (-1)^r f & d_D & (-1)^r\delta\psi_0 \\ 0 & 0 & (-1)^r d_D^* \end{bmatrix}$$

$$C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \longrightarrow C'_r = C_{r-1} \oplus D_r \oplus D^{n-r+2}$$

$$\psi'_0 = \begin{bmatrix} d_C & 0 & 0 \\ (-1)^r f & (-1)^{n-r} T\delta\psi_1 & (-1)^{r(n-r)} \\ 0 & 1 & 0 \end{bmatrix}$$

$$C'^{n-r} = C^{n-r} \oplus D^{n-r+1} \oplus D_{r+1} \longrightarrow C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1}$$

$$\psi'_s = \begin{bmatrix} \psi_s & 0 & 0 \\ (-1)^{n-r} f T\psi_{s+1} & (-1)^{n-r+s} T\delta\psi_{s+1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C'^{n-r+s} = C^{n-r+s+1} \oplus D_{r-s+1} \to C'_r = C_r \oplus D_{r-1} \oplus D^{n-r+1} \quad (s \geqq 1)$$

We call (C', ψ') the n-dimensional symmetric Poincaré G-complex obtained from (C, ψ) by symmetric G-surgery on a connected $(n+1)$-dimensional symmetric G-pair $(f: C \to D, (\delta\psi, \psi))$. It may be verified that performing symmetric G-surgery using a different cycle representative of $(\delta\psi, \psi) \in Q_G^{n+1}(f)$ leads to an isomorphic symmetric Poincaré G-complex (C', ψ').

In the above situation, the following two conditions are equivalent,

(1) C' is acyclic,

(2) the relative homology class $(\delta\psi_0, \psi_0) \in H_{n+1}(C(\mathrm{Hom}^G(f))$ induces the isomorphisms $: H^r(D, C) = H^r(C(f)) \to H_{n+1-r}(D) (0 \leqq r \leqq n+1)$. In such a case, $(f: C \to D, (\delta\psi, \psi))$ is called an $(n+1)$-dimensional Poincaré G-pair with bonudary (C, ψ), and (C, ψ) is called G-null-cobordant.

The direct sum of n-dimensional symmetric Poincaré G-complexes (C, ψ) and (C', ψ') is an n-dimensional symmetric Poincaré G-complex $(C \oplus C', \psi \oplus \psi')$, where $(\psi \oplus \psi')_s = \psi_s \oplus \psi'_s : C^{n-r+s} \oplus C'^{n-r+s} \to C_r \oplus C'_r$ $(s, r \geqq 0)$.

Lemma 4.1. *Let (C', ψ') be an n-dimensional symmetric Poincaré G-complex obtained from an $(n+1)$-dimmensional connected symmetric G-pair $(f: C \to D, (\delta\psi, \psi))$. Then the direct sum $(C, \psi) \oplus (C', -\psi')$ is G-null-cobordant.*

Proof. Define an $(n+1)$-dimensional symmetric G-pair $(h: C \oplus C' \to D', (0, (0, \psi \oplus (-\psi'))$ by $D'_r = C_r \oplus D^{n-r+1}$,

$$d_D = \begin{bmatrix} d_C & (-1)^{n+1}\psi_0 f^* \\ 0 & (-1)^r d_D^* \end{bmatrix} : D_r \longrightarrow D_{r-1},$$

and

$$h(c) = (c, 0) \quad \text{for } c \in C$$
$$h(c') = (c_1, c_3) \text{ for } c' = (c_1, c_2, c_3) \in C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1}.$$

Let (C'', ψ'') be the symmetric Poincaré G-complex obtained from $(C, \psi) \oplus (C', -\psi')$ by symmetric G-surgery on the above G-pair. Then one can verify that C'' is acyclc. Hence the above G-pair is an $(n+1)$-dimensional Poincaré G-pair and $(C, \psi) \oplus (C', -\psi')$ is G-null-cobordant. q.e.d.

We form the half group of the isomorphism classes of n-dimensional symmetric Poincaré G-complexes with respect to orthogonal sums and its associated universal group $X^n_{G,\chi}(\boldsymbol{Z})$. Let $U^n_{G,\chi}(\boldsymbol{Z})$ be the subgroup of $X^n_{G,\chi}(\boldsymbol{Z})$ generated by the isomorphism classes of n-dimensional symmetric Poincaré G-complexes which are G-quasi equivalent to n-dimensional G-null-cobordant symmetric Poincaré G-complexes. Let us define the n-dimensional G-χ-equivariant symmetric algebraic cobordism group $L^n_{G,\chi}(\boldsymbol{Z})$ by $L^n_{G,\chi}(\boldsymbol{Z}) = X^n_{G,\chi}(\boldsymbol{Z})/U^n_{G,\chi}(\boldsymbol{Z})$. Not that by Lemma 4.1., if (C', ψ') is obtained from (C, ψ) by symmetric G-surgery, they represent the same element in $L^n_{G,\chi}(\boldsymbol{Z})$.

5. The map Φ. We define a homomorphism $\Phi: W^\chi_*(G, \boldsymbol{Z}) \to L^*_{G,\chi}(\boldsymbol{Z})$.

(1) Even dimensional case. Let (V, α) be a $(-1)^k$-symmetric G-χ-equivariant form as in §3 $(k \geqq 0)$. Define a $2k$-dimensional symmetric Poincaré G-complex (C_V, ψ_α) by

$$(C_V)_r = \begin{cases} V^* & (r = k) \\ 0 & (r \neq k) \end{cases}, \quad d_{C_V} = 0$$

and

$$(\psi_\alpha)_0 = \alpha : (C_V)^k = V \to (C_V)_k = V^*,$$
$$(\psi_\alpha)_s = 0 \qquad (s \geqq 1).$$

Lemma 5.1. *If (V, α) is split, then (C_V, ψ_α) is G-null-cobordant.*

Proof. Let P be a G-lagrangean of V, and $i: P \to V$ the inclusion. Let $(f: C \to D, (0, \psi_\alpha))$ be the $(2k+1)$-dimensional connected symmetric G-pair defined by

$$f = \begin{cases} i^* : (C_V)k = V^* \longrightarrow D_k = P^* \\ 0 : (C_V)_r = 0 \quad \longrightarrow D_r = 0 \quad (r \neq k). \end{cases}$$

The conditions (*) and (**) in §4 are verified, because the composition $i^*\alpha i$ is trivial and hence $\mathrm{Hom}^G(f) = 0$. And we have easily $H_0(C(f)) = 0$ and the condition (***) in §4. Let (C', ψ') be the $2k$-dimensional symmetric Poincaré G-complex obtained from (C_V, ψ_α) by symmetric G-surgery on the above G-pair. Then C' has the form

$$\cdots \to 0 \to P \xrightarrow{-\alpha i} V^* \xrightarrow{(-1)^k i^*} P^* \to 0 \to \cdots$$

and it is acyclic. q.e.d.

By the adove lemma, we obtain a well defined homomorphism Φ: $W_{2k}^{\chi}(G, \mathbf{Z}) \to L_{G,\chi}^{2k}(\mathbf{Z})$ by putting $\Phi((V, \alpha)) = [(C_V, \psi_\alpha)]$.

(2) Odd dimensional case. Let (S, λ) be a $(-1)^{k+1}$-symmetric equivariant linking form. Take a resolution $0 \to U \xrightarrow{\beta} V \xrightarrow{\gamma} S \to 0$ and a bilinear pairng $\Lambda: V \times V \to \mathbf{Q}$ satisfying (i), (ii) and (iii) of (2) in §3. For $v, v' \in V$, put $\mu(v, v') = \Lambda(v, v') - (-1)^{k+1}\Lambda(v', v)$. Then $\mu(v, v') \in \mathbf{Z}$ and $\mu(v, v') = (-1)^k \mu(v', v)$. Let $\Lambda_U: U \to V^*$ and $\Lambda_V: V \to U^*$ be the maps defined by $(\Lambda_U(u))(v) = \Lambda(\beta(u), v)$ and $(\Lambda_V(v))(u) = \Lambda(v, \beta(u))$ for $u \in U$ and $v \in V$ respectively.

Let (C, ψ) be the $(2k+1)$-dimensional symmetric Poincarê G-complex defined by

$$C_r = \begin{cases} V^* & (r = k+1) \\ U^* & (r = k) \\ 0 & (r \neq k, k+1) \end{cases} \qquad d_r = \begin{cases} \beta^* & (r = k+1) \\ 0 & (r \neq k+1) \end{cases}$$

$$\psi_0 = \begin{cases} \Lambda_U: U \to V^* & (r = k+1) \\ \Lambda_V: V \to U^* & (r = k) \\ 0 & (r \neq k, k+1) \end{cases}$$

$$\psi_1 = \begin{cases} ad\ \mu: V \to V^* & (r = k+1) \\ 0 & (r \neq k+1) \end{cases}$$

$$\psi_s = 0 \qquad (s \geqq 2).$$

Lemma 5.2. *If* (S, λ) *satisfies the condition* (b) *in* (2), §3, *then* (C, ψ) *is* G*-null-cobordant.*

Proof. Let Q be a $\mathbf{Z}[G]$-submodule of S as in (2), §3. Put $V_1 = \gamma^{-1}(Q)$ and $U_1 = \gamma^{-1}(V_1)$. Then there is a short exact sequence $0 \to U_1 \xrightarrow{\beta_1} V_1 \xrightarrow{\gamma_1} Q \to 0$, where $\beta_1 = \beta | U_1$ and $\gamma_1 = \gamma | V_1$ are the restrictions. Let $i: Q \to S$, $i_V: V_1 \to V$ and $i_U: U_1 \to U$ be the inclusions. Since $i^*\lambda i$: $Q \to Q^*$ is trivial, the pairing takes integral values on $V_1 \times V_1$. Denote this pairing by $\Lambda_1: V_1 \times V_1 \to \mathbf{Z}$. Define the adjoint map ad $\Lambda_1: V_1 \to V_1^*$ by $(ad\ \Lambda_1(v))(v') = \Lambda_1(v, v')(v, v' \in V_1)$. Let $(f: C \to D, (\delta\psi, \psi))$ be the $(2k+2)$-dimensional connected symmetric G-pair defined by

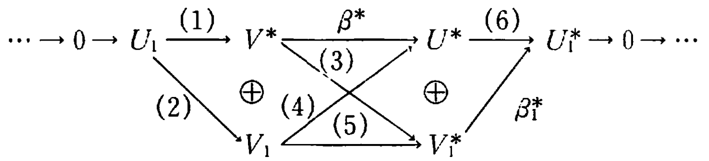
$$f = \begin{cases} i_V^*: C_{k+1} = V^* \to D_{k+1} = V_1^* & (r = k+1) \\ i_U^*: C_k = U^* \to D_k = U_1^* & (r = k) \\ 0: C_r = 0 \to D_r = 0 & (r \neq k, k+1) \end{cases}$$

$$d_D = \beta_1^* \ (r = k+1) \text{ and } 0 \ (r \neq k+1),$$

$$(\delta\psi)_0 = \begin{cases} ad\ \Lambda_1 : D^{k+1} = V_1 \to D_{k+1} = V_1^* & (r = k+1) \\ 0 & (r \neq k+1) \end{cases}$$

$$(\delta\psi)_s = 0 \quad (s \geqq 1).$$

Let (C', ψ') be the $(2k+1)$-dimensional symmetric Poincaré G-complex obtained from (C, ψ) by G-surgery on the above G-pair. Then C' has the form



where the maps are given as follows: $(1) = \Lambda_U i_U$, $(2) = (-1)^{k+2}\beta$, $(3) = (-1)^{k+1} i_V^*$, $(4) = (-1)^{k+1}\Lambda_V i_V$, $(5) = (-1)^{k+1} ad\ \Lambda_1$ and $(6) = (-1)^k\ i_U^*$. Since (S, λ) is a non-singular linking form, this chain complex is acyclic. q.e.d.

Lemma 5.3. *The class* $[(C, \psi)]$ *in* $L^{2k+1}_{G,\chi}(\boldsymbol{Z})$ *does not depend on the particular choice of a resolution of* (S, λ).

Proof. Let $0 \to U \xrightarrow{\beta} V \xrightarrow{\gamma} S \to 0$ and $0 \to U' \xrightarrow{\beta'} V' \xrightarrow{\gamma'} S \to 0$ be two resolutions of S with the associated bilinear pairings $\Lambda : V \times V \to \boldsymbol{Q}$ and $\Lambda' : V' \times V' \to Q$, respectively. Let (C, ψ) and (C', ψ') be the $(2k+1)$-dimensional symmetric Poincaré G-complexes corresponding to the two resolutions respectively constructed as before (Lemma 5.2.). The exact sequence $0 \to U \oplus U' \xrightarrow{\beta \oplus \beta'} V \oplus V' \xrightarrow{\gamma \oplus \gamma'} S \oplus S \longrightarrow 0$ and the bilinear pairing $\Lambda \oplus (-\Lambda') : V \times V' \oplus V \times V' \to \boldsymbol{Q}$ gives a resolution of $(S, \lambda) \oplus (S, -\lambda)$. The $(2k+1)$-dimensional symmetric Poincaré G-complex corresponding to this resolution is $(C, \psi) \oplus (C', -\psi')$ which is G-null-cobordant by Lemma 5.2, since $(S, \lambda) \oplus (S, -\lambda)$ has a $\boldsymbol{Z}[G]$-submodule $Q = \{(s, s) \in S \oplus S \mid s \in S\}$ satisfying (b) in (2), §3. This implies that $[(C, \psi)] = [(C', \psi')]$ in $L^{2k+1}_{G,\chi}(\boldsymbol{Z})$. q.e.d.

For a $(-1)^{k-1}$-symmetric G-χ-equivariant linking form (S, λ), the above $(2k+1)$-dimensional symmetric Poincaré G-complex (C, ψ) is denoted by (C_S, ψ_λ).

Lemma 5.4. *If* (S, λ) *satisfies the condition* (a) *in* (2), §3, *then*

(C_S, ψ_λ) *is G-null-cobordant.*

Proof. Let $0 \to U \xrightarrow{\beta} V \xrightarrow{\gamma} S \to 0$ be a resolution of S satisfying the condition (a). Let $(f : C_S \to D, (0, \psi_\lambda))$ be the $(2k+2)$-dimensional connected symmetric G-pair defined by

$$f = \begin{cases} 1 : (C_S)_{k+1} = V^* \longrightarrow D_{k+1} = V^* & (r = k+1) \\ 0 : (C_S)_k \;\; = U^* \longrightarrow D_k \;\; = 0 & (r = k) \\ 0 : (C_S)_r \;\; = 0 \;\;\; \longrightarrow D_r \;\; = 0 & (r \neq k, k+1). \end{cases}$$

Let (C', ψ') be the $(2k+1)$-dimensional symmetric Poincaré G-complex obtained from (C_S, ψ_λ) by G-surgery on the above G-pair. Then C' Has the form

$$\begin{array}{ccccccc} \cdots \longrightarrow 0 \longrightarrow & V & \xrightarrow{\quad \Lambda_V \quad} & U^* & \longrightarrow 0 \longrightarrow \cdots \\ & \oplus & \nearrow_{\beta^*} & \oplus & \\ & V^* & \xrightarrow{\quad (-1)^{k+1} \quad} & V^* & \end{array}$$

and it is acyclic. Hence (C_S, ψ_λ) is G-null-cobordant. q.e.d.

From Lemma 5.2., 5.3. and 5.4., we obtain a well-defined homomorphism $\Phi : W^{\chi}_{2k+1}(G, Z) \to L^{2k+1}_{G,\chi}(Z)$ by putting $\Phi((S, \lambda)) = [(C_S, \psi_\lambda)]$.

6. The map Ψ. We define a homomorphism $\Psi : L^{*}_{G,\chi}(Z) \to W^{\chi}_{*}(G, Z)$.

(1) Even dimensional case. Let (C, ψ) be a $2k$-dimensional Poincaré G-complex. Put $\hat{H}^k(C) = H^k(C)/\mathrm{Tor}$, where $H^k(C)$ is the k-th cohomology group of C and Tor is its torsion subgroup. Let $\alpha : \hat{H}^k(C) \to (\hat{H}^k(C))^*$ be the map defined by $\alpha(x)(y) = c'(\psi_0(c))$, where $x, y \in \hat{H}^k(C)$ and $x = [c]$, $y = [c']$ for $c, c' \in C^k$. Then, the pair $(\hat{H}^k(C), \alpha)$ defines a $(-1)^k$-symmetric G-χ-equivariant form.

Lemma 6.1. *The correspondence* $(C, \psi) \to (\hat{H}^k(C), \alpha)$ *induces a well-defined homomorphism* $\Psi : L^{2k}_{G,\chi}(Z) \to W^{\chi}_{2k}(G, Z)$.

Proof. Let us assume that (C, ψ) is G-null-cobordant. There is a $(2k+1)$-dimensional symmetric Poincaré G-pair with boundary (C, ψ), $(f : C \to D, (\delta\psi, \psi))$. By (*) in §4,

$$d\delta\psi_0 + (-1)^r \delta\psi_0 d^* = (-1)\mathrm{Hom}^G(f)(\psi_0) : D^{n-r} \to D_r \quad (0 \leqq r \leqq n+1),$$

the following diagram is commutative up to sign:

$$\begin{array}{ccccccc}
\cdots \to H^k(D) & \xrightarrow{f^*} & H^k(C) & \longrightarrow & H^{k+1}(D,C) & \to \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
\cdots \to H_{k+1}(D,C) & \longrightarrow & H_k(C) & \xrightarrow{f_*} & H_k(D) & \longrightarrow \cdots
\end{array}$$

where the two horizontal sequences are the exact sequences of the homology and cohomology groups of the pair $(f: C \to D)$, and the vertical maps are the isomorphisms induced by $\delta\psi_0$ and ψ_0. The standard argument of the Poincaré duality shows that $f^*(H^k(D))/\mathrm{Tor}$ is a G-lagrangean of $(\hat{H}^k(C), \alpha)$.

Finally it is clear that, if (C, ψ) and (C', ψ') are G-quasi equivalent, then $(\hat{H}^k(C), \alpha)$ and $(\hat{H}^k(C'), \alpha)$ are mutually isomorphic. q.e.d.

(2) Odd dimensional case. Let (C, ψ) be a $(2k+1)$-dimensional symmetric Poincaré G-complex. Put $Z^r = \ker\ (d^*: C^r \to C^{r+1})$, $V = \ker\ (q: Z^{k+1} \to H^{k+1}(C)/\mathrm{Tor})$ (q is the projection) and $U = C^k/Z^k$. Then d^* induces a G-homomorphism $\beta: U \to V$, and $V/\beta(U)$ is isomorphic to Tor $H^{k+1}(C)$. Define a bilinear pairing $\Lambda: V \times V \to \boldsymbol{Q}$ by $\Lambda(v, v') = (1/m)c(\psi_0(v))$, where $v, v' \in V$, $mv = d^*c$ (m : integer $\neq 0$, $c \in C^k = (C_k)^*$) and $\psi_0: C^{k+1} \to C_k$. If v or v' is in the image $d^*(C^k)$, then $\Lambda(v, v') \in \boldsymbol{Z}$. Hence $\Lambda(\beta(u), v) \in \boldsymbol{Z}$ and $\Lambda(v, \beta(u)) \in \boldsymbol{Z}$ for $u \in U$ and $v \in V$, and Λ induces a well-defined pairing $\tilde{\lambda}$: Tor $H^{k+1}(C) \times$ Tor $H^{k+1}(C) \to \boldsymbol{Q}/\boldsymbol{Z}$ by $\tilde{\lambda}(x, y) = \omega(\Lambda(v, v'))$ for $x = \gamma(v)$ and $y = \gamma(v')$, where $\gamma: V \to V/\beta(U) =$ Tor $H^{k+1}(C)$ is the projection and $\omega: \boldsymbol{Q} \to \boldsymbol{Q}/\boldsymbol{Z}$ is the quotient map. From the equation

$$\psi_0 + (-1)^k\psi_0^* = -(d\psi_1 + (-1)^k\psi_1 d^*): C^{k+1} \to C_k.$$

it follows that $\Lambda(v, v') + (-1)^k\Lambda(v', v) \in \boldsymbol{Z}$ for $v, v' \in V$. Hence $\tilde{\lambda}(x, y) = (-1)^{k+1}\tilde{\lambda}(y, x)$ for $x, y \in$ Tor $H^{k+1}(C)$. Since ψ_0 is a G-map, $\Lambda(gv, gv') = \chi(g)\Lambda(v, v')$ and $\tilde{\lambda}(gx, gy) = \chi(g)\tilde{\lambda}(x, y)$ for $g \in G$, $v, v' \in V$ and $x, y \in$ Tor $H^{k+1}(C)$. Let λ: Tor $H^{k+1}(C) \to$ (Tor $H^{k+1}(C))^*$ be the adjoint map of $\tilde{\lambda}$. Then λ is an isomorphism, since ψ_0 induces an isomorphism $H^{k+1}(C) \to H_k(C)$ and Tor $H_k(C) \to$ (Tor $H^{k+1}(C))^*$ by the universal coefficient theorem. Consequently the pair (Tor $H^{k+1}(C), \lambda$) is a $(-1)^{k+1}$-symmetric G-χ-equivariant linking form in the sense of §3.

Lemma 6.2. *The correspondence* $(C, \psi) \to$ (*Tor* $H^{k+1}(C), \lambda$) *induces a well-defined homomorphism* $\Psi: L^{2k+1}_{G,\chi}(\boldsymbol{Z}) \to W^{\chi}_{2k+1}(G, \boldsymbol{Z})$.

Proof. Let us assume that (C, ψ) is G-null-cobordant. There is a $(2k+2)$-dimensional symmetric Poincaré G-pair with boundary (C, ψ), $(f: C \to D, (\delta\psi, \psi))$. Consider the following commutative diagram,

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{Tor}\ H^{k+1}(D) & \xrightarrow{f^*} & \mathrm{Tor}\ H^{k+1}(C) & \longrightarrow & \mathrm{Tor}\ H^{k+2}(D,C) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to H^{k+1}(D,C) & \to & H^{k+1}(D) & \xrightarrow{f^*} & H^{k+1}(C\ \ ) & \longrightarrow & H^{k+2}(D,C) \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & H^{k+1}(D)/\mathrm{Tor} & \xrightarrow{f^*} & H^{k+1}(C)/\mathrm{Tor} & \longrightarrow & H^{k+2}(D,C) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}
$$

where the horizontal middle sequence is the cohomology exact sequence of the pair $(f: C \to D)$, and the upper vertical maps are the iuclusions and the lower ones are the quotient maps. Let Q be the image $f^*(\mathrm{Tor}\ H^{k+1}(D))$ and $j: Q \to \mathrm{Tor}\ H^{k+1}(C)$ the inclusion. The orthogonal complement of Q with respect to λ, $Q^\perp = \ker (j^*\lambda: \mathrm{Tor}\ H^{k+1}(C) \to Q^*)$, coincides with $f^*(H^{k+1}(D)) \cap \mathrm{Tor}\ H^{k+1}(C)$. Let S be the quotient module $Q^\perp/Q$. Let $\lambda_S: S \to S^* = \mathrm{Hom}_Z(S, \boldsymbol{Q}/\boldsymbol{Z})$ be the map defined by $(\lambda_S(s))(s') = \lambda(s)(s')$ for $s, s' \in S$. This is well-defined and a G-isomorphism by the duality. Put $V = \ker(f^*: H^{k+1}(D)/\mathrm{Tor} \to H^{k+1}(C)/\mathrm{Tor})$. There is an epimorphism $\gamma: V \to S$. Put $U = f^*(H^{k+1}(D)/\mathrm{Tor}) \cap V$. Let $\beta: U \to V$ be the inclusion. There is a short exact sequence of G-modules $0 \longrightarrow U \xrightarrow{\beta} V \xrightarrow{\gamma} S \longrightarrow 0$. The duality maps $(\psi_0)_*: H^{k+1}(D) \to H_{k+1}(D,C)$ and $(\psi_0)_*: H^{k+1}(D,C) \to H_{k+1}(D)$ induce G-isomorphisms $\Lambda_U: U \to V^*$ and $\Lambda_V: V \to U^*$ such that $\Lambda_V = (-1)^{k+1}\Lambda_U^*$. Since $\beta \otimes \boldsymbol{Q}: U \otimes \boldsymbol{Q} \to V \otimes \boldsymbol{Q}$ is an isomorphism, these define a bilinear pairing $\Lambda: V \times V \to \boldsymbol{Q}$ such that $\Lambda = (-1)^{k+1}\Lambda^*$ and $\Lambda(gv, gv') = \chi(g)\Lambda(v, v')$ for $v, v' \in V$ and $g \in G$. By the construction, $\omega(\Lambda(v, v')) = \lambda_S(\gamma(v), \gamma(v')) \in \boldsymbol{Q}/\boldsymbol{Z}$, for $v, v' \in V$. Hence these give a G-resolution of length 1 of (S, λ_S) satisfying the condition (a) in (2), §3. Now the direct sum $(S, -\lambda_S) \oplus (\mathrm{Tor}\ H^{k+1}(C), \lambda)$ has a $\boldsymbol{Z}[G]$-submodule $Q' = \{(\bar{x}, x) \in S \oplus \mathrm{Tor}\ H^{k+1}(C)\}$ ($x \in Q$ and $\bar{x}$ = the class of x in $Q^\perp/Q$) and it satisfies the condition (b) in (2), §3. Therefore $(\mathrm{Tor}\ H^{k+1}(C), \lambda)$ represents 0 in $W^{\chi}_{2k+1}(G, \boldsymbol{Z})$.

Finally it is clear that if (C, ψ) and (C', ψ') are G-quasi equivalent, then $(\mathrm{Tor}\ H^{k+1}(C), \lambda)$ and $(\mathrm{Tor}\ H^{k+1}(C'), \lambda')$ are isomorphic. q.e.d.

7. Φ and Ψ are mutual inverses. In the preceding two sections, we have constructed the two homomorphisms $\Phi: W^{\chi}_*(G, \boldsymbol{Z}) \to L^*_{G,\chi}(\boldsymbol{Z})$ and $\Psi: L^*_{G,\chi}(\boldsymbol{Z}) \to W^{\chi}_*(G, \boldsymbol{Z})$. By the definitions, $\Psi\Phi$ = the identity. In this section, we shall prove $\Phi\Psi$ = the identity.

(1) Even dimensional case. Let (C, ψ) be a $2k$-dimensional symmetric

Poincaré G-complex. Put $Z_k = \ker(d: C_k \to C_{k-1})$ $B = C_k/Z_k$. These are f.g. $\boldsymbol{Z}$-free G-modules. Let $p: C_k \to B$ be the projection. There is an injective G-homomorphism $\bar{d}: B \to C_{k-1}$ such that $d = \bar{d}p: C_k \to C_{k-1}$.

Lemma 7.1. *Let* $p_*: \mathrm{Hom}_Z^G(C^k, C_k) \to \mathrm{Hom}_Z^G(B^*, B)$ *be the map induced by* p. *Then* $p_*\psi_0 = 0$, *where* $\psi_0: C^k \to C_k$ *is the k-th component of* ψ_0.

Proof. Since coker $(d^*: (C_{k-1})^* \to B^*)$ is a torsion group, for each $c \in B^*$ there is an integer $m \neq 0$ and $c' \in C^{k-1}$ such that $mc = \bar{d}^*c'$ and so $mp^*c = d^*c'$. Then $m\psi_0(p^*c) = \psi_0(d^*c') = (-1)^{k+1}d(\psi_0(c'))$ is in Z_k. Hence $mp\psi_0(p^*c) = 0$, and so $p\psi_0(p^*c) = 0$. q.e.d.

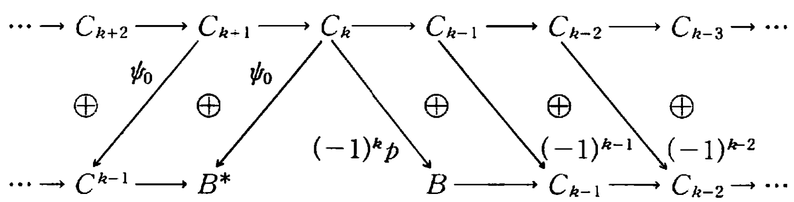
Consider the $(2k+1)$-dimensional connected symmetric G-pair $(f: C \to D, (0, \psi))$ defined by

$$f = \begin{cases} 0: C_r \to D_r = 0 & (r \geqq k+1) \\ p: C_k \to D_k = B & (r = k) \\ 1: C_r \to D_r = C_r & (r \leqq k-1) \end{cases}$$

where the differential of D, d_D, is given by $(d_D)_r = 0$ $(r = k+1)$, $(d_D)_k = \bar{d}$ and $(d_D)_r = (d_C)_r (r \leqq k-1)$. By the above lemma, this G-pair is well defined. Let (C', ψ') be the $2k$-dimensional symmetric Poincaré G-complex obtained from (C, ψ) by G-surgery on the above G-pair.

Lemma 7.2. $C'_k = C_k$ *and* $\psi'_0 = \psi_0: C'^k = C^k \to C'_k = C_k$. *The homology groups of* C' *are given by* $H_k(C') = H_k(C)/Tor$ *and* $H_r(C') = 0$ $(r \neq k)$.

Proof. The first assertion is clear from the definition. Now, C' has the form



and it follows that $H_r(C') = 0$ for $r \leqq k-1$, hence by duality $H_r(C') = 0$ for $r \geqq k+1$, and $H_k(C')$ is $\boldsymbol{Z}$-free. The k-th cycle group $Z'_k = \ker(d_{C'}: C'_k \to C'_{k-1})$ coincides with Z_k. Therefore $H_k(C')$ is isomorphic to some quotient module of $H_k(C)$. But it may be seen that $\psi_0(B^*) \subset \ker$ (the quotient map: $Z_k \to H_k(C)/\mathrm{Tor}$), hence $H_k(C')$ must be isomorphic to $H_k(C)/\mathrm{Tor}$. q.e.d.

Put $Z'^k = \ker(d^*_{C'} : C'_k \to C'^{k+1})$. Let $i : Z'^k \to C'^k$ be the inclusion, and $i^* : C'_k = (C'^k)^* \to (Z'^k)^*$ its dual map. There is a G-homomorphism $\bar{d}' : (Z'^k)^* \to C'_{k-1}$ such that $d_{C'} = \bar{d}' i^* : C'_k \to C'_{k-1}$. Let $q : Z'^k \to H^k(C') = H^k(C)/\mathrm{Tor}$ be the projection. Then the sequence $0 \to (H^k(C)/\mathrm{Tor})^* \xrightarrow{q^*} (Z'^k)^* \xrightarrow{\bar{d}'} C'_{k-1}$ is exact. Define the G-chain map $h : C' \to C''$ by

$$h = \begin{cases} 0 : C'_r \to C''_r = 0 & (r \geqq k+1) \\ i^* : C'_k = C_k \to C''_k = (Z'^k)^* & (r = k) \\ 1 : C'_r \to C''_r = C'_r & (r \leqq k-1) \end{cases}$$

where the differential of C'' is given by $(d_{C''})_r = 0\ (r \geqq k+1)$, $(d_{C''})_k = \bar{d}'$ and $(d_{C''})_r = (d_{C'})_r (r \leqq k-1)$. Then h induces the isomorphisms of the homology groups. Put $\psi'' = h^{\%}\psi' \in Q^{2k}_G(C'')$. Then $\psi''_0 = i^*\psi_0 i : C''^k = Z'^k \to C''_k = (Z'^k)^*$ and $\psi''_s = 0$ $(s \geqq 1)$. (C'', ψ'') is a $2k$-dimensional Poincaré G-complex and $h : (C', \psi') \to (C'', \psi'')$ is a G-quasi equivalence. Now, set $\Phi\Psi(C, \psi) = (\bar{C}, \bar{\psi})$. By definition, $\bar{C}_k = (H^k(C)/\mathrm{Tor})^* = (H^k(C))^*$, and $\bar{\psi}_0 = (\psi_0)_* : H^k(C) \to H_k(C)/\mathrm{Tor} = (H^k(C))^*$ and $\bar{\psi}_s = 0$ for $s \geqq 1$. Define the G-chain map $e : \bar{C} \to C''$ by

$$e = \begin{cases} 0 : \bar{C}_r = 0 \to C''_r = 0 & (r \geqq k+1) \\ q^* : \bar{C}_k = (H^k(C))^* \to C''_k = (Z'^k)^* & (r = k) \\ 0 : \bar{C}_r = 0 \to C''_r & (r \leqq k-1). \end{cases}$$

Then e induces the isomorphisms of the homology groups.

Lemma 7.3. $e^{\%}\bar{\psi} = \psi''$.

Proof. It suffices to show that the right hand square of the following diagram is commutative:

$$\begin{array}{ccccc} C^k & \xleftarrow{\ i\ } & Z'^k & \xrightarrow{\ q\ } & H^k(C) \\ \downarrow \psi_0 & & \psi''_0 = \downarrow i^*\psi_0 i & \cong & \downarrow \psi_0. \\ C_k & \xrightarrow{\ i^*\ } & (Z'_k)^* & \xleftarrow{\ q^*\ } & (H^k(C))^*. \end{array}$$

Clearly the left hand square is commutative. For each $b \in \ker q \subset Z'^k$, therer exist an integer $m \neq 0$ and $c \in C'^{k-1}$ such that $d^*c = mb$. Then, for each $a \in Z'^k$, $ma(\psi''_0(b)) = a(\psi''_0(d^*c)) = (-1)^{k+1}a(d\psi''_0(c)) = (-1)^{k+1}(d^*a)(\psi''_0(c)) = 0$. Hence $a(\psi''_0(b)) = 0$, and so $\psi''_0(\ker q) = 0$. Similarly, $mb\,(\psi''_0\,(a)) = (d^*c)\,(\psi''_0\,(a)) = c\,(d\psi''_0\,(a)) = (-1)^k c\,(\psi''_0\,(d^*a)) = 0$. This implies $b(\psi''_0(a)) = 0$, and so $\psi''_0(Z'^k) \subset \mathrm{im}\ q^*$, since the sequence $(H^k(C))^* \xrightarrow{q^*} (Z'^k)^* \to (\ker q)^* \to 0$ is exact. Consequently, $\psi''_0 = q^*\tilde{\psi}q$ for some

$\bar{\psi} : H^k(C) \to (H^k(C))^*$. Since ψ_0'' induces the same map as ψ_0 on $H^k(C)$, we get $\bar{\psi} = \bar{\psi}_0$. q.e.d.

By the above lemma, e gives a G-quasi equivalence from $\Phi\Psi(C, \psi)$ to (C'', ψ''), and this proves that $\Phi\Psi =$ the dientity.

(2) Odd dimensional case. Let (C, ψ) be a $(2k+1)$-dimensional symmetric Poincaré G-complex. Put $Z_k = \ker(d : C_k \to C_{k-1})$, $R = \ker$ (the quotient map: $Z_k \to H_k(C)/\mathrm{Tor}$), and $K = C_k/R$. K is a f.g. $\boldsymbol{Z}$-free G-module. Let $p : C_k \to K$ be the quotient map. There is a G-homomorphism $\bar{d} : K \to C_{k-1}$ such that $d = \bar{d}p : C_k \to C_{k-1}$. Let $(f : C \to D, (0, \psi))$ be the $(2k+2)$-dimensional connected symmetric G-pair defined by

$$f = \begin{cases} 0 : C_r \to D_r = 0 & (r \geqq k+1) \\ p : C_k \to D_k = K & (r = k) \\ 1 : C_r \to D_r = C_r & (r \leqq k-1) \end{cases}$$

where the differential of D, d_D, is given by $(d_D)_r = 0\ (r \geqq k+1)$, $(d_D)_k = \bar{d}$, and $(d_D)_r = (d_C)_r\ (r \leqq k-1)$. Let (C', ψ') be the $(2k+1)$-dimensional symmetric Poincaré G-complex obtaind from (C, ψ) by G-surgery on the above G-pair.

Lemma 7.4. *$C'_k = C_k$, $C'_{k+1} = C_{k+1}$, and $\psi'_0 = \psi_0 : C^{k+1} \to C_k$ and $\psi'_0 = \psi_0 : C^k \to C_{k+1}$. The homology groups of C' are given by $H_k(C') =$ Tor $H_k(C)$ and $H_r(C') = 0$ $(r \neq k)$.*

Proof. The first assertion is clear from the definition. Now, C' has the form

$$\begin{array}{ccccccccccccc}
\cdots \to & C_{k+3} & \longrightarrow & C_{k+2} & \longrightarrow & C_{k+1} & \longrightarrow & C_k & \longrightarrow & C_{k-1} & \longrightarrow & C_{k-2} & \to \cdots \\
& \oplus & \nearrow_{\psi_0} & \oplus & \nearrow_{\psi_0} & & & & \searrow^{(-1)^k p} & & \searrow^{(-1)^{k-1}} & \oplus & \\
\cdots \to & C^{k-1} & \longrightarrow & K^* & & & & & & K & \longrightarrow & C_{k-1} & \to \cdots
\end{array}$$

and it follows that $H_r(C') = 0$ for $r \leqq k-1$. By duality, $H_r(C') = 0$ for $r \geqq k+2$. Now, $\ker\ (d_{C'} : C'_k \to C'_{k-1}) = \ker$ (the quotient map: $Z_k \to H_k(C)/\mathrm{Tor}$), so that $H_k(C')$ is isomorphic to Tor $H_k(C)$, and $H_{k+1}(C') = 0$. q.e.d.

Put $Z'^r = \ker\ (d^*_{C'} : C'^r \to C'^{r+1})$, and $V = Z'^{k+1}$. Let $i : V \to C'^{k+1}$ be the inclusion, and $i^* : C'_{k+1} = (C'^{k+1}_k)^* \to V^*$ its dual map. There is an injective G-homomorphism $\bar{d}' : V^* \to C'_k$ such that $d_{C'} = \bar{d}' i^* : C'_{k+1} \to C'_k$. Let $h : C' \to C''$ be the G-chain map defined by

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$$h = \begin{cases} 0 : C'_r \to C''_r = 0 & (r \geqq k+2) \\ i^* : C'_{k+1} \to C''_{k+1} = V^* & (r = k+1) \\ 1 : C'_r \to C''_r = C'_r & (r \leqq k) \end{cases}$$

where the differential of C'', $d_{C''}$, is given by $(d_{C''})_r = 0$ $(r \geqq k+2)$, $(d_{C''})_{k+1} = \bar{d}'$, and $(d_{C''})_r = (d_{C'})$ $(r \leqq k)$. Then h induces the isomorphism of the homology groups. Put $\psi'' = h^{\%}\psi' \in Q_C^{2k-1}(C'')$. Then $\psi''_0 = i^*\psi_0 : C''^k = C^k \to C''_{k+1} = V^*$ and $\psi''_0 = \psi_0 i : C''^{k+1} = V \to C''_k = C_k$, and $\psi''_s = 0$ for $s \geqq 1$. (C'', ψ'') is a $(2k+1)$-dimensional symmetric Poincaré G-complex and $h : (C', \psi') \to (C'', \psi'')$ is a G-quasi equivalence.

Note here that $C''_k = C'_k = C_k$. Put $U = C^k/\boldsymbol{Z}^k$. U is a f.g. Z-free G-module. Let $q : C^k \to U$ be the quotient map, and $q^* : U^* \to (C^k)^* = C_k$ its dual map. There is an injective G-homomorphism $\beta : V^* \to U^*$ such that $\bar{d}' = q^*\beta : V^* \to C'_k = C_k$ and $U^*/\beta(V^*) = \mathrm{Tor}\ H_k(C)$.

Lemma 7.5. *There is a G-homomorphism $\bar{\psi}_0 : U \to V^*$ such that $\psi''_0 = \bar{\psi}_0 q : C'^k = C^k \to C''_{k+1} = V^*$.*

Proof. Since $H^{k-1}(C'') = H^{k+1}(C')$ is a torsion group, for each $v \in V$ there is an integer $m \neq 0$ and $c \in C''^k = C^k$ such that $mv = d^*c$. For each $z \in Z^k = \ker q$, $mv(\psi''_0(z)) = (d^*c)(\psi''_0(z)) = c(d\psi''_0(z)) = (-1)^{k+1}c(\psi''_0(d^*z)) = 0$. Hence $v(\psi''_0(z)) = 0$. This proves the lemma. q.e.d.

Since $\psi''_s = 0$ for $s \geqq 1$, $T\psi''_0 = \psi''_0$, where T is the transposition involution. This implies that $\psi''_0 = \psi_0 i : C''^{k+1} = V \to C''_k = C_k$ is equal to $(i^*\psi_0)^*$. Hence $\psi''_0 = \psi_0 i = q^*\bar{\psi}_0^*$, where $\bar{\psi}_0^* : V \to U^*$ is the dual map of $\bar{\psi}_0$. Define the G-chain map $e : \bar{C} \to C''$ by

$$e = \begin{cases} 0 : C_r = 0 \to C''_r = 0 & (r \geqq k+2) \\ 1 : \bar{C}_{k+1} = V^* \to C''_{k+1} = V^* & (r = k+1) \\ q^* : \bar{C}_k = U^* \to C''_k = C_k & (r = k) \\ 0 : \bar{C}_r = 0 \to C''_r = C_r & (r \leqq k-1) \end{cases}$$

where the differential of $\bar{C}$ is given by $(d_{\bar{C}})_{k+1} = \beta$ and $(d_{\bar{C}})_r = 0$ $(r = k+1)$. Then e induces the isomorphisms of the homology groups. Define $\bar{\psi} \in Q_{\bar{C}}^{2k+1}(\bar{C})$ by $(\bar{\psi})_0 = \{\bar{\psi}_0 : U \to V^*,\ \bar{\psi}_0^* : V \to U^*\}$ and $(\bar{\psi})_s = 0$ for $s \geqq 1$. Then $(\bar{C}, \bar{\psi})$ is a $(2k+1)$-dimensional symmetric Poincaré G-complex. By Lemma 7.5 and the above remark, it may be seen that $e^{\%}\bar{\psi} = \psi''$. Hence e induces a G-qnasi equivalence from $(\bar{C}, \bar{\psi})$ to (C'', ψ''). Now $(\bar{C}, \bar{\psi})$ represents the class $\Phi\Psi(C, \psi)$ in $L_{G,Z}^{2k+1}(\boldsymbol{Z})$. This proves that $\Phi\Psi =$ the identity.

Consequently, we obtain the following

Theorem 1. *The maps* $\Phi : W_*^\natural(G, Z) \to L_{G,x}^*(Z)$ *and* $\Psi : L_{G,x}^*(Z) \to W_*^\natural(G, Z)$ *are isomorphisms.*

8. The action of $W_*^\natural(G, Z)$ on Wall groups. First we describe the Wall group $L_*^s(\pi, w)$ by Ranicki's quadratic Poincaré complexes [5]. Let π be a multiplicative group with a homomorphism $w : \pi \to \{\pm 1\}$. Then the integral group ring $Z[\pi]$ has an involution $^-$ defined by $\sum n_h h \to \sum n_h w(h) h^{-1}$, where $n_h \in Z$ and $h \in \pi$. An n-dimensional based f.g. free $Z[\pi]$ chain complex is a chain complex

$$C_* : C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots \longrightarrow C_1 \xrightarrow{d} C_0$$

such that each C_r is a based f.g. free $Z[\pi]$-module and each d is a $Z[\pi]$-homomorphism. The cochain group C^* of C_* is a based f.g. free $Z[\pi]$ chain complex

$$C^* : C^0 \xrightarrow{d^*} C^1 \xrightarrow{d^*} \cdots \longrightarrow C^{n-1} \xrightarrow{d^*} C^n$$

such that $C^r = \mathrm{Hom}_{Z[\pi]}(C_r, Z[\pi])$ $(1 \leqq r \leqq n)$ and d^* is the dual homomorphism of d, where C_r is a $Z[\pi]$-module by the action $(sf)(c) = f(\bar{s}c)$ $(s \in Z[\pi],\ c \in C_r,\ f \in C^r)$ and it is based by the dual base of C_r. The generator $T \in Z_2$ acts on $\mathrm{Hom}_{Z[\pi]}(C^*, C_*)$ by

$$\begin{aligned} T : \mathrm{Hom}_{Z[\pi]}(C^p, C_q) &\to \mathrm{Hom}_{Z[\pi]}(C^q, C_p) \\ f &\longmapsto (-1)^{pq} f^* \end{aligned}$$

For a based f.g. free $Z[\pi]$-module chain complex C, define the Z_2-hyperhomology group $Q_n(C) = H_n(W \otimes_{Z[Z_2]} \mathrm{Hom}_{Z[\pi]}(C^*, C_*))$, where W is the free $Z[Z_2]$-resolution of Z. An element of $Q_n(C)$ is an equivalence class of collection

$$\{\theta_s \in \mathrm{Hom}_{Z[\pi]}(C^{n-r-s}, C_r) \mid r \geqq 0,\ s \geqq 0\}$$

such that

$$d\theta_s + (-1)^r \theta_s d^* + (-1)^{n-s-1}(\theta_{s+1} + (-1)^{s+1} T\theta_{s+1}) = 0 \quad (s \geqq 0).$$

An n-dimensional quadratic Poincaré complex over $Z[\pi]$ (C, θ) is an n-dimensional based f.g. free $Z[\pi]$-module chain complex together with an element $\theta \in Q_n(C)$ such that the cycle $(1+T)\theta_0 \in \{\mathrm{Hom}_{Z[\pi]}(C^{n-r}, C_r),\ r \geqq 0\}$ gives a simple chain equivalence $C^{n-*} \to C_*$. The quadratic L-groups $L_n(\pi)$ $(n \geqq 0)$ are defined to be the algebraic Poincaré cobordism groups of n-dimensional quadratic Poincaré complexes over $Z[\pi]$. The quadratic L-groups are 4-periodic, $L_n(\pi) = L_{n+4}(\pi)$, being equal to the Wall sergery

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obstruction groups $L_n^s(\pi, \ w)$.

Let (G, χ) be a pair of a finite group G and a homomorphism $\chi : G \to \{\pm 1\}$. Let $\phi : \pi \to G$ be an epimorphism. We denote the composite map $\chi\phi$ by χ. If M is an f.g. $\boldsymbol{Z}$-free G-module, then M is also a f.g. $\boldsymbol{Z}$-free $\boldsymbol{Z}[\pi]$-module by $hu = \phi(h)u$ $(h \in \pi,\ u \in M)$.

Lemma 8.1. *Let M be a f.g. $\boldsymbol{Z}$-free G-module, and P a f.g. free $\boldsymbol{Z}[\pi]$-module. Then $H \otimes_Z P$ with the diagoual π-module structure, $h(\sum u \otimes b) = \sum hu \otimes hb$ $(h \in \pi,\ u \in M,\ \ b \in P)$ is a f.g. free $\boldsymbol{Z}[\pi]$-module.*

Proof. Let $\{u_1, \cdots, u_s\}$ be a base over $\boldsymbol{Z}$ of M, and $\{b_1, \cdots, b_t\}$ a base over $\boldsymbol{Z}[\pi]$ of P. Then $\{u_i \otimes b_j,\ 1 \leqq i \leqq s,\ 1 \leqq j \leqq t\}$ form a base over $\boldsymbol{Z}[\pi]$ of $M \otimes_Z P$. q.e.d.

Let (C, ψ) be an m-dimensional symmetric Poincaré G-complex. Let (D, θ) be an n-dimensional quadratic Poincaré complex over $\boldsymbol{Z}[\pi]$. Consider the chain complex $C \otimes_Z D$,

$$(C\otimes_Z D)_r = \sum_k C_k \otimes_Z D_{r-k},\ d(x\otimes y) = x\otimes dy + (-1)^k dx \otimes y (x\otimes y \in C_k \otimes_Z D_{r-k}).$$

We consider $C \otimes_Z D$ as a $\boldsymbol{Z}[\pi]$-module chain complex by diagonal π-action. Then $C \otimes_Z D$ is a f.g. free $\boldsymbol{Z}[\pi]$ chain complex by Lemma 8.1. Now D_{r-k} has a preferred base over $\boldsymbol{Z}[\pi]$, $\{b_1, \cdots, b_t\}$. Let $\{u_1, \cdots, u_s\}$ be a base over $\boldsymbol{Z}$ of C_k. Then $\{u_i \otimes b_j,\ 1 \leqq i \leqq s,\ 1 \leqq j \leqq t\}$ form a base over $\boldsymbol{Z}[\pi]$ of $C_k \otimes_Z D_{r-k}$. We take this base as a preferred base of $C_k \otimes_Z D_{r-k}$. The simple equivalence class of $C \otimes_Z D$ endowed with these bases does not depend on the particular choice of the base over $\boldsymbol{Z}$ of C_k, for $Wh(\boldsymbol{Z}) = 0$.

Now, let

$$\{\psi_s \in \mathrm{Hom}_Z^G(C^{m-r+s},\ C_r) \mid r \geqq 0,\ s \geqq 0\} \text{ and}$$
$$\{\theta_s \in \mathrm{Hom}_{Z[\pi]}(D^{n-r-s},\ D_r) \mid r \geqq 0,\ s \geqq 0\}$$

be collections of chains resenting ψ and θ, respectively. Put

$$(\psi \otimes \theta)_s = \sum_{r=0}^{\infty} (-1)^{(m+r)s} \psi_r \otimes T^r \theta_{s+r} \quad (s \geqq 0)$$

where $\psi_r \otimes T^r \theta_{s+r} \in \mathrm{Hom}_Z^G(C^*,\ C_*)_{m+r} \otimes \mathrm{Hom}_{Z[\pi]}(D^*,\ D_*)_{n-s-r}$. There is a natural inclusion

$$\kappa : \mathrm{Hom}_Z^G(C^*,\ C_*)_{m+r} \otimes \mathrm{Hom}_{Z[\pi]}((D^*,\ D_*)_{n-s-r}$$
$$\to \mathrm{Hom}_{Z[\pi]}(C^* \otimes_Z D^*,\ C_* \otimes_Z D_*)_{m-n-s}$$

defined by $(\kappa(u \otimes v))(c \otimes d) = u(c) \otimes v(d)$ $(u \in \mathrm{Hom}_Z^G(C^{m+r-k},\ \ C_k),$

$v \in \mathrm{Hom}_{Z[\pi]}(D^{n-s-r-j}, D^j)$, $c \in C^{m-r-k}$, $d \in D^{n-s-r-j}$), and π acts on $C^* \otimes_Z D^*$ by $(h(u \otimes v))(c \otimes d) = (u \otimes v)(\chi(h)h^{-1}c \otimes w(h)h^{-1}d) = (u \otimes v)(\chi(h)w(h)h^{-1}(c \otimes d)) = (u \otimes v)(w\chi(h)h^{-1}(c \otimes d))$ $(h \in \pi)$. The collection of chains $\{\kappa((\psi \otimes \theta)_s), s \geqq 0\}$ represents an element of $Q_{m+n}(C \otimes_Z D)$, where the involution of $Z[\pi]$ is given by $\sum n_h h \to \sum w\chi(h) n_h h^{-1}$ $(n_h \in Z, h \in \pi)$. Hence we obtain an $(m+n)$-dimensional quadratic Poincaré complex over $Z[\pi]$, $(C \otimes_Z D, \kappa(\psi \otimes \theta))$. We denote this quadratic Poincaré complex by $(C, \psi) \otimes (D, \theta)$.

Lemma 8.2. *Let $f:(C, \psi) \to (C', \psi')$ be a G-quasi equivalence between m-dimensional symmetric Poincaré G-complexes. Let (D, θ) be an n-dimensional quadratic Poincaré complex over $Z[\pi]$. Then f induces a simple chain equivalence over $Z[\pi]$, $f \otimes 1$, from $(C, \psi) \otimes (D, \theta)$ to $(C', \psi') \times (D, \theta)$.*

Proof. Since $f: C \to D$ induces the isomorphisms of the homology groups, $f \otimes 1 : C \otimes_Z D \to C' \otimes_Z D$ also induces the isomorphisms of the homology groups by Künneth formula. Since the chain complexes $C \otimes_Z D$ and $C' \otimes_Z D$ are both f.g. free $Z[\pi]$-module chain complexes, $f \otimes 1$ is a chain equivalence over $Z[\pi]$ which is clearly simple, as $Wh(Z) = 0$. q.e.d.

Now it can be seen that if an m-dimensional symmetric Poincaré G-complex (C, ψ) is a boundary of an $(m+1)$-dimensional symmetric Poincaré G-pair, then for any n-dimensional quadratic Poincaré complex over $Z[\pi]$, (D, θ), $(C, \psi) \otimes (D, \theta)$ is a Poincaré boundary of an $(m+n+1)$-dimensional quadratic Poincaré pair over $Z[\pi]$.

Consequently, the above construction gives a pairing $L^m_{G,\chi}(Z) \otimes L^s_n \pi, w) \to L^s_{m+n}(\pi, w\chi)$. By the isomorphism $\Phi: W^\chi_*(G, Z) \to L^*_{G,\chi}(Z)$, we obtain a pairing

$$W^\chi_m(G, Z) \otimes L^s_n(\pi, w) \to L^s_{m+n}(\pi, w\chi).$$

9. Main theorem. We return to the surgery problem in §1. Let L^m be an m-dimensional closed PL (or smooth) G-χ-manifold. By equivariant triangulation theorem [3], we can assume that there is a triangulation $t(L^m) = \{\tau \mid \tau : \text{open simplex}\}$ of L^m such that (1) for each $\tau \in t(L^m)$ and $g \in G$, $g\tau \in t(L^m)$ and (2) if $g\tau = \tau$, then g fixes each point of τ. We choose and fix such a triangulation $t(L^m)$. Let $\{C_*(L), \partial_*\}$ be the G-chain complex defined by this triangulation. The manifold $L^m \times L^m$ has the G-CW structure $\{\tau \times \upsilon \mid \tau, \upsilon \in t(L^m)\}$. Let the generator $T \in Z_2$ acts on

$L^m \times L^m$ by $T(x, y) = (y, x)((x, y) \in L^m \times L^m)$. Then $L^m \times L^m$ is a $\boldsymbol{Z}_2 \times G$-manifold and the above CW structure is a $\boldsymbol{Z}_2 \times G$-$CW$ structure. Let S^∞ be the infinite dimensional sphere on which $\boldsymbol{Z}_2$ acts by the antipodal involution. Let $\{e_s, Te_s \mid s = 1, 2, \cdots\}$ be the standard $\boldsymbol{Z}_2$-CW structure of S^∞, where dim $e_s = s$ and Te_s is e_s transformed by T. The manifold $S^\infty \times L^m$ has the $\boldsymbol{Z}_2 \times G$-action defined by $(T, g)(a, x) = (Ta, gx)(T \in \boldsymbol{Z}_2, g \in G, (a, x) \in S^\infty \times L^m)$, and $\{e_s \times \tau, Te_s \times \tau \mid s = 1, 2, \cdots, \tau \in t(L^m)\}$ gives a $\boldsymbol{Z}_2 \times G$-$CW$ structure of $S^\infty \times L^m$. Let $F' : S^\infty \times L^m \to L^m \times L^m$ be the map defincd by $F'(a, x) = (x, x)((a, x) \in S^\infty \times L^m)$. F' is a $\boldsymbol{Z}_2 \times G$-equivariant map. Let $F : S^\infty \times L^m \to L^m \times L^m$ be a $\boldsymbol{Z}_2 \times G$-equivariant cellular approximation of F' with respect to the above $\boldsymbol{Z}_2 \times G$-$CW$ structures of $S^\infty \times L^m$ and $L^m \times L^m$. Then F induces the $\boldsymbol{Z}_2 \times G$-equivariant chain map $F_\sharp : W_* \otimes C_*(L^m) \to C_*(L^m) \otimes C_*(L^m)$, where W_* is the chain complex $C_*(S^\infty)$ defined by the above standard CW structure. W_* is a free $\boldsymbol{Z}[\boldsymbol{Z}_2]$-resolution of $\boldsymbol{Z}$. Now, the chain complex $C_*(L^m) \otimes C_*(L^m)$ is identified with the chain complex $\mathrm{Hom}_Z(C^*(L^m), C_*(L^m))$ by the map $c \otimes d \to (u \to u(c)d)$ $(c, d \in C_*(L^m), u \in C^*(L^m))$. Hence $F_\sharp$ induces the map denoted by the same letter,

$$F_\sharp : W_* \otimes C_*(L^m) \to \mathrm{Hom}_Z(C^*(L^m), C_*(L^m)).$$

Let $[L^m] \in C_m(L^m)$ be the fundamental cycle of L^m. Then $g[L^m] = \chi(g)[L^m]$ for $g \in G$. Hence, for each $s \geqq 0$, $F_\sharp(e_s \otimes [L^m]) = \psi_s$ is an element of $\mathrm{Hom}_Z^G(C^*(L^m), C_*(L^m))_{m+s}$, where G acts on $C^*(L^m)$ by $(gu)(c) = u(\chi(g)g^{-1}c)$ $(g \in G, u \in C^*(L^m), c \in C_*(L^m))$. The pair $(C_*(L^m), \psi = \{\psi_s\})$ is an m-dimensional symmetric poincaré G-complex.

Let $\Omega_*^\chi(G)$ be the equivariant PL (resp. smooth) bordism group of closed PL (resp. smooth) G-χ-manifolds. Then the above construction induces a well-defined homomorphism $\rho' : \Omega_m^\chi(G) \to L_{G,\chi}^m(\boldsymbol{Z})$. By the isomorphism $\Psi : L_{G,\chi}^*(\boldsymbol{Z}) \to W_*^\chi(G, \boldsymbol{Z})$, we obtain the homomorphism $\rho = \Psi\rho'$ $: \Omega_m^\chi(G) \to W_m^\chi(G, \boldsymbol{Z})$ which is given by

$$[L^{2k}] \to \langle H^k(L^{2k})/\mathrm{Tor}, \text{ the intersection form}\rangle$$

and

$$[L^{2k+1}] \to \langle \mathrm{Tor}\ H^{k+1}(L^{2k+1}), \text{ the linking form}\rangle.$$

Theorem 2. *Let π be a finitely presented group with a homomorphism $w : \pi \to \{\pm 1\}$. Let (G, χ) be a pair of a finite group G and a homomorphism $\chi : G \to \{\pm 1\}$. Let $\phi : \pi \to G$ be an epimorphism, and denote the composite $\chi\phi$ by χ. Then the following diagram is commutative:*

$$\begin{array}{ccc} \Omega_m^\chi(G)\otimes L_n^s(\pi,\ w) & \longrightarrow & L_{m+n}^s(\pi,\ w\chi) \\ \downarrow \rho\otimes 1 & & \| \\ W_m^\chi(G,\ \boldsymbol{Z})\otimes L_n^s(\pi,\ w) & \longrightarrow & L_{m+n}^s(\pi,\ w\chi) \end{array} \quad (m,\ n\geqq 0)$$

where the upper map is the map defined by the construction in §1 *and the lower map is the map defined in* §8.

10. Proof of Theorem 2. The proof of Therorem 2 is essentially same as that of the product formula of Ranicki [6, §8]. For each element $x\in L_n^s(\pi,\ w)$ $(n\geqq 5)$, there is an n-dimensional normal map of degree one, $f: M^n\to N^n$, such that $\pi_1(N^n)=\pi$ and $\sigma(f)=x$. We consider the map $f\times_\pi 1: M^n\times_\pi L^m\to M^n\times_\pi L^m$ defined in §1, where L^m is an m-dimensional G-χ-manifold.

First we assume that M^n and N^n are closed manifolds. Let $F:\nu_M\to\xi$ be the bundle map associated to f, where ν_M is the stable normal bundle of M^n and ξ is some bundle over N^n. F induces the bundle map $\tilde F:\tilde\nu_M\to\xi$, where $\tilde\nu_M$ (resp. $\tilde\xi$) is the bundle over $\tilde M^n$ (resp. $\tilde N^n$) induced by the projection from ν_M (resp. ξ). Let $T(\tilde F): T(\tilde\nu_M)\to T(\tilde\xi)$ be the map induced by $\tilde F$ between the Thom spaces of $\tilde\nu_M$ and $\tilde\xi$. Following [6], there is a stable π-map $H:\Sigma^\infty\tilde N_+^n\to\Sigma^\infty\tilde M_+^n$ which is an equivariant S-dual of $T(\tilde F)$, where $\tilde N_+^n$ (resp. $\tilde M_+^n$) denotes the disjoint union of $\tilde N^n$ (resp. $\tilde M^n$) and one π-fixed point. The map H is called a geometric Umker map for the normal map f. H defines the composite π-map

$$\theta_H: N_+^n \xrightarrow{\text{adjoint}(H)} \Omega^\infty\Sigma^\infty\tilde M_+^n \xrightarrow{\text{stable homotopy projection}} S_+^\infty\wedge_{Z_2}\tilde M_+^n\wedge\tilde M_+^n,$$

where the generator $T\in\boldsymbol{Z}_2$ acts on S^∞ by the antipodal map and on $\tilde M_+^n\wedge\tilde M_+^n$ by the transposition $(a,\ b)\to(b,\ a)$. Let $C(\tilde M)$ be the f.g. free $\boldsymbol{Z}[\pi]$-chain complex of $\tilde M^n$. Then θ_H induces the homomorphism

$$\theta_H: H_n(N^n,\ {}^w\boldsymbol{Z})\to H_n(W\otimes_{Z[Z_2]}C(\tilde M)\otimes_{Z[\pi]}C(\tilde M))$$

where ${}^w\boldsymbol{Z}$ denotes the twisted coefficient associated to $w:\pi\to\{\pm 1\}$, W is the $\boldsymbol{Z}[\boldsymbol{Z}_2]$-chain complex $C(S^\infty)$ defined in §9, and $T\in\boldsymbol{Z}_2$ acts on $C(\tilde M)\otimes_{Z[\pi]}C(\tilde M)$ by the signed transposition $a\otimes b\to(-1)^{pq}b\otimes a$ $(a\in C_p(\tilde M)$ and $b\in C_q(\tilde M))$. θ_H depends only on the stable π-equivariant homotopy class of H.

Define the Umker $\boldsymbol{Z}[\pi]$-module chain map $f^!: C(\tilde N)\to C(M)$ to be the composite $\boldsymbol{Z}[\pi]$-chain map

$$f^!: C(\tilde N)\xrightarrow{([N]\cap -)^{-1}} C(\tilde N)^{n-*}\xrightarrow{\tilde f^*} C(\tilde M)^{n-*}\xrightarrow{[M]\cap -} C(\tilde M)$$

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where $C(\tilde{N})$ is the f.g. free $Z[\pi]$-chain complex of $\tilde{N}^n$. $[M]\in H_n(M^n, {}_wZ)$ and $[N]\in H_n(N^n, {}^wZ)$ are the fundamental classes of M^n and N^n respectively, and $\cap$ — denotes the cap products. Let $C(f^!)$ be the algebraic mapping cone of $f^!$.

Let $e: C(\tilde{M})\to C(f^!)$ be the projection. Then e induces the map

$$e_\%: H_n(W\otimes_{Z[Z_2]}C(M)\otimes_{Z[\pi]}C(M))\to H_n(W\otimes_{Z[Z_2]}(C(f^!)\otimes_{Z[\pi]}C(f^!)).$$

Put $\theta=e_\%\theta_H([N])$. Now the chain complex $C(f^!)\otimes_{Z[\pi]}C(f^!)$ is isomorphic to the chain complex $\mathrm{Hom}_{Z[\pi]}(C(f^!), C(f^!)_*)$, and θ is considered as an element of $Q_n(C(f^!))$. The pair $(C(f^!), \theta)$ is an n-dimensional quadratic Poincaré complex over $Z[\pi]$ and represents the surgery obstruction of f.

Next we consider the surgery obstruction of the normal map $\tilde{f}\times_\pi 1: \tilde{M}^n\times_\pi L^m\to\tilde{N}^n\times_\pi L^m$. Note that π acts on the covering spaces $\tilde{M}^n\times L^m$ and $\tilde{N}^n\times L^m$ by the diagonal actions. If $H:\Sigma^\infty\tilde{N}_+\to\Sigma^\infty\tilde{M}_+$ is a geometric Umker map for f, then

$$H\wedge 1:\Sigma^\infty(\tilde{N}^n\times L^m)_+=\Sigma^\infty\tilde{N}^n\wedge L^m_+\to\Sigma^\infty(\tilde{M}^n\times L^m)_+=\Sigma^\infty\tilde{M}^n_+\wedge L^m_+$$

is a geometric Umker map for $\tilde{f}\times_\pi 1$. The composite π-map

$$\begin{aligned}\theta_{H\wedge 1}:(\tilde{N}^n\times L^m)_+ &\xrightarrow{\text{adjoint }(H\wedge 1)}\Omega^\infty\Sigma^\infty(\tilde{M}^n\times L^m)_+\\ &\xrightarrow{\text{stable projection}} S^\infty_+\wedge_{Z_2}(\tilde{M}^n\times L^m)_+\wedge(\tilde{M}^m\times L^m)\\ &= S^\infty_+\wedge_{Z_2}M^n_+\wedge M^n_+\wedge L^m_+\wedge L^m_+\end{aligned}$$

is given by $\theta_H\wedge d$, where d is the diagonal map $L^m\to L^m\times L^m$.

Let $C(L)$ be the G-chain complex defined by an equivariant triangulation $t(L^m)$ as in §9. Then $\theta_{H\wedge 1}$ induces the homomorphism

$$\begin{aligned}\theta_{H\wedge 1}: &H_{n+m}((\tilde{N}^n\times_\pi L^m)\ {}^{w\chi}Z)\\ &\to H_{n+m}(W\otimes_{Z[Z_2]}(C(\tilde{M})\otimes C(L))\otimes_{Z[\pi]}(C(\tilde{M})\otimes C(L))).\end{aligned}$$

The Umker $Z[\pi]$-chain map for $f\times_\pi 1$ is given by

$$f^!\otimes 1: C(\tilde{N})\otimes C)L)\to C(\tilde{M})\otimes C(L)$$

and the algebraic mapping cone $C((\tilde{f}\times_\pi 1)^!)$ is $Z[\pi]$-chain equivariant to $C(f^!)\otimes C(L)$ on which π acts diagonally. The chain map $e\otimes 1: C(\tilde{M})\otimes C(L)\to C(f^!)\otimes C(L)$ induces the homomorphism

$$\begin{aligned}(e\otimes 1)_\%: &H_{n+m}(W\otimes_{Z[Z_2]}(C(\tilde{M})\otimes C(L))\otimes_{Z[\pi]}(C(\tilde{M})\otimes C(L)))\\ &\to H_{n+m}(W\otimes_{Z[Z_2]}(C(f^!)\otimes C(L))\otimes_{Z[\pi]}(C(f^!)\otimes C(L)).\end{aligned}$$

Put $\theta'=(e\otimes 1)_{\% H\wedge 1}([\tilde{N}\times_\pi L])$, where $[\tilde{N}\times_\pi L]$ is the fundamental class of $\tilde{N}^n\times_\pi L^m$. Then the pair $(C(f^!)\otimes C(L), \theta')$ is an $(n+m)$-dimensional quadratic Poincaré complex over $Z[\pi]$, and it represents the surgery obstruction $\sigma(\bar{f}\times_{\pi}1)$. By the same argument as in [6, §8] we see that $(C(f^!)\otimes C(L), \theta')$ is equivalent to $(C(f^!), \theta)\otimes(C(L), \psi)$.

When M^m and N^n have non-empty boundaries, a similar construction can be made by the use of the homotopy Umker map pair

$$H : (\Sigma^\infty \tilde{N}^n_+, \Sigma^\infty \partial\tilde{N}^n_+) \to (\Sigma^\infty \tilde{M}^n_+, \Sigma^\infty \partial\tilde{M}^n_+).$$

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