Mathematical Journal of Okayama University

Volume 28, *Issue* 1 1986 *Article* 19

JANUARY 1986

Equivalence of module categories

Shoji Kyuno*

*Tohoku Gakuin University

Math. J. Okayama Univ. 28 (1986), 147−150

EQUIVALENCE OF MODULE CATEGORIES

To the memory of Professor Tadao Tannaka

Shoji KYUNO

Morita contexts and gamma rings are equivalent concepts ([2]). Therefore, the duality theory obtained in Morita contexts is interpreted in the terms of gamma rings and vice versa. Nobusawa [2] proved directly that when R and L contain the unities the categories of all R-modules and L-modules are equivalent, where R and L are the right operator ring and the left operator ring of a gamma ring of homomorphisms respectively. Furthermore, in [3] he obtained a generalization of one of Morita duality theorems, that is, if $R^2 = R$ and $L^2 = L$, then the categories of properly generated R-modules and L-modules are equivalent.

In this note, without the assumption $R^2 = R$ and $L^2 = L$, we shall prove the following theorem :

Theorem. *Let $(R, L, M, \Gamma, \tau, \mu)$ be a Morita context, in which τ and μ are surjective. It is not assumed that the rings R, L have unities nor that the modules are unitary. The categories of properly generated R-modules and L-modules are equivalent.*

We refer to Jacobson [1, p. 166] for the definition of a Morita context $(R, L, M, \Gamma, \tau, \mu)$, where R, L are rings, $M = {}_LM_R$ is an L-R-bimodule, $\Gamma = {}_R\Gamma_L$ is an R-L-bimodule. We shall use his notations, except that the products $\Gamma \times M$ to R and $M \times \Gamma$ to L will be denoted by γx and $x\gamma$ ($x \in M$ and $\gamma \in \Gamma$), since all relevant associative laws hold. It is not assumed that the rings have unities nor that the modules are unitary. But, we assume that τ and μ are surjective.

Let R be a ring and M be a right R-module. If it satisfies (1) $MR = M$, (2) $\{x \in M \mid xR = 0\} = \{0\}$, then according to Nobusawa [3] we say M is properly generated over R.

Let PGM(R) be a category of properly generated right modules over R where the morphisms are R-module homomorphisms. Similarly, PGM(L) denotes a category of properly generated right modules over L where the morphisms are L-module homomorphisms.

Proof of the theorem. Let $G \in$ ob PGM(R). Let A be a free additive

Mathematical Journal of Okayama University, Vol. 28 [1986], Iss. 1, Art. 19

abelian group generated by the set of ordered pairs (g, γ), where $g \in G$, $\gamma \in \Gamma$, and let B be the subgroup of elements $\sum_i m_i(g_i, \gamma_i) \in A$, where m_i are integers such that $\sum_i m_i g_i(\gamma_i x) = 0$ for all $x \in M$. Denote by $[G, \Gamma]$ the factor group A/B and, without causing any ambiguity, by $[g, \gamma]$ the coset $(g, \gamma)+B$. Every element in $[G, \Gamma]$ therefore can be expressed as a finite sum $\sum_i [g_i, \gamma_i]$. $[G, \Gamma]$ forms a right L-module with definition

$$\sum_i [g_i, \gamma_i] \sum_j x_j\beta_j = \sum_{i,j} [g_i(\gamma_i x_j), \beta_j]$$

for $\sum_i [g_i, \gamma_i] \in [G, \Gamma]$ and $\sum_j x_j\beta_j \in L$. It is well-defined, because $\sum_i [g_i, \gamma_i] = \sum_j [g'_j, \gamma'_j]$ means $\sum_i g_i(\gamma_i x) = \sum_j g'_j(\gamma'_j x)$ for any $x \in M$.

To see $[G, \Gamma] \in$ ob PGM(L), let $\sum_i [g_i, \gamma_i]$ be an element in $[G, \Gamma]$ such that $(\sum_i [g_i, \gamma_i])L = 0$, that is, $(\sum_i [g_i, \gamma_i])M\Gamma = 0$. By the definition, $\sum_i [g_i(\gamma_i M), \Gamma] = 0$, which implies $(\sum_i g_i(\gamma_i M))\Gamma M = 0$, that is, $(\sum_i g_i(\gamma_i M))R = 0$. Since $G \in$ ob PGM(R), $\sum_i g_i(\gamma_i M) = 0$. Hence, $\sum_i [g_i, \gamma_i] = 0$.

In addition, $[G, \Gamma]L = [G, \Gamma]M\Gamma = [G\Gamma M, \Gamma] = [GR, \Gamma] = [G, \Gamma]$. Therefore, $[G, \Gamma] \in$ ob PGM(L). Similarly, for $U \in$ ob PGM(L) we can define a right R-module $[U, M]$ and show that $[U, M] \in$ ob PGM(R).

An R-module homomorphism $f : A_R \to B_R$ determines an L-module homomorphism $g : [A, \Gamma]_L \to [B, \Gamma]_L$ by

$$g(\sum_i [a_i, \gamma_i]) = \sum_i [f(a_i), \gamma_i].$$

To see that g is well-defined, let $\sum_i [a_i, \gamma_i] = 0$. Then for any $\sum_j x_j\omega_j \in L$, $\sum_i [f(a_i), \gamma_i] \sum_j x_j\omega_j = \sum_{i,j} [f(a_i)(\gamma_i x_j), \omega_j] = \sum_{i,j} [f(a_i(\gamma_i x_j)), \omega_j] = \sum_j [f(\sum_i a_i(\gamma_i x_j)), \omega_j] = 0$. Hence, $\sum_i [f(a_i), \gamma_i] = 0$.

It is easy to see that g is an L-module homomorphism.

Similarly, an L-module homomorphism $h : U_L \to V_L$ determines an R-module homomorphism $k : [U, M]_R \to [V, M]_R$ by $k(\sum_j [u_j, x_j]) = \sum_j [h(u_j), x_j]$.

Let f_1 and f_2 be R-module homomorphisms such that $f_1 : A \to B$ and $f_2 : B \to C$. Let g_1 and g_2 be L-module homomorphisms determined by f_1 and f_2 respectively. Then, $f_2 f_1 : A \to C$ determines an L-module homomorphism $p : [A, \Gamma] \to [C, \Gamma]$ such that $p = g_2 g_1$. Indeed, for any $\sum_i [a_i, \gamma_i] \in [A, \Gamma]$ we have $p(\sum_i [a_i, \gamma_i]) = \sum_i [f_2 f_1(a_i), \gamma_i] = \sum_i [f_2(f_1(a_i)), \gamma_i] = g_2(\sum_i [f_1(a_i), \gamma_i]) = g_2 g_1(\sum_i [a_i, \gamma_i])$.

Clearly, $1_A : A \to A$ determines $1_{[A, \Gamma]} : [A, \Gamma] \to [A, \Gamma]$. Thus, we have functors F : PGM$(R) \to$ PGM(L) and H : PGM$(L) \to$ PGM(R), where for $A \in$ ob PGM(R) $F(A) = [A, \Gamma]$ and for $U \in$ ob PGM(L) $H(U)$

$= [U, M]$.

For any $A \in$ ob PGM(R) and $U \in$ ob PGM(L),

$$HF(A) = H([A,\Gamma]) = [[A,\Gamma],M]$$
$$\text{and } FH(U) = F([U,M]) = [[U,M],\Gamma].$$

Define the mapping $\eta_A : A = A(\Gamma M) \to [[A,\Gamma],M]$ by

$$a = \textstyle\sum_i a_i(\gamma_i x_i) \mapsto \sum_i [[a_i,\gamma_i],x_i].$$

We show that η_A is an isomorphism.

$$\begin{aligned}
a = \textstyle\sum_i a_i(\gamma_i x_i) = 0 &\iff (\textstyle\sum_i a_i(\gamma_i x_i))R = 0\\
&\qquad (\text{By } A \in \text{ob PGM}(R).)\\
&\iff (\textstyle\sum_i a_i(\gamma_i x_i))\Gamma M = 0\\
&\qquad (\text{By } R = \Gamma M.)\\
&\iff [\textstyle\sum_i a_i(\gamma_i x_i),\Gamma] = 0\\
&\qquad (\text{By the definition that a coset is } 0.)\\
&\iff \textstyle\sum_i [a_i,\gamma_i](x_i\Gamma) = 0\\
&\qquad (\text{By the definition of an } L\text{-module}\\
&\qquad\ \text{imposed on } [A,\Gamma].)\\
&\iff \textstyle\sum_i [[a_i,\gamma_i],x_i] = 0\\
&\qquad (\text{By the definition that a coset is } 0.)
\end{aligned}$$

Hence, η_A is a bijection of A onto $HF(A)$. It is easy to see that η_A is an R-module homomorphism.

For an R-module homomorphism $f: A_R \to B_R$ and for $a = \sum_i a_i(\gamma_i x_i) \in A$, we have

$$\begin{aligned}
HF(f)\eta_A(a) &= HF(f)\eta_A(\textstyle\sum_i a_i(\gamma_i x_i)) = HF(f)\sum_i [[a_i,\gamma_i],x_i]\\
&= \textstyle\sum_i [F(f)([a_i,\gamma_i]),x_i] = \sum_i [[f(a_i),\gamma_i],x_i]\\
&= \eta_B f(a).
\end{aligned}$$

Therefore, we have the following commutative diagram :

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] & HF(A) \arrow[d, "HF(f)"] \\
B \arrow[r, "\eta_B"'] & HF(B)
\end{tikzcd}$$

Thus, $HF \cong 1_{\mathrm{PGM}(R)}$.

Similarly, we have $FH \cong 1_{\mathrm{PGM}(L)}$.
This completes the proof.

REFERENCES

[1] N. JACOBSON: Basic Algebra II, Freeman, San Francisco, 1980.
[2] N. NOBUSAWA: On duality in Γ-rings, Math. J. Okayama Univ. **25** (1983), 69−73.
[3] N. NOBUSAWA: Γ-rings and Morita equivalence of rings, Math. J. Okayama Univ. **26** (1984), 151−156.

DEPARTMENT OF MATHEMATICS
TOHOKU GAKUIN UNIVERSITY
TAGAJO, MIYAGI 985, JAPAN

(*Received October 28, 1985*)