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THE S^1 -TRANSFER MAP AND HOMOTOPY GROUPS OF SUSPENDED COMPLEX PROJECTIVE SPACES

JUNO MUKAI

1. Introduction. We denote by P^n the complex n -dimensional projective space. Let $SU(n)$ and $SO(n)$ be the special unitary and rotation groups, respectively. As is well known, the suspended space EP^{n-1} is canonically embedded in $SU(n)$ [17]. We consider the composition of the canonical mappings

$$EP^{n-1} \longrightarrow SU(n) \longrightarrow SO(2n) \longrightarrow \Omega^{2n}S^{2n},$$

where $\Omega^{2n}S^{2n}$ denotes the space of base point preserving maps from S^{2n} to S^{2n} . We define the S^1 -transfer map

$$g_n : E^{2n+1}P^{n-1} \longrightarrow S^{2n}$$

by taking the adjoint of the above.

We denote by $B_{k,n}$ the image of the induced homomorphism

$$g_{n*} : \pi_{2n+k}(E^{2n+1}P^{n-1}) \longrightarrow \pi_{2n+k}(S^{2n})$$

and

$$B_k = \lim_{n \rightarrow \infty} B_{k,n}.$$

The purpose of the present paper is to determine the group structure of B_k for $k \leq 11$.

Our method is essentially to compute stable homotopy groups of EP^{n-1} from the unstable viewpoint. The main tools are the works on the homotopy groups of $SU(n)$ [13] and the composition methods in the homotopy groups of S^n [15].

Our main result is stated as follows :

Theorem 1.1. B_k for $k \leq 11$ and generators of their 2-primary components are listed in the following table.

| $k =$ | 1, 2 | 3 | 4, 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-----------------|------|-------------------|------|----------------|--------------------|----------------|-------------------------------|----------------|--------------------|
| $B_k \approx$ | 0 | \mathbb{Z}_{24} | 0 | \mathbb{Z}_2 | \mathbb{Z}_{120} | \mathbb{Z}_2 | $\mathbb{Z}_2 + \mathbb{Z}_2$ | \mathbb{Z}_3 | \mathbb{Z}_{504} |
| gen. of 2-comp. | | ν | | ν^2 | 2σ | $\bar{\nu}$ | $\eta^2\sigma, \nu^3$ | | ζ |

Let $\pi_n: S^{2n+1} \longrightarrow P^n$ be the natural projection. The following theorem plays an important role in proving Theorem 1.1.

Theorem 1.2. *The homotopy class of $E^{k+1}\pi_{n-1}$ is of order $n!$ for $k \geq 0$.*

The result about the k -th stable homotopy group of EP^∞ for $k \leq 11$ will be stated as Theorem 1.3 in the last section.

2. The S^1 -transfer map and the complex J homomorphism. Let S^{2n-1} be the unit sphere in the complex n -dimensional space C^n and let $U(n)$ be the unitary group. Let $\phi: S^{2n-1} \times S^1 \longrightarrow U(n)$ be a mapping defined as follows: $\phi(u, q)v = v + u(q-1)\langle u, v \rangle$ for $u \in S^{2n-1}$, $q \in S^1$ and $v \in C^n$, where $\langle u, v \rangle = \sum_{k=1}^n \bar{u}_k v_k$ for $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$. Then it gives the matrix form $\phi(u, q) = (\delta_{ij} + (q-1)\bar{u}_i u_j)$ for $1 \leq i, j \leq n$.

According to [5], the complex quasi-projective space EP^{n-1}_+ is the space obtained from $S^{2n-1} \times S^1$ by imposing the equivalence relation: $(u, q) \sim (ug, q)$ for $g \in S^1$ and $(u, 1) \sim$ a point for $u \in S^{2n-1}$. The generalized reflection

$$j' = j'_n: EP^{n-1}_+ \longrightarrow U(n)$$

is defined as the mapping induced from ϕ .

Let $e_1 = (1, 0, \dots, 0) \in S^{2n-1}$ and take $[e_1] \in P^{n-1}$ as the base point of P^{n-1} . Then we define

$$j = j_n: EP^{n-1} \longrightarrow SU(n)$$

by

$$j([u] \wedge q) = j'([u, q]) \begin{bmatrix} q^{-1} & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{bmatrix},$$

where $[u] \wedge q \in P^{n-1} \wedge S^1 = EP^{n-1}$. Obviously, j' and j are homeomorphisms into.

According to [17], $SU(n)$ is a cell complex composed of 2^{n-1} cells: 0-dim. cell e^0 and $\{2(k_1 + k_2 + \dots + k_m) - n\}$ -dim. cells $e^{2k_1-1, 2k_2-1, \dots, 2k_m-1}$ for $n \geq k_1 \geq k_2 \geq \dots \geq k_m \geq 2$ and $m \geq 1$. In particular, e^{2k-1} is identified with the $(2k-1)$ -dim. cell of EP^{n-1} by j_n , where $k=2, \dots, n$.

The S^1 -transfer map

$$g_n: E^{2n+1}P^{n-1} \longrightarrow S^{2n}$$

is defined by taking the adjoint of the composition

$$EP^{n-1} \xrightarrow{j} SU(n) \subset U(n) \xrightarrow{r} SO(2n) \xrightarrow{i} \Omega^{2n} S^{2n},$$

where r is the realization and i is the canonical inclusion.

It follows from the definition that

$$(1) \quad g_n | E^{2n+1} P^{n-2} = E^2 g_{n-1}.$$

Let $J: \pi_r(SO(n)) \longrightarrow \pi_{n+r}(S^n)$ (resp. $J_c: \pi_r(SU(n)) \longrightarrow \pi_{2n+r}(S^{2n})$) be the J homomorphism (resp. complex J homomorphism). Then it is easy to prove

Proposition 2.1. *The following diagram is commutative for $k \leq n$:*

$$\begin{array}{ccccc} \pi_{2n+2k-1}(E^{2n+1}P^{n-1}) & \xrightarrow{g_n*} & \pi_{2n+2k-1}(S^{2n}) & & \\ \uparrow \wr & \nearrow E^{2(n-k)}g_{k*} & \uparrow J & & \\ \pi_{2n+2k-1}(E^{2n+1}P^{k-1}) & & \pi_{2k-1}(SU(n)) & \xrightarrow{r_*} & \pi_{2k-1}(SO(2n)) \\ \uparrow E^{2n} & \nearrow J_c & \uparrow & & \\ \pi_{2k-1}(EP^{k-1}) & \xrightarrow{j_k*} & \pi_{2k-1}(SU(k)) & \xrightarrow{r_*} & \pi_{2k-1}(SO(2k)). \end{array}$$

We recall that $\pi_k(SU(n))$ is isomorphic to \mathbb{Z} (resp. 0) for $k < 2n$ if k is odd (resp. even) by [4].

3. The characteristic map of a unitary bundle. We regard S^{2n} as a subspace of S^{2n+1} consisting of points $z = (z_0, \dots, z_n)$ such that $\operatorname{Re}(z_n) = 0$ and $\sum_{k=0}^n |z_k|^2 = 1$. According to § 24.2 of [11], the characteristic map $T'_n: S^{2n} \longrightarrow U(n)$ for the normal form of a unitary bundle is given by

$$T'_n(z) = (\delta_{ij} - 2\bar{z}_i z_j / (1 + z_n^2)), \quad 0 \leq i, j \leq n-1.$$

Put $u_{k+1} = z_k / \sqrt{1 + z_n^2}$ for $0 \leq k \leq n-1$, where $z_n \neq \pm \sqrt{-1}$. Put $u = (u_1, \dots, u_n) \in S^{2n-1}$ and $q = -(1 - z_n)^2 / (1 + z_n)^2 \in S^1$. Then it follows immediately that $T'_n(z) = j'_n([u, q])$. So we have the following

Lemma 3.1. *The characteristic map $T'_n: S^{2n} \longrightarrow U(n)$ is homotopic to the composition*

$$S^{2n} \xrightarrow{\pm E\pi_{n-1}} EP^{n-1} \xrightarrow{j_n} SU(n) \subset U(n).$$

From now on, we use often the same letter for a map and its homotopy class.

As is well known, $\pi_{2n}(SU(n)) \approx \mathbb{Z}_{n!}$ for $n \geq 2$. So Theorem 25.2 of [11] and Lemma 3.1 lead us to the following

Proposition 3.2. $j_n E \pi_{n-1}$ generates $\pi_{2n}(SU(n)) \approx \mathbb{Z}_{n!}$ and $j_* : \pi_{2n}(EP^{n-1}) \longrightarrow \pi_{2n}(SU(n))$ is an epimorphism for $n \geq 2$.

4. The Toda map $\zeta^{m,n}$. According to [13] and [14], there exists a cellular mapping $\zeta : E^3 P^\infty \longrightarrow EP^\infty$ such that the degree of

$$\zeta_* : H_{2i+1}(E^3 P^\infty) \longrightarrow H_{2i+1}(EP^\infty)$$

is i for $i \geq 2$. We define a mapping

$$\zeta^{m,n} : E^{2m+1} P^n \longrightarrow EP^{m+n}$$

as the composition $\zeta \circ E^2 \zeta \circ \dots \circ E^{2(m-1)} \zeta : E^{2m+1} P^n \longrightarrow E^{2m-1} P^{n+1} \longrightarrow \dots \longrightarrow E^3 P^{m+n-1} \longrightarrow EP^{m+n}$. ($\zeta^{0,n}$ = the identity map). In particular, we denote

$$\zeta_n = \zeta^{n-1,1} : S^{2n+1} \longrightarrow EP^n.$$

Hideyuki Kachi pointed the author out the following [13]:

Theorem 4.1 (Toda). $j_{n+1} \zeta_n$ generates $\pi_{2n+1}(SU(n+1)) \approx \mathbb{Z}$ and $j_* : \pi_{2n+1}(EP^n) \longrightarrow \pi_{2n+1}(SU(n+1))$ is a split epimorphism for $n \geq 1$.

By Proposition 3.2 and Theorem 4.1, we have a split exact sequence for $n \geq 1$:

$$(2) \quad 0 \longrightarrow \pi_{2n+2}(SU(n+1), EP^n) \xrightarrow{\delta} \pi_{2n+1}(EP^n) \xrightarrow{j_*} \pi_{2n+1}(SU(n+1)) \longrightarrow 0.$$

Proposition 4.2. Let H be the infinite cyclic subgroup of $\pi_{2n+1}(EP^n)$ generated by ζ_n . Then, the restriction homomorphism

$$E^k | H : H \longrightarrow \pi_{2n+k+1}(E^{k+1} P^n)$$

is a split monomorphism for $k \geq 1$.

Proof. Let m be an integer such that $k \leq 2m$. Consider the composition of the homomorphisms $j_{m+n+1} \zeta_*^{m,n} E^{2m} : \pi_{2n+1}(EP^n) \longrightarrow \pi_{2m+2n+1}(SU(m+n+1)) \approx \mathbb{Z}$. Then this is a split epimorphism by Theorem 4.1, since $\zeta_*^{m,n} E^{2m} \zeta_n = \zeta_{m+n}$. This completes the proof.

Professor Hirosi Toda suggested the present form of Proposition 4.2 to the author.

By Theorem 4.1, Propositions 2.1 and 4.2, we have the following

Theorem 4.3. $J_C(\pi_{2k-1}(SU(n))) = \{E^{2(n-k)} g_k \circ E^{2n} \zeta_{k-1}\} \subset B_{2k-1,n}$ for $k \leq n$.

Let $p_n: P^n \longrightarrow P^n/P^{n-1} = S^{2n}$ be the collapsing map and let ι_n be the identity class of $\pi_n(S^n) \approx \mathbb{Z}$ for $n \geq 1$. From the definition, we have

$$(3) \quad Ep_n \circ \zeta_n = n! \iota_{2n+1} \text{ for } n \geq 1.$$

5. Proof of Theorem 1.2. Let $i_n: P^{n-1} \longrightarrow P^n$ and $i'_n: SU(n-1) \longrightarrow SU(n)$ be the inclusions and let $p'_n: SU(n) \longrightarrow SU(n)/SU(n-1) = S^{2n-1}$ be the projection. Then we have a commutative diagram

$$(4) \quad \begin{array}{ccccc} EP^{n-2} & \xrightarrow{Ei_{n-1}} & EP^{n-1} & \xrightarrow{Ep_{n-1}} & S^{2n-1} \\ \downarrow j_{n-1} & & \downarrow j_n & & \parallel \\ SU(n-1) & \xrightarrow{i'_n} & SU(n) & \xrightarrow{p'_n} & S^{2n-1}, \end{array}$$

where the upper sequence is the cofiber and the lower is the fiber.

Hereafter we use simply i and p to denote the natural inclusion and the collapsing map respectively, unless otherwise stated.

Now we give a proof of Theorem 1.2. Let $p: (EP^n, EP^{n-1}) \longrightarrow (S^{2n+1}, *)$ be the collapsing map. Then $p_*: \pi_r(EP^n, EP^{n-1}) \longrightarrow \pi_r(S^{2n+1})$ for $r \leq 2n+2$ is an isomorphism by [3]. So we have an exact sequence for $n \geq 2$:

$$\pi_{2n+1}(EP^n) \xrightarrow{Ep_{n*}} \pi_{2n+1}(S^{2n+1}) \xrightarrow{\Delta} \pi_{2n}(EP^{n-1}) \xrightarrow{Ei_{n*}} \pi_{2n}(EP^n),$$

where $\Delta = \delta \circ p_*^{-1}: \pi_{2n+1}(S^{2n+1}) \longrightarrow \pi_{2n+1}(EP^n, EP^{n-1}) \longrightarrow \pi_{2n}(EP^{n-1})$.

Obviously, $(Ei_n)_*(E\pi_{n-1}) = 0$, and so $E\pi_{n-1} = \Delta(at_{2n+1})$ for some integer a . By (3), $n! E\pi_{n-1} = a\Delta(Ep_{n*}\zeta_n) = 0$. By Proposition 3.2, $j_{n*}E\pi_{n-1}$ is of order $n!$. This concludes the assertion of Theorem 1.2 for $k=0$. It is noted that $(a, n!) = 1$.

Next assume that $k \geq 1$ and let m be an integer such that $2m \geq k$. We consider homomorphisms between the exact sequences

$$\begin{array}{ccccc} \pi_{2n+1}(EP^n) & \xrightarrow{Ep_{n*}} & \pi_{2n+1}(S^{2n+1}) & \xrightarrow{\Delta} & \pi_{2n}(EP^{n-1}) \\ \downarrow E^{2m} & & \Downarrow E^{2m} & & \downarrow E^{2m} \\ \pi_r(E^{2m+1}P^n) & \xrightarrow{E^{2m+1}p_{n*}} & \pi_r(S^r) & \xrightarrow{\Delta'} & \pi_{r-1}(E^{2m+1}P^{n-1}), \end{array}$$

where $r = 2m + 2n + 1$. Using the case that $k=0$, we have

$$E^{2m+1}\pi_{n-1} = a\Delta'(\iota_{2m+2n+1}).$$

So it is sufficient to prove that $\mathcal{A}'(\iota_{2m+2n+1})$ is of order $n!$.

Let x be the order of the above element. Assume that $x < n!$ and put $y = n!/x \geq 2$. Then there exists an element $\alpha \in \pi_r(E^{2m+1}P^n)$ such that $E^{2m+1}p_{n*}\alpha = x\iota_r$ for $r = 2m+2n+1$. By (3), there exists an element $\beta \in \pi_r(E^{2m+1}P^{n-1})$ such that $E^{2m}\zeta_n = y\alpha + i_*\beta$, where $i = E^{2m+1}i_n$. Consider the commutative diagram

$$\begin{array}{ccccc} \pi_r(E^{2m+1}P^{n-1}) & \xrightarrow{\zeta_*^{m,n-1}} & \pi_r(EP^{m+n-1}) & \xrightarrow{j_*} & \pi_r(SU(m+n)) \\ \downarrow i_* & & \downarrow i'_* & & \downarrow i''_* \\ \pi_r(E^{2m+1}P^n) & \xrightarrow{\zeta_*^{m,n}} & \pi_r(EP^{m+n}) & \xrightarrow{j_*} & \pi_r(SU(m+n+1)). \end{array}$$

Then, $j_*\zeta_*^{m,n}i_*\beta = i''_*j_*\zeta_*^{m,n-1}\beta = 0$, since $\pi_{2m+2n+1}(SU(m+n)) \approx \mathbb{Z}_2$ or 0 by Theorem 4.4 of [13]. Therefore, $j_*\zeta_{m-n} = j_*\zeta_*^{m,n}E^{2m}\zeta_n = yj_*\zeta_*^{m,n}\alpha$. This contradicts Theorem 4.1. Hence $\mathcal{A}'(\iota_{2m+2n+1})$ is of order $n!$. This completes the proof of Theorem 1.2.

6. A characterization of ζ_n and $E\pi_n$. We shall prepare a lemma concerning the Toda bracket. Let X, Y and Z be spaces with base points. Let $[X, Y]$ be the set of homotopy classes of base point preserving maps from X to Y . We denote by $W = Z \cup_a CY$ the mapping cone of $a \in [Y, Z]$. Let $i: Z \rightarrow W$ be the inclusion and let $p: W \rightarrow EY$ be the mapping which shrinks Z to a point. We denote by ι_X the homotopy class of the identity map of X .

Lemma 6.1. i) $\iota_{EY} \in \{p, i, \alpha\} \bmod p_*[EY, W] + (E\alpha)^*[EZ, EY]$ if $Y = EY'$.

ii) Assume that $0 \notin \{p, i, \alpha\}$ and $\alpha\beta = 0$, where $\beta \in [X, Y]$ and $X = EX'$. Suppose given an element $\tilde{\beta} \in [EX, W]$ such that $p_*\tilde{\beta} = -E\beta$. Then

$$a) \quad \tilde{\beta} \in \{i, \alpha, \beta\} \bmod \text{Ker } p_* + (E\beta)^*[EY, W].$$

Furthermore assume that the sequence

$$[EX, Z] \xrightarrow{i_*} [EX, W] \xrightarrow{p_*} [EX, EY]$$

is exact. Then

$$b) \quad \tilde{\beta} \in \{i, \alpha, \beta\} \bmod i_*[EX, Z] + (E\beta)^*[EY, W].$$

Proof. By (1.14) and Lemma 1.1 of [15], we have i).

By Proposition 1.4 of [15], $p_*\{i, \alpha, \beta\} = -\{p, i, \alpha\} \circ E\beta$. So, by i) and the assumption, there exists an element $\gamma \in \{i, \alpha, \beta\}$ such that $p_*\gamma = -E\beta$.

Therefore, $\tilde{\beta} - \gamma \in \text{Ker } p_*$. By the definition, $\{i, \alpha, \beta\}$ is a double coset of $i_*[EX, Z] \subset \text{Ker } p_*$ and $(E\beta)^*[EY, W]$. This completes the proof.

Lemma 6.2. $\iota_{2n+1} \in \{Ep_n, Ei_n, E\pi_{n-1}\} \bmod n! \iota_{2n+1}$ for $n \geq 2$.

Proof. The bracket is a coset of the subgroup

$$Ep_{n*}\pi_{2n+1}(EP^n) + (E^2\pi_{n-1})^*[E^2P^{n-1}, S^{2n+1}].$$

By (3), (4) and Theorem 4.1, $Ep_{n*}\pi_{2n+1}(EP^n) = \{n! \iota_{2n+1}\}$. On the other hand, $[E^2P^{n-1}, S^{2n+1}] \approx 0$. This completes the proof.

By Theorem 1.2, (3), Lemmas 6.1 and 6.2, we have the following

Proposition 6.3. For $n \geq 2$,

$$\zeta_n \in -\{Ei_n, E\pi_{n-1}, n! \iota_{2n}\} \bmod Ei_{n*}\pi_{2n+1}(EP^{n-1}) + n! \pi_{2n+1}(EP^n).$$

Hereafter we use the following [15]:

$$\pi_{n+1}(S^n) = \{\eta_n\} \approx \mathbf{Z}_2 \text{ (resp. } \mathbf{Z} \text{) for } n \geq 3 \text{ (resp. } n=2\text{)}.$$

By Theorem 24.3 of [11], (4) and Proposition 3.2, we have the following

Lemma 6.4. $p_n\pi_n = n\eta_{2n}$ for $n \geq 1$.

Proposition 6.5. i) For odd $n \geq 3$,

$$E\pi_n \in \{Ei_n, E\pi_{n-1}, \eta_{2n}\} \bmod Ei_{n*}\pi_{2n+2}(EP^{n-1}) + \pi_{2n+1}(EP^n) \circ \eta_{2n+1}.$$

ii) There exists an element $\lambda_n \in \pi_{2n+2}(EP^{n-1})$ such that

$$Ei_n \circ \lambda_n = E\pi_n \text{ (resp. } 2E\pi_n \text{)}$$

for even (resp. odd) $n \geq 2$.

Proof. By Lemma 6.4, $E\pi_{n-1} \circ \eta_{2n} = E\pi_{n-1} \circ (p_n\pi_n) = (E\pi_{n-1} \circ p_n) \circ \pi_n = 0$ for odd $n \geq 3$. Obviously, the sequence

$$\pi_r(EP^{n-1}) \xrightarrow{Ei_{n*}} \pi_r(EP^n) \xrightarrow{Ep_{n*}} \pi_r(S^{2n+1})$$

is exact for $r=2n+2$. So we have i) by Lemmas 6.1 and 6.2. We have also ii) by Lemma 6.4 and the above exact sequence. This completes the proof.

7. Generators of $\pi_{2n+k}(SU(n))$ for $1 \leq k \leq 3$. We consider the exact sequence

$$(5) \quad \pi_{r+1}(S^{2n+1}) \xrightarrow{\mathcal{A}'} \pi_r(SU(n)) \xrightarrow{i_*} \pi_r(SU(n+1)) \xrightarrow{p_*} \pi_r(S^{2n+1}) \xrightarrow{\mathcal{A}'} \pi_{r-1}(SU(n)).$$

By (2.2) of [9], $\mathcal{A}'(\alpha \circ E\beta) = \mathcal{A}'(\alpha) \circ \beta$ for $\alpha \in \pi_{t+1}(S^{2n+1})$ and $\beta \in \pi_q(S^t)$, where $2n \leq t \leq q$ and $q=r$ or $r-1$. Using the proof of Theorem 1.2 for $k=0$, we have an integer b with $(b, n!) = 1$ such that

$$(5)' \quad \mathcal{A}'(\iota_{2n+1}) = bj_n E\pi_{n-1}.$$

Using [4], Theorem 4.4 of [13], (5) and (5)' for $r=2n+1$, we see that $\mathcal{A}' : \pi_{2n+2}(S^{2n+1}) \longrightarrow \pi_{2n+1}(SU(n))$ is an epimorphism and $\mathcal{A}'(\eta_{2n+1}) = j_n \circ E\pi_{n-1} \circ \eta_{2n}$. This leads us to the following

Proposition 7.1. i) $\pi_{2n+1}(SU(n)) = \{j_n \circ E\pi_{n-1} \circ \eta_{2n}\} \approx \mathbf{Z}_2$ (resp. 0) for even (resp. odd) $n \geq 2$.

ii) $j_* : \pi_{2n+1}(EP^{n-1}) \longrightarrow \pi_{2n+1}(SU(n))$ is an epimorphism for $n \geq 2$.

By Theorem 1.2, Propositions 3.2 and 7.1, we have a split exact sequence for $n \geq 2$:

$$(6) \quad 0 \longrightarrow \pi_{2n+1}(SU(n), EP^{n-1}) \xrightarrow{\delta} \pi_{2n}(EP^{n-1}) \xrightarrow{j_*} \pi_{2n}(SU(n)) \longrightarrow 0.$$

Hereafter we use the following ([13] and [15]): $\pi_{2n+2}(SU(n)) \approx \mathbf{Z}_{(n+1)!} + \mathbf{Z}_2$ (resp. $\mathbf{Z}_{(n+1)!/2}$) for even (resp. odd) $n \geq 3$, $\pi_{n+2}(S^n) = \{\eta_n^2\} \approx \mathbf{Z}_2$ for $n \geq 2$ and $\pi_6(S^3) \approx \mathbf{Z}_{12}$.

Proposition 7.2. i) $j_{n*} : \pi_{2n+2}(EP^{n-1}) \longrightarrow \pi_{2n+2}(SU(n))$ is an epimorphism for $n \geq 2$.

ii) $\pi_{2n+2}(SU(n)) = \{j_n \lambda_n, j_n \circ E\pi_{n-1} \circ \eta_{2n}^2\} \approx \mathbf{Z}_{(n+1)!} + \mathbf{Z}_2$ for even $n \geq 4$ and $\pi_6(SU(2)) = \{j_2 \lambda_2\} \approx \mathbf{Z}_{12}$.

iii) $\pi_{2n+2}(SU(n)) = \{j_n \lambda_n\} \approx \mathbf{Z}_{(n+1)!/2}$ for odd $n \geq 3$.

Proof. By inspecting the proof of Theorem 4.3 of [13], we have a commutative diagram for $n \geq 3$:

$$\begin{array}{ccc} \pi_{2n+3}(EP^{n+1}, EP^{n-1}) & \xrightarrow{\delta} & \pi_{2n+2}(EP^{n-1}) \\ \downarrow p_* & & \downarrow j_{n*} \\ \pi_{2n+3}(EP^{n+1}/EP^{n-1}) & & \\ \Downarrow & & \\ \pi_{2n+3}(SU(n+2)/SU(n)) & \xrightarrow{\mathcal{A}'} & \pi_{2n+2}(SU(n)). \end{array}$$

By [4], $\pi_{2n+2}(SU(n+2)) \approx 0$, and so \mathcal{A}' is an epimorphism. By [3], p_* is an epimorphism. This leads us to i).

Using Proposition 7.1. i), (5) and (5)' for $r=2n+2$, we see that

$$i'_* : \pi_{2n+2}(SU(n)) \longrightarrow \pi_{2n+2}(SU(n+1))$$

is an epimorphism for even $n \geq 2$. By considering the group structure of $\pi_{2n+2}(SU(k))$ for $k=n$ and $n+1$, $\Delta'(\eta_{2n+1}^2) = j_n \circ E\pi_{n-1} \circ \eta_{2n}^2 \neq 0$ (resp. $=0$) for even (resp. odd) $n \geq 3$. So, i'_* is a monomorphism for odd $n \geq 3$. Hence (4), Propositions 3.2 and 6.5.ii) lead us to ii) and iii). This completes the proof.

By Propositions 7.1 and 7.2, we have a split exact sequence for even $n \geq 2$:

$$(7) \quad 0 \longrightarrow \pi_{2n+2}(SU(n), EP^{n-1}) \xrightarrow{\delta} \pi_{2n+1}(EP^{n-1}) \xrightarrow{j'_*} \pi_{2n+1}(SU(n)) \longrightarrow 0.$$

We have also an isomorphism for odd $n \geq 3$:

$$(7)' \quad \delta : \pi_{2n+2}(SU(n), EP^{n-1}) \xrightarrow{\approx} \pi_{2n+1}(EP^{n-1}).$$

From now on, we use the same symbol to denote generators of $\pi_{n+k}(S^n)$ and its 2-primary component.

Hereafter we use the following ([8] and [15]): $\pi_{2n+3}(SU(n)) \approx \mathbf{Z}_{a(n)}$, where $a(n) = (24, n)$ (resp. $(24, n+3)/2$) for even (resp. odd) $n \geq 2$. $\pi_6(S^3) = \{\nu'\} \approx \mathbf{Z}_{12}$, $\pi_7(S^4) = \{\nu_4, E\nu'\} \approx \mathbf{Z} + \mathbf{Z}_{12}$ and $\pi_{n+3}(S^n) = \{\nu_n\} \approx \mathbf{Z}_{24}$ for $n \geq 5$. $6\nu' = \eta_3^3$ and $2\nu_n = E^{n-3}\nu'$ for $n \geq 5$.

Proposition 7.3. i) $\pi_{2n+3}(SU(n)) = \{j_n \circ E\pi_{n-1} \circ \nu_{2n}\} \approx \mathbf{Z}_{a(n)}$ for $n \geq 2$.
ii) $j'_* : \pi_{2n+3}(EP^{n-1}) \longrightarrow \pi_{2n+3}(SU(n))$ is an epimorphism for $n \geq 2$.

Proof. We consider the exact sequence (5) for $r=2n+3$:

$$\pi_{r+1}(S^{2n+1}) \xrightarrow{\Delta'} \pi_r(SU(n)) \xrightarrow{i'_*} \pi_r(SU(n+1)) \xrightarrow{p'_*} \pi_r(S^{2n+1}).$$

By Proposition 7.1. i), (4) and Lemma 6.4, p'_* is a monomorphism for $n \geq 1$. So Δ' is an epimorphism for $n \geq 1$. By (5)', $\Delta'(\nu_{2n+1}) = b(j_n \circ E\pi_{n-1} \circ \nu_{2n})$ for $n \geq 2$. This completes the proof.

By Propositions 7.2 and 7.3, we have a short exact sequence for $n \geq 2$:

$$(8) \quad 0 \longrightarrow \pi_{2n+3}(SU(n), EP^{n-1}) \xrightarrow{\delta} \pi_{2n+2}(EP^{n-1}) \xrightarrow{j'_*} \pi_{2n+2}(SU(n)) \longrightarrow 0.$$

8. Determination of $\pi_{n+k}(E^{n+1}P^m)$ for $k \leq 7$. We consider an exact sequence

$$\pi_{n+k+1}(E^{n+1}P^m, E^{n+1}P^{m-1}) \xrightarrow{\delta} \pi_{n+k}(E^{n+1}P^{m-1}) \xrightarrow{i_*} \pi_{n+k}(E^{n+1}P^m) \\ \xrightarrow{j_*} \pi_{n+k}(E^{n+1}P^m, E^{n+1}P^{m-1}).$$

By [3], $p'_*: \pi_{n+k+1}(E^{n+1}P^m, E^{n+1}P^{m-1}) \longrightarrow \pi_{n+k+1}(S^{n+2m+1})$ is an isomorphism for $n \geq k-2m-1$ and $m \geq 2$. So we have the exact sequence for $n \geq k-2m-1$ and $m \geq 2$:

$$(9)_{m,k} \quad \pi_{n+k+1}(S^{n+2m+1}) \xrightarrow{\Delta} \pi_{n+k}(E^{n+1}P^{m-1}) \xrightarrow{i_*} \pi_{n+k}(E^{n+1}P^m) \\ \xrightarrow{p_*} \pi_{n+k}(S^{n+2m+1}),$$

where $\Delta = \Delta_m = \delta \circ p'_*^{-1}$.

We have also the exact sequence for $n \geq k-2m-2$ and $m \geq 2$:

$$(9)'_{m,k} \quad \pi_{n+k}(E^{n+1}P^{m-1}) \xrightarrow{i_*} \pi_{n+k}(E^{n+1}P^m) \xrightarrow{p_*} \pi_{n+k}(S^{n+2m+1}) \\ \xrightarrow{\Delta} \pi_{n+k-1}(E^{n+1}P^{m-1}).$$

As is well known, $\Delta(\alpha \circ E\beta) = \Delta(\alpha) \circ \beta$, where $\alpha \in \pi_{i+1}(S^{n+2m+1})$ and $\beta \in \pi_{n+k}(S')$ for $n \geq k-2m-1$ and $m \geq 2$. By inspecting the proof of Theorem 1.2, $\Delta(\iota_{n+2m+1}) = bE^{n+1}\pi_{m-1}$, and so $\Delta(E\alpha) = b(E^{n+1}\pi_{m-1} \circ \alpha)$ for $\alpha \in \pi_{n+k}(S^{n+2m})$, where b is an integer such that $(b, n!) = 1$. For the simplicity, we use the following expression, because it makes no difference to continuation of the subsequent arguments.

$$(10)_m \quad \Delta_m(E\alpha) = E^{n+1}\pi_{m-1} \circ \alpha \text{ for } n \geq k-2m-1 \text{ and } m \geq 2.$$

Now, we start to compute $\pi_{n+k}(E^{n+1}P^m)$.

By Lemma 6.4, we have

$$(11) \quad \pi_1 = \eta_2.$$

By (10)₂ and (11), we have

$$(12) \quad \Delta_2(\iota_{n+5}) = \eta_{n+3} \text{ and } \Delta_2(\eta_{n+5}) = \eta_{n+3}^2 \text{ for } n \geq 0.$$

Using (3), (12) and (9)_{2,k} for $k=4$ and 5, we have the following

- Proposition 8.1.** i) $\pi_{n+k}(E^{n+1}P^1) \approx 0$ for $k \leq 2$ and $n \geq 0$.
 ii) $\pi_{n+3}(E^{n+1}P^1) = \{E^n \zeta_1\} \approx \mathbf{Z}$ for $n \geq 0$.
 iii) $\pi_{n+4}(E^{n+1}P^2) \approx 0$ for $n \geq 0$.
 iv) $\pi_{n+5}(E^{n+1}P^2) = \{E^n \zeta_2\} \approx \mathbf{Z}$ for $n \geq 0$.

By [17], $SU(3) = EP^2 \cup e^{5,3}$, and so $j_{3*}: \pi_k(EP^2) \longrightarrow \pi_k(SU(3))$ is an isomorphism for $k \leq 6$. Therefore, by Proposition 3.2, $\pi_6(EP^2) = \{E\pi_2\} \approx \mathbf{Z}_6$. By (9)_{2,6} for $n=0$ and by (12), $i_*: \pi_6(S^3) \longrightarrow \pi_6(EP^2)$ is an epimorphism,

where $i = Ei_2 : S^3 \longrightarrow EP^2$. By (11), $i\eta_3 = 0$. Hence we have

$$(13) \quad E\pi_2 = \pm i\nu'.$$

By (10)₂ and (12), we have

$$(14) \quad \Delta_2(\eta_{n+5}^2) = 12\nu_{n+3} \text{ (resp. } 6E\nu') \text{ for } n \geq 2 \text{ (resp. } n=1).$$

Using (9)_{2,6}, (9)_{2,6}, (14) and the above argument, we have the following

Proposition 8.2. i) $\pi_6(EP^2) = \{E\pi_2\} \approx \mathbf{Z}_6$.

ii) $\pi_7(E^2P^2) = \{i\nu_4, iE\nu'\} \approx \mathbf{Z} + \mathbf{Z}_6$.

iii) $\pi_{n+6}(E^{n+1}P^2) = \{i\nu_{n+3}\} \approx \mathbf{Z}_{12}$ for $n \geq 2$.

Proposition 8.3. Let ξ be a generator of $\pi_7(EP^2)$, and $i' = Ei_3 : EP^2 \longrightarrow EP^3$.

i) $\pi_7(EP^2) = \{\xi\} \approx \mathbf{Z}$.

ii) $\pi_7(EP^3) = \{i'\xi, \xi_3\} \approx \mathbf{Z} + \mathbf{Z}$.

Proof. By [3] and [17], $\pi_8(SU(k), EP^{k-1}) \approx \pi_8(S^8) \approx \mathbf{Z}$ for $k \geq 3$. So, by (2) and (7)' for $n=3$, we have the assertion.

Hereafter we use the following [15] : $\pi_8(S^4) = \{\nu_4\eta_7, E\nu'\eta_7\} \approx \mathbf{Z}_2 + \mathbf{Z}_2$, $\pi_9(S^5) = \{\nu_5\eta_8\} \approx \mathbf{Z}_2$ and $\pi_{n+4}(S^n) \approx 0$ for $n \geq 6$. $\nu'\eta_6 = \eta_3\nu_4$, $\eta_n\nu_{n+1} = 0$ and $\nu_{n+1}\eta_{n+4} = 0$ for $n \geq 5$.

By (10)₂ and (11), $\Delta_2(\nu_{n+5}) = \eta_{n+3}\nu_{n+4} = 0$ for $n \geq 2$. So we have

$$(15) \quad \Delta_2(\nu_{n+5}) = 0 \text{ for } n \geq 2.$$

By (9)_{2,7}, (9)_{2,7}, (14) and (15), we have the following

Proposition 8.4. i) $\pi_{n+7}(E^{n+1}P^2) = \{i\nu_{n+3}\eta_{n+6}\} \approx \mathbf{Z}_2$ for $n=0$ or 1.

ii) $\pi_{n+7}(E^{n+1}P^2) \approx 0$ for $n \geq 3$.

Lemma 8.5. $E\xi = i\nu_4\eta_7$.

Proof. Using the EHP-sequence (cf. Theorem 2.2 of [16, Chap. 12]), we have the exact sequence

$$\pi_7(EP^2) \xrightarrow{E} \pi_8(E^2P^2) \xrightarrow{H} \pi_8(EP^2 * EP^2) \xrightarrow{P} \pi_6(EP^2),$$

where $EP^2 * EP^2$ denotes the reduced join.

Clearly, $\pi_8(EP^2 * EP^2) \approx \pi_8(E^3(P^2 \wedge P^2)) \approx \pi_8(E^5P^2 \vee S^9) \approx \pi_8(E^5P^2) \approx 0$ by Proposition 8.1. iii). So E is an epimorphism. Hence we have the assertion by Propositions 8.3 and 8.4.

Lemma 8.6. i) $\Delta_3(\iota_{n+7}) = \pm 2i\nu_{n+3}$ (resp. $\pm iE^n\nu'$) for $n \geq 2$ (resp. $n=0$ or 1).

ii) $\Delta_3(\eta_{n+7}) = 0$ for $n \geq 0$.

Proof. By (10)₃ and (13), we have i). So, $\Delta_3(\eta_{n+7}) = 2i\nu_{n+3}\eta_{n+6} = 0$ for $n \geq 2$ and $\Delta_3(\eta_{n+7}) = iE^n(\nu'\eta_6) = iE^n(\eta_3\nu_4) = 0$ for $n=0$ or 1 . This completes the proof.

Using (9)_{3,6}, (9)_{3,7}, Lemma 8.6, (3), Propositions 8.2 and 8.4, we have the following

Proposition 8.7. i) $\pi_6(EP^3) \approx 0$.

ii) $\pi_7(E^2P^3) = \{i\nu_4\} \approx \mathbf{Z}$.

iii) $\pi_{n+6}(E^{n+1}P^3) = \{i\nu_{n+3}\} \approx \mathbf{Z}_2$ for $n \geq 2$.

iv) $\pi_{n+7}(E^{n+1}P^3) = \{E^n\zeta_3, i\nu_{n+3}\eta_{n+6}\} \approx \mathbf{Z} + \mathbf{Z}_2$ for $n=1$ or 2 .

v) $\pi_{n+7}(E^{n+1}P^3) = \{E^n\zeta_3\} \approx \mathbf{Z}$ for $n \geq 3$.

9. Determination of $\pi_8(EP^m)$ for $m=2$ and 3 . In this section we use the following [15]: $3\nu' \in \{\eta_3, 2\iota_4, \eta_4\} \bmod 6\nu'$, $\pi_7(S^3) = \{\nu'\eta_6\} \approx \mathbf{Z}_2$ and $\pi_8(S^3) = \{\nu'\eta_6^2\} \approx \mathbf{Z}_2$.

By (11), Propositions 6.3 and 8.1. iv), we have the following

Lemma 9.1. $\zeta_2 \in \{i, \eta_3, 2\iota_4\} \bmod 2\zeta_2$.

Proposition 9.2. i) $\lambda_3 \equiv \pm \zeta_2\nu_5 \bmod \xi\eta_7$.

ii) $\pi_8(EP^2) = \{\zeta_2\nu_5, \xi\eta_7\} \approx \mathbf{Z}_{12} + \mathbf{Z}_2$.

Proof. By (4), we have the commutative diagram

$$\begin{array}{ccccccc} & & & \pi_8(EP^2) & & & \\ & i_* \nearrow & & \downarrow j_* & \nwarrow p_* & & \\ \pi_8(S^3) & \xrightarrow{i'_*} & \pi_8(SU(3)) & \xrightarrow{p'_*} & \pi_8(S^5) & \xrightarrow{\Delta'} & \pi_7(S^3), \end{array}$$

where the horizontal sequence is exact.

Since $i'_*\pi_8(S^3) = \{j_*(i_*\nu'\eta_6^2)\} = 0$, p'_* is a monomorphism. Therefore, $\text{Ker } p_* = \text{Ker } j_*$. By (5)' and (11), $\Delta'(\nu_5) = \eta_3\nu_4$, and so $\text{Im } p'_* = \{2\nu_5\}$. Therefore, by (3) and Proposition 7.2,

$$p_*(\zeta_2\nu_5) = 2\nu_5 = p_*(\pm\lambda_3).$$

On the other hand, we consider the exact sequence (8) for $n=3$:

$$0 \longrightarrow \pi_9(SU(3), EP^2) \xrightarrow{\delta} \pi_8(EP^2) \xrightarrow{j_*} \pi_8(SU(3)) \longrightarrow 0.$$

By [3] and [17], $\pi_9(SU(3), EP^2) \approx \pi_9(S^8) \approx \mathbb{Z}_2$. By inspecting the proof of Proposition 8.3, we obtain that $\text{Im } \delta = \{\xi\eta_7\}$. This leads us to i).

By Proposition 1.4 of [15] and by Lemma 9.1, $12\zeta_2\nu_5 = \zeta_2\eta_5^3 \in \{i, \eta_3, 2\iota_4\} \circ \eta_5^3 = -i\{\eta_3, 2\iota_4, \eta_4\} \circ \eta_5^3 = i\nu'\eta_5^3 = 0 \pmod{2\zeta_2\eta_5^3} = 0$. So the order of $\zeta_2\nu_5$ is 12 by i) and by Proposition 7.2. iii). This completes the proof.

Lemma 9.3. i) $\zeta_3 \in \pm\{i', i\nu', 6\iota_6\} \pmod{\{i'\xi, 6\zeta_3\}}$.
ii) $E\pi_3 \in \{i', i\nu', \eta_6\} \pmod{\{i'\zeta_2\nu_5, i'\xi\eta_7\}}$.

Proof. By Proposition 1.2 of [15] and by (13), Propositions 6.3 and 8.3, we have i).

By Proposition 1.4 of [15] and by i), $\zeta_3\eta_7 \in \{i', i\nu', 6\iota_6\} \circ \eta_7 = -i'\{i\nu', 6\iota_6, \eta_6\} \subset i'_*\pi_8(EP^2)$. By Proposition 1.2 of [15], (13) and Proposition 6.5. i), $E\pi_3 \in \{i', \pm i\nu', \eta_6\} \supset \{i', i\nu', \eta_6\} \pmod{i'_*\pi_8(EP^2) + \pi_7(EP^3) \circ \eta_7}$. So we have ii) by Propositions 8.3. ii) and 9.2. ii). This completes the proof.

To determine the group structure of $\pi_8(EP^3)$, we need the following

Lemma 9.4. *The 10-skeleton of the complex $SU(n)/EP^{n-1}$ for $n \geq 4$ is $S^8 \vee S^{10}$.*

Proof. As is well known, the cohomology ring $H^*(SU(n); \mathbb{Z}_2)$ is isomorphic to the exterior algebra $E_{\mathbb{Z}_2}[x_2, \dots, x_n]$, where $\deg. x_k = 2k - 1$. By abuse of notation, we denote by $e^3, e^5, \dots, e^{2n-1}$ the cohomology classes corresponding to the generators x_2, x_3, \dots, x_n .

Let $Sq^2: H^*(X; \mathbb{Z}_2) \rightarrow H^*(X; \mathbb{Z}_2)$ be the squaring operation [12]. Since the 7-skeleton of $SU(n)$ is EP^3 , $Sq^2 e^3 = e^5$ and $Sq^2 e^5 = 0$. By the Cartan formula, $Sq^2(e^3 e^5) = (e^5)^2 = 0$. This completes the proof.

Proposition 9.5. i) $2E\pi_3 \equiv \pm i'\zeta_2\nu_5 \pmod{i'\xi\eta_7}$.
ii) $\pi_8(EP^3) = \{E\pi_3, i'\xi\eta_7\} \approx \mathbb{Z}_{24} + \mathbb{Z}_2$.

Proof. By [3] and Lemma 9.4, $\pi_9(SU(4), EP^3) \approx \pi_9(SU(4)/EP^3) \approx \pi_9(S^8 \vee S^{10}) \approx \mathbb{Z}_2$. So, by (6) for $n=4$, $\pi_8(EP^3) \approx \mathbb{Z}_{24} + \mathbb{Z}_2$. Hence, by Theorem 1.2, Propositions 6.5. ii) and 9.2, we have the assertion.

10. Determination of $\pi_{n+k}(E^{n+1}P^m)$ for $k=8, 9$ and $n \geq 3$. Hereafter we use the following [15]: $\pi_{10}(S^5) = \{\nu_5\eta_5^8\} \approx \mathbb{Z}_2$, $\pi_{11}(S^6) = \{[\iota_6, \iota_6]\} \approx \mathbb{Z}$ and $\pi_{n+5}(S^n) \approx 0$ for $n \geq 7$, where $[\iota_6, \iota_6]$ denotes the Whitehead product.

We consider the exact sequence (9)_{2,8} for $n \geq 2$:

$$\pi_{n+8}(S^{n+3}) \xrightarrow{i_*} \pi_{n+8}(E^{n+1}P^2) \xrightarrow{p_*} \pi_{n+8}(S^{n+5}) \xrightarrow{\Delta} \pi_{n+7}(S^{n+3}).$$

By (15), p_* is an epimorphism. So there exists an element $\alpha \in \pi_{10}(E^3P^2)$ such that $p_*\alpha = -\nu_7$. We write $\tilde{\nu}_{n+7} = E^n\alpha$ for $n \geq 0$. By Lemmas 6.1, 6.2 and Proposition 8.1. iii), we have

$$(16) \quad \begin{aligned} p_*\tilde{\nu}_{n+5} &= -\nu_{n+5} \text{ for } n \geq 2, \\ \tilde{\nu}_7 &\in \{i, \eta_5, \nu_6\} \bmod \{i\nu_5\eta_8^2, E^2(\zeta_2\nu_5)\}, \text{ and} \\ \tilde{\nu}_{n+5} &\in \{i, \eta_{n+3}, \nu_{n+4}\} \bmod E^n(\zeta_2\nu_5) \text{ for } n \geq 4. \end{aligned}$$

Lemma 10.1. $\tilde{\nu}_n$ is of order 24 for $n \geq 7$.

Proof. It is sufficient to show that $24\tilde{\nu}_7=0$. By Proposition 1.4 of [15] and by (16), $24\tilde{\nu}_7 \in \{i, \eta_5, \nu_6\} \circ 24\iota_{10} = -i\{\eta_5, \nu_6, 24\iota_9\}$. By Proposition 1.2 of [15], $\{\eta_5, \nu_6, 24\iota_9\} \subset \{\eta_5, 12\nu_6, 2\iota_9\} = \{\eta_5, \eta_8^2, 2\iota_9\} \supset \{\eta_5^2, \eta_8, 2\iota_9\} = \{12\nu_5, \eta_8, 2\iota_9\} \supset 2\{6\nu_5, \eta_8, 2\iota_9\} \subset 2\pi_{10}(S^5) = 0$. So we have $\{i\eta_5, \nu_6, 24\iota_9\} \ni 0 \bmod \eta_5 \circ \pi_{10}(S^6) + 2\pi_{10}(S^5) = 0$. Therefore $24\tilde{\nu}_7=0$. This completes the proof.

By (9)_{2,8}, (16) and Lemma 10.1, we have the following

Proposition 10.2. i) $\pi_{11}(E^4P^2) = \{i[\iota_6, \iota_6], \tilde{\nu}_8\} \approx \mathbf{Z} + \mathbf{Z}_{24}$.
ii) $\pi_{n+8}(E^{n+1}P^2) = \{\tilde{\nu}_{n+5}\} \approx \mathbf{Z}_{24}$ for $n \geq 4$.

Lemma 10.3. i) $2E^3\pi_3 \equiv \pm 2i'\tilde{\nu}_7 \bmod i\nu_5\eta_8^2$.
ii) $2E^{n+1}\pi_3 = \pm 2i'\tilde{\nu}_{n+5}$ for $n \geq 3$.

Proof. By (3) and (16), $p_*(2\tilde{\nu}_7 + E^2(\zeta_2\nu_5)) = 0$. So, by (9)_{2,8} for $n=2$, $2\tilde{\nu}_7 \equiv -E^2(\zeta_2\nu_5) \bmod i\nu_5\eta_8^2$. Therefore, by Proposition 9.5. i) and Lemma 8.5, we have the assertion.

By Lemma 8.6. ii), we have

$$(17) \quad \Delta_3(\eta_{n+7}^2) = 0 \text{ for } n \geq 1.$$

By (9)_{3,8}, (9)_{3,8}, Lemma 8.6. ii) and (17), we have a short exact sequence for $n \geq 1$:

$$0 \longrightarrow \pi_{n+8}(E^{n+1}P^2) \xrightarrow{i'_*} \pi_{n+8}(E^{n+1}P^3) \xrightarrow{p'_*} \pi_{n+8}(S^{n+7}) \longrightarrow 0,$$

where $p' = E^{n+1}p_3$. So, by Proposition 10.2 and Lemma 10.3, we have the following

Proposition 10.4. i) $\pi_{11}(E^4P^3) = \{i[\iota_6, \iota_6], i'\tilde{\nu}_8, E^4\pi_3\} \approx \mathbf{Z} + \mathbf{Z}_{24} + \mathbf{Z}_2$.
ii) $\pi_{n+8}(E^{n+1}P^3) = \{i'\tilde{\nu}_{n+5}, E^{n+1}\pi_3\} \approx \mathbf{Z}_{24} + \mathbf{Z}_2$ for $n \geq 4$.

Hereafter we use the following [15] : $\pi_{n+6}(S^n) = \{\nu_n^2\} \approx \mathbf{Z}_2$ for $n \geq 5$. By (9)_{2,9} and (9)_{2,9}', we have the following

Proposition 10.5. $\pi_{n+9}(E^{n+1}P^2) = \{i\nu_{n+3}^2\} \approx \mathbf{Z}_2$ for $n \geq 3$.

By Lemma 8.6. i), we have

$$(18) \quad \Delta_3(\nu_{n+7}) = 0 \text{ for } n \geq 2.$$

By (9)_{3,9}, (9)_{3,9}', (17) and (18), we have a short exact sequence for $n \geq 2$:

$$0 \longrightarrow \pi_{n+9}(E^{n+1}P^2) \xrightarrow{i'_*} \pi_{n+9}(E^{n+1}P^3) \xrightarrow{p'_*} \pi_{n+9}(S^{n+7}) \longrightarrow 0.$$

So, by Lemma 6.4 and Proposition 10.5, we have the following

Proposition 10.6. $\pi_{n+9}(E^{n+1}P^3) = \{i\nu_{n+3}^2, E^{n+1}\pi_3 \circ \eta_{n+8}\} \approx \mathbf{Z}_2 + \mathbf{Z}_2$ for $n \geq 3$.

By (10)₄, we have

$$(19) \quad \Delta_4(\iota_{n+9}) = E^{n+1}\pi_3 \text{ and } \Delta_4(\eta_{n+9}) = E^{n+1}\pi_3 \circ \eta_{n+8} \text{ for } n \geq 0.$$

Using (9)_{4,8}, (9)_{4,9}, (19), Lemma 10.3, (3), Propositions 9.5. ii), 10.4 and 10.6, we have the following

Proposition 10.7. i) $\pi_8(EP^4) = \{i'\xi\eta_7\} \approx \mathbf{Z}_2$.

ii) $\pi_{11}(E^4P^4) = \{i[\iota_6, \iota_6], i'\nu_8\} \approx \mathbf{Z} + \mathbf{Z}_2$.

iii) $\pi_{n+8}(E^{n+1}P^4) = \{i'\nu_{n+5}\} \approx \mathbf{Z}_2$ for $n \geq 4$.

iv) $\pi_{n+9}(E^{n+1}P^4) = \{E^n\zeta_4, i\nu_{n+3}^2\} \approx \mathbf{Z} + \mathbf{Z}_2$ for $n \geq 3$.

11. Determination of $\pi_{n+10}(E^{n+1}P^m)$ for $n \geq 4$. Hereafter we use the following [15] : $\pi_9(S^3) = \{\alpha_1(3)\alpha_1(6)\} \approx \mathbf{Z}_3$, where $\alpha_1(3) = 4\nu'$ and $\alpha_1(n) = E^{n-3}\alpha_1(3)$ for $n \geq 3$. $\nu'\nu_6 = \alpha_1(3)\alpha_1(6)$. $\pi_{10}(S^3) = \{\alpha_2(3), \ddot{\alpha}_1(3)\} \approx \mathbf{Z}_{15}$, where $\alpha_2(3)$ (resp. $\ddot{\alpha}_1(3)$) denotes the generator of the direct summand \mathbf{Z}_3 (resp. \mathbf{Z}_5) of $\pi_{10}(S^3)$.

Lemma 11.1. i) $p'_*\lambda_4 = \pm 3\nu_7$.

ii) $\lambda_4 \in \pm\{i', i\nu', 3\nu_6\} \bmod \{3\zeta_3\nu_7\} + \text{Ker } p'_*$.

iii) $8\lambda_4 \equiv 0 \bmod 8(\text{Ker } p'_*)$.

Proof. We consider the commutative diagram

$$\begin{array}{ccccccc} \pi_{10}(EP^2) & \xrightarrow{i'_*} & \pi_{10}(EP^3) & & & & \\ \downarrow j_{3*} & & \downarrow j_{4*} & \searrow p'_* & & & \\ 0 \longrightarrow & \pi_{10}(SU(3)) & \xrightarrow{i''_*} & \pi_{10}(SU(4)) & \xrightarrow{p''_*} & \pi_{10}(S^7) & \xrightarrow{\Delta'} \pi_9(SU(3)), \end{array}$$

where the horizontal sequence is exact.

By [8], $\pi_9(SU(3)) \approx \mathbf{Z}_3$ and $\pi_{10}(SU(3)) \approx \mathbf{Z}_{30}$. By the proof of Proposition 7.3, by (5)' and (13), $\mathcal{A}'(\nu_7) = \pm j_3 i \nu' \nu_6$ generates $\pi_9(SU(3))$. So, by Proposition 7.2, we have i).

Using Proposition 1.2 of [15], (13), Lemmas 6.1, 6.2 and Proposition 8.3. ii), we have ii).

By Propositions 1.2 and 1.4 of [15], $\pm 8\lambda_4 \in \{i', i\nu', 3\nu_6\} \circ 8\iota_{10} = -i'\{i\nu', 3\nu_6, 8\iota_9\} \supset -i'i\{\nu', 3\nu_6, 8\iota_9\} \subset i_*\pi_{10}(S^3) = 8i_*\pi_{10}(S^3) \subset 8(\text{Ker } p'_*)$. This leads us to iii). This completes the proof.

Hereafter we use the following [15]: $\pi_{14}(S^7) = \{\sigma'\} \approx \mathbf{Z}_{120}$, $\pi_{15}(S^8) = \{\sigma_8, E\sigma'\} \approx \mathbf{Z} + \mathbf{Z}_{120}$ and $\pi_{n+7}(S^n) = \{\sigma_n\} \approx \mathbf{Z}_{240}$ for $n \geq 9$. $40\sigma' = \alpha_2(7)$ and $24\sigma' = \check{\alpha}_1(7)$, where $\alpha_2(n) = E^{n-3}\alpha_2(3)$ and $\check{\alpha}_1(n) = E^{n-3}\check{\alpha}_1(3)$ for $n \geq 3$. $2\sigma_n = E^{n-7}\sigma'$ for $n \geq 9$. $\alpha_2(n) \in 2\{\alpha_1(n), \alpha_1(n+3), 3\iota_{n+6}\}$ for $n \geq 5$ (cf. (13.8) of [15]).

By (15), we have

$$(20) \quad \mathcal{A}_2(\nu_{n+5}^2) = 0 \text{ for } n \geq 5.$$

By $(9)_{2,10}$, $(9)'_{2,10}$ and (20), we have the following

Proposition 11.2. i) $\pi_{14}(E^5 P^2) = \{i\sigma'\} \approx \mathbf{Z}_{120}$.

ii) $\pi_{15}(E^6 P^2) = \{i\sigma_8, iE\sigma'\} \approx \mathbf{Z} + \mathbf{Z}_{120}$.

iii) $\pi_{n+10}(E^{n+1} P^2) = \{i\sigma_{n+3}\} \approx \mathbf{Z}_{240}$ for $n \geq 6$.

We consider the exact sequence $(9)'_{3,10}$ for $n \geq 2$:

$$\pi_{n+10}(E^{n+1} P^2) \xrightarrow{i'_*} \pi_{n+10}(E^{n+1} P^3) \xrightarrow{p'_*} \pi_{n+10}(S^{n+7}) \xrightarrow{\Delta} \pi_{n+9}(E^{n+1} P^2).$$

By (18), p'_* is an epimorphism. So there exists an element $\beta \in \pi_{12}(E^3 P^3)$ such that $p'_*\beta = -\nu_9$. We write $\check{\nu}_{n+9} = E^n \beta$ for $n \geq 0$. Using Proposition 1.2 of [15], Lemmas 6.1, 6.2, (13) and Proposition 8.7, we have

$$(21) \quad \begin{aligned} & p'_*\check{\nu}_{n+7} = -\nu_{n+7}, \text{ and} \\ & \check{\nu}_{n+7} \in \pm\{i', 2i\nu_{n+3}, \nu_{n+6}\} \bmod \{E^n(\xi_3\nu_7)\} + i'_*\pi_{n+10}(E^{n+1} P^2) \text{ for } n \geq 2. \end{aligned}$$

Lemma 11.3. i) $3\check{\nu}_{n+7} \equiv \pm E^n \lambda_4 \bmod iE^{n-4}\sigma'$ for $n \geq 4$.

ii) $24\check{\nu}_{n+7} \equiv \pm i\alpha_2(n+3) \bmod i\check{\alpha}_1(n+3)$ for $n \geq 4$.

Proof. By (21) and Lemma 11.1, $3p'_*\check{\nu}_{11} = p'_*(\pm E^4 \lambda_4) = -3\nu_{11}$. So, by Proposition 11.2, $3\check{\nu}_{11} \pm E^4 \lambda_4 \in \text{Ker } p'_* = i'_*\pi_{14}(E^5 P^2) = \{i\sigma'\}$. This leads us to i).

By Proposition 1.4 of [15] and by (21), $\pm 24\check{\nu}_{n+7} \in \{i', 2i\nu_{n+3}, \nu_{n+6}\}$

$\circ 24\iota_{n+10} = -i'\{2i\nu_{n+3}, \nu_{n+6}, 24\iota_{n+9}\} \bmod 24i'_*\pi_{n+10}(E^{n+1}P^2)$ for $n \geq 2$. By Proposition 1.2 of [15], $\{2i\nu_{n+3}, \nu_{n+6}, 24\iota_{n+9}\} \subset \{2i\nu_{n+3}, 8\nu_{n+6}, 3\iota_{n+9}\} = \{2i\alpha_1(n+3), \alpha_1(n+6), 3\iota_{n+9}\} \supset 2i\{\alpha_1(n+3), \alpha_1(n+6), 3\iota_{n+9}\} \ni i\alpha_2(n+3) \bmod 3\pi_{n+10}(E^{n+1}P^2)$. Therefore, $24\tilde{\nu}_{n+7} \equiv \pm i\alpha_2(n+3) \bmod 3i'_*\pi_{n+10}(E^{n+1}P^2)$ for $n \geq 2$. On the other hand, $24\tilde{\nu}_{11} \equiv 0 \bmod 8i\sigma'$ by i), Lemma 11.1. iii) and Proposition 11.2. i). This leads us to ii).

By $(9)_{3,10}$, $(9)_{3,10}$, Proposition 11.2, (21) and Lemma 11.3, we have the following

- Proposition 11.4.** i) $\pi_{14}(E^5P^3) = \{i\sigma', \tilde{\nu}'_{11}\} \approx \mathbf{Z}_8 + \mathbf{Z}_{360}$.
 ii) $\pi_{15}(E^6P^3) = \{i\sigma_8, iE\sigma', \tilde{\nu}'_{12}\} \approx \mathbf{Z} + \mathbf{Z}_8 + \mathbf{Z}_{360}$.
 iii) $\pi_{n+10}(E^{n+1}P^3) = \{i\sigma_{n+3}, \tilde{\nu}'_{n+7}\} \approx \mathbf{Z}_{16} + \mathbf{Z}_{360}$ for $n \geq 6$.

- Lemma 11.5.** i) $\mathcal{A}_4(\eta_{n+9}^2) = E^{n+1}\pi_3 \circ \eta_{n+8}^2$ for $n \geq 1$.
 ii) $E^{n+1}\pi_3 \circ \eta_{n+8}^2 = 180(\tilde{\nu}'_{n+7} + aiE^{n-4}\sigma')$ for $n \geq 4$, where $a=0$ or 1 .

Proof. By $(10)_4$, we have i). By Lemma 6.4 and (21),

$$E^{n+1}\pi_3 \circ \eta_{n+8}^2 - 12\tilde{\nu}_{n+7} \in \text{Ker } p'_* = i'_*\pi_{n+10}(E^{n+1}P^2)$$

for $n \geq 2$. So, by Proposition 11.2 and Lemma 11.3, $E^5\pi_3 \circ \eta_{12}^2 \equiv 12\tilde{\nu}'_{11} \bmod 4i\sigma'$. Therefore, $E^5\pi_3 \circ \eta_{12}^2 \equiv 180\tilde{\nu}'_{11} \bmod 60i\sigma' = 180i\sigma'$. This completes the proof.

Let $i'' = E^{n+1}i_4: E^{n+1}P^3 \longrightarrow E^{n+1}P^4$ for $n \geq 0$. Then we have the following

- Proposition 11.6.** i) $\pi_{14}(E^5P^4) = \{i\sigma', i''\tilde{\nu}'_{11}\} \approx \mathbf{Z}_8 + \mathbf{Z}_{180}$.
 ii) $\pi_{15}(E^6P^4) = \{i\sigma_8, iE\sigma', i''\tilde{\nu}'_{12}\} \approx \mathbf{Z} + \mathbf{Z}_8 + \mathbf{Z}_{180}$.
 iii) $\pi_{n+10}(E^{n+1}P^4) = \{i\sigma_{n+3}, i''\tilde{\nu}'_{n+7}\} \approx \mathbf{Z}_{16} + \mathbf{Z}_{180}$ for $n \geq 6$.

Proof. Using $(9)_{4,10}$, $(9)_{4,10}$, (19), Proposition 10.6 and Lemma 11.5. i), we have a short exact sequence for $n \geq 1$:

$$0 \longrightarrow \pi_{n+11}(S^{n+9}) \xrightarrow{\Delta} \pi_{n+10}(E^{n+1}P^3) \xrightarrow{i''_*} \pi_{n+10}(E^{n+1}P^4) \longrightarrow 0.$$

Hence, by Proposition 11.4 and Lemma 11.5. ii), we have the assertion.

- Lemma 11.7.** i) $\mathcal{A}_5(\iota_{n+11}) = E^{n+1}\pi_4$ for $n \geq 0$.
 ii) $E^{n+1}\pi_4 = \pm 3(i''\tilde{\nu}'_{n+7} + aiE^{n-4}\sigma') + 15biE^{n-4}\sigma'$ for $n \geq 4$, where a is same as in Lemma 11.5 and b is an odd integer.

Proof. By $(10)_5$, we have i). By Proposition 6.5. ii) and Lemma 11.3.

i), we have

$$E^{n+1}\pi_4 \equiv \pm 3i''\tilde{\nu}'_{n+7} \bmod iE^{n-4}\sigma'$$

for $n \geq 4$. By Theorem 1.2, $E^{n+1}\pi_4$ is of order 120. Hence, by Lemma 11.5. ii), Proposition 11.6 and its proof, we have ii). This completes the proof.

By (9)_{5,10}, Proposition 11.6 and Lemma 11.7, we have the following

Proposition 11.8. i) $\pi_{14}(E^5P^5) = \{i\sigma', i''\tilde{\nu}'_{11}\} \approx \mathbf{Z}_{12}$.

ii) $\pi_{15}(E^6P^5) = \{i\sigma_8, iE\sigma', i''\tilde{\nu}'_{12}\} \approx \mathbf{Z} + \mathbf{Z}_{12}$.

iii) $\pi_{n+10}(E^{n+1}P^5) = \{i\sigma_{n+3}, i''\tilde{\nu}'_{n+7}\} \approx \mathbf{Z}_{24}$ for $n \geq 6$.

12. Determination of $\pi_{n+11}(E^{n+1}P^m)$ for $n \geq 6$. Hereafter we use the following [15]: $\pi_{n+8}(S^n) = \{\varepsilon_n\} \approx \mathbf{Z}_2$ for $3 \leq n \leq 5$, $\pi_{17}(S^9) = \{\sigma_9\eta_{16}, \varepsilon_9, \tilde{\nu}_9\} \approx \mathbf{Z}_2 + \mathbf{Z}_2 + \mathbf{Z}_2$ and $\pi_{n+8}(S^n) = \{\varepsilon_n, \tilde{\nu}_n\} \approx \mathbf{Z}_2 + \mathbf{Z}_2$ for $n \geq 10$. $\sigma'\eta_{14} + \varepsilon_7 + \tilde{\nu}_7 = \eta_7\sigma_8$ and $\varepsilon_n + \tilde{\nu}_n = \eta_n\sigma_{n+1}$ for $n \geq 9$. $\tilde{\nu}_n = \{\nu_n, \eta_{n+3}, \nu_{n+4}\}$ for $n \geq 7$.

The following secondary compositions contain ε_n : $\{\eta_n, E^{n-2}\nu', 3\nu_{n+4}\}$ for $n \geq 3$; $\{\eta_n, 2\nu_{n+1}, \nu_{n+1}^2\}$ and $\{\eta_n, 2\nu_{n+1}, \nu_{n+4}\}$ for $n \geq 4$; $\{\eta_n, \nu_{n+1}, 2\nu_{n+4}\}$ and $\{2\nu_n, \nu_{n+3}, \eta_{n+6}\}$ for $n \geq 5$.

Lemma 12.1. $2\tilde{\nu}_{n+5}\nu_{n+8} = i\varepsilon_{n+3}$ for $n \geq 2$.

Proof. By Proposition 1.4 of [15] and by (16),

$$2\tilde{\nu}_7\nu_{10} \in \{i, \eta_5, \nu_6\} \circ 2\nu_{10} = -i\{\eta_5, \nu_6, 2\nu_9\} \ni i\varepsilon_5$$

$\bmod (i\eta_5)_*\pi_{13}(S^6) + i_*\pi_{10}(S^5) \circ 2\nu_{10} = 0$. So we have the assertion for $n=2$. This completes the proof.

By (10)₂ and (11), we have

$$(22) \quad \Delta_2(\sigma_{n+5}) = \eta_{n+3}\sigma_{n+4} \text{ for } n \geq 6.$$

Using (9)_{2,11}, (9)_{2,11}, (16), (20), (22) and Lemma 12.1, we have the following

Proposition 12.2. i) $\pi_{17}(E^7P^2) = \{i\sigma_9\eta_{16}, \tilde{\nu}_{11}\nu_{14}\} \approx \mathbf{Z}_2 + \mathbf{Z}_4$.

ii) $\pi_{n+11}(E^{n+1}P^2) = \{\tilde{\nu}_{n+5}\nu_{n+8}\} \approx \mathbf{Z}_4$ for $n \geq 7$.

By (9)_{3,11} and Proposition 12.2, we have the following

Proposition 12.3. i) $\pi_{17}(E^7P^3) = \{i\sigma_9\eta_{16}, i'\tilde{\nu}_{11}\nu_{14}\} \approx \mathbf{Z}_2 + \mathbf{Z}_4$.

ii) $\pi_{n+11}(E^{n+1}P^3) = \{i'\tilde{\nu}_{n+5}\nu_{n+8}\} \approx \mathbf{Z}_4$ for $n \geq 7$.

- Lemma 12.4.** i) $E\pi_3 \circ \nu_8 \in i'\{i, \eta_3, \nu_4^2\} \bmod \{i\varepsilon_3, i'\zeta_2\nu_5^2\}$.
 ii) $\Delta_3(\nu_{n+9}) = E^{n+1}\pi_3 \circ \nu_{n+8} = \pm i'\tilde{\nu}_{n+5}\nu_{n+8}$ for $n \geq 2$.

Proof. By Proposition 1.4 of [15] and by Lemma 9.3. ii),

$$E\pi_3 \circ \nu_8 \in \{i', i\nu', \eta_6\} \circ \nu_8 = -i'\{i\nu', \eta_6, \nu_7\}$$

$\bmod i'\zeta_2\nu_5^2$. By Proposition 1.2 of [15], $\{i\nu', \eta_6, \nu_7\} \subset \{i, \nu'\eta_6, \nu_7\} = \{i, \eta_3\nu_4, \nu_7\} \supset \{i, \eta_3, \nu_4^2\} \bmod i_*\pi_{11}(S^3) + \pi_8(EP^2) \circ \nu_8$. So, by Proposition 9.2. ii), we have i).

By (10)₄, we have the first equality of ii). By Proposition 1.2 of [15], by (16) and Proposition 8.1. iv),

$$\tilde{\nu}_7\nu_{10} \in \{i, \eta_5, \nu_6\} \circ \nu_{10} \subset \{i, \eta_5, \nu_6^2\}$$

$\bmod i_*\pi_{13}(S^5) + \pi_7(E^3P^2) \circ \nu_7^2 = \{i\varepsilon_5, E^2(\zeta_2\nu_5^2)\}$. So, by Proposition 1.3 of [15] and by i),

$$E^3\pi_3 \circ \nu_{10} \equiv i'\tilde{\nu}_7\nu_{10} \bmod \{i\varepsilon_5, i'E^2(\zeta_2\nu_5^2)\}.$$

By Propositions 1.3, 1.4 of [15] and by Lemma 9.1,

$$E(\zeta_2\nu_5^2) \in -\{i, \eta_4, 2\varepsilon_5\} \circ \nu_5^2 = i\{\eta_4, 2\varepsilon_5, \nu_5^2\} \ni i\varepsilon_4 \bmod 0.$$

So we obtain that $E(\zeta_2\nu_5^2) = i\varepsilon_4$. Therefore, by Lemma 12.1, we have the second equality of ii) for $n=2$. This completes the proof.

Using (9)_{4,11}, (9)_{4,11}', Lemmas 11.5, 12.4 and Proposition 12.3, we have the following

- Proposition 12.5.** i) $\pi_{17}(E^7P^4) = \{i\sigma_9\eta_{16}\} \approx \mathbb{Z}_2$.
 ii) $\pi_{n+11}(E^{n+1}P^4) \approx 0$ for $n \geq 7$.

- Lemma 12.6.** i) $\Delta_5(\eta_{n+11}) = E^{n+1}\pi_4 \circ \eta_{n+10}$ for $n \geq 0$.
 ii) $E^5\pi_4 \circ \eta_{14} \equiv 0 \bmod i\sigma'\eta_{14}$.
 iii) $E^{n+1}\pi_4 \circ \eta_{n+10} = 0$ for $n \geq 6$.

Proof. By (10)₅, we have i). By Lemma 11.7. ii),

$$E^5\pi_4 \circ \eta_{14} \equiv i''\tilde{\nu}_{11}'\eta_{14} \bmod i\sigma'\eta_{14}.$$

By Propositions 1.2, 1.4 of [15] and by (21),

$$\begin{aligned} \tilde{\nu}_{11}'\eta_{14} &\in \{i', 2i\nu_7, \nu_{10}\} \circ \eta_{14} \\ &= -i'\{2i\nu_7, \nu_{10}, \eta_{13}\} \\ &\supset -i'i\{2\nu_7, \nu_{10}, \eta_{13}\} \\ &\ni i\varepsilon_7 \bmod i_*\pi_{14}(E^5P^2) \circ \eta_{14}. \end{aligned}$$

So, by Proposition 11.2. i), $E^5 \pi_4 \circ \eta_{14} \equiv i\epsilon_7 \pmod{i\sigma' \eta_{14}}$. By Lemmas 12.1, 12.4. ii) and by the exact sequence (9)_{4,11} for $n=2$, $i\epsilon_5 = 2i'' i' \tilde{\nu}_7 \nu_{10} = 2i'' \Delta_4(\nu_{11}) = 0$. This completes the proof.

Using (9)_{5,11}, (9)_{5,11}', Lemmas 11.7. i), 12.6, (3), Theorem 1.2 and Proposition 12.5, we have the following

Proposition 12.7. i) $\pi_{17}(E^7 P^5) = \{E^6 \zeta_5, i\sigma_9 \eta_{16}\} \approx \mathbf{Z} + \mathbf{Z}_2$.
 ii) $\pi_{n+11}(E^{n+1} P^5) = \{E^n \zeta_5\} \approx \mathbf{Z}$ for $n \geq 7$.

13. Proof of Theorem 1.1. In this section we use the following ([1], [4] and [15]): $J_C \pi_3(SU(n)) = \pi_{2n+3}(S^{2n}) = \{\nu_{2n}\} \approx \mathbf{Z}_{24}$ for $n \geq 3$, $J_C \pi_7(SU(n)) = 2\pi_{2n+7}(S^{2n}) = \{2\sigma_{2n}\} \approx \mathbf{Z}_{120}$ for $n \geq 5$, $J_C \pi_8(SU(n)) \approx 0$ for $n \geq 5$, $J_C \pi_9(SU(n)) = \{\eta_{2n}^2 \sigma_{2n+2}\} \approx \mathbf{Z}_2$ for $n \geq 6$. $J_C \pi_{11}(SU(n)) = \pi_{2n+11}(S^{2n}) = \{\check{\zeta}_{2n}\} \approx \mathbf{Z}_{504}$ for $n \geq 7$. $\pi_{n+9}(S^n) = \{\eta_n^2 \sigma_{n+2}, \nu_n^3, \mu_n\} \approx \mathbf{Z}_2 + \mathbf{Z}_2 + \mathbf{Z}_2$ for $n \geq 11$ and $\pi_{n+10}(S^n) = \{\eta_n \mu_{n+1}, \beta_1(n)\} \approx \mathbf{Z}_6$ for $n \geq 12$. $\nu_n \sigma_{n+3} = \sigma_n \nu_{n+7} = 0$ and $\{\nu_n, 6\nu_{n+3}, \nu_{n+6}\} = 0$ for $n \geq 12$ (cf. (7.13) and (7.14) of [15]). $\beta_1(n) \in \{\alpha_1(n), \alpha_1(n+3), \alpha_1(n+6)\}$ for $n \geq 5$.

Let c be an integer such that $(c, 24) = 1$. Then, by Theorem 4.3, we have the following

Proposition 13.1. i) $E^{2(n-2)} g_2 = c\nu_{2n}$ for $n \geq 3$.
 ii) $E^{2(n-3)} g_3 \circ E^{2n} \zeta_2 = 0$ for $n \geq 4$.
 iii) $\{E^{2(n-4)} g_4 \circ E^{2n} \zeta_3\} = \{2\sigma_{2n}\}$ for $n \geq 5$.
 iv) $E^{2(n-5)} g_5 \circ E^{2n} \zeta_4 = \eta_{2n}^2 \sigma_{2n+2}$ for $n \geq 6$.
 v) $\{E^{2(n-6)} g_6 \circ E^{2n} \zeta_5\} = \{\check{\zeta}_{2n}\}$ for $n \geq 7$.

Now, we are ready to prove Theorem 1.1.

It is trivial that $B_k \approx 0$ for $k=1, 2, 4$ or 5 .

By (1), Propositions 8.1. ii) and 13.1. i),

$$B_{3,n} = g_{n*} i_* \pi_{2n+3}(E^{2n+1} P^1) = \{E^{2(n-2)} g_2\} = \{\nu_{2n}\} \approx \mathbf{Z}_{24} \text{ for } n \geq 3.$$

Using (1), Propositions 8.7, 10.7. iv), 12.7. ii) and 13.1, we have the following: $B_{6,n} = \{\nu_{2n}^2\} \approx \mathbf{Z}_2$ for $n \geq 4$, $B_{7,n} = \{2\sigma_{2n}\} \approx \mathbf{Z}_{120}$ for $n \geq 5$, $B_{9,n} = \{\eta_{2n}^2 \sigma_{2n+2}, \nu_{2n}^3\} \approx \mathbf{Z}_2 + \mathbf{Z}_2$ for $n \geq 6$ and $B_{11,n} = \{\check{\zeta}_{2n}\} \approx \mathbf{Z}_{504}$ for $n \geq 7$.

By (1) and Proposition 10.7. iii), $B_{8,n}$ is generated by

$$E^{2(n-5)} g_5 \circ i' \tilde{\nu}_{2n+5} = E^{2(n-3)} g_3 \circ \tilde{\nu}_{2n+5} \text{ for } n \geq 5.$$

By (1) and Proposition 13.1. i), $E^{2(n-3)} g_3 | S^{2n+3} = c\nu_{2n}$ for $n \geq 3$. By Propositions 1.2, 1.7 of [15], by (11) and (16),

$$E^{2(n-3)}g_3 \circ \tilde{\nu}_{2n+5} \in \{c\nu_{2n}, \eta_{2n+3}, \nu_{2n+4}\} = \tilde{\nu}_{2n} \bmod 0 \text{ for } n \geq 4.$$

Therefore, $B_{8,n} = \{\tilde{\nu}_{2n}\} \approx \mathbf{Z}_2$ for $n \geq 5$.

Next we shall prove that $B_{10,n} = \{\beta_1(2n)\} \approx \mathbf{Z}_3$ for $n \geq 6$.

By (1) and Proposition 11.8. iii), $B_{10,n}$ is generated by

$$E^{2(n-2)}g_2 \circ \sigma_{2n+3} \text{ and } E^{2(n-4)}g_4 \circ \tilde{\nu}'_{2n+7}$$

for $n \geq 6$. By Proposition 13.1. i), $E^{2(n-2)}g_2 \circ \sigma_{2n+3} = c\nu_{2n}\sigma_{2n+3} = 0$ for $n \geq 6$. Using Propositions 1.2, 1.7 of [15], (1), (13), (21) and Proposition 13.1. i), we see that for $n \geq 4$

$$\begin{aligned} E^{2(n-4)}g_4 \circ \tilde{\nu}'_{2n+7} &\in \{E^{2(n-3)}g_3, \pm 2i\nu_{2n+3}, \nu_{2n+6}\} \\ &\supset \{\pm E^{2(n-3)}g_3 \circ i, 2\nu_{2n+3}, \nu_{2n+6}\} \\ &\supset \pm c\{\nu_{2n}, 2\nu_{2n+3}, \nu_{2n+6}\} \\ &= \pm c\{\alpha_1(2n), 2\alpha_1(2n+3), \alpha_1(2n+6)\} \\ &\ni \pm 2c\beta_1(2n) \\ &\bmod E^{2(n-3)}g_3 * \pi_{2n+10}(E^{2n+1}P^2) + \pi_{2n+7}(S^{2n}) \circ \nu_{2n+7}. \end{aligned}$$

By (1), Propositions 11.2. iii) and 13.1, the indeterminacy is equal to $\{\nu_{2n}\sigma_{2n+3}\} + \{\sigma_{2n}\nu_{2n+7}\} = 0$ for $n \geq 6$. Therefore we have

$$E^{2(n-4)}g_4 \circ \tilde{\nu}'_{2n+7} = \pm 2c\beta_1(2n) \text{ for } n \geq 6.$$

Hence, $B_{10,n} = \{\beta_1(2n)\} \approx \mathbf{Z}_3$ for $n \geq 6$. Thus the proof of Theorem 1.1 is complete.

Remark. The result of Theorems 1.1 and 4.3 overlaps with the ones of Becker-Schultz [2] and Knapp [6].

Finally, we state the result about generators of the stable homotopy groups of EP^∞ . We set

$$\pi_k^S(EP^\infty) = \lim_{n \rightarrow \infty} \pi_{n+k}(E^{n+1}P^\infty).$$

By summarizing Propositions 8.1, 8.7, 10.7, 11.8 and 12.7, we have the following

Theorem 1.3. $\pi_k^S(EP^\infty)$ for $k \leq 11$ and generators of them are listed in the following table.

| $k =$ | 1,2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|------------------------------|-----|--------------|---|---------------------|----------------|---------------------|-----------------|-----------------------------|----------------------------|---------------------|
| $\pi_k^S(EP^\infty) \approx$ | 0 | \mathbf{Z} | 0 | \mathbf{Z} | \mathbf{Z}_2 | \mathbf{Z} | \mathbf{Z}_2 | $\mathbf{Z} + \mathbf{Z}_2$ | \mathbf{Z}_{24} | \mathbf{Z} |
| <i>gen.</i> | | i | | $iE^\infty \zeta_2$ | $i\nu$ | $iE^\infty \zeta_3$ | $i'\tilde{\nu}$ | $iE^\infty \zeta_4, i\nu^2$ | $i\sigma, i''\tilde{\nu}'$ | $iE^\infty \zeta_5$ |

Remark. The above result about the group structure of $\pi_k^S(EP^\infty)$ overlaps with the results of Liulevicius [7] and Mosher [10].

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Additional remark, added in proof. Using the notion of the C -projectivity, Hideaki Ōshima obtained the following: $B_{13} = \pi_{13}^S(S^0)$ and $B_{15} = 2\pi_{15}^S(S^0) + H$, where $H=0$ or $\{\eta\kappa\} \approx \mathbb{Z}_2$.