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CENTRALIZERS OF A MODULE OVER A QUASI-FROBENIUS EXTENSION

Dedicated to Professor Kiiti Morita on his 60th birthday

YOSHIMI KITAMURA

Introduction. This paper is a natural sequel to our previous paper [3]. The notation and terminology employed there will be used here.

In his paper [4], Y. Miyashita developed a commutator theory of a Frobenius extension. Successively we study a commutator theory of a Quasi-Frobenius (QF) extension in this paper.

Our main result of the present paper is the following :

Theorem A (cf. [4, Th. 2.10]). *Let A/B be a ring extension, M a right A -module such that $M \otimes_B A_A | M_A$, that is, $M \otimes_B A$ is A -isomorphic to an A -direct summand of a finite direct sum of copies of A , $A' = \text{End}(M_A)$, the A -endomorphism ring of M , and $B' = \text{End}(M_B)$. If A/B is a QF extension, then the following statements hold.*

- 1) B'/A' is a QF extension.
- 2) ${}_{B'}B' \otimes_{A'} M | {}_{B'}M$.
- 3) $\widetilde{A}/\widetilde{B}$ is a QF extension such that $\widetilde{A} \cong \widetilde{A} \otimes_B \widetilde{B} \cong \widetilde{B} \otimes_B A$ canonically. Here \widetilde{A} and \widetilde{B} denote respectively the double centralizers of M as A and B -modules.

As a consequence of the above theorem, we can obtain the following in §2.

Theorem B (cf. [4, Th. 2.6]). *Let A be a ring, B a subring of A and T an intermediate subring of A/B . If T is a QF extension of B such that ${}_{\tau}T \otimes_B A_A | {}_{\tau}A_A$, then B' , the centralizer of B in A , is a QF extension of T' , the centralizer of T in A , such that ${}_{A}A \otimes_{T'} B'_{B'} | {}_{A}A_{B'}$, and moreover, T'' , the double centralizer of T in A , is a QF extension of B'' such that $T'' \cong T \otimes_B B'' \cong B'' \otimes_B T$ canonically.*

Also a direct proof of Morita [6, Ths. 1.1 and 1.3] is given in §1.

1. Throughout this note, we shall denote by $\text{End}(X_R)$ the R -endomorphism ring of a right R -module X and consider $\text{End}(X_R)$ as a left operator domain of X . Similarly, for a left R -module ${}_R Y$, we shall consider $[\text{End}({}_R Y)]^\circ$, the opposite ring of $\text{End}({}_R Y)$, as a right operator domain of Y . Further if X is a left R -right S -bimodule, we shall say X a

R - S -module simply.

Let $\varphi : B \rightarrow A$ be a ring homomorphism.¹⁾ Every A -module may be regarded as a B -module via φ . Especially $\text{Hom}(A_B, B_B)$ has a B - A -module structure in a natural way :

$$(b \cdot f)(x) = b \cdot f(x), (f \cdot a)(x) = f(ax) \quad (a, x \in A, b \in B, f \in \text{Hom}(A_B, B_B)).$$

Similarly $\text{Hom}({}_B A, {}_B B)$ may be regarded as an A - B -module.

Let X be an A' - A -module and Y an A - A'' -module. If A is finitely generated (f. g., briefly) and projective as a left B -module, then a mapping

$$(1) \quad X \otimes_B A \rightarrow \text{Hom}(\text{Hom}({}_B A, {}_B B)_B, X_B), x \otimes a \mapsto (f \mapsto xf(a))$$

is an A' - A -isomorphism whose inverse is given by $F \mapsto \sum_i F(f_i) \otimes a_i$, where $(f_i, a_i)_{1 \leq i \leq t}$ is a dual basis for ${}_B A$, that is, $f_i \in \text{Hom}({}_B A, {}_B B)$, $a_i \in A (i = 1, \dots, t)$ such that $\sum_i f_i(a) \cdot a_i = a$ for all $a \in A$. Also, a mapping

$$(2) \quad \text{Hom}({}_B A, {}_B B) \otimes_B Y \rightarrow \text{Hom}({}_B A, {}_B Y), g \otimes y \mapsto (a \mapsto g(a)y)$$

is an A - A'' -isomorphism whose inverse is given by $u \mapsto \sum_i f_i \otimes u(a_i)$. Similarly if A is f. g. projective as a right B -module, then a mapping

$$(3) \quad A \otimes_B Y \rightarrow \text{Hom}({}_B \text{Hom}({}_B A, {}_B B), {}_B Y), a \otimes y \mapsto (f \mapsto f(a)y)$$

is an A - A'' -isomorphism whose inverse is given by $G \mapsto \sum_j d_j \otimes G(g_j)$ and a mapping

$$(4) \quad X \otimes_B \text{Hom}({}_B A, {}_B B) \rightarrow \text{Hom}(A_B, X_B), x \otimes f \mapsto (a \mapsto xf(a))$$

is an A' - A -isomorphism whose inverse is given by $v \mapsto \sum_j v(d_j) \otimes g_j$, where $(g_j, d_j)_{1 \leq j \leq s}$ ($g_j \in \text{Hom}({}_B A, {}_B B)$, $d_j \in A$) is a dual basis for A_B .

Now a ring extension $[A/B, \varphi]$, namely, a ring homomorphism $\varphi : B \rightarrow A$, is called a Frobenius extension if A is f. g. projective as a right B -module and isomorphic to $\text{Hom}({}_B A, {}_B B)$ as a B - A -module (cf. Kasch [2]). If $[A/B, \varphi]$ is a Frobenius extension then there exist a B - B -homomorphism h of A to B and a finite number of elements r_i, l_i in $A (i = 1, \dots, n)$ such that

$$(5) \quad a = \sum r_i h(l_i a) = \sum h(ar_i) l_i \quad (a \in A),$$

and conversely (cf. Onodera [8]). When this is the case, we shall call such $(h; r_i, l_i)_{1 \leq i \leq n}$ a Frobenius system for $[A/B, \varphi]$.

Next a ring extension $[A/B, \varphi]$ is called a right QF extension if

1) By a ring homomorphism, we shall mean one which sends an identity element to an identity element.

A is f. g. projective as a right B -module and isomorphic to a direct summand of a finite direct sum of copies of $\text{Hom}(A_B, B_B)$ as a B - A -module (cf. Müller [7]). A left QF extension is defined symmetrically, and a QF extension is defined as an extension which is right QF as well as left QF.

If A_B is f. g. projective then from (3), we have an A - A -isomorphism

$$(6) \quad A \otimes_B A \longrightarrow \text{Hom}({}_B\text{Hom}(A_B, B_B), A_B), \quad x \otimes y \longmapsto (f \longmapsto f(x)y),$$

and under this isomorphism, $\text{Hom}({}_B\text{Hom}(A_B, B_B), {}_B A_A)$ is corresponding to the set $(A \otimes_B A)^d$ of casimir elements in $A \otimes_B A$. Here an element $\gamma \in A \otimes_B A$ is said to be casimir if $a\gamma = \gamma a$ for all $a \in A$. Thus $[A/B, \varphi]$ is right QF if and only if A_B is f. g. projective and there exist a finite number of B - B -homomorphisms α_i of A to B and casimir elements $\sum_j x_{ij} \otimes y_{ij} \in A \otimes_B A$ ($i = 1, \dots, p$) such that

$$(7) \quad \sum_i \sum_j \alpha_j (ax_{ij}) y_j = a \quad (a \in A).$$

Such $(\alpha_i; \sum_j x_{ij} \otimes y_{ij})_{1 \leq i \leq p}$ is called a right QF system for $[A/B, \varphi]$. Similarly a left QF system for $[A/B, \varphi]$ is defined as a system $(\beta_k; \sum_l w_{kl} \otimes z_{kl})_{1 \leq k \leq q}$ where each β_k is a B - B -homomorphism of A to B and each $\sum_l w_{kl} \otimes z_{kl}$ is a casimir element in $A \otimes_B A$ such that

$$(8) \quad \sum_k \sum_l w_{kl} \beta_k (z_{kl} a) = a \quad (a \in A).$$

Needless to say, a ring extension is QF if and only if it has a right QF system and a left QF system.

Let M be a right A -module, and set $A' = \text{End}(M_A)$, $B' = \text{End}(M_B)$, $\tilde{A} = [\text{End}({}_{A'}M)]^\circ$ and $\tilde{B} = [\text{End}({}_{B'}M)]^\circ$. Then M may be considered as a B' - \tilde{A} -module, and further an inclusion mapping $\varphi' : A' \longrightarrow B'$ and ring homomorphisms $\rho_A : A \longrightarrow \tilde{A}$ and $\rho_B : B \longrightarrow \tilde{B}$ defined by $m \rho_A(a) = ma$ and $m \rho_B(b) = mb$ ($m \in M$, $a \in A$, $b \in B$) respectively are induced. Moreover $\varphi' : A' \longrightarrow B'$ induces a ring homomorphism $\tilde{\varphi} : \tilde{B} \longrightarrow \tilde{A}$ in a natural way. The meaning of the above notations will be retained throughout this section.

Let us now assume that M satisfies the following condition :

$$(9) \quad M \otimes_B A_A \mid M_A.$$

First suppose that $[A/B, \varphi]$ is a Frobenius extension with a Frobenius system $(h; r_i, l_i)_i$. Then, by Miyashita [4, Prop. 2.3], (9) is equivalent to the existence of $f_j, g_j \in B'$ ($j = 1, \dots, t$) such that

$$(10) \quad \sum_j g_j (f_j(m) a) = mh(a) \quad (a \in A, m \in M).$$

So, let $f_j, g_j \in B'$ ($j = 1, \dots, t$) be as above and define an A' - A' -homomorphism $Tr: B' \rightarrow A'$ by

$$(11) \quad [Tr(b')](m) = \sum_i (b'(mr_i)) l_i \text{ for } b' \in B', m \in M.$$

Then we have

$$(12) \quad \sum_j g_j \cdot Tr(f_j b') = b' \quad (b' \in B').$$

Indeed, operating the left hand to $m \in M$, we have by (5) and (10)

$$\sum_j g_j (\sum_i f_j (b'(mr_i)) l_i) = \sum_i b'(mr_i) h(l_i) = b'(m).$$

Similarly we have

$$(13) \quad \sum_j Tr(b' g_j) \cdot f_j = b' \quad (b' \in B').$$

Thus we have seen $(Tr; g_j, f_j)_{1 \leq j \leq t}$ a Frobenius system for $[B'/A', \varphi']$. Furthermore, (11) implies ${}_B B' \otimes_{A'} M|_B M$. Therefore, applying the above argument to the left module M over the Frobenius extension $[B'/A', \varphi']$ with the Frobenius system $(Tr; g_j, f_j)_j$, we have by (11) that $(\tilde{h}; \rho_A(r_i), \rho_A(l_i))_i$ is a Frobenius system for $[\tilde{A}/\tilde{B}, \tilde{\varphi}]$, where $\tilde{h} \in \text{Hom}(\tilde{A}_{\tilde{B}}, \tilde{B}_{\tilde{B}})$ is defined by

$$m\tilde{h}(\tilde{a}) = \sum g_j(f_j(m) \tilde{a}) \text{ for } \tilde{a} \in \tilde{A}, m \in M \quad (\text{cf. (11)}).$$

Then, (10) shows $\tilde{h}(\rho_A(a)) = \rho_B(h(a))$ for $a \in A$.

It follows that a canonical mapping $\tilde{B} \otimes_B \tilde{A} \rightarrow \tilde{A}, \tilde{b} \otimes a \mapsto \tilde{b}a$, is an isomorphism whose inverse is given by $\tilde{A} \ni \tilde{a} \mapsto \sum_i \tilde{h}(\tilde{a}\rho_A(r_i)) \otimes l_i \in \tilde{B} \otimes_B \tilde{A}$. Similarly a canonical mapping $A \otimes_B \tilde{B} \rightarrow \tilde{A}$ defined by $a \otimes \tilde{b} \mapsto a\tilde{b}$ is an isomorphism.

Next suppose that $[A/B, \varphi]$ is a QF extension having a right QF system $(\alpha_i; \sum_j x_{ij} \otimes y_{ij})_{1 \leq i \leq p}$ and a left QF system $(\beta_k; \sum_l w_{kl} \otimes z_{kl})_{1 \leq k \leq q}$. Then $(x_{ij} \cdot \alpha_i, y_{ij})_{i,j}$ and $(\beta_k \cdot z_{kl}, w_{kl})_{k,l}$ are dual bases for ${}_B A$ and A_B by (7) and (8) respectively. By (9), there exist $u_n: M \otimes_B A_A \rightarrow M_A$ and $v_n: M_A \rightarrow M \otimes_B A_A$ ($n = 1, \dots, t$) such that

$$(14) \quad \sum v_n \cdot u_n = Id_{A \otimes_B M}.$$

Define $U_n \in B'$ and $V_n \in \text{Hom}({}_A B', {}_A A')$ by

$$(15) \quad U_n(m) = u_n(m \otimes 1) \text{ for } m \in M; U_n = u_n \circ \iota$$

$$(16) \quad V_n(b') = \pi \cdot (b' \otimes Id_A) \cdot v_n \text{ for } b' \in B',$$

where ι and π denote respectively the mappings given by

$$\iota : M \longrightarrow M \otimes_B A, \quad \iota(m) = m \otimes 1,$$

$$\pi : M \otimes_B A \longrightarrow M, \quad \pi(m \otimes a) = ma.$$

It is then easy to see that $(V_n, U_n)_n$ is a dual basis for ${}_A B'$. Similarly let us define $c_{k,n} \in B'$ and $c_{k,n}^* \in \text{Hom}(B'_{A'}, A'_{A'})$ by

$$(17) \quad c_{k,n} = \gamma \cdot (Id_M \otimes \beta_k) \cdot v_n$$

$$(18) \quad [c_{k,n}^*(b')] (m) = u_n(\sum_l b'(mw_{kl}) \otimes z_{kl}) \quad (b' \in B', m \in M),$$

where $\gamma : M \otimes_B B \longrightarrow M, \gamma(m \otimes b) = mb$.

Since $[\sum_{k,n} c_{k,n} \cdot c_{k,n}^*(b')] (m) = \sum_k \{\gamma \cdot (Id_M \otimes \beta_k)\} \{(\sum_n v_n \cdot u_n) (\sum_l b'(mw_{kl}) \otimes z_{kl})\} = b'(m \sum_{k,l} w_{kl} \beta_k(z_{kl})) = b'(m), (c_{k,n}^*, c_{k,n})_{k,n}$ is a dual basis for $B'_{A'}$. Further let us define $B'-A$ -homomorphisms

$$\Phi_i : M \otimes_B A \longrightarrow \text{Hom}(A_B, M_B), \quad \Psi_i : \text{Hom}(A_B, M_B) \longrightarrow M \otimes_B A$$

by

$$(19) \quad [\Phi_i(m \otimes a)] (x) = m\alpha_i(ax) \text{ for } m \in M, a, x \in A,$$

$$(20) \quad \Psi_i(g) = \sum_j g(x_{ij}) \otimes y_{ij} \text{ for } g \in \text{Hom}(A_B, M_B)$$

and mappings

$$\Psi'_i : B' \longrightarrow \text{Hom}(B'_{A'}, A'_{A'}), \quad \Phi'_i : \text{Hom}(B'_{A'}, A'_{A'}) \longrightarrow B'$$

by

$$(21) \quad \Phi'_i(f')(m) = \sum_n [f'(\gamma \cdot (Id_M \otimes \alpha_i) \cdot v_n)] (u_n(m \otimes 1))$$

for $f' \in \text{Hom}(B'_{A'}, A'_{A'}), m \in M,$

$$(22) \quad [\Psi'_i(b')(b'_i)] (m) = \sum_j ((b' b'_i)(mx_{ij})) y_{ij} \text{ for } b', b'_i \in B', m \in M.$$

Then it is easy to see

$$(23) \quad \sum_i \Phi'_i \cdot \Psi'_i = Id_{B'}.$$

Obviously, every Φ'_i is a left A' -homomorphism. Moreover we shall claim that for any $h \in \text{Hom}({}_B A_B, {}_B B_B), \sum_n (\gamma \cdot (Id_M \otimes h) \cdot v_n) \otimes (u_n \cdot \iota)$ is casimir in $B' \otimes_{A'} B'$. In fact, noting $(V_n, U_n)_n$ to be a dual basis for ${}_A B'$ and an isomorphism $B' \otimes_{A'} B' \cong x' \otimes y' \longmapsto (g' \longmapsto x' g'(y')) \in \text{Hom}(\text{Hom}({}_A B', {}_A A'_{A'}), B'_{A'})$, we have the conclusion if we show the

following equation for all $b' \in B'$ and all n' :

$$\begin{aligned} \sum_n (\gamma \cdot (Id_M \otimes h) \cdot v_n) \cdot V_{n'}, (u_n \cdot \iota \cdot b') \\ = \sum_n b' \cdot (\gamma \cdot (Id_M \otimes h) \cdot v_n) \cdot V_{n'} (u_n \cdot \iota). \end{aligned}$$

However this equation can be verified directly. Especially $\sum_n (\gamma \cdot (Id_M \otimes \alpha_i) \cdot v_n) \otimes (u_n \cdot \iota)$ is casimir in $B' \otimes_{A'} B'$, which implies ϕ'_i a right B' -homomorphism, and so, it is an A' - B' -homomorphism. On the otherhand, ψ'_i is obviously an A' - B' -homomorphism. Accordingly we have ${}_{A'}B'_{B'} \mid_{A'} \text{Hom}(B'_{A'}, A'_{A'})_{B'}$ by (23). Therefore, recalling $(c_{k,n}^*, c_{k,n})$ to be a dual basis for $B'_{A'}$, if we set for each i, k, n

$$\begin{aligned} \alpha'_i &= \psi'_i (1_{B'})^2, \\ d_{k,n}^{(\circ)} &= \phi'_i (c_{k,n}^*), \end{aligned}$$

then we can see that

$$(24) \quad (\alpha'_i ; \sum_{k,n} c_{k,n} \otimes d_{k,n}^{(\circ)})_{1 \leq i \leq p}$$

is a right QF system for $[B'/A', \varphi']$ (see (3)). We have further

$$(25) \quad \alpha'_i (b') (m) = \sum_j (b' (m x_{ij})) y_{ij}$$

$$(26) \quad d_{k,n}^{(\circ)}(m) = u_n (m \otimes \sum_i \alpha_i (w_{kl}) z_{kl}) \quad (b' \in B', m \in M).$$

In fact, (25) is obvious by (22), and further the left hand of (26) is equal to $\sum_{n'} [c_{k,n}^* (\gamma \cdot (Id_M \otimes \alpha_i) \cdot v_{n'})] (u_{n'} (m \otimes 1))$

$$\begin{aligned} &= u_n (\sum_i (\gamma \cdot (Id_M \otimes \alpha_i)) ((\sum_{n'} v_{n'} \cdot u_{n'}) (m \otimes w_{kl})) \otimes z_{kl}) \quad (\text{by (18)}) \\ &= u_n (m \otimes \sum_i \alpha_i (w_{kl}) z_{kl}) \quad (\text{by (14)}). \end{aligned}$$

This shows (26).

Similarly we shall define B' - A -homomorphisms

$$\psi_k : \text{Hom}(A_B, M_B) \longrightarrow M \otimes_B A, \quad \varphi_k : M \otimes_B A \longrightarrow \text{Hom}(A_B, M_B)$$

by

$$(19)_i \quad [\varphi_k (m \otimes a)] (x) = m \beta_k (ax) \text{ for } a, x \in A, m \in M,$$

$$(20)_i \quad \psi_k (g) = \sum_i g(w_{kl}) \otimes z_{kl} \text{ for } g \in \text{Hom}(A_B, M_B).$$

Then $\sum_k \varphi_k \cdot \psi_k = Id$.

Further if mappings

$$\varphi'_k : \text{Hom}(B'_{A'}, A'_{A'}) \longrightarrow B', \quad \psi'_k : B' \longrightarrow \text{Hom}(B'_{A'}, A'_{A'})$$

are defined by

2) Here $1_{B'}$ denotes the identity element of a ring B' .

$$(21)_1 \quad \varphi'_k(f') = \sum_n f'(\gamma \cdot (Id_M \otimes \beta_k) \cdot v_n) \cdot (u_n \cdot \iota) \text{ for } f' \in \text{Hom}(B'_{A'}, A'_{A'})$$

$$(22)_1 \quad [\psi'_k(b')(b'_i)](m) = \sum_l ((b' \cdot b'_i)(mw_{kl})) z_{kl} \text{ for } b', b'_i \in B', m \in M,$$

then φ'_k and ψ'_k are $A' \cdot B'$ -homomorphisms such that

$$(23)_1 \quad \sum_k \psi'_k \cdot \varphi'_k = Id.$$

Indeed, the last equation can be seen as follows :

$$\begin{aligned} & [\{ (\sum_k \psi'_k \cdot \varphi'_k)(c^*_{k',n'}) \} (b')] (m) \\ &= \sum_{k,l} [\varphi'_k(c^*_{k',n'}) \cdot b'] (mw_{kl}) \cdot z_{kl} \text{ (by (22))}_1 \\ &= \sum_{k,l,n} [c^*_{k',n'}(\gamma \cdot (Id_M \otimes \beta_k) \cdot v_n)] (u_n(b'(mw_{kl}) \otimes 1)) \cdot z_{kl} \text{ (by (21))}_1 \\ &= \sum_{k,l,n} u_n [\sum_{l'} (\gamma \cdot Id_M \otimes \beta_k) \cdot v_n] (u_n(b'(mw_{kl}) \otimes 1) \cdot w_{k'l'}) \otimes z_{k'l'}] \cdot z_{kl} \\ & \hspace{20em} \text{(by (18))} \\ &= \sum_{k,l,l'} u_n [(b'(mw_{kl}) \cdot \beta_k(w_{k'l'})) \otimes z_{k'l'}] \cdot z_{kl} \text{ (by (14))} \\ &= \sum_{l'} u_n (b'(m \sum_{k,l} w_{kl} \beta_k(z_{kl}w_{k'l'})) \otimes z_{k'l'}) \\ &= u_n (\sum_{l'} b'(mw_{k'l'}) \otimes z_{k'l'}) = [c^*_{k',n'}(b')] (m), \end{aligned}$$

where the fifth equation is followed from the fact that $\sum_{l'} w_{k'l'} \otimes z_{k'l'} \in A \otimes_B A$ is casimir. Since $\{c^*_{k',n'}\}_{k',n'}$ is a generating set for the left A' -module $\text{Hom}(B'_{A'}, A'_{A'})$, the above equation yields (23)₁.

Now let us consider the following sequence

$$B' \xrightarrow{\sigma'} \text{Hom}({}_{A'}\text{Hom}(B'_{A'}, A'_{A'}), {}_{A'}A') \xrightleftharpoons[\varphi_k^*]{\psi_k^*} \text{Hom}({}_{A'}B', {}_{A'}A'),$$

where $\varphi_k^* = \text{Hom}(\varphi'_k, Id_{A'})$, $\psi_k^* = \text{Hom}(\psi'_k, Id_{A'})$ and σ' denotes a canonical isomorphism. Since $(V_n, U_n)_n$ is a dual basis for ${}_{A'}B'$, if we set

$$\begin{aligned} \beta'_k &= (\psi_k^* \cdot \sigma')(1_{B'}) \\ T_n^{(k)} &= (\sigma'^{-1} \cdot \varphi_k^*)(V_n), \\ (24)_1 \quad & (\beta'_k ; \sum_n T_n^{(k)} \otimes U_n)_{1 \leq k \leq q} \end{aligned}$$

is then a left QF system for B'/A' (see (1)). Further we have

$$(25)_1 \quad [\beta'_k(b')](m) = \sum_l (b'(mw_{kl})) z_{kl} \quad (b' \in B', m \in M),$$

$$(26)_1 \quad T_n^{(k)} = \gamma \cdot (Id_M \otimes \beta_k) \cdot v_n (=c_{k,n}).$$

In fact, (25)₁ is obvious and (26)₁ can be seen as follows: It is enough to show that

$$\text{Hom}(\varphi'_k, Id_{A'}) (V_n) = \sigma' (\gamma \cdot (Id_M \otimes \beta_k) \cdot v_n).$$

Since $[\{\text{Hom}(\varphi'_k, Id_{A'}) (V_n)\} (c_{k',n'}^*)] (m)$

$$= [\pi \cdot (\varphi'_k (c_{k',n'}^*) \otimes Id_A) \cdot v_n] (m) \quad (\text{by (16)})$$

$$= \sum_r [\varphi'_k (c_{k',n'}^*)] (m_r) \cdot a_r \quad (v_n(m) = \sum_r m_r \otimes a_r \text{ in } M \otimes_B A)$$

$$= u_{n'} (\sum_{r,l'} m_r \otimes \beta_k (w_{k'l'}) z_{k'l'}) \cdot a_r \quad (\text{by (18) and (21),})$$

$$= u_{n'} (\sum_{r,l'} m_r \beta_k (a_r w_{k'l'}) \otimes z_{k'l'})$$

$$= [c_{k',n'}^* (\gamma \cdot (Id_M \otimes \beta_k) \cdot v_n)] (m)$$

(where the fourth equation is followed from the casimirness of $\sum_{l'} w_{k'l'} \otimes z_{k'l'} \in A \otimes_B A$) the fact that $\{c_{k',n'}^*\}_{k',n'}$ is a generating set of ${}_{A'}\text{Hom}(B'_{A'}, A'_{A'})$ yields the desired equation.

Let us consider the following :

$$B' \otimes_{A'} M \xrightarrow{\lambda \otimes Id_M} \text{Hom}(M_A, \text{Hom}(A_B, M_B)_A) \otimes_{A'} M$$

$$\xrightarrow{\nu} \text{Hom}(A_B, M_B)$$

$$\xrightleftharpoons[\text{Hom}(\beta_k \cdot z_{kl}, Id_M)]{\text{Hom}(\hat{w}_{kl}, Id_M)} \text{Hom}(B_B, M_B) \xrightarrow{\nu_1} M,$$

where λ, ν and ν_1 are canonical isomorphisms, and for each $a \in A$, \hat{a} denotes a mapping $B \rightarrow A$ given by $\hat{a}(b) = ab$.

Setting

$$(28) \quad f_{k,l} = \nu_1 \cdot \text{Hom}(\hat{w}_{kl}, Id_M) \cdot \nu \cdot (\lambda \otimes Id_M) : B' \otimes_{A'} M \rightarrow M$$

$$(29) \quad g_{k,l} = (\lambda^{-1} \otimes Id_M) \cdot \nu^{-1} \cdot \text{Hom}(\beta_k \cdot z_{kl}, Id_M) \cdot \nu_1^{-1} : M \rightarrow B' \otimes_{A'} M,$$

$f_{k,l}$ and $g_{k,l}$ are left B' -homomorphisms such that

$$(14') \quad \sum_{k,l} g_{k,l} \cdot f_{k,l} = Id ; {}_{B'} B' \otimes_{A'} M |_{B'} B',$$

$$(30) \quad f_{k,l} (b' \otimes m) = b' (mw_{kl}) \quad (b' \in B', m \in M),$$

$$(31) \quad g_{k,l} (m) = \sum_{k',l',n} (\gamma \cdot (Id_M \otimes \beta_{k'}) \cdot v_n) \otimes u_n(m \otimes \beta'_k (z_{kl} w_{k'l'}) z_{k'l'})$$

From now on, we shall denote ρ_A and ρ_B by ρ simply. Corresponding to $U_n, T_n^{(k)} \in B'$, let us define $U_{k,l}, T_{k,l}^{(k)} \in \text{End}({}_{A'}M)$ by

$$(15') \quad U_{k,l} (m) = f_{k,l} (1 \otimes m) \text{ for } m \in M,$$

$$(26') \quad T_{k,i}^{(i)} = \gamma' \cdot (\alpha'_i \otimes Id_M) \cdot g_{k,i},$$

where $\gamma' : A' \otimes_{A'} M \rightarrow M$ is the canonical isomorphism. Then we have

$$(32) \quad [U_{k,i}]^\circ = \rho(w_{ki})$$

$$(33) \quad [T_{k,i}^{(i)}]^\circ = \rho(\sum_j \beta_k(z_{ki} x_{ij}) y_{ij}).$$

In fact, (32) is obvious by (30), and (33) can be seen as follows :

$$\begin{aligned} (T_{k,i}^{(i)})(m) &= \sum_{k',v,n,j} [\gamma' \cdot (Id_M \otimes \beta_{k'}) \cdot v_n (u_n (m \otimes \beta_k(z_{kl} w_{k'v}) z_{k'v}) x_{ij})] y_{ij} \\ &= \sum_{k',v,j} m \beta_{k'}(\beta_k(z_{kl} w_{k'v}) z_{k'v}) x_{ij} y_{ij} \\ &= \sum_j m \beta_k(z_{kl} \sum_{k',v} w_{k'v} \beta_{k'}(z_{k'v}) x_{ij}) y_{ij} \\ &= m \sum_j \beta_k(z_{kl} x_{ij}) y_{ij} \quad (\text{by (8)}), \end{aligned}$$

which shows (33). Further using (8), (32) and (33), we have

$$\sum_{k,i} [U_{k,i}]^\circ \otimes [T_{k,i}^{(i)}]^\circ = \sum_j \rho(x_{ij}) \otimes \rho(y_{ij}) (\in \tilde{A} \otimes_{\tilde{B}} \tilde{A}).$$

Similarly, corresponding to $d_{k,n}^{(i)} \in B'$, let us define $d_{i,k,i}^{(k')} \in \text{End}({}_{A'}M)$ by

$$[d_{i,k,i}^{(k')}] (m) = f_{k,i}((\sum_{k_1,n} c_{k_1,n} \beta_{k'}(d_{k_1,n}^{(i)})) \otimes m) \text{ for } m \in M \quad (\text{cf. (26)}).$$

Then a calculation using (17), (26), (25)₁ and (30) shows that the right hand of the above equation is equal to $m \sum_{k'} w_{ki} w_{k'v} \alpha_i(z_{k'v})$, and so,

$$(35) \quad [d_{i,k,i}^{(k')}]^\circ = \rho(\sum_{k'} w_{ki} w_{k'v} \alpha_i(z_{k'v})).$$

Accordingly, $\sum_{i,k,i} [d_{i,k,i}^{(k')}]^\circ \otimes [T_{k,i}^{(i)}]^\circ = \sum_{k'} \rho(w_{k'v}) \otimes \rho(z_{k'v}) (\in \tilde{A} \otimes_{\tilde{B}} \tilde{A})$.

Therefore, applying the above argument that leads a left QF and a right QF systems for A/B to those for B'/A' to the left modules M over the QF extension B'/A' with the left QF system (24)₁ and the right QF system (24), we know that

$$(24^*) \quad (\tilde{\alpha}_i ; \sum_j \rho(x_{ij}) \otimes \rho(y_{ij}))_i$$

is a right QF system and

$$(24^*)_1 \quad (\tilde{\beta}_k ; \sum_i \rho(w_{ki}) \otimes \rho(z_{ki}))_k$$

is a left QF system for \tilde{A}/\tilde{B} , where $\tilde{\alpha}_i, \tilde{\beta}_k \in \text{Hom}(\tilde{B} \tilde{A}_{\tilde{B}}, \tilde{B} \tilde{B}_{\tilde{B}})$ are defined

$$\text{by} \quad m \tilde{\alpha}_i(\tilde{a}) = \sum_{k,n} c_{k,n} (d_{k,n}^{(i)}(m) \tilde{a})$$

$$m\tilde{\beta}_k(\tilde{a}) = \sum_n T_n^{(k)}(U_n(m)\tilde{a}) \quad (\tilde{a} \in \tilde{A}, m \in M).$$

To be easily seen, we have

$$(36) \quad \tilde{\alpha}_i(\rho(a)) = \rho(\alpha_i(a)), \quad \tilde{\beta}_k(\rho(a)) = \rho(\beta_k(a)) \quad (a \in A).$$

It follows therefore that mappings $A \otimes_B \tilde{B} \rightarrow \tilde{A}$, $a \otimes \tilde{b} \mapsto a\tilde{b}$, and $\tilde{A} \rightarrow A \otimes_B \tilde{B}$, $\tilde{a} \mapsto \sum_{k,l} w_{kl} \otimes \tilde{\beta}_k(z_{kl}\tilde{a})$, are mutually inverse isomorphisms. The same holds for mappings $\tilde{B} \otimes_{B'} A \rightarrow \tilde{A}$, $\tilde{b} \otimes a \mapsto \tilde{b}a$, and $\tilde{A} \rightarrow \tilde{B} \otimes_{B'} A$, $\tilde{a} \mapsto \sum_{i,j} \tilde{\alpha}_i(\tilde{a}x_{ij}) \otimes y_{ij}$.

Summarizing the above, we obtain the following :

Theorem 1.1 (cf. [4, Th. 2.10]). *Suppose that $[A/B, \varphi]$ is a QF (resp. Frobenius) extension and M is a right A -module with $M \otimes_B A_A \mid M_A$. Then there holds the following :*

- 1) $[B'/A', \varphi']$ is a QF (resp. Frobenius) extension.
- 2) ${}_{B'}B' \otimes_{A'} M \mid {}_{B'}M$.
- 3) $[\tilde{A}/\tilde{B}, \tilde{\varphi}]$ is a QF (resp. Frobenius) extension such that $\tilde{A} \cong A \otimes_B \tilde{B} \cong \tilde{B} \otimes_{B'} A$ canonically.

Proposition 1.2. *Suppose that $[A/B, \varphi]$ is a QF extension. Then the following statements hold.*

- 1) If A/B is H -separable (i. e., ${}_A A \otimes_B A_A \mid {}_A A_A$) then ${}_{A'}B'A' \mid {}_{A'}A'A'$. Furthermore assume $M \otimes_B A_A \mid M_A$.
- 2) If A/B is separable (i. e., ${}_A A_A \mid {}_A A \otimes_B A_A$), then ${}_{A'}A'A' \mid {}_{A'}B'A'$.
- 3) If ${}_B A_B \mid {}_B B_B$ then B'/A' is H -separable.
- 4) If ${}_B B_B \mid {}_B A_B$ then B'/A' is separable.

Proof. Let $(\alpha_i : \sum_j x_{ij} \otimes y_{ij})_i$ and $(\beta_k : \sum_l w_{kl} \otimes z_{kl})_k$ be respective right QF and left QF systems for A/B . 1): Since A/B is H -separable, there exist $f_n : {}_A A \otimes_B A_A \rightarrow {}_A A_A$ and $g_n : {}_A A_A \rightarrow {}_A A \otimes_B A_A$ ($n = 1, \dots, n_0$) such that $\sum g_n \cdot f_n = Id$. Let us consider a sequence of $B'A$ -modules

$$M \otimes_B A \xrightarrow{\gamma_1} M \otimes_A A \otimes_B A \xrightleftharpoons[\bar{g}_n = Id_M \otimes g_n]{f_n = Id_M \otimes f_n} M \otimes_{A'} A \xrightarrow{\gamma_2} M$$

where γ_1 and γ_2 are both the canonical isomorphisms.

Setting $u_n = \gamma_2 \cdot \bar{f}_n \cdot \gamma_1$, $v_n = \gamma_1^{-1} \cdot \bar{g}_n \cdot \gamma_2^{-1}$

then
$$\sum v_n \cdot u_n = Id.$$

Therefore we can use Theorem 1.1 and its proof in any case. So, we shall employ the notations as before throughout the proof. First a right QF system (24) yields

$$\sum_i \sum_{k,n} (\alpha'_i(b' \cdot c_{k,n})) \cdot d_{k,n}^{(i)} = b' \quad (b' \in B').$$

However, to be easily verified, $c_{k,n}$ and $d_{k,n}^{(i)}$ are contained in the centralizer $V_{B'}(A')$ of A' in B' at the present case. It follows that ${}_{A'}B'_{A'} \mid {}_{A'}A'_{A'}$. Furthermore assume $M \otimes_B A_A \mid M_A$. Hence B'/A' is QF by Theorem 1.1. 2): Suppose ${}_A A_A \mid A \otimes_B A_A$. Thus

$$\begin{aligned} {}_{A'}M_A &\cong {}_{A'}M \otimes_A A_A \mid {}_{A'}M \otimes_A A \otimes_B A_A \cong {}_{A'}M \otimes_B A_A \sim {}_{A'}\text{Hom}(A_B, M_B)_{A'} \\ \text{which yields } {}_{A'}\text{Hom}(M_A, M_A)_{A'} &\mid {}_{A'}\text{Hom}(M_A, \text{Hom}(A_B, M_B)_{A'})_{A'} \\ &\cong {}_{A'}\text{Hom}(M_B, M_B)_{A'}. \end{aligned}$$

This shows 2). 3): Let us assume ${}_B A_B \mid {}_B B_B$. Thus there exist $\delta_r \in \text{Hom}({}_B A_B, {}_B B_B)$ and $a_r \in V_A(B)$ ($r = 1, \dots, s$) such that $\sum a_r \delta_r(a) = a$ for all $a \in A$. Replacing $f_{k,n}$ and $g_{k,n}$ in the proof of Theorem 1.1 respectively by

$$\begin{aligned} f_r &= \nu_1 \cdot \text{Hom}(\hat{a}_r, Id_M) \cdot \nu \cdot (\lambda \otimes Id_M) \\ g_r &= (\lambda^{-1} \otimes Id_M) \cdot \nu^1 \cdot \text{Hom}(\delta_r, Id_M) \cdot \nu_1^{-1}, \end{aligned}$$

we have ${}_B B' \otimes_{A'} M_B \mid {}_{B'} M_B$ (see (27)). This yields ${}_{B'} B' \otimes_{A'} B'_{B'} \sim {}_{B'}\text{Hom}(B'_{A'}, B'_{A'}) \cong {}_{B'}\text{Hom}(B' \otimes_{A'} M_B, M_B)_{B'} \mid {}_{B'}\text{Hom}(M_B, M_B)_{B'} = {}_{B'} B'_{B'}$, which shows 3). 4): Finally suppose ${}_B B_B \mid {}_B A_B$. Then there exist $f_r \in \text{Hom}({}_B A_B, {}_B B_B)$ and $a_r \in V_A(B)$ ($r = 1, \dots, t$) such that $\sum f_r(a_r) = 1$.

Setting
$$c_i = \sum_{j,r} a_r x_{ij} f_r(y_{ij})$$
 then
$$c_i \in V_A(B), \quad \sum_i \alpha_i(c_i) = 1.$$

Thus a mapping $\rho_{c_i} : M \rightarrow M$ defined by $\rho_{c_i}(m) = mc_i$ is in $V_{B'}(A')$, and so,
$$\sum_i \sum_{k,n} c_{k,n} (\rho_{c_i} \cdot d_{k,n}^{(i)}) = 1_{B'} \quad (\text{see (36)}).$$

On the other hand, every $\sum_{k,n} c_{k,n} \otimes d_{k,n}^{(i)}$ is casimir in $B' \otimes_{A'} B'$ and hence so is $\sum_i \sum_{k,n} c_{k,n} \otimes \rho_{c_i} \cdot d_{k,n}^{(i)}$. It follows that B'/A' is separable. Thus our proof is complete.

Proposition 1.3. *Assume that $[A/B, \varphi]$ is right QF (resp. left QF, Frobenius) and M_A is a generator. Then $[A(= \tilde{A})/\tilde{B}, \tilde{\varphi}]$ is right QF (resp. left QF, Frobenius), and moreover right QF (resp. left QF, Frobenius) system for $[A/B, \varphi]$ gives a right QF (resp. left QF, Fro-*

benius) system for $[A/\tilde{B}, \tilde{\varphi}]$ naturally.

Proof. First we shall show that any $f \in \text{Hom}(A_B, B_B)$ is contained in $\text{Hom}(A_{\tilde{B}}, B_{\tilde{B}})$ (note that M_A is a generator). In fact, $mf(u(m')\tilde{b}) = (\lambda_m \cdot f \cdot u)(m'\tilde{b}) = m(f(u(m'))\tilde{b})$ ($m, m' \in M, \tilde{b} \in \tilde{B}, u \in \text{Hom}(M_A, A_A)$) means the conclusion, where λ_m denotes a mapping $A \rightarrow M$ given by $\lambda_m(a) = ma$. Let $(\alpha_i; \sum_j x_{ij} \otimes y_{ij})_i$ be a right QF system for A/B . The mention just above shows that every α_i is in $\text{Hom}(A_{\tilde{B}}, \tilde{B}_{\tilde{B}})$ and $A_{\tilde{B}} | \tilde{B}_{\tilde{B}}$. We claim further every α_i is contained in $\text{Hom}({}_{\tilde{B}}A, {}_{\tilde{B}}\tilde{B})$. To see this, let $x \in A, \tilde{b} \in \tilde{B}$ and $f \in \text{Hom}(A_B, B_B)$ be arbitrary elements. Then $f(\alpha_i(\tilde{b}x) - \tilde{b}\alpha_i(x)) = f(1)\alpha_i(\tilde{b}x) - f(\tilde{b})(\alpha_i(x)) = \alpha_i(f(1)\tilde{b}x - f(\tilde{b})x) = 0$, which implies $\alpha_i(\tilde{b}x) - \tilde{b}\alpha_i(x) = 0$ as desired, since A_B torsionless. On the other hand, every casimir element in $A \otimes_B A$ is mapped canonically into a casimir element in $A \otimes_{\tilde{B}} A$. Therefore we have shown the proposition for a right QF extension.

Next let $(\beta_k; \sum_l w_{kl} \otimes z_{kl})_k$ be a left QF system for A/B . By the mention at the beginning of the proof, every β_k is in $\text{Hom}(A_{\tilde{B}}, \tilde{B}_{\tilde{B}})$. Further we can regard as $\text{Hom}({}_B A, {}_B B) \subset \text{Hom}({}_{\tilde{B}} A, {}_{\tilde{B}} \tilde{B})$; $[(g(\tilde{b}x) - \tilde{b}g(x))\beta_k](y) = \beta_k(yg(\tilde{b}x)) - \beta_k(y\tilde{b}g(x)) = g(\beta_k(y)\tilde{b}x) - g(\beta_k(y)\tilde{b})x = 0$ ($g \in \text{Hom}({}_B A, {}_B B), x, y \in A$) $\implies (g(\tilde{b}x) - \tilde{b}g(x))\beta_k = 0 \implies g(\tilde{b}x) - \tilde{b}g(x) = 0$ as desired. In particular, $\beta_k \in \text{Hom}({}_{\tilde{B}} A_{\tilde{B}}, {}_{\tilde{B}} \tilde{B}_{\tilde{B}})$ and ${}_{\tilde{B}} A | {}_{\tilde{B}} \tilde{B}$. It follows that the proposition for a left QF extension has been shown. The case of Frobenius extension is now obvious. Thus our proof is complete.

Let B_0 be the subring of B generated as a ring by the identity element of B and by all the elements of the form $f(a)$ ($a \in A, f \in \text{Hom}(A_B, B_B)$), \bar{B} the double centralizer of A_B (i. e., $\bar{B} = [\text{End}({}_{\text{End}(A_B)} A)]^\circ$), and T an arbitrary intermediate subring of $\bar{B}/\varphi(B_0)$. Then the proof of the above proposition (taking A_A as M_A) shows the following assertion obtained in Morita [6, Ths 1.1 and 1.3].

Proposition 1.4. *Under the same notations as above, if A/B is right QF (resp. left QF, Frobenius) then so is A/T .*

2. In this section, B denotes a subring of a ring $A, \varphi : B \rightarrow A$ the inclusion mapping, T an intermediate subring of A/B and \mathbb{Z} the ring of integers. Moreover, for a subset S of A, S' denotes the centra-

lizer of S in A .

Under the above notations, we shall prove the following as an application of Theorem 1. 1.

Theorem 2.1 (cf. [4, Th. 2. 6]). *If A/B is a QF extension with ${}_T T \otimes \otimes_B A_A | {}_T A_A$ then B'/T' is a QF extension with ${}_A A \otimes {}_T B'_B | {}_A A_B$, and moreover $T'' (= (T')')/B''$ is a QF extension such that*

$$T \otimes_B B'' \longrightarrow T'', \quad t \otimes b'' \longmapsto tb''$$

and

$$B'' \otimes_B T \longrightarrow T'', \quad b'' \otimes t \longmapsto b''t$$

are isomorphisms.

Before going to prove the theorem, we note some facts. For a T - A -module X , we can give X a left $T \otimes_{\mathbf{Z}} A^\circ$ -module structure in a natural way: $(t \otimes a^\circ) \cdot x = txa$ ($a \in A, t \in T, x \in X$). Conversely if X is a left $T \otimes_{\mathbf{Z}} A^\circ$ -module then X has a T - A -module structure. Hence the $T \otimes_{\mathbf{Z}} A^\circ$ -endomorphism ring of left $T \otimes_{\mathbf{Z}} A^\circ$ -module A can be identified with the centralizer T' of T in A naturally. The following lemma can be easily verified.

Lemma 2.2. 1) $T \otimes_B A \cong (T \otimes A^\circ) \otimes_{B \otimes A} A^\circ$ as left $T \otimes A^\circ$ -modules by the correspondence $t \otimes a \longmapsto t \otimes 1^\circ \otimes a$, where the unspecified tensor products of the right hand are taken over \mathbf{Z} .

2) For each element $\sum t_i \otimes a_i \in (T \otimes_B A)^T = \{\gamma \in T \otimes_B A \mid t\gamma = \gamma t \text{ for all } t \in T\}$, a mapping

$$\gamma (\sum t_i \otimes a_i): B' \longrightarrow T', \quad [\gamma (\sum t_i \otimes a_i)](b') = \sum t_i b' a_i$$

is a left T' -homomorphism.

Lemma 2.3. 1) ${}_T T \otimes_B A_A | {}_T A_A$ is equivalent to the existence of a finite number of elements $b'_m \in B'$ and $\sum_n t_{mn} \otimes a_{mn} \in (T \otimes_B A)^T$ ($m = 1, \dots, m_0$) such that

$$(37) \quad \sum_m \sum_n t_{mn} \otimes a_{mn} b'_{m'} = 1 \otimes 1 \quad (\in T \otimes_B A).$$

2) Assume ${}_T T \otimes_B A_A | {}_T A_A$ and let $(b'_m; \sum_n t_{mn} \otimes a_{mn})_m$ be as above

1). Then there holds the following:

i) $\sum_{m,n} f(tt_{mn}) a_{mn} t_1 b'_m = f(tt_1) \quad (t, t_1 \in T, f \in \text{Hom}(T_B, T_B))$.

ii) ${}_T B'$ is f.g. projective with a dual basis $(\gamma_m, b'_m)_m$, where $\gamma_m = \gamma (\sum_n t_{mn} \otimes a_{mn})$ ($m=1, \dots, m_0$).

Proof. 1) and 2-i) can be verified easily. Let $b' \in B'$ be an arbitrary element. A mapping $\lambda_{b'} : A \rightarrow A$ defined by $\lambda_{b'}(a) = b'a$ is a left B -homomorphism obviously. Operating $Id_T \otimes \lambda_{b'} : T \otimes_B A \rightarrow T \otimes_B A$ to (37), we have

$$\sum_{m,n} t_{mn} \otimes b'a_{mn}b'_m = 1 \otimes b',$$

which yields $b' = \sum_{m,n} t_{mn} b'_m = \sum_m \gamma_m(b')b'_m$.

This implies ii) from Lemma 2. 2 2).

We are now ready for proving the theorem.

Proof of Theorem 2. 1. Assume that T/B is a QF extension with ${}_T T \otimes_B A_A \mid {}_T A_A$. Let $(\alpha_i; \sum_j x_{ij} \otimes y_{ij})_i$ and $(\beta_k; \sum_l w_{kl} \otimes z_{kl})_k$ be respective right QF and left QF systems for T/B and let $(b'_m; \sum_n t_{mn} \otimes a_{mn})_m$ be as Lemma 2. 3 1). Since ${}_T B'$ is f. g. projective by Lemma 2. 3 2-ii), a mapping $\sigma : A \otimes_{{}_T B'} \rightarrow \text{Hom}(\text{Hom}({}_T B', {}_T T')_{{}_T}, A_{{}_T})$ defined by $[\sigma(a \otimes b')](g) = a.g(b')$ is an A - B' -isomorphism (see (1)).

Setting $\gamma_{k,l} = \sum_{m,n} \beta_k(z_{kl} t_{mn})a_{mn} \otimes b'_m \in A \otimes_{{}_T B'}$,

we have by Lemma 2. 3 2-i)

$$[\sigma(\gamma_{k,l} b')](\eta_{m'}) = \sum_{n'} \beta_k(z_{kl} t_{m'n'}) b'a_{m'n'} = [\sigma(b' \gamma_{k,l})](\eta_{m'}).$$

However, $\{\eta_{m'}\}_{m'}$ is a generating set of $\text{Hom}({}_T B', {}_T T')_{{}_T}$ by Lemma 2. 3 2-ii). Therefore the above equation shows $\sigma(\gamma_{k,l} b') = \sigma(b' \gamma_{k,l})$, that is, $\gamma_{k,l} \in (A \otimes_{{}_T B'})^B$ for each k, l .

Similarly we have $[\sigma(\sum_{k,l} w_{kl} \gamma_{k,l})](\eta_{m'}) = [\sigma(1 \otimes 1)](\eta_{m'})$, and so,

$$\sum_{k,l} w_{kl} \gamma_{k,l} = 1 \otimes 1 \in A \otimes_{{}_T B'}.$$

It follows that ${}_A A \otimes_{{}_T B'} \mid {}_A A_{B'}$ and $(\eta_{k,l} w_{kl})_{k,l}$ is a dual basis for $T''_{B''}$ by Lemma 2. 3 (left and right are replaced), where $\eta_{k,l} : T'' \rightarrow B''$ is defined by $\eta_{k,l}(t'') = \sum_{m,n} \beta_k(z_{kl} t_{mn}) a_{mn} t'' b'_m$ ($t'' \in T''$). Then $\eta_{k,l}(t) = \beta_k(z_{kl} t)$ for $t \in T$. Thus, if $\sum t_r \otimes a_r \in T \otimes_B B''$ is an element with $\sum t_r a_r = 0$, then

$$\begin{aligned} \sum t_r \otimes a_r &= \sum \sum_{k,l} w_{kl} \beta_k(z_{kl} t_r) \otimes a_r \\ &= \sum_{k,l} w_{kl} \otimes \eta_{k,l}(\sum t_r a_r) = 0, \end{aligned}$$

which implies that a canonical mapping $T \otimes_B B'' \ni t \otimes b'' \mapsto tb'' \in T''$ is an injection. However, recalling the above dual basis for $T''_{B''}$, it is obviously surjective, and so, it is a bijection. Moreover our assump-

tions, T/B being QF and ${}_{\tau}T \otimes {}_B A_A | {}_{\tau}A_A$, yield

$${}_A A \otimes {}_B T_{\tau} \sim {}_A \text{Hom}(T_B, A_B)_{\tau} \cong {}_A \text{Hom}(T \otimes {}_B A_A, A_A)_{\tau} \\ | {}_A \text{Hom}(A_A, A_A)_{\tau} \cong {}_A A_{\tau}.$$

Therefore a symmetric argument shows that a mapping $B'' \otimes {}_B T \ni b'' \otimes t \mapsto b'' t \in T''$ is a bijection.

On the other hand, it is easy to see that $(\alpha_i \otimes Id_{A^{\circ}}; \sum_j (x_{ij} \otimes 1^{\circ}) \otimes (y_{ij} \otimes 1^{\circ}))_i$ and $(\beta_k \otimes Id_{A^{\circ}}; \sum_l (w_{kl} \otimes 1^{\circ}) \otimes (z_{kl} \otimes 1^{\circ}))_k$ are respective right QF and left QF systems for $T \otimes {}_z A^{\circ} / B \otimes {}_z A^{\circ}$. Since $(T \otimes {}_z A^{\circ}) \otimes {}_{B \otimes {}_z A^{\circ}} A \cong T \otimes {}_B A | A$ as left $T \otimes {}_z A^{\circ}$ -modules by Lemma 2.2 1) and our assumption, we can apply Theorem 1.1 to the left module A over the QF extension $T \otimes {}_z A^{\circ} / B \otimes {}_z A^{\circ}$. Thus $\text{End}({}_{B \otimes {}_z A^{\circ}} A) / \text{End}({}_{\tau \otimes {}_z A^{\circ}} A)$, and hence B'/T' is a QF extension. Finally, a similar argument shows that T''/B'' is QF. Thus our proof is complete.

3. Let $\varphi : B \rightarrow A$ be a ring homomorphism, U a right B -module and V a right A -module. Put $B^* = \text{End}(U_B)$ and $A^* = \text{End}(V_A)$. In a natural way, $\text{Hom}(A_B, U_B)$ is a B^* - A -module. In the subsequent study, we assume always the following conditions.

(38) There exists an A -isomorphism $\alpha : V \rightarrow \text{Hom}(A_B, U_B)$.

(39) $V_B | U_B; \sum g_t \cdot f_t = Id_V$ for some $f_t : V_B \rightarrow U_B$ and $g_t : U_B \rightarrow V_B$ ($t=1, \dots, t_0$).

By (38), V may be regarded as a B^* - A -module. Further α induces a ring homomorphism $\varphi^* : B^* \rightarrow A^*$ defined by $\varphi^*(b^*)(v^*) = \alpha^{-1}(b^* \alpha(v))$ or $\varphi^*(b^*) = \alpha^{-1} \cdot b^* \cdot \alpha$ ($b^* \in B^*, v \in V$). On the other hand, (39) gives the canonical isomorphism

$$\pi : V \rightarrow \text{Hom}({}_{B^*} \text{Hom}(V_B, U_B), {}_{B^*} U), [\pi(v)](f) = f(v).$$

Moreover, we have a chain of B^* - A^* -isomorphisms

$$\begin{array}{ccc} A^* & \xrightarrow{\text{Hom}(Id_V, \alpha)} & \text{Hom}(V_A, \text{Hom}(A_B, U_B)_A) \\ & \searrow \xi & \downarrow \\ & & \text{Hom}(V \otimes {}_A A_B, U_B) \\ & & \downarrow \\ & & \text{Hom}(V_B, U_B). \end{array}$$

where the vertical mappings are canonical. Then the composite $\xi : A^* \rightarrow \text{Hom}(V_B, U_B)$ of these isomorphisms is a B^* - A^* -isomorphism such that

$$[\xi(a^*)](v) = [\alpha(a^*v)] \quad (1) \quad (a^* \in A^*, v \in V).$$

Therefore, we have a left A^* -isomorphism

$$(38^*) \quad \alpha^* = \text{Hom}(\xi, Id_V) \cdot \pi : V \longrightarrow \text{Hom}({}_{B^*}A^*, {}_{B^*}U) \\ [\alpha^*(v)](a^*) = [\alpha(a^*v)] \quad (1).$$

Accordingly, a ring homomorphism $\tilde{\varphi} : \tilde{B} = [\text{End}({}_{B^*}U)]^\circ \longrightarrow \tilde{A} = [\text{End}({}_{A^*}V)]^\circ$ can be induced by α^* in a similar way. For convenience, the canonical ring homomorphisms $A \longrightarrow \tilde{A}$ and $B \longrightarrow \tilde{B}$ will be denoted by ρ . Under the above notations, we can obtain the following which corresponds to Theorem 1.1.

Theorem 3.1. *Assume that U_B and V_A satisfy (38) and (39). If $[A/B, \varphi]$ is QF (resp. Frobenius), then there hold the followings:*

- 1) $[A^*/B^*, \varphi^*]$ is QF (resp. Frobenius).
- 2) ${}_{A^*}V \cong {}_{A^*}\text{Hom}({}_{B^*}A^*, {}_{B^*}U)$.
- 3) ${}_{B^*}V \mid {}_{B^*}U$.
- 4) $[\tilde{A}/\tilde{B}, \tilde{\varphi}]$ is QF (resp. Frobenius) such that

$$\tilde{B} \otimes_B A \longrightarrow \tilde{A}, \quad \tilde{b} \otimes a \longmapsto \tilde{b}\rho(a)$$

and

$$A \otimes_B \tilde{B} \longrightarrow \tilde{A}, \quad a \otimes \tilde{b} \longmapsto \rho(a)\tilde{b}$$

are isomorphisms.

Proof. The assertion 2) has been shown already. Let us set $V' = \text{Hom}({}_{A_B}, U_B)$ and $\Delta = \text{End}({}_{A_B})$. Then we can consider A as a subring of Δ and V' a B^* - Δ -module in a natural way. Put $\Lambda = \text{End}(V'_\Delta)$ and $\Gamma = \text{End}(V'_\Delta)$. We have then a ring homomorphism $\psi : B^* \longrightarrow \Gamma$ defined by $\psi(b^*)(v') = b^*v'$ ($b^* \in B^*, v' \in V'$) and a ring isomorphism $\bar{\psi} : A^* \longrightarrow \Lambda$ induced by α . Then it is easy to see $\bar{\psi} \cdot \psi^* = \iota \cdot \psi$, where ι denotes the inclusion mapping of Γ to Λ .

Let $(\alpha_i ; \sum_j x_{ij} \otimes y_{ij})_i$ and $(\beta_k ; \sum_l w_{kl} \otimes z_{kl})_k$ be respective right QF and left QF systems for A/B and let $(\delta_n, a_n)_{1 \leq n \leq n_0}$ be a dual basis for A_B . Define $u_n : A \otimes_B A_A \longrightarrow A_A$ and $v_n : A_A \longrightarrow A \otimes_B A_A$ by $u_n(x \otimes y) = \delta_n(x)y$ and $v_n(x) = a_n \otimes x$ respectively. Then $\sum_n v_n \cdot u_n = Id$, and so $A \otimes_B A_A \mid A_A$. Therefore we can apply Theorem 1.1 to the right module A over the QF extension A/B . Thus we know by its

proof that Δ/A is a QF extension with a right QF system $(\alpha'_i; \sum_{k,n} c_{k,n} \otimes d_{k,n}^{(i)})$ and a left QF system $(\beta'_k; \sum_n c_{k,n} \otimes U_n)_k$,

$$\begin{aligned} \text{where} \quad [\alpha'_i(\delta)](a) &= \sum_j \delta(ax_{ij})y_{ij} && \text{(see (25))} \\ [\beta'_k(\delta)](a) &= \sum_l \delta(aw_{kl})z_{kl} && \text{(see (25))} \\ c_{k,n}(a) &= a_n \beta_k(a) && \text{(see (17))} \\ d_{k,n}^{(i)}(a) &= \delta_n(a) \sum_l \alpha_l(w_{kl})z_{kl} && \text{(see (26))} \\ U_n(a) &= \delta_n(a) \quad (\delta \in \Delta, a \in A) && \text{(see (15)).} \end{aligned}$$

Further define $\bar{f}_i: V' \otimes_A \Delta_A \longrightarrow V'_A$ and $\bar{g}_i: V'_A \longrightarrow V' \otimes_A \Delta_A$ by

$$\begin{aligned} [\bar{f}_i(w \otimes \delta)](a) &= (f_i \cdot \alpha^{-1})(w\delta(a)) \\ \bar{g}_i(w) &= \sum_n ((\alpha \cdot g_i)(w(a_n))) \otimes \delta_n \quad (w \in V', \delta \in \Delta, a \in A) \end{aligned}$$

Then
$$\sum \bar{g}_i \cdot \bar{f}_i = Id; \quad V' \otimes_A \Delta_A | V'.$$

Hence we can apply Theorem 1.1 to the right module V' over the QF extension Δ/A with the above left QF and right QF systems. Thus we know that A/Γ is a QF extension with a right QF system

$$(40) \quad (\alpha_i^+; \sum_{k,t} c_{k,t}^+ \otimes d_{k,t}^{+(i)})$$

and a left QF system

$$(41) \quad (\beta_k^+; \sum_i c_{k,t}^+ \otimes U_i^+)_k$$

where

$$\begin{aligned} [\alpha_i^+(\lambda)](w) &= \sum_{k,n} (\lambda(w \cdot c_{k,n})) \cdot d_{k,n}^{(i)} \\ [\beta_k^+(\lambda)](w) &= \sum_n (\lambda(w \cdot c_{k,n})) \cdot U_n \\ c_{k,t}^+(w) &= (\gamma \cdot (Id_{V'} \otimes \beta_k^+) \cdot \bar{g}_i)(w) \\ d_{k,t}^{+(i)}(w) &= \bar{f}_i(w \otimes \sum_n \alpha_i^+(c_{k,n}) \cdot U_n) \\ U_i^+(w) &= \bar{f}_i(w \otimes 1) \end{aligned}$$

$$(w \in V', \lambda \in A, \gamma: V' \otimes_A A \ni w \otimes a \longmapsto wa \in V').$$

Moreover we can see that

$$\begin{aligned} [\alpha_i^+(\lambda)(w)](a) &= [\lambda(w \cdot a \cdot \alpha_i)](1) \\ [\beta_k^+(\lambda)(w)](a) &= [\lambda(w \cdot a \cdot \beta_k)](1) \\ c_{k,t}^+(w) &= \sum_l ((\alpha \cdot g_l)(w(w_{kl}))) \cdot z_{kl} \\ [d_{k,t}^{+(i)}(w)](a) &= (f_i \cdot \alpha^{-1})(w \cdot a \sum_j \beta_k(x_{ij})y_{ij}) \\ [U_i^+(w)](a) &= (f_i \cdot \alpha^{-1})(w \cdot a) \quad (w \in V', a \in A). \end{aligned}$$

Let us define further $\alpha_i^*, \beta_k^* \in \text{Hom}(B^*A^*_{B^*}, B^*B^*_{B^*})$ by

$$(42) \quad [\alpha_i^* (a^*)] (u) = [\alpha (a^* \alpha^{-1} (\lambda_u \cdot \alpha_i))] (1)$$

$$(43) \quad [\beta_k^* (a^*)] (u) = [\alpha (a^* \alpha^{-1} (\lambda_u \cdot \beta_k))] (1),$$

where for $u \in U$, λ_u denotes a mapping $B \ni b \longmapsto ub \in U$. We have then $\psi \cdot \alpha_i^* = \alpha_i^+ \cdot \bar{\psi}$ and $\psi \cdot \beta_k^* = \beta_k^+ \cdot \bar{\psi}$. It follows that

$$(40^*) \quad (\alpha_i^* ; \sum_{k,t} c_{k,t}^* \otimes d_{k,t}^{*(\phi)})_i$$

is a right QF system and

$$(41^*) \quad (\beta_k^* ; \sum_t c_{k,t}^* \otimes U_t^*)_k$$

is a left QF system for $[A^*/B^*, \varphi^*]$, where $c_{k,t}^* = \bar{\psi}^{-1}(c_{k,t})$, $d_{k,t}^{*(\phi)} = \bar{\psi}^{-1}(d_{k,t}^{(\phi)})$ and $\bar{\psi}^{-1}(U_t^*)$. Moreover we can see that

$$(44) \quad c_{k,t}^* (v) (= \alpha^{-1}(c_{k,t}^+ (\alpha (v)))) = \sum_l (g_l (\alpha(v) (w_{kl}))) z_{kl}$$

$$(45) \quad (\alpha (d_{k,t}^{*(\phi)} (v))) (a) (= [d_{k,t}^{*(\phi)} (a (v))] (a)) = f_t (va \sum_j \beta_k (x_{ij}) y_{ij})$$

$$(46) \quad [\alpha (U_t^* (v))] (a) (= [U_t^+ (\alpha(v))] (a)) = f_t(va).$$

On the other hand, if we define $f_n^* : B^*V \longrightarrow B^*U$ and $g_n^* : B^*U \longrightarrow B^*V$ by $f_n^*(v) = [\alpha(v)] (a_n)$ and $g_n^*(u) = \alpha^{-1} (\lambda_u \cdot \delta_n)$, then $\sum g_n^* \cdot f_n^* = Id_v$, which shows 3).

Now 1), 2) and 3) enable us to use the above argument to the QF extension A^*/B^* with the left QF system (41*) and the right QF system (40*). So, let $\tilde{c}_{i,n}$, $\tilde{d}_{i,n}^{(k)}$ and \tilde{a}_n be in \tilde{A} such that

$$(44') \quad v\tilde{c}_{i,n} = \sum_{k,t} c_{k,t}^* \{g_n^* ((\alpha^* (v)) (d_{k,t}^{*(\phi)}))\}$$

$$(45') \quad [\alpha^* (v\tilde{d}_{i,n}^{(k)})] (a^*) = f_n^* (\sum_t (c_{k,t}^* \cdot \alpha_i^* (U_t^*)) (a^*v))$$

$$(46') \quad [\alpha^* (v\tilde{a}_n)] (a^*) = f_n^* (a^*v).$$

Then
$$[\alpha (a^*v\tilde{a}_n)] (1) = \alpha (a^*va_n) (1),$$

and so,
$$[\xi (a^*)] (v\tilde{a}_n) = [\xi (a^*)] (va_n).$$

Since $\xi (A^*) = \text{Hom}(V_B, U_B)$ and $V_B | U_B$, the last implies $\tilde{a}_n = \rho (a_n)$. Furthermore, the right hands of (44') and (45') are equal to $v \sum_j \delta_n (x_{ij}) y_{ij}$ and $v \sum_l w_{kl} \alpha_l (z_{kl} a_n)$, respectively, and so,

$$\tilde{c}_{i,n} = \rho (\sum_j \delta_n (x_{ij}) y_{ij}),$$

$$\tilde{d}_{i,n}^{(k)} = \rho (\sum_l w_{kl} \alpha_l (z_{kl} a_n)).$$

It follows that

$$\begin{aligned} \sum_{i,n} \widetilde{a}_{i,n}^{(k)} \otimes \widetilde{c}_{i,n} &= \sum_l \rho(w_{kl}) \otimes \rho(z_{kl}), \\ \sum_n \widetilde{a}_n \otimes \widetilde{c}_{i,n} &= \sum_j \rho(x_{ij}) \otimes \rho(y_{ij}) \text{ in } \widetilde{A} \otimes_{\widetilde{B}} \widetilde{A}. \end{aligned}$$

Therefore, if we define $\widetilde{\alpha}_i, \widetilde{\beta}_k \in \text{Hom}(\widetilde{B}\widetilde{A}\widetilde{B}, \widetilde{B}\widetilde{B}\widetilde{B})$ by

$$(42') \quad u\widetilde{\alpha}_i(\widetilde{a}) = \{\alpha^*((\alpha^{*-1}(\rho_u \cdot \alpha_i^*))\widetilde{a})\} (1_{A^*})$$

$$(43') \quad u\widetilde{\beta}_k(\widetilde{a}) = \{\alpha^*((\alpha^{*-1}(\rho_u \cdot \beta_k^*)) \cdot \widetilde{a})\} (1_{A^*})$$

($u \in U, \rho_u : B^* \ni b^* \mapsto b^*u \in U$), we know that $(\widetilde{\beta}_k : \sum_l \rho(w_{kl}) \otimes \rho(z_{kl}))_k$ and $(\widetilde{\alpha}_i : \sum_j \rho(x_{ij}) \otimes \rho(y_{ij}))_i$ are respective left QF and right QF systems for $[\widetilde{A}/\widetilde{B}, \widetilde{\varphi}]$. Moreover it is easy to see $\widetilde{\alpha}_i(\rho(a)) = \rho(\alpha_i(a))$ and $\widetilde{\beta}_k(\rho(a)) = \beta_k(a)$ for $a \in A$. Therefore, as was mentioned in the proof of Theorem 1.1, we have the latter half of 4). Similarly we can show the assertion for a Frobenius extension. Thus our proof is complete.

Remark. Under the same assumption as the theorem, let us consider the following diagram of functors :

$$\begin{array}{ccc} & \overline{D}_1 & \\ & \text{Mod}_A \xleftrightarrow{\quad} {}_{A^*}\text{Mod} & \\ S, T \uparrow & \overline{D}_2 & \uparrow S^*, T^* \\ & D_1 & \\ & \text{Mod}_B \xleftrightarrow{\quad} {}_{B^*}\text{Mod} & \\ & D_2 & \end{array}$$

where Mod_B (resp. ${}_{B^*}\text{Mod}$) denotes the category of right B (resp. left B^*)-modules and

$$\begin{aligned} D_1 &= \text{Hom}_B(-, U_B), & D_2 &= \text{Hom}_{B^*}(-, {}_{B^*}U) \\ \overline{D}_1 &= \text{Hom}_A(-, V_A), & \overline{D}_2 &= \text{Hom}_{A^*}(-, {}_{A^*}V) \\ S &= - \otimes_B A_A, & T &= \text{Hom}_B(A_B, -) \\ S^* &= {}_{A^*}A^*{}_{B^*} \otimes -, & T^* &= \text{Hom}_{B^*}({}_{B^*}A^*, -). \end{aligned}$$

Then the functors $\overline{D}_1 \circ S, T^* \circ D_1 : \text{Mod}_B \longrightarrow {}_{A^*}\text{Mod}$ are equivalent. The same holds for the functors $\overline{D}_2 \circ S^*, T \circ D_2 : {}_{B^*}\text{Mod} \longrightarrow \text{Mod}_A$. In fact, $(\overline{D}_1 \circ S)(X) = \text{Hom}(X \otimes_B A_A, V_A)$

$$\begin{aligned}
 &\cong \text{Hom}(X_B, \text{Hom}(A_A, V_A)_B) \\
 &\cong \text{Hom}(X_B, V_B) \\
 \text{Hom}(Id_X, \alpha^*) & \\
 &\cong \text{Hom}(X_B, \text{Hom}({}_B A^*, {}_B U)_B) \\
 &\cong \text{Hom}({}_B A, {}_B \text{Hom}(X_B, U_B)) \\
 &= (\mathbf{T}^* \circ \mathbf{D}_1)(X) \qquad (X \in \text{Mod}_B).
 \end{aligned}$$

Thus, if we define $\eta_X : (\bar{\mathbf{D}}_1 \circ \mathbf{S})(X) \longrightarrow (\mathbf{T}^* \circ \mathbf{D}_1)(X)$ by the composite of these isomorphisms, then η_X is a left A^* -isomorphism which is natural in $X \in \text{Mod}_B$ obviously. The second assertion can be shown similarly.

Corollary. *Suppose that U_B is an f.g. injective cogenerator and B is right artinian. Put $V = \text{Hom}(A_B, U_B)$. If $[A/B, \varphi]$ is QF (resp. Frobenius) then $[A^*/B^*, \varphi^*]$ is QF (resp. Frobenius).*

Proof. By our assumption A is right artinian and V is an f.g. injective cogenerator as an A , hence as a B -module. Thus $V_B | U_B$, and so this corollary is a direct consequence of the theorem.

Remark 1) This corollary also can be seen as follows: In what follows, for a ring R , we shall denote by \mathfrak{D}_R (resp. ${}_R \mathfrak{D}$) the full subcategory of Mod_R (resp. ${}_R \text{Mod}$) consisting of all f.g. right (resp. left) R -modules. By our assumption, A^* and B^* are both left artinian, ${}_B U$ and ${}_A V$ are both f.g. injective cogenerators, $A = [\text{End}({}_A V)]^\circ$ and $B = [\text{End}({}_B U)]^\circ$, and further $\mathbf{D} = (\mathbf{D}_{10}, \mathbf{D}_{20})$ (resp. $\bar{\mathbf{D}} = (\bar{\mathbf{D}}_{10}, \bar{\mathbf{D}}_{20})$) gives a duality between \mathfrak{D}_B and \mathfrak{D}_{B^*} (resp. between \mathfrak{D}_A and ${}_A \mathfrak{D}$), where $(\)_0$ denotes the restriction of $(\)$ to \mathfrak{D} .³⁾ However, as was mentioned in the proof of the corollary, we have $V_B | U_B$ and ${}_B V | {}_B U$, and so, Remark to Theorem 3.1 yields that

$$\bar{\mathbf{D}}_{20} \circ \mathbf{S}^*_0 \cong \mathbf{T}_0 \circ \bar{\mathbf{D}}_{20}, \quad \bar{\mathbf{D}}_{10} \circ \mathbf{S}_0 \cong \mathbf{T}^*_0 \circ \bar{\mathbf{D}}_{10}.$$

Moreover the assumption that A/B is QF (resp. Frobenius) yields $\mathbf{S}_0 \sim$ (resp. \cong) \mathbf{T}_0 .⁴⁾ Thus

$$\begin{aligned}
 \mathbf{S}^*_0 \cong \bar{\mathbf{D}}_{10} \circ \bar{\mathbf{D}}_{20} \circ \mathbf{S}^*_0 \cong \bar{\mathbf{D}}_{10} \circ \mathbf{T}_0 \circ \bar{\mathbf{D}}_{20} &\sim (\text{resp. } \cong) \bar{\mathbf{D}}_{10} \circ \mathbf{S}_0 \circ \bar{\mathbf{D}}_{20} \\
 &\cong \mathbf{T}^*_0 \circ \bar{\mathbf{D}}_{10} \circ \bar{\mathbf{D}}_{20} \cong \mathbf{T}^*_0,
 \end{aligned}$$

and so A^*/B^* is QF (resp. Frobenius).

3) See [1, Th. 6].

4) See [6, Th. 5.1].

2) With the same assumptions as in the corollary, if $[A/B, \varphi]$ is QF then φ and φ^* are both monic: Let $b \in B$ and $b^* \in B^*$ be elements with $\varphi(b) = 0$ and $\varphi^*(b^*) = 0$. Since B is right artinian, U_B is f. g. injective and V_B is a cogenerator, we have therefore $U_B|V_B$. As $Vb = V\varphi(b) = 0$, $Ub = 0$. Thus the faithfulness of U_B implies $b = 0$. Finally $\varphi^*(b^*) = 0 \implies b^* \text{Hom}(A_B, U_B) = 0 \implies b^* (\text{Hom}(A_B, U_B)(A)) = b^* U = 0$ as desired.

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