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ON THE NILPOTENCY INDEX OF THE RADICAL OF A GROUP ALGEBRA. II

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Throughout this paper, let G be a p-solvable group with a p-Sylow subgroup P of order p^a , K an algebraically closed field of characteristic p, and t(G) the nilpotency index of the radical of the group algebra KG of G over K.

The next inequality has been shown by D. A. R. Wallace [10].

$$t(G) \ge a(p-1) + 1$$
.

Along with this inequality, the following question arises: If t(G) = a(p-1) + 1, then is P elementary? This question has an affirmative answer for any group of p-length 1 (S. A. Jennings [4] and K. Morita [5]) and for any group with a regular p-Sylow subgroup (Y. Tsushima [9]). However, this question has not always an affirmative answer; $t(S_4) = 4$ for p = 2 (see [6]). In the present paper, we shall give an affirmative answer to this question in case p = 2, P is meta-cyclic and $G/O(G) \neq S_4$ (Proposition 1). Moreover, we shall construct an example which gives a negative answer to Tsushima's problems [8, Problems 3, 4, 5].

Proposition 1. Assume that p = 2, P is meta-cyclic, and $G/O(G) \neq S_4$. Then t(G) = a + 1 if and only if P is elementary abelian.

Proof. If P is elementary, then t(G) = a + 1 (see [6]). Conversely, assume that t(G) = a + 1. Since the center of P is contained in $O_{2',2}(G)$ ([1, Theorem 6. 3. 3]), G is of p-length 1 if $|P| \le 4$. Hence we may assume $|P| \ge 8$. We shall proceed by induction on |G|. If $O(G) \ne 1$, then $a+1=t(G) \ge t(G/O(G)) \ge a+1$, and so P is elementary. If O(G)=1, then $N=O_2(G)\neq 1$ and the inequality $t(G)\geq t(N)+t(G/N)-1$ ([10, Theorem 2.4]) forces N elementary. Since N is meta-cyclic and elementary, we must have $|N| \leq 4$ and $C_{\sigma}(N) = N$ (see [1, Theorem In view of $C_a(N) = N$, G/N is isomorphic to a subgroup of the automorphism group of N, and so $|G/N| \le 6$. Since $|P| \ge 8$ and $|G| \le 24$, the order of G is either 8 or 24. |G| = 8 is contrary to the assumption that P is meta-cyclic. Thus |G| = 24, $N = C_{\sigma}(N)$ is elementary abelian of type (2, 2), and G/N acts faithfully on N by conjugation. Let $S = \langle a \rangle$ be a 3-Sylow subgroup of G, and N = $\{1, x, y, xy\}$. Then by $C_0(N) = N$, we may assume that $x^a = y$ and $y^a = x$

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xy. Consequently $N_{\sigma}(S) \cap N = 1$ and $|N_{\sigma}(S)| = 6$ by Sylow's theorem. We conclude therefore that G is a semi-direct product of N and $N_{\sigma}(S)$, and so $N_{\sigma}(S)$ is the automorphism group of N. Thus, G is the holomorph of N and hence $G = S_4$, a contradiction.

Remark. If p is odd and P is meta-cyclic, then P is regular by [2, Satz 10.2]. Thus, by the result of Tsushima [9], t(G) = a(p-1)+1 forces P elementary.

In the remaining part of this paper, we consider a group G such that $G/O(G) = S_4$. Let $\{e_1, e_2, \dots, e_t\}$ be the set of all centrally primitive idempotents of KO(G), $I(e_s) = \{x \in G | xe_sx^{-1} = e_s\}$, and $H_s = I(e_s)/O(G)$.

Proposition 2. Assume that p=2 and $G/O(G)=S_4$. Then

$$t(G) = \begin{cases} 5 & \text{if } |H_s| = 8 \text{ for some s.} \\ 4 & \text{otherwise.} \end{cases}$$

Proof. By Morita's theorem [5, Theorem 2], $t(G) = \text{Max}_s \{t(T_s)\}$, where T_s is a twisted group algebra of H_s over K with a factor set and $t(T_s)$ denotes the nilpotency index of the radical of T_s . If $|H_s|$ is neither 8 nor 24, then $t(T_s) \leq 4$ by [7, Theorem 1.6]. If $|H_s| = 8$, then $t(T_s) = t(P) = 5$. If $|H_s| = 24$, then $T_s = KS_4$ by [3, Corollary], and so $t(T_s) = 4$ by [6]. This completes the proof.

Example 1. Assume that p=2. Let $\langle d \rangle$ be a cyclic group of order 3. Let G be the semi-direct product of $\langle d \rangle$ by S_4 such that $d^{(12)}=d^{-1}$ and A_4 acts trivially on $\langle d \rangle$. Then t(G)=4 by Proposition 2.

Example 2. Assume that p=2. Let $\langle c \rangle$ and $\langle d \rangle$ be cyclic groups of order 3. Let U be a homomorphism of S_3 into GL(2,3) defined by

$$a = (1 \ 2 \ 3) \longrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
 and $b = (1 \ 2) \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Then U can be regarded as a homomorphism of S_4 into GL(2,3). Let G be the semi-direct product of $M = \langle c \rangle \times \langle d \rangle$ by S_4 with respect to U. Then the following relations hold:

$$c^a = c$$
, $d^a = cd$, $c^b = c$ and $d^b = d^{-1}$.

On the other hand, $e = (1 + \omega c + \omega^2 c^2)(1 + d + d^2)$ is a primitive idempotent of KM, where ω is a primitive 3rd root of 1 in K. We obtain that

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 $e^a \neq e$ and $e^b = e$. Thus |I(e)/M| = 8, and consequently t(G) = 5 by Proposition 2. In Example 1, we have already showed that $t(G/\langle c \rangle) = 4$. Since $\langle c \rangle$ is contained in the center of G, this gives a negative answer to Tsushima's Problems [8, Problems 3, 4, 5].

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