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OPERATOR AVERAGES FOR SUBSEQUENCES

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1. Introduction. In this paper we shall show that if T is a (not necessarily positive) *Dunford-Schwartz operator* on L_1 of a σ-finite measure space and $k_1, k_2, \cdots$ is a *uniform sequence* (in the sense of Brunel and Keane [1]), then the ergodic average

$$\frac{1}{n}\sum_{i=1}^{n} T^{k_i} f$$

converges almost everywhere for every $f \in L_p$ with $1 \leq p < \infty$. Let us then write

$$f^* = \lim_n \frac{1}{n}\sum_{i=1}^{n} T^{k_i} f \quad \text{almost everywhere.}$$

We next show that if $1 < p < \infty$ and $f \in L_p$ then

$$\lim_n \left\| \frac{1}{n}\sum_{i=1}^{n} T^{k_i} f - f^* \right\|_p = 0.$$

Let $(\Omega, \mathscr{B}, m)$ be a σ-finite measure space and T a linear contraction on $L_1(\Omega) = L_1(\Omega, \mathscr{B}, m)$, i.e. T is a linear operator on $L_1(\Omega)$ such that $\|T\|_1 \leq 1$. We call T a *Dunford-Schwartz operator* if T satisfies, in addition, that $\|Tf\|_\infty \leq \|f\|_\infty$ for every $f \in L_1(\Omega) \cap L_\infty(\Omega)$. By the Riesz convexity theorem, the Dunford-Schwartz operator T is uniquely extended to a linear contraction on each $L_p(\Omega)$ with $1 < p < \infty$. It follows from [6] that if T is a *positive* Dunford-Schwartz operator on $L_1(\Omega)$ and $k_1, k_2, \cdots$ is a uniform sequence, then the average

$$\frac{1}{n}\sum_{i=1}^{n} T^{k_i} f(\omega)$$

converges almost everywhere for every $f \in L_1(\Omega)$, and thus also for every $f \in L_p(\Omega)$ with $1 < p < \infty$, because the inequalities

$$\sup_{n\geq 1} \frac{k_n}{n} < \infty \qquad \text{(see e.g. [1])}$$

and

$$\left\| \sup_{n\geq 1} \left| \frac{1}{n}\sum_{i=1}^{n} T^{k_i} f \right| \right\|_p \leq \left(\sup_{n\geq 1} \frac{k_n+1}{n}\right) \left\| \sup_{n\geq 1} \frac{1}{n}\sum_{i=0}^{n-1} T^i |f| \right\|_p$$
$$< \infty \qquad \text{(see e.g. [3], p. 678)}$$

enable us to apply Banach's convergence theorem ([3], p. 332) to obtain the latter. But so far as we know it has not been known whether the positivity of T is necessary in the above result. Therefore it would be interesting to investigate this point, and that is the starting point for the work in this paper. We shall see, as mentioned in the beginning of this section, that the positivity of T is not necessary.

2. Preliminaries. Let X be a compact Hausdorff space and $C(X)$ the Banach space of all continuous complex functions on X with the uniform norm $\|f\|_u=\sup\{|f(x)| : x\in X\}$. Suppose φ is a continuous mapping of X into itself. Since X is compact, there exists a unique uniformity $\mathfrak{U}$ on X compatible with the topology of X (see e. g. [5], p. 188). We will assume, throughout this paper, that the powers φ^n, $n\geq 0$, form an equicontinuous family with respect to $\mathfrak{U}$, i. e. for every $x\in X$ and all $U\in\mathfrak{U}$ there exists a neighborhood W of x such that $(\varphi^n x, \varphi^n y)\in U$ for every $y\in W$ and $n\geq 0$.

Put for $f\in C(X)$ and $n\geq 1$

$$f_n(x)=\frac{1}{n}\sum_{i=0}^{n-1} f(\varphi^i x) \qquad (x\in X).$$

Since X is compact and so $f\in C(X)$ is uniformly continuous with respect to $\mathfrak{U}$, the subset $\{f_n : n\geq 1\}\subset C(X)$ is equicontinuous. Thus by the Arzela-Ascoli theorem ([3], p. 266), $\{f_n : n\geq 1\}$ is relatively compact, and therefore by a mean ergodic theorem (see e. g. [3], p. 661), there exists $f_\infty\in C(X)$ such that

$$\lim_n \|f_n - f_\infty\|_u = 0.$$

Since the dual space of $C(X)$ is the space of all bounded regular measures on $(X, \mathscr{X})$, where $\mathscr{X}$ stands for the σ-field of all Borel subsets of X, it may be readily seen that f_∞ is a constant function for each $f\in C(X)$ if and only if φ leaves invariant a unique regular probability measure on $(X, \mathscr{X})$ which we will denote by μ. The system (X, φ) is therefore called *uniquely ergodic* if f_∞ is a constant function for each $f\in C(X)$. Since $f_\infty(\varphi x)=f_\infty(x)$ for all $x\in X$, (X, φ) is uniquely ergodic whenever φ leaves no nontrivial closed subset of X invariant. If the system $(X, \mathscr{X}, \mu, \varphi)$ is uniquely ergodic, then we get

$$f_\infty(x)=\int f_\infty\, d\mu=\lim_n \int f_n\, d\mu=\int f\, d\mu$$

for every $f\in C(X)$ and all $x\in X$. If $(X, \mathscr{X}, \mu, \varphi)$ is uniquely ergodic and supp $\mu=X$, then $(X, \mathscr{X}, \mu, \varphi)$ is called *strictly ergodic.*

Since φ leaves μ invarint, it follows that if we set $Y=\text{supp }\mu$ then

$\varphi Y \subset Y$, and thus the subsystem $(Y, \varphi|_Y)$ is strictly ergodic.

The following definition of a uniform sequence is somewhat more general than that due to Brunel and Keane [1]. They considered a compact metric space and a homeomorphism of the space onto itself to define a uniform sequence.

Definition. A sequence $k_1, k_2, \cdots$ of nonnegative integers is called *uniform* if there exist

(i) a strictly ergodic system $(X, \mathscr{X}, \mu, \varphi)$ (in the above sense),

(ii) a subset Y of X such that $Y \in \mathscr{X}$ and $\mu(Y) > 0 = \mu(\partial Y)$, where ∂Y denotes the boundary of Y, and

(iii) a point $y \in X$ such that

$$k_1 = \min\{i \geq 0 : \varphi^i y \in Y\}$$

and

$$k_n = \min\{i > k_{n-1} : \varphi^i y \in Y\} \qquad (n \geq 2).$$

Lemma 1. *If $k_1, k_2, \cdots$ is a uniform sequence generated by y and Y as above, then*

$$\lim_n \frac{n}{k_n} = \mu(Y).$$

Proof. Since $\mu(\partial Y) = 0$ and μ is regular, for any $\varepsilon > 0$ there exist a compact subset F and an open subset G of X such that $F \subset Y^\circ \subset Y^- \subset G$ and $\mu(G - F) < \varepsilon$, where Y° and Y^- denote, respectively, the interior and closure of Y. Choose f and g in $C(X)$ so that

$$1_F \leq f \leq 1_Y \leq g \leq 1_G.$$

Then we have

$$\begin{aligned}\int f\,d\mu &= \lim_n \frac{1}{n}\sum_{i=0}^{n-1} f(\varphi^i y) \leq \liminf_n \frac{1}{n}\sum_{i=0}^{n-1} 1_Y(\varphi^i y)\\ &\leq \liminf_n \frac{n}{k_n} \leq \limsup_n \frac{n}{k_n}\\ &\leq \lim_n \frac{1}{n}\sum_{i=0}^{n-1} g(\varphi^i y) = \int g\,d\mu,\end{aligned}$$

and hence

$$0 \leq \limsup_n \frac{n}{k_n} - \liminf_n \frac{n}{k_n} \leq \int (g - f)\,d\mu < \varepsilon.$$

Thus the lemma follows, since $0 \leq \mu(Y) - \int f\,d\mu < \varepsilon$ and ε is arbitrary.

Lemma 2. *If k_1, k_2, $\cdots$ is a uniform sequence generated by y and Y as above, then for any $\varepsilon>0$ there exist open subsets Y_1, Y_2 and W of X such that*

(i) $Y_1 \subset Y \subset Y_2$, $\mu(Y_2 - Y_1) < \varepsilon$ *and* $\mu(\partial Y_1) = 0 = \mu(\partial Y_2)$,

(ii) $y \in W$ *and, for every* $x \in W$ *and all* $i \geq 0$,

$$1_{Y_1}(\varphi^i x) \leq 1_Y(\varphi^i y) \leq 1_{Y_2}(\varphi^i x).$$

Proof. Let f be the function in the proof of Lemma 1. Since $\{x : f(x) = \alpha\}$, $0<\alpha<1$, is an uncountable, mutually disjoint collection, there is $0<\alpha<1$ such that the open set $Y_1 = \{x : f(x) > \alpha\}$ satisfies

$$F \subset Y_1 \subset Y^\circ \quad \text{and} \quad \mu(\partial Y_1) \leq \mu(\{x : f(x) = \alpha\}) = 0.$$

On the other hand, since f is uniformly continuous with respect to the uniformity $\mathfrak{U}$ on X and the family $\{\varphi^n : n \geq 0\}$ is equicontinuous with respect to $\mathfrak{U}$ (by assumption), there exists a neighborhood W_1 of y such that

$$| f(\varphi^i x) - f(\varphi^i y) | < \alpha$$

for every $x \in W_1$ and all $i \geq 0$. Therefore if $\varphi^i x \in Y_1$, i. e. $f(\varphi^i x) > \alpha$ for $x \in W_1$ and $i \geq 0$, then $f(\varphi^i y) > 0$ and thus $\varphi^i y \in Y^\circ \subset Y$. This implies that

$$1_{Y_1}(\varphi^i x) \leq 1_Y(\varphi^i y) \qquad (x \in W_1,\ i \geq 0).$$

Similarly, we see that there exist an open subset Y_2 of X and a neighborhood W_2 of y such that $Y \subset Y^- \subset Y_2 \subset G$, $\mu(\partial Y_2) = 0$ and, for every $x \in W_2$ and all $i \geq 0$,

$$1_Y(\varphi^i y) \leq 1_{Y_2}(\varphi^i x).$$

Therefore, putting $W = W_1 \cap W_2$, the lemma follows.

3. Ergodic theorems.

Theorem 1. *Let $(\Omega, \mathscr{B}, m)$ be a σ-finite measure space and T a Dunford-Schwartz operator on $L_1(\Omega)$. If k_1, k_2, $\cdots$ is a uniform sequence then for any $f \in L_p(\Omega)$, with $1 \leq p < \infty$, the limit*

$$f^*(\omega) = \lim_n \frac{1}{n} \sum_{i=1}^n T^{k_i} f(\omega)$$

exists and is finite a. e. on Ω.

Proof. Let $(X, \mathscr{X}, \mu, \varphi)$ and y, Y be the apparatus connected with this sequence. S will denote the operator on $L_1(X)$ induced by φ, i. e.

$$(Sh)(x) = h(\varphi x) \qquad (x \in X)$$

for all $h \in L_1(X)$. Since φ leaves μ invariant, S is a Dunford-Schwartz operator on $L_1(X)$, and thus taking $(\Omega', \mathscr{B}', m')$ to be the direct product of $(\Omega, \mathscr{B}, m)$ and $(X, \mathscr{X}, \mu)$ and T' the tensor product of T and S, it follows from a direct calculation and an approximation argument that T' is a Dunford-Schwartz operator on $L_1(\Omega')$.

First of all we shall consider the case $f \in L_1(\Omega) \cap L_\infty(\Omega)$. Without loss of generality we may assume here that $|f| \leq 1$ a. e. on Ω. Let $\varepsilon > 0$ be given, and choose open subsets Y_1, Y_2 and W of X satisfying conditions (i) and (ii) in Lemma 2. Define

$$g(\omega, x) = f(\omega)1_Y(x),$$
$$g_1(\omega, x) = f(\omega)1_{Y_1}(x),$$

and

$$g_2(\omega, x) = f(\omega)1_{Y_2}(x).$$

Since g_1 and g_2 are in $L_1(\Omega') \cap L_\infty(\Omega')$, by the Dunford-Schwartz individual ergodic theorem ([3], p. 675) we see that the limits

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T'^i g_1(\omega, x) \quad \text{and} \quad \lim_n \frac{1}{n} \sum_{i=0}^{n-1} T'^i g_2(\omega, x)$$

exist and are finite a. e. on Ω'. Thus, by the fact that $\mu(W) > 0$ (which is clear, since W is a nonempty open subset of X and supp $\mu = X$) we may apply Fubini's theorem to infer that there exists a point $x \in W$ at which the limit

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i f(\omega) 1_{Y_1}(\varphi^i x) = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} T'^i g_1(\omega, x)$$

exists and is finite m-a. e. on Ω.

On the other hand, since $\mu(\partial Y_1) = 0 = \mu(\partial Y_2)$, we see, as in the proof of Lemma 1, that

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} 1_{Y_1}(\varphi^i x) = \mu(Y_1)$$

and

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} 1_{Y_2}(\varphi^i x) = \mu(Y_2).$$

Since $0 \leq \mu(Y_2) - \mu(Y_1) < \varepsilon$ and

$$1_{Y_1}(\varphi^i x) \leq 1_Y(\mu^i y) \leq 1_{Y_2}(\varphi\ x) \qquad (i \geq 0),$$

then there exists a positive integer N such that

$$n \geq N \quad \text{implies} \quad 0 \leq \frac{1}{n} \sum_{i=0}^{n-1} [1_Y(\varphi^i y) - 1_{Y_1}(\varphi^i x)] < \varepsilon.$$

Therefore for all $n \geq N$ and almost all $\omega \in \Omega$ we have

$$\left|\frac{1}{n}\sum_{i=0}^{n-1} T^i f(\omega) 1_Y(\varphi^i y) - \frac{1}{n}\sum_{i=0}^{n-1} T^i f(\omega) 1_{Y_1}(\varphi^i x)\right|$$
$$\leq \frac{1}{n}\sum_{i=0}^{n-1} |T^i f(\omega)| [1_Y(\varphi^i y) - 1_{Y_1}(\varphi^i x)]$$
$$\leq \frac{1}{n}\sum_{i=0}^{n-1} [1_Y(\varphi^i y) - 1_{Y_1}(\varphi^i x)] < \varepsilon,$$

which shows that for almost all $\omega \in \Omega$ the sequence $\{\frac{1}{n}\sum_{i=0}^{n-1} T^i f(\omega) 1_Y(\varphi^i y)\}_{n=1}^{\infty}$ is a Cauchy sequence; and thus the limit

$$\lim_n \frac{1}{n}\sum_{i=0}^{n-1} T^i f(\omega) 1_Y(\varphi^i y) = \lim_n \frac{1}{k_n+1}\sum_{i=1}^{n} T^{k_i} f(\omega)$$

exists and is finite a. e. on Ω. Now, by Lemma 1, we observe that the limit

$$\lim_n \frac{1}{n}\sum_{i=1}^{n} T^{k_i} f(\omega) = \lim_n \left[\frac{k_n+1}{n} \cdot \frac{1}{k_n+1}\sum_{i=1}^{n} T^{k_i} f(\omega)\right]$$
$$= \mu(Y)^{-1} \lim_n \frac{1}{n}\sum_{i=1}^{n} T^{k_i} f(\omega)$$

exists and is finite a. e. on Ω.

To complete the proof we intend to apply Banach's convergence theorem ([3], p. 332). To do this, let τ denote the linear modulus of T in the sense of Chacon and Krengel [2]. Thus τ is a positive Dunford-Schwartz operator on $L_1(\Omega)$ satisfying

$$|T^i f| \leq \tau^i |f| \qquad \text{a. e. on } \Omega$$

for every $f \in L_p(\Omega)$ with $1 \leq p < \infty$ and all $i \geq 0$. Then for each $f \in L_p(\Omega)$ with $1 \leq p < \infty$ we have

$$\sup_{n\geq 1}\left|\frac{1}{n}\sum_{i=0}^{n} T^{k_i} f(\omega)\right| \leq \left[\sup_{n\geq 1}\frac{k_n+1}{n}\right] \cdot \left[\sup_{n\geq 1}\frac{1}{n}\sum_{i=0}^{n-1} \tau^i |f|(\omega)\right],$$

and (see e. g. [3], VIII. 6 or [4], Chap. 2) $\sup_{n\geq 1} \frac{1}{n}\sum_{i=0}^{n-1} \tau^i |f|(\omega) < \infty$ a. e. on Ω; in particular, if $1 < p < \infty$ and $f \in L_p(\Omega)$ then

$$\left\|\sup_{n\geq 1} \frac{1}{n}\sum_{i=0}^{n-1} \tau^i |f|(\omega)\right\|_p \leq \frac{p}{p-1} \|f\|_p.$$

This enables us to apply Banach's convergence theorem to complete the proof, because $L_1(\Omega) \cap L_\infty(\Omega)$ is dense in each $L_p(\Omega)$ with $1 \leq p < \infty$.

Using the above-given argument and Lebesgue's dominated convergence theorem, we have at once the following mean ergodic theorem.

Theorem 2. *Let $(\Omega, \mathscr{B}, m)$ be a σ-finite measure space and T a Dunford-Schwartz operator on $L_1(\Omega)$. If $k_1, k_2, \cdots$ is a uniform sequence, $1<p<\infty$ and $f\in L_p(\Omega)$, then the sequence $\{\frac{1}{n}\sum_{i=1}^{n} T^{k_i}f\}_{n=1}^{\infty}$ converges in $L_p(\Omega)$ in the norm topology.*

4. A generalization. For subsets A and B of the nonnegative integers, let $|A|$ denote the cardinal number of A and let $A\triangle B$ denote the symmetric difference of A and B. We will call a strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers *almost uniform* if there exists a uniform sequence $k_1', k_2', \cdots$ such that

$$\lim_n \frac{|\{0, 1, \cdots, n-1\} \cap \{k_i : i\geq 1\} \triangle \{k_i' : i\geq 1\}|}{n} = 0.$$

In this section we remark that, by a routine modification of the argument given in the preceding section, Theorems 1 and 2 can be generalized to almost uniform sequences. That is, we have the following

Theorem 3. *Let $(\Omega, \mathscr{B}, m)$ be a σ-finite measure space and T a Dunford-Schwartz operator on $L_1(\Omega)$. If $k_1, k_2, \cdots$ is an almost uniform sequence then for any $f\in L_p(\Omega)$, with $1\leq p<\infty$, the limit*

$$f^{\sim}(\omega) = \lim_n \frac{1}{n}\sum_{i=1}^{n} T^{k_i}f(\omega)$$

exists and is finite a. e. on Ω. In particular, if $1<p<\infty$ and $f\in L_p(\Omega)$ then $f^{\sim}\in L_p(\Omega)$ and

$$\lim_n \|\frac{1}{n}\sum_{i=1}^{n} T^{k_i}f - f^{\sim}\|_p = 0.$$

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