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On the subgroups H of a group G such that J(KH)KG ⊃ J(KG)

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ON THE SUBGROUPS H OF A GROUP G SUCH THAT $J(KH)KG \supset J(KG)$

Dedicated to Professor KIITI MORITA on his 60th birthday

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Let K be a field of characteristic $p>0$, and G a finite group whose order is divisible by p. $J(KG)$ will denote the Jacobson radical of the group algebra KG, and all modules under consideration will be right modules. For a subset S of G, $\hat{S}$ will denote the sum of all elements of S. If T is a subset of KG then $l(T)$ (resp. $r(T)$) will denote the left (resp. right) annihilator of T in KG.

We consider the following classes of subgroups of G: $\mathfrak{P}(G)=\{H\subset G \mid H$ contains a Sylow p-subgroup of $G\}$, $\mathfrak{R}(G)=\{H\subset G \mid J(KG)\subset J(KH)KG\}$, and $\mathfrak{C}(G)=\{H\subset G \mid$ the induced KG-module $\mathfrak{N}^G=\mathfrak{N}\otimes_{KH} KG$ is completely reducible for every irreducible KH-module $\mathfrak{N}\}$. By [3], it is known that the class $\mathfrak{P}(G)$ coincides with that of subgroups H of G such that every KG-module is (G, H)-projective.

Recently, in his paper [2], D. C. Khatri gave some sufficient conditions under which the above three classes are identical. In the present paper, we shall prove that $\mathfrak{R}(G)=\mathfrak{C}(G)$ without any restriction (Theorem 1) and that G is p-radical, namely, $\mathfrak{P}(G)=\mathfrak{R}(G)$ if and only if $\mathfrak{R}(G)$ contains a Sylow p-subgroup of G (Theorem 3). In §2, one will see that the results in [2] can be easily derived from Theorem 3. Furthermore, in case G is a Frobenius group and the order of its complement is divisible by p, we shall show that G is p-radical if and only if so is H (Theorem 8). Finally, as an application of Theorem 3, we shall present the condition for a p-solvable, p-radical group to have no blocks of p-defect 0 (Theorem 10).

1. At first, we shall prove the following which contains a result in [2]:

Theorem 1. $\mathfrak{R}(G)=\mathfrak{C}(G)$.

Proof. Let $H\in\mathfrak{R}(G)$, and $\mathfrak{N}$ an irreducible KH-module. Then, we have $\mathfrak{N}^G J(KG)=(\mathfrak{N}\otimes KG)J(KG)=\mathfrak{N}\otimes J(KG)\subset\mathfrak{N}\otimes J(KH)KG=\mathfrak{N}J(KH)\otimes KG=0$. Hence, $\mathfrak{N}^G$ is completely reducible, and $H\in\mathfrak{C}(G)$. Conversely, assume that $H\in\mathfrak{C}(G)$. Let $G=\cup_{i=1}^{n} Hx_i$, $x_1=1$, be a coset decomposition of G over H. Let $\alpha = \sum_{i=1}^{n} a_i x_i$ be an arbitrary element of $J(KG)$, where $a_i \in KH$. If $\mathfrak{N}$ is an arbitrary irreducible KH-module then $\mathfrak{N}^G \alpha = 0$, since $\mathfrak{N}^G$ is completely reducible. Therefore, $0 = (\mathfrak{N} \otimes 1)\alpha = \sum_{i=1}^{n} \mathfrak{N} a_i \otimes x_i$. This implies that $\mathfrak{N} a_i = 0$ for all i, and so $a_i \in J(KH)$. Hence $H \in \mathfrak{R}(G)$.

Remark 1. Since there exists a splitting field for G which is finite separable over K, $KG/J(KG)$ is a separable algebra. Therefore, for an arbitrary extension field F of K there holds $J(FG) = FJ(KG)$ and $J(KG) = J(FG) \cap KG$. Now, let L be an arbitrary field of characteristic p, and Z_p the prime field of characteristic p. If $H \in \mathfrak{R}(G)$, that is, $J(KG) \subset J(KH)KG$, then $J(Z_pG) = J(KG) \cap Z_pG \subset J(KH)KG \cap Z_pG = KJ(Z_pH)Z_pG \cap Z_pG = J(Z_pH)Z_pG$. Therefore, $J(LG) = LJ(Z_pG) \subset LJ(Z_pH)Z_pG = J(LH)LG$. This means that $\mathfrak{R}(G)$ is determined by G and p.

Khatri [2] proved that $\mathfrak{R}(G) \subset \mathfrak{P}(G)$ for any p-nilpotent group G. However, as was stated in [6], we obtain the following:

Theorem 2. $\mathfrak{R}(G) \subset \mathfrak{P}(G)$.

Proof. Let $H \in \mathfrak{R}(G)$. Then $\mathfrak{A} = K\hat{H} \otimes_{KH} KG$ is completely reducible by Theorem 1. Let $e = \sum_{i=1}^{n} \hat{H} \otimes x_i \in \mathfrak{A}$, and $\varphi : \mathfrak{A} \longrightarrow Ke$ a KG-epimorphism defined by $\varphi(\sum_{i=1}^{n} k_i \hat{H} \otimes x_i) = \sum_{i=1}^{n} k_i e \ (k_i \in K)$, where $G = \cup_{i=1}^{n} Hx_i$ is a coset decomposition of G over H. Then Ker $\varphi = \{\sum_{i=1}^{n} k_i \hat{H} \otimes x_i \mid \sum_{i=1}^{n} k_i = 0\}$. We can easily see that Ke is the unique trivial KG-submodule of the completely reducible KG-module $\mathfrak{A}$, and so we have $\mathfrak{A} = Ker\ \varphi \oplus Ke$. If p divides $[G : H]$, then $e \in$ Ker φ. This contradiction shows that $[G : H]$ is prime to p, and hence $H \in \mathfrak{P}(G)$.

If $\mathfrak{P}(G)$ coincides with $\mathfrak{R}(G)$, G will be called a *p-radical* group. (Any p'-group is p-radical by definition.)

Theorem 3. *The following conditions are equivalent:*

(1) *G is p-radical.*

(2) *$\mathfrak{R}(G)$ contains a Sylow p-subgroup of G.*

(3) $J(KG) = \cap_{x \in G} J(KP^x)KG$, *where P is a Sylow p-subgroup of G.*

(4) $l(J(KG)) = \sum_{x \in G} KG\widehat{P^x}$, *where P is a Sylow p-subgroup of G.*

Proof. (1) $\Longrightarrow$ (2) is trivial. (2) $\Longrightarrow$ (3): Let σ_x be the inner automorphism of G by $x \in G$, and σ_x^* the algebra automorphism of KG

induced by σ_x. If $\mathfrak{R}(G)$ contains a Sylow p-subgroup P of G then $J(KG) \subset J(KP)KG$. Hence, $J(KG) = \sigma_x^*(J(KG)) \subset \sigma_x^*(J(KP)KG) = J(KP^x)KG$, whence it follows $J(KG) \subset \cap_{x\in G} J(KP^x)KG$. On the other hand, $\cap_{x\in G} J(KP^x)KG$ is a nilpotent ideal of KG by [7, Prop. 2], and so $J(KG) = \cap_{x\in G} J(KP^x)KG$. (3)$\Longrightarrow$(4): Since KG is a Frobenius algebra, we have $rl(R) = R$ for any right ideal R of KG. Hence, we have $l(JKG)) = l(\cap_{x\in G} J(KP^x)KG) = \sum_{x\in G} l(J(KP^x)KG) = \sum_{x\in G} KG\widehat{P^x}$. (4)$\Longrightarrow$(2): Since $l(J(KG)) \supset KG\hat{P}$, we have $J(KG) = rl(J(KG)) \subset r(KG\hat{P}) = J(KP)KG$. Hence, $P \in \mathfrak{R}(G)$. (3)$\Longrightarrow$(1): If $H \in \mathfrak{P}(G)$ then H contains a Sylow p-subgroup P of G. Let $G = \cup_{i=1}^n Hx_i$ be a coset decomposition of G over H. Then, we have $J(KG) = \cap_{x\in G} J(KP^x)KG \subset \cap_{x\in H} J(KP^x)KG = \cap_{x\in H}(\sum_{i=1}^n J(KP^x)KHx_i) = \sum_{i=1}^n(\cap_{x\in H} J(KP^x)KH)x_i = (\cap_{x\in H} J(KP^x)KH)KG \subset J(KH)KG$. Hence $H \in \mathfrak{R}(G)$. By Theorem 2, we have eventually $\mathfrak{P}(G) = \mathfrak{R}(G)$.

Remark 2. In [1], W. E. Deskins asserted that if G is a p-nilpotent group then $\mathfrak{R}(G)$ contains a Sylow p-subgroup of G. But this is false and a counter example is given by A. I. Saksonov [6]: Let $p=3$, and $G=SL(2,3)$. Then, $G=\langle x, y, z \mid x^4=1, x^2=y^2, yxy^{-1}=x^{-1}, z^3=1, zxz^{-1}=y, zyz^{-1}=xy \rangle$. Let c_a be the class sum of the cunjugate class containing $a \in G$. Then, $c_x = x+x^3+y+xy+x^2y+x^3y$, $c_z = z+zx^2y+zx^3+zx^3y$ and $c_z^3 = 1-c_x$. Since c_x is an idempotent, we have $(c_x+c_z-1)^3 = c_x^3+c_z^3-1=0$. Hence, c_x+c_z-1 is a central nilpotent element of KG, and so an element of $J(KG)$. However, it is not contained in $J(K\langle z\rangle)KG$. The above example shows also that an extention of a p-radical group by a p-radical group is not always p-radical (cf. [2, p. 61]).

2. In virtue of Theorem 3, we can prove the results of Khatri [2] without effort.

Theorem 4. ([2, Th. 2]). *Let H be a normal subgroup of G. Then, H is in $\mathfrak{P}(G)$ if and only if H is in $\mathfrak{R}(G)$.*

Proof. If $H \in \mathfrak{P}(G)$ then G/H is a p'-group. Therefore, $J(KG) = J(KH)KG$, and so $H \in \mathfrak{R}(G)$. The converse is contained in Theorem 2.

Theorem 5. ([2, Th. 5]). *Let N be a normal subgroup of G such that G/N is a p'-group. Then, G is p-radical if and only if so is N.*

Proof. If G is p-radical then $J(KG) \subset J(KP)KG$, where P is a Sylow p-subgroup of G. Since G/N is a p'-group, N contains P. There-

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fore, $J(KN)=J(KG)\cap KN\subset J(KP)KG\cap KN=J(KP)KN$. Hence, $P\in \mathfrak{N}(N)$, and so N is p-radical by Theorem 3. Conversely, if N is p-radical then $J(KN)\subset J(KP)KN$, where P is a Sylow p-subgroup of N. Since G/N is a p'-group, P is a Sylow p-subgroup of G and $J(KG)=J(KN)KG\subset J(KP)KG$. Hence, $P\in\mathfrak{N}(G)$, and so G is p-radical again by Theorem 3.

Theorem 6. *If G is a p-radical group, and H a normal subgroup of G, then G/H is p-radical.*

Proof. Let $\nu: G\longrightarrow G/H$ be the natural homomorphism, and $\nu^*: KG\longrightarrow K(G/H)$ the algebra homomorphism induced by ν. Since G is p-radical, $J(KG)\subset J(KP)KG$, where P is a Sylow p-subgroup of G. Hence, $J(K(G/H))=\nu^*(J(KG))\subset\nu^*(J(KP)KG)=J(K(PH/H))\cdot K(G/H)$, and so G/H is p-radical by Theorem 3.

Theorem 7. ([2, Th. 7]). *Let M be a normal p-subgroup of G. Then, G is p-radical if and only if so is G/M.*

Proof. If G/M is p-radical then $J(K(G/M))\subset J(K(P/M))K(G/M)$, where P is a Sylow p-subgroup of G. This together with $J(K(G/M))\cong J(KG)/J(KM)KG$ and $J(K(P/M))K(G/M)\cong J(KP)KG/J(KM)KG$ implies that $J(KG)\subset J(KP)KG$. Hence, G is p-radical by Theorem 3. The converse is contained in Theorem 6.

The next contains [2, Th. 4].

Theorem 8. *Let G be a Frobenius group with kernel N and complement H.*

(1) *If p divides the order of N, then G is p-radical.*

(2) *In case p divides the order of H, G is p-radical if and only if so is H.*

Proof. (1) If p divides the order of N, then any Sylow p-subgroup of G is normal by Thompson's theorem. Hence, G is p-radical by Theorem 7.

(2) By Theorem 6, it remains only to prove the if part. If H is p-radical then $J(KH)\subset J(KP)KH$, where P is a Sylow p-subgroup of H. Since $J(KG)=J(KH)\hat{N}$ by [4, Th. 4], we have $J(KG)\subset J(KP)KH\hat{N}\subset J(KP)KG$, and so G is p-radical by Theorem 3.

Theorem 9. ([2, p. 61]). *Let $G=H_1\times H_2$. Then, G is p-radical if and only if so are H_1 and H_2.*

Proof. If H_1 and H_2 are p-radical then $J(KH_i) \subset J(KP_i)KH_i$, where P_i is a Sylow p-subgroup of H_i. Since $J(KG) = J(KH_1)KH_2 + J(KH_2)KH_1$ by [5, Th.], we have $J(KG) \subset J(KP_1)KG + J(KP_2)KG$. If we put $P = P_1 \times P_2$ then $J(KP_i) \subset J(KP)$. Hence, we have $J(KG) \subset J(KP)KG$, and so G is p-radical by Theorem 3. The converse is contained in Theorem 6.

3. For a moment let G be a permutation group on $\Omega = \{1, 2, \cdots, n\}$, and $G_{i_1, i_2, \ldots, i_t}$ be the stabilizer of the points $\{i_1, i_2, \cdots, i_t\} \subset \Omega$. Then we have

Lemma. *If p divides the order of $G_{\alpha,\beta}$ for every $\{\alpha, \beta\} \subset \Omega$, then $(\sum_{\alpha\in\Omega} KG\hat{G}_\alpha)^2 = 0$, and conversely.*

Proof. Let $G_\beta = \cup_{i=1}^{s} G_{\alpha,\beta}\, x_i$ be a coset decomposition of G_β over $G_{\alpha,\beta}$. If p divides the order of every $G_{\alpha,\beta}$, then $\hat{G}_\alpha \hat{G}_\beta = \hat{G}_\alpha \hat{G}_{\alpha,\beta}\,(x_1 + x_2 + \cdots + x_s) = |G_{\alpha,\beta}|\, \hat{G}_\alpha(x_1 + x_2 + \cdots + x_s) = 0$. For any $x \in G$ we have $xG_\beta x^{-1} = G_{x(\beta)}$, and so $\hat{G}_\alpha x \hat{G}_\beta = 0$. Thus, we obtain $(\sum_{\alpha\in\Omega} KG\hat{G}_\alpha)^2 = 0$. Conversely, if $(\sum_{\alpha\in\Omega} KG\hat{G}_\alpha)^2 = 0$ then $0 = \hat{G}_\alpha \hat{G}_\beta = |\hat{G}_{\alpha,\beta}|\, \hat{G}_\alpha(x + x_2 + \cdots + x_s)$ for every $\{\alpha, \beta\} \subset \Omega$. Since $G_\alpha x_i \cap G_\alpha x_j = \emptyset$ for $i \neq j$, p divides the order of every $G_{\alpha,\beta}$.

Theorem 10. *Let G be a p-solvable and p-radical group. Then, G has no blocks of p-defect 0 if and only if each pair of Sylow p-subgroups of G has a non-trivial intersection.*

Proof. We choose K a splitling field for G. Let c be the sum of all p-elements of G. Then, by [8, Th. 1] and [7, Cor. 4], c^2 is equal to the sum of the block idempotents of p-defect 0 and $l(J(KG)) = KGc$. Since G is p-radical, we have $l(J(KG)) = \sum_{x\in G} KG\widehat{P^x}$ by Theorem 3, where P is a Sylow p-subgroup of G. Therefore, G has no blocks of p-defect 0 if and only if $(\sum_{x\in G} KG\widehat{P^x})^2 = 0$. Let $G = \cup_{i=1}^{n} Px_i$, $x_1 = 1$, be a coset decomposition of G over P. Regarding G as a permutation group on $\Omega = \{P, Px_2, \cdots, Px_n\}$, we see that the stabilizer of Px_i is $x_i^{-1} Px_i$. Hence, by Lemma, $(\sum_{x\in G} KG\widehat{P^x})^2 = 0$ if and only if $x_i^{-1} Px_i \cap x_j^{-1} Px_j \neq 1$ for $1 \leq i, j \leq n$. This completes the proof.

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