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On decompositions into simple rings

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ON DECOMPOSITIONS INTO SIMPLE RINGS

Dedicated to Professor Kiiti Morita on the occasion
of his 60th birthday

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It is the purpose of this paper to give the conditions for a (non-zero) ring to be a direct sum of complete rings of linear transformations of finite rank of vector spaces over division rings, which are motivated by the results in [4], [5] and [6] (Theorem 1). Moreover, we shall give several equivalent conditions for a ring to be a direct sum of division rings (Theorem 2).

A ring R is defined to be *left* (resp. *right*) *s-unital* if $RI=I$ (resp. $IR=I$) for every left (resp. right) ideal I of R. Needless to say, if R is left s-unital then the left R-module ${}_RR$ is unital (or the right R-module R_R is faithful). Every ring with left identity is left s-unital and every regular ring is left and right s-unital.

Lemma 1. *Let R be a left s-unital ring.*

(a) *If A is a proper (two-sided) ideal of R then A is contained in a maximal left ideal, in particular, R contains a maximal left ideal.*

(b) *If every maximal left ideal of R is a direct summand of ${}_RR$ then ${}_RR$ is completely reducible, and conversely.*

Proof. (a) let u be an arbitrary element of R not contained in A. Then, $eu=u$ with some $e\in R$, and by Zorn's lemma there exists a maximal member M in the family of left ideals B of R with $B\supseteq\{x\in R\,|\,xu\in A\}$ $(\supseteq A)$ and $B\not\ni e$. One will easily see that M is a maximal left ideal of R.

(b) Suppose that the socle S of ${}_RR$ does not coincide with R. Then, by (a) the ideal S is contained in some maximal left ideal M, and by hypothesis $R=M\oplus N$ with a minimal left ideal N. However, this is a contradiction.

Lemma 2. *The following conditions are equivalent:*

(1) *R is left non-singular (i.e., the left singular ideal of R is 0).*

(2) *${}_RR$ is unital and every left annihilator is closed in ${}_RR$ i.e., every left annihilator has no proper essential extension in ${}_RR$).*

Proof. Let Z be the left singular ideal of R. If $Z=0$ then ${}_RR$ is

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obviously unital. Let T be an arbitrary subset of R. If J is a left ideal of R in which the left annihilator $l(T)$ is essential, then for every $b \in J$ the left ideal $\{x \in R \mid xb \in l(T)\}$ is essential. Hence, $bT \subseteq Z = 0$, namely, $b \in l(T)$. This proves that $l(T) = J$. The converse will be almost evident.

Now, we can state our main theorem.

Theorem 1. *The following conditions are equivalent*:

(1) $R = \bigoplus_{\lambda \in \Lambda} R_\lambda$, *where* R_λ *is the complete ring of linear transformations of finite rank of a vector space over a division ring.*

(2) *R is a left s-unital semi-prime ring and every left ideal of R is a left annihilator.*

(3) *R is a regular ring and every left ideal of R is a left annihilator.*

Proof. (3) $\Longrightarrow$ (2) is obvious, and (1) $\Longrightarrow$ (3) is a direct consequence of [3, Theorem IV. 16. 3].

(2) $\Longrightarrow$ (1). We shall prove first that the left singular ideal Z of R is 0. To see this, we assume $Z \neq 0$. As is well-known, there exists a left ideal I of R such that $Z \cap I = 0$ and $Z + I$ is essential. By hypothesis, $Z \oplus I = l(T)$ with a subset T of R. If $T = 0$ then $R = Z \oplus I$. Since $I = RI = (Z+I)I = ZI + I^2 \subseteq Z \oplus I$, we obtain $ZI = 0$. Then $(IZ)^2 = 0$, so that $IZ = 0$, which implies that I is an ideal of R. By Lemma 1 (a), there exists then a maximal left ideal M containing I. Again by hypothesis, $M = l(u)$ with some u in the right annihilator $r(I) = Z$. Noting here that Ru is isomorphic to R/M as left R-module, Ru is a minimal left ideal and generated by some non-zero idempotent $e \in Z$. But, this yields a contradiction $Re \cap l(e) = 0$. Hence, $T \neq 0$. Now, let t be an arbitrary non-zero element of T. Then, recalling that $t \in Z$ by $Z \oplus I \subseteq l(t)$, one will readily see that $(Rt)^2 \subseteq Zt = 0$. This contradiction shows that $Z = 0$. Hence, by Lemma 2, R has no proper essential left ideals. Accordingly, every maximal left ideal is a direct summand of ${}_RR$, and hence ${}_RR$ is completely reducible by Lemma 1 (b). Now, let R_λ be an arbitrary homogeneous component of R. Then, as is well-known, R_λ is a (non-trivial) simple ring and every left ideal of R_λ is a left annihilator in R_λ. Hence, again by [3, Theorem IV. 16. 3], R_λ is the complete ring of linear transformations of finite rank of a vector space over a division ring.

Combining Theorem 1 with [3, Theorem IV. 16. 4], we obtain at once

Corollary 1. *The following conditions are equivalent*:

(1) *R is a direct sum of artinian simple rings.*

(2) *R is a left (or right) s-unital semi-prime ring such that every left ideal is a left annihilator and every right ideal is a right annihilator.*

(3) *R is a regular ring such that every left ideal is a left annihilator and every right ideal is a right annihilator.*

The next contains all the results in [4, § 5], [5, Theorem II] and [6].

Corollary 2. *Let R be a ring with* 1. *Then the following conditions are equivalent*:

(1) *R is artinian semi-simple.*

(2) *Every maximal left ideal of R is a direct summand of* ${}_RR$.

(3) *R is left non-singular and every essential left ideal of R is a left annihilator.*

(4) *R is semi-prime and every essential left ideal of R is a left annihilator.*

(5) *R is a regular ring and every essential left ideal of R is a left annihilator.*

(6) *R is semi-prime and every left ideal of R is a left annihilator.*

(7) *R has no proper essential left ideals.*

(2′)—(7′) *The left-right analogues of* (2)—(7).

Proof. Obviously, (6)$\Longrightarrow$(4), (5)$\Longrightarrow$(4), and (1)$\Longrightarrow$(5), (6). Moreover, (2)$\Longrightarrow$(1) is contained in Lemma 1 (b), and (3)$\Longrightarrow$(7)$\Longrightarrow$(2) is easy by Lemma 2. Finally, the argument used in the proof of Theorem 1 will enables us to see that (4)$\Longrightarrow$(3).

A ring without non-zero nilpotent elements is called a *reduced ring*. If R is a reduced ring then the left annihilator $l(T)$ of a subset T of R coincides with $r(T)$ and every idempotent in R is central.

The next is [2, Lemma 2], and is essentially due to R. Yue Chi Ming.

Lemma 3. *The following conditions are equivalent*:

(1) *R is a left non-singular ring and every closed left ideal of R is two-sided.*

(2) *R is a reduced ring and* $I \oplus l(I)$ *is essential in* ${}_RR$ *for every left ideal I of R.*

(3) *R is a reduced ring and every closed left ideal of R is the annihilator of a left ideal.*

Proof. For the sake of completeness, we shall give here the proof. (1)$\Rightarrow$(2). Suppose there exists a non-zero element b with $b^2=0$.

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Then, there exists a non-zero left ideal K which is maximal with respect to $l(b)\cap K=0$. Since K is closed, K is an ideal by hypothesis. Thus $Kb\subseteq K\cap l(b)=0$, which implies a contradiction $K\subseteq l(b)$. Hence, R is a reduced ring and $I\cap l(I)=0$ for every left ideal I. Now, let L be a left ideal of R containing $l(I)$ which is maximal with respect to $I\cap L=0$. Since the closed left ideal L is an ideal, we have then $L\subseteq l(I)$. This proves that $I+l(I)=I+L$ is essential in ${}_RR$.

(2)$\Longrightarrow$(3). Let J be a closed left ideal. If $J\subset l(r(J))$, then there exists a non-zero left subideal K of $l(r(J))$ such that $J\cap K=0$. We have then $l(J)=r(J)=r(l(r(J)))\subseteq r(K\oplus J)\subseteq r(J)=l(J)$, that is, $l(J)=l(K\oplus J)$. But, this yields a contradiction $(K\oplus J)\oplus l(K\oplus J)=K\oplus J\oplus l(J)\supset J\oplus l(J)$. Hence, $J=l(r(J))$.

(3)$\Longrightarrow$(1). Let b be an arbitrary element of the left singular ideal of R. Since $Rb\cap l(b)=Rb\cap r(b)=0$, b has to be 0.

Finally, we shall extremely specialize Theorem 1.

Theorem 2. *The following conditions are equivalent*:

(1) *R is a direct sum of division rings.*

(2) *R is a left s-unital, reduced ring without proper essential left ideals.*

(3) *R is a left s-unital, reduced ring and every maximal left ideal is a direct summand of ${}_RR$.*

(4) *R is a left s-unital, reduced ring and every maximal left ideal has a non-zero annihilator.*

(5) *R is a strongly regular ring and every maximal left ideal has a non-zero annihilator.*

(6) *R is a left V-ring (i. e., $R^2=R$ and every left ideal is an intersection of maximal left ideals) and every maximal left ideal is the left annihilator of a left ideal.*

(7) *R is a reduced ring and every left ideal is an annihilator.*

(8) *R is a reduced ring and $l(l(I))=I$ for every left ideal I of R.*

(9) *$I\cap J=IJ$ and $l(l(I))=I$ for all left ideals I, J of R.*

(10) *R is a left non-singular ring and every left ideal is the left annihilator of a left ideal.*

(11) *R is a regular ring and every left ideal is the left annihilator of a left ideal.*

(12) *Every left ideal of R is the left annihilator of a left ideal and idempotent.*

(2′)—(12′) *The left-right analogues of* (2)—(12).

Proof. We shall give the proof without making use of Theorem 1.

Obviously, (1)$\Longrightarrow$(2)$\Longrightarrow$(3), (5)$\Longrightarrow$(4), and (8)$\Longrightarrow$(7). By [1, Theorem], R is a strongly regular ring if and only if R is a left V-ring and a left duo ring (i. e., a ring having no strictly left-sided ideals). Hence, (1)$\Longrightarrow$(6)$\Longrightarrow$(11)$\Longrightarrow$(10). Moreover, [1, Theorem] also enables us to see that (11)$\Longleftrightarrow$(12) and (1)$\Longrightarrow$(9)$\Longrightarrow$(8).

(3)$\Longrightarrow$(1). By Lemma 1 (b), ${}_RR$ is completely reducible. Since every minimal left ideal in the reduced ring R is generated by a central idempotent, it is a two-sided ideal, and itself a division ring.

(4)$\Longrightarrow$(3). If M is a maximal left ideal then $l(M)\neq 0$ and $M\cap l(M)=0$, so that $R=M\oplus l(M)$.

(7)$\Longrightarrow$(10). Let I be an arbitrary left ideal, and $I=l(T)$ with a subset T of R. Let T' be the left ideal generated by T. Then, $I=r(T)=r(T')=l(T')$, and by Lemma 3 R is left non-singular.

(10)$\Longrightarrow$(5). Since R is a left duo ring, by Lemmas 2 and 3 we see that R is a reduced ring and $R=I\oplus l(I)$ (and $I=l(l(I))$) for every left ideal I of R. Now, let a be an arbitrary element of R. Then, considering I as the left ideal generated by a^3, we have $a=u+v$, $u\in I$, $v\in l(I)$. Since $u^3+v^3=a^3\in I$, it follows then $v^3=0$, and hence $v=0$. This proves that $a\in I$ and R is strongly regular.

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