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Shigemoto Asano* Kaoru Motose†

*Shinshu University
†Shinshu University

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ON QF-2 ALGEBRAS WITH COMMUTATIVE RADICALS

Dedicated to Professor Gorô Azumaya on his 60th birthday

SHIGEMOTO ASANO and KAORU MOTOSE

Group algebras (of finite groups over an algebraically closed field) with commutative radicals have been studied by a number of authors: D. A. R. Wallace [4, 5, 6], S. Koshitani [1] and K. Motose and Y. Ninomiya [2]. In particular, Wallace has given, in [6], a result which determines the structure of blocks of group algebras of this type. The most important part of his result may be stated in the following form: *Let A be a block of a group algebra of the type mentioned above. If the radical N of A is such that $N^2 \neq 0$, then A is a commutative completely primary algebra.*

The purpose of the present note is to generalize this result to the case of QF-2 algebras in the sense of R. M. Thrall [3], over an arbitrary field K.

Theorem. *Let A be a QF-2 algebra over a field K and let A be itself a block. Assume that the radical N of A is commutative and N^2 does not vanish. Then A is a completely primary almost symmetric algebra over K such that the residue class algebra A/N is a (commutative) field. Moreover, if the base field K is perfect, then A is a commutative completely primary symmetric algebra over K.*

Proof. We note that, since N is commutative, N^2 is contained in the center of A. Let us first consider the case that K is an arbitrary field. We begin by proving the following contention: Let e and f be two primitive idempotents. If $eN^2 (= N^2e) \neq 0$, $ef = fe = 0$ and either $eAf \neq 0$ or $fAe \neq 0$, then eA and fA are isomorphic (as right A-modules). To show this, let M denote the left annihilator of N. If $eM = 0$, then $eMf = 0$. On the other hand, assume that $eM \neq 0$. Since eM is the unique minimal right A-submodule of eA, we have $eM \subseteq eN^2$, hence $eMf \subseteq eN^2f = efN^2 = 0$. Thus in either case we have $eMf = 0$. Since $eNf \cdot N = e \cdot N \cdot fN = e \cdot fN \cdot N = 0$, we have $eNf \subseteq eMf$, and therefore $eNf = 0$. Similarly we have $fNe = 0$. The condition that either $eAf \neq 0$ or $fAe \neq 0$ implies now that eA and fA are isomorphic. From the assumption that A is itself a block, together with what we have proved above, it follows that the indecomposable direct summands of the right regular module A are all isomorphic each other. Therefore A is a full matrix ring over a completely primary algebra. But, by the commutativity of N it follows that A itself is a completely primary algebra. By the same fact we have $(ab-ba)N^2=0$ for all $a, b \in A$. Since the left annihilator $l(N^2)$ of N^2 is a proper ideal of A, A/N is a field. Now let t be the nilpotency index of N, and m a nonzero element of N^{t-1}. Then we see that $M = N^{t-1} = Am$. Here m is a central element of A and the mapping $a+N \to am\,(a \in A)$ is an isomorphism of A/N onto M (as left and right A-modules). Therefore A is an almost symmetric algebra.

Now let K be a perfect field. Then there exists a subalgebra L of A which is isomorphic to A/N (as an algebra over K). Thus A is a direct sum of L and N, as a K-space. Since M is isomorphic to A/N as a left L-module, we get $M = Lm$. Let α be a generating element of L over K (i. e. $L=K(\alpha)$), and let $f(x)$ be the defining polynomial of α over K. To prove that A is commutative, it suffices to show that the primitive element α commutes with any $x \in N$. First of all one verifies directly that $x\alpha - \alpha x \in M$; hence one can choose an element λ in L to write $x\alpha = \alpha x + \lambda m$. We can then establish, by induction, the formula $x\alpha^t = \alpha^t x + t\lambda\alpha^{t-1}m$ $(t=1, \cdots, \deg f(x))$. From this it follows that $0 = xf(\alpha) = f(\alpha)x + \lambda f'(\alpha)m = \lambda f'(\alpha)m$, and hence $\lambda = 0$. This proves that A is a commutative symmetric algebra.

Example. If K is not perfect in Theorem, then A is not necessarily commutative. To show this let us construct an example.

Let F be a field of characteristic 2, $P = F(t)$ the field of rational functions in one variable t over F, and $K = F(t^2)$. For an arbitrary element $\alpha = a + bt$ $(a, b \in K)$ of P, let $\tilde{\alpha}$ denote b, the coefficient of t. Then $\widetilde{\alpha\beta} = \tilde{\alpha}\beta + \alpha\tilde{\beta}$ for any two elements α, β in P. Let A be an associative algebra over K defined in the following way :

1) A is a 3-dimensional left vector space over P with a basis $\{1, m, m^2\}$.

2) The multiplication in A is defined by the rule

$$m^3 = 0 \text{ and } m\alpha = \alpha m + \tilde{\alpha} m^2 \text{ for any } \alpha \in P.$$

Then A is a non-commutative almost symmetric algebra over K such that the radical $N = Pm + Pm^2$ is commutative.

Corollary. *Let A be a weakly symmetric algebra over a field K and let A be itself a block. Assume that the radical N of A is commutative.*

Then A is of one of the following three types:

(1) *A is a simple algebra over K.*

(2) *A is a full matrix ring over a completely primary weakly symmetric algebra B over K such that the square of the radical $N'(=N\cap B)$ of B vanishes. (In this case B/N' is a division algebra and N' is one-dimensional as a left B/N'-space as well as a right one.)*

(3) *A is a completely primary almost symmetric algebra over K such that A/N is a field.*

Proof. In view of Theorem, we have only to consider the case that $N\neq 0$ and $N^2=0$. Let e be an arbitrary primitive idempotent. Then Ne is isomorphic to Ae/Ne as a left A-module. Hence the indecomposable left ideal Ae has only one (non-isomorphic) composition factor. Noting that A is itself a block, we can see that A is a full matrix ring over a completely primary algebra B. It is now easy to see that B is an algebra as described in our corollary.

If, in the corollary, we assume moreover that K is perfect, we can say something more.

(i) When A is of type (2), B satisfies the following conditions:

a) B is a 2-dimensional left D-space with a basis $\{1, m\}$, where D is a finite dimensional division subalgebra of B over K.

b) The multiplication in B is given by the rule

$$m^2=0 \text{ and } m\alpha=\sigma(\alpha)m \text{ for any } \alpha\in D,$$

where σ is a K-algebra automorphism of D.
Conversely, let B be an associative algebra over K satisfying the conditions a) and b), and let A be a full matrix ring over B. Then A is a weakly symmetric algebra over K with radical of square zero.

(ii) When A is of type (3), then, by Theorem, A is a commutative completely primary symmetric algebra.

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The TECHNOLOGICAL UNIVERSITY OF NAGAOKA
SHINSHU UNIVERSITY

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