Mathematical Journal of Okayama University

Volume 17, *Issue* 2 1974 *Article* 1

JUNE 1975

On the true maximum order of a class of arithmetical functions

D. Suryanarayana* R. Sita Rama Chandra Rao†

*Andhra University

†Andhra University

ON THE TRUE MAXIMUM ORDER OF A CLASS OF ARITHMETICAL FUNCTIONS

D. SURYANARAYANA
and
R. SITA RAMA CHANDRA RAO

1. Introduction. Let $f(n)$ be an arithmetical function, which is positive and satisfies the condition that $f(n)=O(n^{\beta})$ for some fixed $\beta>0$. Define the arithmetical function $F(n)$ by setting $F(1)=1$ and $F(n)=f(a_1)$ $f(a_2)\cdots f(a_r)$ if $1<n=\prod_{i=1}^{r} p_i^{a_i}$. The main object of this paper is to prove the following theorem which gives a useful and easy way of obtaining the "true maximum order" of $F(n)$.

Theorem. *We have*

$$\limsup_{n\to\infty} \frac{\log F(n) \log\log n}{\log n} = \sup_m \frac{\log f(m)}{m}.$$

The usefulness of the theorem is illustrated in § 3 by applying it to some known divisor functions.

The condition on $f(n)$, namely $f(n)=O(n^{\beta})$ for some fixed $\beta>0$ assures us that $\sup_m \frac{\log f(m)}{m}$ (denoted throughout the rest of the paper by K_f) is finite. We assume throughout the paper that $K_f>0$.

In 1958 A. A. Drozdova and G. A. Freĭman [1] proved the following result, namely

$$\text{(1. 1)} \qquad \log F(n) \leqq K_f \frac{\log n}{\log\log n} + O\left(\frac{\log n}{(\log\log n)^2 \log\log\log n}\right),$$

where $f(n)>0$ and satisfies the condition that

$$f(n)=f(n-1)\left\{1+O\left(\frac{1}{n}\right)\right\}$$

and $F(n)$ is as defined above. It can be easily shown that any arithmetical function $f(n)$ satisfying their condition also satisfies our condition, namely $f(n)=O(n^{\beta})$ for some fixed $\beta>0$, so that our class of functions $f(n)$ is more rich than the class discussed by them. In fact, for the function $F(n)=\tau^{(e)}(n)$ defined in § 3, $f(n)=\cdot(n)$ which satisfies our condition, but not their condition (see Remark in § 3). Moreover, from (1. 1), it only follows that K_f is an upper bound of $\limsup_{n\to\infty} \frac{\log F(n) \log\log n}{\log n}$, whereas our theorem shows that K_f is exactly equal to this limit superior.

2. Proof of the theorem. Throughout the following the letter p with or without suffixes denotes a prime number, p_r denotes the r-th prime, $\pi(x)$ denotes the number of primes $\leqq x$, where x is a real variable $\geqq 2$, and $\theta(x) = \sum_{p\leqq x} \log p$. In the proof of the theorem, we make use of the well-known result that there exists a positive constant $A < 1$ such that $\theta(x) > Ax$ (cf. [2; Theorem 414]).

We first prove that given $\varepsilon > 0$, there are infinitely many positive integers n such that

$$\frac{\log F(n) \log\log n}{\log n} > K_f - \varepsilon. \tag{2.1}$$

For this, choose an integer $l > 1$ such that $\frac{\log f(l)}{l} > K_f - \frac{\varepsilon}{2}$. Such an integer l exists, since $K_f = \sup_m \frac{\log f(m)}{m}$. Putting $n_r = (2\cdot 3\cdot 5 \cdots p_r)^l$, we have

$$F(n_r) = \{f(l)\}^r = \{f(l)\}^{\pi(p_r)}.$$

Also, $Ap_r < \theta(p_r) = \frac{1}{l} \log n_r$ and $\pi(p_r) \log p_r \geqq \theta(p_r) = \frac{1}{l} \log n_r$.

Hence

$$\log F(n_r) = \pi(p_r) \log f(l) \geqq \frac{\log n_r}{\log p_r} \frac{\log f(l)}{l}.$$

But we have

$$\log A + \log p_r < \log\left(\frac{\log n_r}{l}\right) \leqq \log\log n_r,$$

so that

$$\log p_r < \log\log n_r - \log A.$$

Hence

$$\log F(n_r) > \frac{\log n_r}{\log\log n_r - \log A} \frac{\log f(l)}{l}.$$

Now, since $\frac{\log f(l)}{l} > K_f - \frac{\varepsilon}{2}$ and $A<1$, we have

$$\frac{\log F(n_r) \log\log n_r}{\log n_r} > \frac{\log\log n_r}{\log\log n_r - \log A}\left(K_f - \frac{\varepsilon}{2}\right) > K_f - \varepsilon,$$

for $r \geqq r_0(\varepsilon)$. Hence (2. 1) follows.

We next prove that given $\varepsilon > 0$,

$$\frac{\log F(n) \log\log n}{\log n} < (1+\varepsilon) K_f, \tag{2. 2}$$

for all $n \geqq N(\varepsilon)$. For this, we choose a number δ such that $0<\delta<\varepsilon$ and a number η such that $0<\eta<\frac{\delta}{1+\delta}$. For $n \geqq 3$, we define

$$\omega = \omega(n) = \frac{(1+\delta)K_f}{\log\log n} \text{ and } \Omega = \Omega(n) = (\log n)^{1-\eta}.$$

Then by the choice of η, we have

$$\Omega^\omega = e^{\omega \log \Omega} = e^{(1-\eta)(1+\delta)k_f} > e^{k_f}.$$

Now, if $n = \prod_{p|n} p^{a_p}$, then

$$\frac{F(n)}{n^\omega} = \prod_{p|n} \frac{f(a_p)}{p^{a_p\omega}} = \prod_{\substack{p\leq\Omega \\ p|n}} \frac{f(a_p)}{p^{a_p\omega}} \cdot \prod_{\substack{p>\Omega \\ p|n}} \frac{f(a_p)}{p^{a_p\omega}} = \Pi_1 \cdot \Pi_2, \tag{2. 3}$$

say. Since

$$\Omega^\omega > e^{K_f} \text{ and } K_f \geqq \frac{\log f(a_p)}{a_p},$$

we find that each factor in the product Π_2 is $\leqq 1$, for

$$\frac{f(a_p)}{p^{a_p\omega}} < \frac{f(a_p)}{\Omega^{a_p\omega}} < \frac{f(a_p)}{e^{k_f a_p}} \leqq 1.$$

Also, in the product Π_1, since $f(n) = O(n^\beta)$, we have

$$\frac{f(a_p)}{p^{a_p\omega}} \leqq \frac{f(a_p)}{2^{a_p\omega}} = \frac{f(a_p)}{e^{a_p\omega \log 2}} \leqq \frac{B(a_p)^\beta}{(a_p\omega)^\beta} = \frac{B}{\omega^\beta},$$

where B is an absolute positive constant. Thus

$$\log \Pi_1 \leqq \Omega \log\left(\frac{B}{\omega^\beta}\right) \sim \beta(\log n)^{1-\eta} \log\log\log n = o\left(\frac{\log n}{\log\log n}\right).$$

Hence by (2. 3)

$$\begin{aligned} \log F(n) &= \omega \log n + \log \Pi_1 + \log \Pi_2 \\ &< \frac{(1+\delta) K_f \log n}{\log\log n} + \frac{(\varepsilon-\delta)K_f \log n}{\log\log n}, \end{aligned}$$

for $n \geqq N(\varepsilon)$. Hence (2. 2) follows.

Thus the theorem is completely proved.

3. Applications. First of all, let us apply the theorem to determine the "true maximum order" of $\tau(n)$, where $\tau(n)$ is the number of divisors of the integer n. Let us take $f(n)=n+1$, then $F(n)=\tau(n)$. It is clear that $f(n)=O(n)$. Since

$$\sup_m \frac{\log f(m)}{m} = \sup_m \frac{\log (m+1)}{m} = \log 2,$$

in virtue of the theorem we have

$$\limsup_{n\to\infty} \frac{\log \tau(n) \log \log n}{\log n} = \log 2. \tag{3.1}$$

This result is well known (f. [2; Theorem 317]).

Let us now take $f(n)=n$, then $F(n)=\alpha(n)$, where $\alpha(n)$ is the number of square-full divisors of n. A divisor d of n is called square-full, if a prime p divides d then p^2 also divides d (cf. [6]). In this case

$$\sup_m \frac{\log f(m)}{m} = \sup_m \frac{\log m}{m} = \frac{1}{3} \log 3.$$

Hence in virtue of the theorem, we have

$$\limsup_{n\to\infty} \frac{\log \alpha(n) \log \log n}{\log n} = \frac{1}{3} \log 3. \tag{3.2}$$

Let us take $f(n)=\tau(n)$, then $F(n)=\tau^{(e)}(n)$, where $\tau^{(e)}(n)$ is the number of exponential divisors of n. A divisor $d=\prod_{i=1}^{r} p_i^{b_i}$ of $n=\prod_{i=1}^{r} p_i^{a_i}$ is called an exponential divisor of n, if $b_i \mid a_i$ for each i (cf. [3; p. 257]). Since $f(n)=\tau(n)<n$, the condition of the theorem is satisfied with $\beta=1$. In this case

$$\sup_m \frac{\log f(m)}{m} = \sup_m \frac{\log \tau(m)}{m} = \frac{1}{2} \log 2,$$

since $\tau(m) \leqq 2^{m/2}$ for $m \geqq 1$ and $\frac{\log \tau(2)}{2} = \frac{1}{2} \log 2$. Hence in virtue of the theorem, we have

$$\limsup_{n\to\infty} \frac{\log \tau^{(e)}(n) \log \log n}{\log n} = \frac{1}{2} \log 2. \tag{3.3}$$

This is a recently known result. A proof of this result due to P. Erdös may be found in [3; Theorem 6.2]. However, his proof is on different lines and is rather complicated (at least, not as straight forward as it is given here).

Remark. The function $f(n)=\tau(n)$ does not satisfy the condition laid down by A. A. Drozdova and G. A. Freĭman [1], namely $\frac{f(n)}{f(n-1)} = 1 + O\left(\frac{1}{n}\right)$, since $\frac{\tau(p)}{\tau(p-1)} \leqq \frac{2}{4} = \frac{1}{2}$ for every prime $p \geqq 7$.

Let k be a fixed integer $\geqq 2$. Let $\tau_k(n)$ denote the number of ordered k-tuples of positive integers, whose product equals n. Let $\theta_k(n)$ denote the number of ordered k-tuples of positive integers which are pairwise relatively prime and whose product equals n. Let $t_k(n)$ denote the number of ordered k-tuples of positive integers whose l. c. m. equals n. It is known (cf. [7; p. 5]) that

$$\tau_k(n)=\prod_{i=1}^{r}\binom{k+a_i-1}{a_i} \text{ if } n=\prod_{i=1}^{r} p_i^{a_i}$$

and (cf. [8; p. 587]) $\theta_k(n)=k^{\omega(n)}$, where $\binom{u}{v}$ is the binomial coefficient and $\omega(n)$ is the number of distinct prime factors of n. It can be easily shown that $\sum_{d|n} t_k(d)=(\tau(n))^k$, so that

$$t_k(n)=\prod_{i=1}^{r}\{(a_i+1)^k-a_i^k\} \text{ if } n=\prod_{i=1}^{r} p_i^{a_i}.$$

Let us now apply the theorem for the functions $\tau_k(n)$, $\theta_k(n)$ and $t_k(n)$. Taking $f(n)=\binom{k+n-1}{n}$, $f(n)=k$ and $f(n)=(n+1)^k-n^k$, we see that the condition of the theorem is satisfied with $\beta=k$, $\beta=1$ and $\beta=k-1$ respectively. Also

$$\sup_m \frac{\log\binom{k+m-1}{m}}{m}=\log k,$$

since $\{\log\binom{k+m-1}{m}\}/m$ is monotonically decreasing for $m\geqq 1$,

$$\sup_m \frac{\log k}{m}=\log k$$

and

$$\sup_m \frac{\log\{(m+1)^k-m^k\}}{m}=\log(2^k-1).$$

Hence in virtue of the theorem, we have

$$\limsup_{n\to\infty}\frac{\log \tau_k(n)\log\log n}{\log n}=\log k, \tag{3. 4}$$

$$\limsup_{n\to\infty}\frac{\log \theta_k(n)\log\log n}{\log n}=\log k \tag{3. 5}$$

and

$$\limsup_{n\to\infty}\frac{\log t_k(n)\log\log n}{\log n}=\log(2^k-1). \tag{3. 6}$$

As a particular case of (3. 5) for $k=2$, we have

$$\limsup_{n\to\infty}\frac{\log \tau^{*}(n)\log\log n}{\log n}=\log 2, \tag{3.7}$$

where $\tau^{*}(n)$ denotes the number of unitary divisors of n. By a unitary divisor of n, we mean as usual, a divisor d of n such that $(d, n/d)=1$.

Let us now take $f(n)=n$ if n is even and $f(n)=n+1$ if n is odd. Then $F(n)=\tau^{**}(n)$, where $\tau^{**}(n)$ is the number of bi-unitary divisors of n (cf. [5; §1]). By a bi-unitary divisor of n, we mean a divisor d of n such that $(d, n/d)^{**}=1$, where the symbol $(a, b)^{**}$ stands for the greatest unitary divisor of both a and b. In this case

$$\sup_{m}\frac{\log f(m)}{m}=\log 2.$$

Hence in virtue of the theorem, we have

$$\limsup_{n\to\infty}\frac{\log \tau^{**}(n)\log\log n}{\log n}=\log 2. \tag{3.8}$$

Similarly, we can establish the following results, by making use of the theorem:

$$\limsup_{n\to\infty}\frac{\log \tau(n^{k})\log\log n}{\log n}=\log (k+1), \tag{3.9}$$

$$\limsup_{n\to\infty}\frac{\log \tau^{(e)}(n^{k})\log\log n}{\log n}=\log \tau(k), \text{ if } k\geqq 2, \tag{3.10}$$

$$\limsup_{n\to\infty}\frac{\log \tau^{**}(n^{k})\log\log n}{\log n}=\begin{cases}\log k, & \text{if } k \text{ is even},\\ \log (k+1), & \text{if } k \text{ is odd}.\end{cases} \tag{3.11}$$

It should be remarked that the result (3. 8) and the result (3. 11) in case $k=2$, were proved earlier by M. V. Subbarao and the first-named author (cf. [4; Theorem 3]) using the method adopted by P. Erdös in proving (3. 3).

REFERENCES

[1] A. A. DROZDOVA and G. A. FREĬMAN: The estimation of certain arithmetic functions (in Russian). Elabuž. Gos. Ped. Inst. Učen. Zap. **3** (1958), 160—165. MR 40 #7213.

[2] G. H. HARDY and E. M. WRIGHT: An introduction to the theory of numbers. 4 th ed. Clarendon Press, Oxford, 1965

[3] M. V. SUBBARAO: On some arithmetic convolutions. Lecture Notes in Mathematics, Vol.

251. The theory of arithmetic functions. Springer-Verl., Berlin etc., 1972, pp. 247—271.

[4] M. V. SUBBARAO and D. SURYANARAYANA: Arithmetical functions associated with the bi-unitary divisors of an integer. Abstract 71T-A241. Notices of the Amer. Math. Soc., **18** (1971), 946.

[5] D. SURYANARAYANA: The number of bi-unitary divisors of an integer. Lecture Notes in Mathematics, Vol. **251.** The theory of arithmetic functions. Springer-Verl., Berlin etc., 1972, pp. 273—282.

[6] D. SURYANARAYANA and R. SITA RAMA CHANDRA RAO: The number of square-full divisors of an integer. Proc. Amer. Math. Soc., **34** (1972), 79—80.

[7] E. C. TITCHMARSH: The theory of the Riemann zeta function. Clarendon Press, Oxford, 1951.

[8] R. VAIDYANATHASWAMY: The theory of multiplicative arithmetical functions. Trans. Amer. Math. Soc., **33** (1931), 579—662.

DEPARTMNT OF MATHEMATICS,
ANDHRA UNIVERSITY,
WALTAIR, INDIA

(*Received October 18, 1973*)

Authors' remarks, added on July 18, 1975 at the time of proof correction: While the present paper was in the course of publication, the main theorem of this paper (in a more precise form) under yet weaker assumption, namely $f(n)=o(n/\log n)$ has been published by E. Heppner in Archiv der Mathematik 24 (1973), 63–66, under the title "Die maximale Ordunng primzahl-unabhängiger multiplikativer Funktionen". However, our method of proof of the theorem is elementary and does not make use of the 'Prime Number Theorem' with or without an error term; where as E. Heppner's proof is not as elementary as ours and moreover makes use of 'Prime Number Theorem' with an error term. We also remark that a proof of the result (3. 2) has been published as Theorem 3 by J. Knopfmacher in Proc. Amer. Math. Soc. 40 (1973), 373–377, in his paper under the title "A prime-divisor function". The main theorem with its proof as presented in this could be included in any of the forthcoming text books on Number Theory.