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On generalized Harrison cohomology and Galois object

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ON GENERALIZED HARRISON COHOMOLOGY AND GALOIS OBJECT

Dedicated to Professor Kiiti Morita on his 60th birthday

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Let R be a commutative ring with identity, and G a finite abelian group. Let $H^2(R, G)$ be the second Harrison cohomology group defined in [6], and $E(GR)$ the group of isomorphism classes of G-Galois extensions of R. In [1] and [8], S. U. Chase and M. Orzech proved that there exists a group isomorphism

$$j: H^2(R, G) \longrightarrow N(GR)$$

where $N(GR)$ is the subgroup of $E(GR)$ consisting of those extensions which have normal bases.

In this paper we generalize the notion of Harrison cohomology and push the idea of [1] and [3] to obtain the information concerning the relation between the generalized Harrison cohomology groups and Galois objects over commutative rings in the sense of [4]. In § 1, we shall introduce the notions of Galois coalgebra and weak Galois algebra, which generalize those in [3, §4]. In §2, the generalized Harrison cohomology group Harr-$H^n(R, H)$ for a commutative Hopf R-algebra H will be concerned with Galois coalgebras and weak Galois algebras. Then under some reasonable assumptions, we can expand j to an isomorphism Harr-$H^2(R, H) \longrightarrow NX_{\mathbf{A}_0}(R, H)$, where $\mathbf{A}_0$ is the category of R-algebras whose objects are finitely generated projective R-modules and $NX_{\mathbf{A}_0}(R, H)$ is the group of isomorphism classes of Galois H-algebras in the category $\mathbf{A}_0$. In § 3, we show that our generalized Harrison cohomology group is a special case of the right H-comodule algebra cohomology group introduced by Y. Doi in [5].

Throughout this paper, R will denote a commutative ring with identity and unadorned $\otimes$ will mean $\otimes_R$. Moreover we shall assume, unless explicitly stated otherwise, that every ring has an identity which is preserved by every homomorphism, every module is unital, and every algebra is an R-algebra. We shall denote by $-^*$ the functor $\mathrm{Hom}_R(-, R)$. As to other notations we shall refer to [4], [7] and [9].

1. Galois coalgebra and weak Galois algebra. In this section, H

Mathematical Journal of Okayama University, Vol. 17 [1974], Iss. 2, Art. 5

will represent always a commutative Hopf algebra, and Δ the diagonal map of H. We define the algebra homomorphisms $\Delta_i^2 : H\otimes H \longrightarrow H\otimes H \otimes H$ (i=0, 1, 2, 3) by

$$\Delta_0^2(x)=1\otimes x,\ \Delta_1^2(x)=(\Delta\otimes 1)(x),\ \Delta_2^2(x)=(1\otimes \Delta)(x),\ \Delta_3^2(x)=x\otimes 1.$$

Let $\mathfrak{H}$ be the set of elements of u in $H\otimes H$ such that

$$\Delta_1^2(u)\ \Delta_3^2(u)=\Delta_0^2(u)\ \Delta_2^2(u). \tag{1.1}$$

In $\mathfrak{H}$, we write $u\sim u'$ if there exists a unit element v in H such that

$$\Delta(v)\ u=u'(v\otimes v). \tag{1.2}$$

Then the relation $\sim$ is an equivalence relation and $\bar{\mathfrak{H}}$ will mean the set of equivalence classes determined by this relation. If $\bar{u}_1$ and u_2 are the equivalence classes containing u_1 and u_2, respectively, then $u_1\ u_2=\overline{u_1 u_2}$. Hence $\bar{\mathfrak{H}}$ is a semi-group with the identity $\bar{1}$.

For $H\otimes H$ and $H\otimes H\otimes H$, we define an H-module structure via the diagonal action, i. e.,

$$h(x_1\otimes x_2)=\textstyle\sum_{(h)} h_{(1)}\,x_1\otimes h_{(2)}\,x_2$$

and

$$h(x_1\otimes x_2\otimes x_3)=\textstyle\sum_{(h)} h_{(1)}\,x_1\otimes h_{(2)}\,x_2\otimes h_{(3)}\,x_3$$

where $\Delta(h)=\sum_{(h)} h_{(1)}\otimes h_{(2)}$ and $(\Delta\otimes 1)\Delta(h)=\sum_{(h)} h_{(1)}\otimes h_{(2)}\otimes h_{(3)}$ (Sweedler's notation).

By a *Galois coalgebra* (H, D), we mean H together with an H-module homomorphism $D: H\longrightarrow H\otimes H$ which satisfies

$$(D\otimes 1)D=(1\otimes D)D : H\longrightarrow H\otimes H\otimes H. \tag{1.3}$$

Two Galois coalgebras (H, D) and $(H, \widetilde{D})$ are defined to be ***isomorphic*** if there is an H-module automorphism φ of H such that the diagram below commutes

$$\begin{array}{ccc} H & \xrightarrow{\ \varphi\ } & H \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle \widetilde{D}} \\ H\otimes H & \xrightarrow[\varphi\otimes\varphi]{} & H\otimes H \end{array} \ . \tag{1.4}$$

Let (H, D_1), (H, D_2) be Galois coalgebras. Then the mapping $D: H \longrightarrow H\otimes H$ defined by $D(x)=\Delta(x)\ D_1(1)\ D_2(1)$ is an H-module homomorphism which satisfies (1.3). Therefore we obtain a Galois coalgebra

(H, D). Moreover if $\psi: (H, D_1) \longrightarrow (H, D_1')$ is an isomorphism of Galois coalgebras, then the diagram below commutes

$$\begin{array}{ccc} H & \xrightarrow{\ \psi\ } & H \\ {\scriptstyle D_1}\downarrow & & \downarrow{\scriptstyle D_1'} \\ H \otimes H & \xrightarrow[\psi \otimes \psi]{} & H \otimes H \end{array}$$

which means that (H, D) and (H, D') are isomorphic, where $D'(x) = \Delta(x)$ $D_1'(1)$ $D_2(1)$. Hence the set of isomorphism classes of Galois coalgebras $\mathbf{C}(R, H)$ is a commutative semi-group with addition

$[(H, D)] + [(H, D_2)] = [(H, D)]$ $([(H, D_1)], [(H, D_2)] \in \mathbf{C}(R, H))$ where $D(x) = \Delta(x)\ D_1(1)\ D_2(1)$. Obviously $[(H, \Delta)]$ is the zero element in $\mathbf{C}(R, H)$.

For $\bar{\mathfrak{H}}$, and $\mathbf{C}(R, H)$, we have the following

Proposition 1.1. *Let* $\theta_1: \mathbf{C}(R, H) \longrightarrow \bar{\mathfrak{H}}$ *be the mapping defined by* $\theta_1([(H, D)]) = \overline{D(1)}$. *Then* θ_1 *is a semi-group isomorphism.*

Proof. Let (H, D) be a Galois coalgebra, and $D(1) = u$. Then, noting that $D(x) = \Delta(x) D(1)$, we see that D satisfies (1. 3) if and only if u satisfies (1. 1). Moreover if $\varphi: (H, D) \longrightarrow (H, \widetilde{D})$ is an isomorphism of Galois coalgebras, then $\varphi(x) = x\ \varphi(1)$ $(x \in H)$ and $\varphi(1)$ is a unit in H, because φ is an H-module automorphism. Since all mappings in the diagram (1. 4) are H-module homomorphisms, the commutativity of (1. 4) is equivalent to the condition (1. 2) with $u = D(1)$ and $u' = \widetilde{D}(1)$. Thus θ_1 is well defined. By the definition of addition in $\mathbf{C}(R, H)$, θ_1 is a homomorphism and $\theta_1([(H, \Delta)]) = \bar{1}$. Now let $\bar{u}$ be in $\bar{\mathfrak{H}}$ and let $D(u): H \longrightarrow H \otimes H$ be the mapping defined by $D(u)(x) = \Delta(x) u$. Then by (1. 1) and (1. 2), $(H, D(u))$ is a Galois coalgebra and is uniquely determined up to isomorphism of Galois coalgebras. Therefore, if we define $\theta_1'(\bar{u}) = [(H, D(u))]$, then θ_1' is the inverse homomorphism of θ_1. Thus θ_1 is an isomorphism, completing the proof.

Definition 1.2. Let S be an algebra (not necessarily with identity), and $\varphi: H \otimes S \longrightarrow S$ an R-module homomorphism. Then (φ, H) *measures* S *to* S if

(1) $\varphi(h \otimes xy) = \sum_{(h)} \varphi(h_{(1)} \otimes x)\ \varphi(h_{(2)} \otimes y)$

(2) $\varphi(h \otimes 1) = \varepsilon(h) 1$ (in case S has 1)

where $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$ and ε is the counit map in H.

For brevity, we shall denote the pair (φ, H) by the symbol H alone. If S is an H-module and H measures S to S, then S is called an *H-module algebra.*

Definition 1.3. An algebra S (not necessarily with identity) is called a *weak Galois algebra* if S is an H-module algebra and there exists an H-module isomorphism $\tau : H \longrightarrow S$. Two weak Galois algebras S and $\widetilde{S}$ are said to be *isomorphic* if there exists an algebra isomorphism $S \longrightarrow \widetilde{S}$ which is also an H-module isomorphism.

In case H has an antipode λ, given an arbitrary H-module M, M^* will be understood always as an H-module defined by

$$(hf)(m)=f(\lambda(h)m) \qquad (h \in H,\ f \in M^*,\ m \in M).$$

In the subsequent study of this section, we shall restrict our attention to a fixed finite, commutative, cocommutative Hopf algebra (cf. [4, p. 55]) *such that H^* is isomorphic to H as H-module.*

Let (H, D) be a Galois coalgebra. Then H^* is a weak Galois algebra canonically. Moreover, since $(H \otimes H)^* \cong H^* \otimes H^*$, the mapping $D^* : (H \otimes H)^* \longrightarrow H^*$ yields a multiplication on H^*, namely,

$$(1.5) \qquad (fg)(x)=(f \otimes g)D(x)=(f \otimes g)\varDelta(x)D(1) \qquad (f, g \in H^*,\ x \in H).$$

Then the multiplication is associative by (1.3) and H^* becomes an algebra, which we denote by $H(D)$. Since D is an H-module homomorphism, H measures $H(D)$ to $H(D)$. Thus $H(D)$ is a weak Galois algebra.

Let S be an arbitrary weak Galois algebra, and μ the multiplication of S. By Def. 1.3 and $H^* \cong H$, there exists an H-module isomorphism η: $S^* \longrightarrow H$. Hence we have an H-module homomorphism $D(S, \eta)$: $H \longrightarrow H \otimes H$ which is the composition

$$H \xrightarrow{\eta^{-1}} S^* \xrightarrow{\mu^*} (S \otimes S)^* \cong S^* \otimes S^* \xrightarrow{\eta \otimes \eta} H \otimes H.$$

Now let T be another weak Galois algebra. If we define a product on H^* by

$$(fg)(x)=(f \otimes g)(\varDelta(x)\ D(S, \eta)(1)\ D(T, \zeta)(1)) \quad (x \in H,\ f,\ g \in H^*)$$

then it is easy to see that H^* is a weak Galois algebra, which will be denoted by $H^{\triangle}$. For $H^{\triangle}$, we have the following

Lemma 1.4. *$H^{\triangle}$ is uniquely determined by the isomorphism classes of S and T up to isomorphism of weak Galois algebras.*

Proof. Let $\psi : S \longrightarrow S_1$ be an isomorphism of weak Galois algebras,

$\eta_1 : S_1^* \longrightarrow H$ an H-module homomorphism, and $H_1^\triangle$ the weak Galois algebra defined by S_1 and T as above. Then there exists an H-module isomorphism $\theta : H \longrightarrow H$ such that the following diagram is commutative

$$\begin{array}{ccccccc} H & \xrightarrow{\eta^{-1}} & S^* & \xrightarrow{\mu^*} & (S\otimes S)^* \cong S^*\otimes S^* & \xrightarrow{\eta\otimes\eta} & H\otimes H \\ \theta\downarrow & & \downarrow\phi^* & & \downarrow\phi^*\otimes\phi^* & & \downarrow\theta\otimes\theta \\ H & \xrightarrow[\eta_1^{-1}]{} & S_1^* & \xrightarrow[\mu_1^*]{} & (S_1\otimes S_1)^* \cong S_1^*\otimes S_1^* & \xrightarrow[\eta_1\otimes\eta_1]{} & H\otimes H \end{array}$$

where μ_1 is the multiplication of S_1, and thus $(\theta\otimes\theta)\, D(S, \eta)=D(S_1, \eta_1)\,\theta$. If we define $\varphi : H_1^\triangle \longrightarrow H^\triangle$ by $\varphi(f)=f\theta$, then φ is an H-module and algebra isomorphism, completing the proof.

Let $\mathbf{A}(R, H)$ be the set of isomorphism classes of weak Galois algebras. Then by Lemma 1.4, we can define the sum of the isomorphism class of S and the isomorphism class of T as that of H. Thus $\mathbf{A}(R, H)$ is a commutative semigroup and the isomorphism class of the canonical weak Galois algebra H^* is the zero element in $\mathbf{A}(R, H)$. Now, let φ : $(H, D) \longrightarrow (H, \widetilde{D})$ be an isomorphism of Galois coalgebras. Then, φ^* : $H(\widetilde{D}) \longrightarrow H(D)$ is an H-module and algebra isomorphism. We can define therefore the mapping θ_2 : $\mathbf{C}(R, H) \longrightarrow \mathbf{A}(R, H)$ by $\theta_2([(H, D)])=(H(D))$. Moreover, by the definition of additions in $\mathbf{C}(R, H)$ and $\mathbf{A}(R, H)$, it is easy to see that θ_2 is a monomorphism.

Proposition 1.5. *θ_2 is a semi-group isomorphism.*

Proof. Let S be a weak Galois algebra, and η: $S^* \longrightarrow H$ an H-module isomorphism. Then, we have the following commutative diagram

$$(1.6)\qquad \begin{array}{ccc} S^* & \xrightarrow{\mu^*} & S^*\otimes S^* \\ \eta\downarrow & & \downarrow\eta\otimes\eta \\ H & \xrightarrow[D(S,\eta)]{} & H\otimes H\,. \end{array}$$

Noting that μ is associative and η is an isomorphism, we have $(D(S, \eta)\otimes 1)D(S, \eta)=(1\otimes D(S, \eta))D(S, \eta)$. Hence $(H, D(S, \eta))$ is a Galois coalgebra. Transposing the commutative diagram (1.6), it follows that $\eta^*\, D(S, \eta)^* = \mu(\eta^*\otimes\eta^*)$. Since η is an H-module isomorphism, η^* : $H(D(S, \eta)) \longrightarrow S$ is an H-module and algebra isomorphism. Therefore θ_2 is onto, completing the proof.

Remark 1.6. For some useful finite, commutative, cocommutative

Mathematical Journal of Okayama University, Vol. 17 [1974], Iss. 2, Art. 5

Hopf algebras, the supplementary assumption that H^* is isomorphic to H as H-module is automatically satisfied.

(1) Let G be a finite abelian group. Then the group algebra RG is a finite Hopf algebra such that $(RG)^* \cong RG$ as RG-module.

(2) Let H be a finite, commutative, cocommutative Hopf algebra. Then H^* is an H-Hopf module in the sense of [9, p. 93] with the left H-module structure

$$(hf)(x) = f(\lambda(h)x) \quad (h, x \in H, \ f \in H^*)$$

and with the left H-comodule structure

$$\psi : H^* \longrightarrow H \otimes H^*$$

defined by $\psi(g) = \sum_{i=1}^{n} x_i \otimes gf_i$ $(g \in H^*)$, where $\{x_i, f_i\}_{1 \leq i \leq n}$ is an R-projective coordinate system of H. By [4, p. 128] or [9, p. 84],

$$H^* \cong H \otimes I$$

as left H-module and I is a projective R-module of rank one. Therefore if $\mathrm{Pic}(R) = 0$, then $H^* \cong H$ as left H-module.

(3) Let R be a commutative algebra over $GF(p)$ $(p \neq 0)$ and let $H = Rd_0 \oplus Rd_1 \oplus \cdots \oplus Rd_{p-1}$ be a free R-module with a free basis $\{d_0 = 1, d_1, \cdots, d_{p-1}\}$. Then H is a finite, commutative, cocommutative Hopf algebra with antipode λ :

$$\begin{aligned} & d_i d_j = \binom{i+j}{i} d_{i+j} && (1 \leq i, \ j \leq p-1) \\ & \Delta(d_n) = \textstyle\sum_{i=0}^{n} d_i \otimes d_{n-i} && (0 \leq n \leq p-1) \\ & \varepsilon(d_i) = \delta_{i,0} && \text{(Kronecker's delta)} \\ & \lambda(d_i) = (-1)^i d_i. \end{aligned}$$

Hence H^* is also a finite, commutative, cocommutative Hopf algebra with the dual basis $\{d_0^*, d_1^*, \cdots, d_{p-1}^*\}$. We set $f = d_0^* + d_1^* + \cdots + d_{p-1}^*$. Then it is easily seen that f is a free basis of H^* as H-module. Therefore $H \cong H^*$ as H-module.

2. Galois object and cohomology. Throughout this section we shall assume, unless explicitly stated otherwise, H is a commutative Hopf algebra with the diagonal map Δ and the counit map ε.

Let $\otimes^n H$ denote the tensor product $H \otimes \cdots \otimes H$ (n-times), and $\otimes^0 H = R$. Let $U(\ \ )$ be the multiplicative group of the ring $(\ \ )$. We define $\delta^n : U(\otimes^n H) \longrightarrow U(\otimes^{n+1} H)$ by the formula

$$\delta^n(u) = \Delta_0^n(u) \{ \textstyle\prod_{i=1}^{n} \Delta_i^n(u)^{(-1)^i} \} \Delta_{n+1}^n(u)^{(-1)^{n+1}} \qquad (u \in U(\otimes^n H)) \tag{2.1}$$

where the algebra homomorphisms Δ_0^n, Δ_i^n, $\Delta_{n+1}^n : \otimes^n H \longrightarrow \otimes^{n+1} H$ be defined by the conditions

$$\Delta_0^n(x) = 1 \otimes x, \qquad \Delta_{n+1}^n(x) = x \otimes 1 \qquad (x \in \otimes^n H)$$

$$\Delta_i^n(h_1 \otimes \cdots \otimes h_n) = h_1 \otimes \cdots \otimes h_{i-1} \otimes \Delta(h_i) \otimes h_{i+1} \otimes \cdots \otimes h_n$$

$(i=1, 2, \cdots n;\ h_i \in H)$. Then one can easily check that $\delta^{n+1}\ \delta^n = 0$, which enables us to define a cochain complex $\mathfrak{H}(R, H) = \{U(\otimes^n H), \delta^n\}_{n \geq 0}$. The n-th cohomology group of $\mathfrak{H}(R, H)$ is denoted by Harr-H^n (R, H), and will be called the generalized Harrison cohomology group.

Let M be an H-module (resp. $\otimes^2 H$-module). Since $\otimes^2 H$(resp. $\otimes^3 H$) can be viewed as a right H-module (resp. $\otimes^2 H$-module) via the diagonal map Δ(resp. Δ_i^2, $i=1, 2$), we have an $\otimes^2 H$-module (resp. $\otimes^3 H$-module) $\triangle(M) = (\otimes^2 H) \otimes_H M$ (resp. $\triangle_i (M) = (\otimes^3 H) \otimes_{H \otimes H} M$). If X, Y are finitely generated projective faithful H-module, then $X \otimes Y$ may be viewed in the obvious way as a finitely generated projective faithful $\otimes^2 H$-module, and there exist isomorphisms $\triangle_1 (X \otimes Y) \cong \triangle(X) \otimes Y$, $\triangle_2(X \otimes Y) \cong X \otimes \triangle(Y)$, $\triangle_1 \triangle(H) \cong \triangle_2 \triangle(H)$. We shall treat these isomorphisms as identifications.

Let $\triangle^H : H \longrightarrow \triangle(H)$ be a mapping defined by $\triangle^H(h) = 1 \otimes 1 \otimes h$. For an element u in $H \otimes H$, we define $\alpha_u : (\otimes^2 H) \otimes_H H \longrightarrow \otimes^2 H$ as the composition $(\otimes^2 H) \otimes_H H \cong \otimes^2 H \xrightarrow{m_u} \otimes^2 H$, where $\otimes^2 H \otimes_H H \cong \otimes^2 H$ denotes the natural isomorphism and m_u denotes the multiplication by u. Then α_u is an $\otimes^2 H$-module homomorphism and $\alpha_u \triangle^H(x) = \Delta(x)u$ $(x \in H)$.

The following lemma is easily proved.

Lemma 2.1. *Let u be an element in $H \otimes H$. Then u satisfies* (1.1) *if and only if the diagram below commutes*

$$\begin{array}{ccc}
\triangle_2 (H \otimes H) = H \otimes \triangle(H) & \xrightarrow{\ 1 \otimes \alpha_u\ } & H \otimes H \otimes H \\
{\scriptstyle \triangle_2(\alpha_u)} \uparrow & & \uparrow {\scriptstyle \alpha_u \otimes 1} \\
\triangle_2 \triangle(H) = \triangle_1 \triangle(H) & \xrightarrow[\ \triangle_1(\alpha_u)\ ]{} & \triangle_1(H \otimes H) = \triangle(H) \otimes H
\end{array}$$

where $\triangle_i\ (\alpha_u) = 1 \otimes 1 \otimes 1 \otimes \alpha_u$.

Definition 2.2 ([5, 1. 1. Def.]). A *right H-comodule algebra* is a pair (S, α), where S is an algebra and $\alpha : S \longrightarrow S \otimes H$ is an algebra homomorphism such that $(\alpha \otimes 1)\alpha = (1 \otimes \Delta)\alpha$ and $(1 \otimes \varepsilon)\alpha = 1_S$. For brevity, if there is no confusion, we shall denote the pair (S, α) by the symbol S alone. A right H-comodule algebra S will be called a *Galois*

Mathematical Journal of Okayama University, Vol. 17 [1974], Iss. 2, Art. 5

H-object if S is a faithfully flat R-module and $\gamma: S\otimes S \longrightarrow S\otimes H$ defined by $\gamma(x\otimes y)=(x\otimes 1)\,\alpha(y)$ is an algebra isomorphism.

Remark 2.3. A commutative right H-comodule algebra is an H-object in the sense of [4, p. 55].

Now let H be a finite Hopf algebra, and (S, α) a right H^*-comodule algebra. Then S has a left H-module structure which is defined by

$$h(x)=\sum_{(x)} x_{(1)}\otimes\langle h,\ x_{(2)}\rangle \qquad (x\in S,\ h\in H)$$

where $\alpha(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$ in $S\otimes H^*$ (Sweedler's notation), and $\langle\ ,\ \rangle$: $H\otimes H^* \longrightarrow R$ denotes the duality pairing. Hence S is an H-module algebra (cf. [4, p. 56]). Conversely, if S is an H-module algebra, then we obtain a map $\alpha: S\longrightarrow S\otimes H^*$ such that

$$\alpha(s)=\sum_{i=1}^{n} h_i\, s\otimes h_i^* \qquad (s\in S,\ h_i\in H,\ h_i^*\in H^*)$$

where $\{h_i,\ h_i^*\}_{1\leq i\leq n}$ is an R-projective coordinate system of H. Since S is an H-module algebra, S is a right H^*-comodule algebra with respect to α. *In the subsequent study, every right H^*-comodule algebra (resp. H-module algebra) will be regarded as an H-module algebra (resp. right H^*-comodule algebra) in the above way.*

Definition 2.4. Let H be be a finite Hopf algebra. A weak Galois algebra S is called a *Galois H-algebra* if S is a Galois H^*-object such that $S\cong H$ as H-module. (Needless to say, every Galois H-algebra is a weak Galois algebra.)

Now let S be an H-module algebra, and $F(S)=\mathrm{Hom}_H(\otimes^2 H,\ S)$, where $\otimes^2 H$ is viewed as an H-module via $h(x\otimes y)=\Delta(h)(x\otimes y)$ $(h,\ x,\ y\in H)$. Then $F(S)$ is a $\otimes^2 H$-module via the formula

$$[(h_1\otimes h_2)f](x\otimes y)=f(xh_1\otimes yh_2) \qquad (h_i,\ x,\ y\in H,\ f\in F(S))$$

and we can define a mapping $\varphi: S\otimes S\longrightarrow F(S)$ by

$$[\varphi(s\otimes t)](h_1\otimes h_2)=h_1(s)\ h_2(t) \qquad (h_i\in H,\ s,\ t\in S).$$

Lemma 2.5. *Let H be a finite Hopf algebra, and S a faithfully flat R-module. Then S is a Galois H^*-object if and only if S is an H-module algebra such that the mapping $\varphi: S\otimes S\longrightarrow F(S)$ defined above is a $\otimes^2 H$-module isomorphism.*

Proof. Assume that S is an H-module algebra. Let $\{h_i,\ h_i^*\}_{1\leq i\leq n}$ be an R-projective coordinate system of H, and consider the following diagram

$$\begin{array}{ccc} S\otimes S & \xrightarrow{\varphi} & \mathrm{Hom}_H(\otimes^2 H, S) \\ \gamma\downarrow & & \downarrow\psi \\ S\otimes H^* & \xrightarrow[\delta]{} & \mathrm{Hom}_R(H, S) \end{array}$$

where γ, ψ, δ are defined as follows

$$\gamma(s\otimes t)=\textstyle\sum_{i=1}^n h_i st\otimes h_i^* \qquad (s,\ t\in S)$$
$$\psi(f)(h)=f(1\otimes h) \qquad (f\in \mathrm{Hom}_H(\otimes^2 H,\ S),\ h\in H)$$
$$\delta(s\otimes h^*)(h)=h^*(h)s \qquad (s\in S,\ h\in H).$$

Then the diagram commutes and δ is an isomorphism. We define a map ψ': $\mathrm{Hom}_R(H,\ S)\longrightarrow \mathrm{Hom}_R(\otimes^2 H, S)$ by $\psi'(g)(h\otimes h')=\sum_{(h)} h_{(1)}\, g(\lambda(h_{(2)})h')$, where λ is the antipode of H ($h,\ h'\in H,\ g\in \mathrm{Hom}_R(H,\ S)$). Since Δ is an algebra homomorphism, we have

$$\begin{aligned}\psi'(g)(x(h\otimes h')) &= \textstyle\sum_{(x),(h)} x_{(1)} h_{(1)}\, g(\lambda(x_{(2)} h_{(2)})\, x_{(3)} h') \\ &= \textstyle\sum_{(x),(h)} x_{(1)}\, \varepsilon(x_{(2)})\, h_{(1)}\, g(\lambda(h_{(2)})\, h') \\ &= \textstyle\sum_{(h)} x h_{(1)}\, g(\lambda(h_{(2)})\, h') = x\psi'(g)(h\otimes h')\end{aligned}$$

($x\in H$), which means ψ' is in $\mathrm{Hom}_H(\otimes^2 H,\ S)$. Moreover,

$$\psi\psi'(g)(h)=\psi'(g)(1\otimes h)=g(h)$$

and

$$\begin{aligned}\psi'\psi(f)(h\otimes h') &= \textstyle\sum_{(h)} h_{(1)}\, \psi(f)(\lambda(h_{(2)})\, h') = \sum_{(h)} h_{(1)} f(1\otimes\lambda(h_{(2)})\, h') \\ &= f(\textstyle\sum_{(h)} h_{(1)}\otimes h_{(2)}\, \lambda(h_{(3)})\, h') \\ &= f(\textstyle\sum_{(h)} h_{(1)}\otimes \varepsilon(h_{(2)})\, h') = f(h\otimes h')\end{aligned}$$

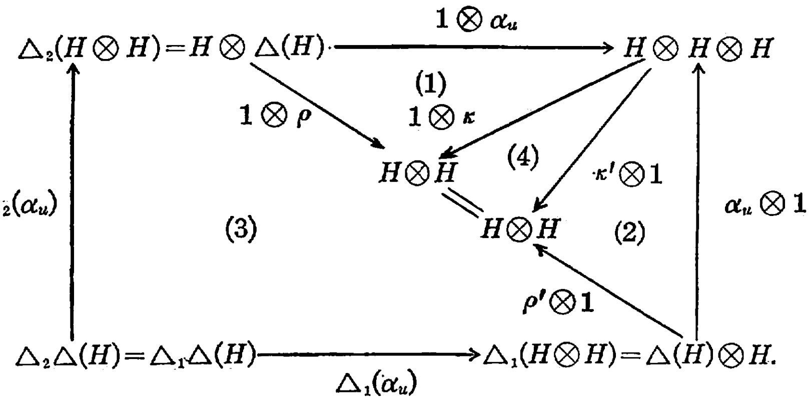
($h,\ h'\in H,\ g\in \mathrm{Hom}_R(H,\ S),\ f\in \mathrm{Hom}_H(\otimes^2 H,\ S)$). Hence ψ is an isomorphism. The lemma is then easily seen.

Theorem 2.6. *Let H be a finite Hopf algebra. Then a weak Galois algebra S is a Galois H-algebra if and only if $\theta_1\,\theta_2^{-1}$ $((S))$ is a unit in $\bar{\mathfrak{H}}$.*

Proof. By Prop. 1. 5, we may assume that $S=H(D(u))$ with $\bar{u}$ in $\bar{\mathfrak{H}}$. Assume that $\bar{u}$ is a unit in $\bar{\mathfrak{H}}$. Then by (1. 2), u is a unit in $\mathfrak{H}$. First, we shall show that S possesses an identity element. We consider the R-module homomorphisms ρ, ρ': $(\otimes^2 H)\otimes_H H\longrightarrow H$ defined by

$$\rho(a\otimes b\otimes c)=\varepsilon(a)bc, \qquad \rho'(a\otimes b\otimes c)=\varepsilon(b)ac.$$

Setting $\kappa=\rho\alpha_u^{-1}$ and $\kappa'=\rho'\alpha_u^{-1}$, we obtain the diagram



Parts (1) and (2) of this diagram are commutative by the definition of κ, κ' and a routine computation shows that part (3) is commutative. Since $\triangle_i(\alpha_u)$ and α_u are isomorphisms, we obtain from Lemma 2. 1 that part (4) of the above diagram commutes. Let $u = \sum_i u_{1i} \otimes u_{2i}$, and $u^{-1} = \sum_i v_{1i} \otimes v_{2i}$. Then $(1 \otimes \kappa)(1) = (\kappa' \otimes 1)(1)$, i. e.,

$$1 \otimes \sum_i \varepsilon(v_{1i})\, v_{2i} = \sum_i v_{1i}\, \varepsilon(v_{2i}) \otimes 1.$$

Therefore

$$\sum_i \varepsilon(v_{1i})\, \varepsilon(v_{2i}) = \sum_i v_{1i}\, \varepsilon(v_{2i}) = \sum_i \varepsilon(v_{1i})\, v_{2i}.$$

If we define $\bar{\varepsilon}: H \longrightarrow R$ by $\bar{\varepsilon}(h) = \sum_i \varepsilon(h)\, v_{1i}\, \varepsilon(v_{2i})$ $(h \in H)$, then we have

$$\begin{aligned}(f\bar{\varepsilon})(x) &= \textstyle\sum_{(x),i} f(x_{(1)}\, u_{1i})\, \bar{\varepsilon}(x_{(2)}\, u_{2i}) \\ &= \textstyle\sum_{(x),i,j} f(x_{(1)}\, \varepsilon(x_{(2)})\, u_{1i}\, v_{1j}\, \varepsilon(u_{2i})\, \varepsilon(v_{2j})) = f(x)\end{aligned}$$

$(x \in H)$. In other words, $\bar{\varepsilon}$ is a right identity in S. By symmetry, $\bar{\varepsilon}$ is a left identity. Next, we show that S is an H-module algebra. Recalling that Δ is an algebra homomorphism and $\alpha_u \triangle^H(x) = \Delta(x)u$ $(x \in H)$, we then have

$$h(fg)(x) = (fg)(\lambda(h)x) = (f \otimes g)\alpha_u \triangle^H(\lambda(h)x) = \textstyle\sum_{(h)} ((h_{(1)} f)(h_{(2)} g))(x)$$

and

$$(h\bar{\varepsilon})(x) = \bar{\varepsilon}(\lambda(h)x) = \varepsilon(h)\, \bar{\varepsilon}(x)$$

where h, $x \in H$, f, $g \in S$. Thus S is an H-module algebra. Now we consider the following diagram

(2. 2)
$$\begin{array}{ccc} S \otimes S & \xrightarrow{\ \varphi\ } & \mathrm{Hom}_H(\otimes^2 H,\, S) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\theta} \\ (H \otimes H)^* & \xrightarrow[\alpha^*_{\lambda(u)}]{} & (\otimes^2 H \otimes_H H)^* \end{array}$$

where φ is as in Lemma 2. 5, and

$$\rho(f \otimes g)=(f \otimes g)\,(\lambda \otimes \lambda)$$
$$\theta(\tau)\,(h_1 \otimes h_2 \otimes h_3)=\tau(h_1 \otimes h_2)\,[\lambda(h_3)].$$

A routine computation then shows that this diagram commutes, and ρ, θ, $\alpha^*_{\lambda(u)}$ are isomorphisms. Thus φ is an isomorphism. By the definition of S, it is easy to see that S is a faithfully flat R-module, and S is a Galois H^*-object by Lemma 2. 5. Conversely, if $H(D(u))$ is a Galois H-algebra, then the diagram (2. 2) commutes, and by Lemma 2. 5, φ is an isomorphism. Thus $\alpha^*_{\lambda(u)}$ s an isomorphism, that is, u is a unit, completing the proof.

Let $\mathbf{A}_0$ (resp. $\mathbf{C}_0$) be the category of R-algebras (resp. the category of R-coalgebras) whose objects are finitely generated projective R-modules. Then the functor $*$: $\mathbf{C}_0^{op} \longrightarrow \mathbf{A}_0$ enables us to obtain the theory of Galois H^*-objects in $\mathbf{C}_0$. A Galois H-object in $\mathbf{C}_0$ which is a Galois coalgebra as well will be called a Galois H-coalgebra. Henceforth $N_{\mathbf{A}_0}(H^*)$ (resp. $N_{\mathbf{C}_0}(H)$) denotes the set of isomorphism classes of Galois H-algebras (resp. the set of isomorphism classes of Galois H-coalgebras). Then we have the following

Proposition 2.7. (1) *Let S be a weak Galois H-algebra with an H-module isomorphism η : $S^* \longrightarrow H$. Then (S) is in $N_{\mathbf{A}_0}(H^*)$ if and only if $D(S, \eta)$ (1) is a unit.* (2) *Let (H, D) be a Galois coalgebra. Then $[(H, D)]$ is in $N_{\mathbf{C}_0}(H)$ if and only if $D(1)$ is a unit.*

Proof. (1) By Th. 2. 6, S is a Galois H-algebra in $\mathbf{A}_0$ if and only if $D(S, \eta)(1)$ is a unit. Therefore, (S) is in $N_{\mathbf{A}_0}(H^*)$ if and only if $D(S, \eta)$ (1) is a unit. (2) To be easily seen $D(H(D), 1)=D$. Accordingly by Th. 2. 6 and (1), $H(D)$ is a Galois H-algebra if and only if $D(1)$ is a unit. Hence, (H, D) is a Galois H-object in $\mathbf{C}_0$ if and only if $D(1)$ is a unit. Thus $[(H, D)]$ is in $N_{\mathbf{C}_0}(H)$ if and only if $D(1)$ is a unit, completing the proof.

Let $X_{\mathbf{A}_0}(H^*)$ (resp. $X_{\mathbf{C}_0}(H)$) be the set of isomorphism classes of Galois H^*-objects in $\mathbf{A}_0$ (resp. the set of isomorphism classes of Galois H-objects in $\mathbf{C}_0$). Patterning after the proof of [4, Proposition and Remarks 4. 7], we can introduce an abelian group structure into $X_{\mathbf{A}_0}(H^*)$. Clearly, $N_{\mathbf{A}_0}(H^*)$ (resp. $N_{\mathbf{C}_0}(H)$) is a subgroup of $X_{\mathbf{A}_0}(H^*)$ (resp. $X_{\mathbf{C}_0}(H)$), and by the duality we have $X_{\mathbf{A}_0}(H^*) \cong X_{\mathbf{C}_0}(H)$. By Prop. 2. 7, $N_{\mathbf{A}_0}(H^*)$(resp. $N_{\mathbf{C}_0}(H)$) may be regarded as a subset of $\mathbf{A}(R, H)$ (resp. $\mathbf{C}(R, H)$). Moreover, we can easily see that $N_{\mathbf{C}_0}(H)$ (resp. $N_{\mathbf{A}_0}(H^*)$) is a subgroup of $\mathbf{C}(R, H)$ (resp. $\mathbf{A}(R, H)$). Therefore by Prop. 1. 1 and Prop. 1. 5, we have the following

Theorem 2.8. *If H is a finite, commutative, cocommutative Hopf algebra such that $H \cong H^*$ as H-module, then*

$$\text{Harr-}H^2(R, H) \cong N_{\mathrm{A}_0}(H^*) \cong N_{\mathrm{C}_0}(H).$$

3. Comparison with other cohomologies. In this section, we shall assume that H is a commutative Hopf algebra and A is a commutative right H-comodule algebra with the structure map $\alpha : A \longrightarrow A \otimes H$. In [5], Y. Doi defined several cohomology groups of comodule algebras, comodule coalgebras, etc. We recall here the definition of the cohomology group of comodule algebras [5, § 2].

Let F be an additive functor from the category of commutative R-algebras to the category of abelian groups. We define algebra homomorphisms $d_i : A \otimes (\otimes^n H) \longrightarrow A \otimes (\otimes^{n+1} H)$ as follows :

$$d_0(a \otimes h_1 \otimes \cdots \otimes h_n) = \alpha(a) \otimes h_1 \otimes \cdots \otimes h_n,$$
$$d_i(a \otimes h_1 \otimes \cdots \otimes h_n) = a \otimes h_1 \otimes \cdots \otimes h_{i-1} \otimes \Delta(h_i) \otimes h_{i+1} \otimes \cdots \otimes h_n,$$
$$d_{n+1}(a \otimes h_1 \otimes \cdots \otimes h_n) = a \otimes h_1 \otimes \cdots \otimes h_n \otimes 1$$

$(a \in A,\ h_i \in H;\ i = 1, 2, \cdots, n)$. Then we have a cochain complex $\{F(A \otimes (\otimes^n H)),\ D^n\}_{n \geq 0}$ with coboundary $D^n = \sum_{i=0}^{n+1} (-1)^i F(d_i)$ and denote the n-th cohomology group by $\text{Alg}_R\text{-}H^n(A, H, F)$. Since R is a commutative right H-comodule algebra via the inclusion $R \longrightarrow R \otimes H \cong H$, we have

$$\text{Alg}_R\text{-}H^n(R, H, U) = \text{Harr-}H^n(R, H),$$

where U is the functor from the category of commutative algebras to the category of abelian groups defined by $(\ \ ) \longrightarrow U(\ \ )$.

Let $\{F(\otimes^{n+1} A),\ E\}$ be the Amitsur complex of A with coboundary $E^n : F(\otimes^{n+1} A) \longrightarrow F(\otimes^{n+1} A)$ defined by $E^n = \sum_{i=1}^{n+1} (-1)^i F(e_i)$, where $e_i : \otimes^{n+1} A \longrightarrow \otimes^{n+2} A$ is defined by $e_i(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_{n+1}$. Since A is a right H-comodule algebra, we have an algebra homomorphism $\Omega^n : \otimes^{n+1} A \longrightarrow A \otimes (\otimes^n H)$ which is given by

$$\Omega^n(a_1 \otimes \cdots \otimes a_{n+1}) = \sum a_1 a_{2(0)} a_{3(0)} \cdots a_{n+1(0)} \otimes a_{2(1)} a_{3(1)} \cdots a_{n+1(1)} \otimes \cdots \otimes a_{n(n-1)} a_{n+1(n-1)} \otimes a_{n+1(n)},$$

where $\alpha(a) = \sum_{(a)} a_{(0)} \otimes a_{(1)}$ in $A \otimes H$, and inductively

$$\sum a_{(0)} \otimes a_{(1)} \otimes \cdots \otimes a_{(n)} = (\alpha \otimes 1 \otimes \cdots \otimes 1)(\sum a_{(0)} \otimes a_{(1)} \otimes \cdots \otimes a_{(n-1)})$$

(Sweedler's notation) [5, 3. 5]. Then we have the following

Theorem 3.1. *Let A be a Galois H-object. Then Ω^n induces an isomorphism of complexes*

$$\tilde{\Omega}: \{F(\otimes^{n+1} A),\ E^n\}_{n\geq 0} \longrightarrow \{F(A\otimes(\otimes^n H)),\ D^n\}_{n\geq 0}.$$

Especially, $H^n(A/R,\ F) \cong \mathrm{Alg}_R\text{-}H^n(A,\ H,\ F)$.

Proof. By [5, §4, Prop.], $\tilde{\Omega}$ is a morphism of these complexes. Since $\gamma: A\otimes A \longrightarrow A\otimes H$ ($\gamma(a\otimes b)=(a\otimes 1)\alpha(b)$) is an algebra isomorphism, we can easily see that $\Omega^n=(\gamma\otimes 1\otimes\cdots\otimes 1)(1\otimes\gamma\otimes 1\otimes\cdots\otimes 1)\cdots(1\otimes 1\otimes\cdots\otimes\gamma)$. This implies $\tilde{\Omega}$ is an isomorphism.

By the last theorem and [2, Th. 7.6],

Corollary 3.2. *Let A be a Galois H-object which is a finitely generated projective R-module. Then there exists an exact sequence*

$$0\longrightarrow \mathrm{Alg}_R\text{-}H^1(A,\ H,\ U)\longrightarrow P(R)\longrightarrow P(A)\longrightarrow \mathrm{Alg}_R\text{-}H^2(A,\ H,\ U)$$
$$\longrightarrow B(A/R)\longrightarrow \mathrm{Alg}_R\text{-}H^1(A,\ H,\ P)\longrightarrow \mathrm{Alg}_R\text{-}H^3(A,\ H,\ U),$$

where P( ) is the Picard group of ( ) and B(A/R) is the Brauer group of Azumaya R-algebras split by A.

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Added in proof. Recently, the auther has found that the result of Corollary 3.2 was obtained by K. Yokogawa in a different point of view in Appendix of his paper: On $S \otimes_R S$-module structures of S/R-Azumaya algebras, to appear.