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Takeshi Kataoka*

*Okayama University

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NOTE ON A MEAN ERGODIC THEOREM

TAKESHI KATAOKA

S.-Y. Shaw [2] showed an interesting mean ergodic theorem, and recently R. Sato [1] improved his result. In this paper we shall show that an analogous result holds for k -parameter semigroups of operators. To do this we apply the method of Sato.

The result is the following

Theorem. *Let $\{T(t_1, \dots, t_k); t_1, \dots, t_k \geq 0\}$ be a strongly measurable k -parameter semigroup of uniformly bounded linear operators on a Banach space X . Suppose there exist $\delta_i > 0$ ($i = 1, \dots, k$) such that $\|T(0, \dots, 0, t_i, 0, \dots, 0) - I\| < 2$ for all $0 < t_i < \delta_i$. Then for each $0 < t_i < \delta_i$ ($i = 1, \dots, k$) we have*

$$\begin{aligned} \lim_{\alpha_1, \dots, \alpha_k \rightarrow \infty} \frac{1}{\alpha_1 \cdots \alpha_k} \int_0^{\alpha_1} \cdots \int_0^{\alpha_k} T(s_1, \dots, s_k)x \, ds_1 \cdots ds_k \\ = \lim_{n \rightarrow \infty} \frac{1}{n^k} \sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} T(i_1 t_1, \dots, i_k t_k)x \end{aligned}$$

whenever one of these limits exists.

Proof Let X_0 (resp. X_{t_1, \dots, t_k}) be the set of x for which

$$\begin{aligned} P_0 x \equiv \lim_{\alpha_1, \dots, \alpha_k \rightarrow \infty} \frac{1}{\alpha_1 \cdots \alpha_k} \int_0^{\alpha_1} \cdots \int_0^{\alpha_k} T(s_1, \dots, s_k)x \, ds_1 \cdots ds_k \\ (\text{resp. } P_{t_1, \dots, t_k} x \equiv \lim_{n \rightarrow \infty} \frac{1}{n^k} \sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} T(i_1 t_1, \dots, i_k t_k)x) \end{aligned}$$

exists. Then by the uniform boundedness of the semigroup we get

$$X_0 = \bigcap_{s_1, \dots, s_k > 0} N[T(s_1, \dots, s_k) - I] \oplus \overline{\text{sp}} \bigcup_{s_1, \dots, s_k > 0} R[T(s_1, \dots, s_k) - I]$$

and

$$\begin{aligned} X_{t_1, \dots, t_k} = \bigcap_{i_1, \dots, i_k=1}^{\infty} N[T(i_1 t_1, \dots, i_k t_k) - I] \\ \oplus \overline{\text{sp}} \bigcup_{i_1, \dots, i_k=1}^{\infty} R[T(i_1 t_1, \dots, i_k t_k) - I], \end{aligned}$$

where $\overline{\text{sp}} U$ denotes the closed linear space spanned by U . It is clear that if $x \in \bigcap_{i_1, \dots, i_k=1}^{\infty} N[T(i_1 t_1, \dots, i_k t_k) - I]$ then $P_0 x$ exists, thus $X_{t_1, \dots, t_k} \subset X_0$. On the other hand, we have

$$\begin{aligned} & \frac{1}{n^k} \sum_{i_1=0}^{2n-1} \sum_{i_2=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} T\left(\frac{i_1 t_1}{2}, i_2 t_2, \dots, i_k t_k\right)x \\ &= [2I + (T(\frac{t_1}{2}, 0, \dots, 0) - I)] \left[\frac{1}{n^k} \sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} T(i_1 t_1, \dots, i_k t_k)x \right], \end{aligned}$$

so that for any x in $\overline{\text{sp}} \bigcup_{i_1, \dots, i_k=1}^{\infty} R[T(\frac{i_1 t_1}{2}, i_2 t_2, \dots, i_k t_k) - I]$, namely for any x with $P_{t_1/2, t_2, \dots, t_k} x = 0$,

$$\begin{aligned} & \left\| \frac{1}{n^k} \sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} T(i_1 t_1, \dots, i_k t_k)x \right\| \\ & \leq (2 - \|T(\frac{t_1}{2}, 0, \dots, 0) - I\|)^{-1} \left\| \frac{1}{n^k} \sum_{i_1=0}^{2n-1} \sum_{i_2=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} T\left(\frac{i_1 t_1}{2}, i_2 t_2, \dots, i_k t_k\right)x \right\| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Thus $P_{t_1, \dots, t_k} x = 0$, that is,

$$\begin{aligned} & \overline{\text{sp}} \bigcup_{i_1, \dots, i_k=1}^{\infty} R\left[T\left(\frac{i_1 t_1}{2}, i_2 t_2, \dots, i_k t_k\right) - I\right] \\ & \subset \overline{\text{sp}} \bigcup_{i_1, \dots, i_k=1}^{\infty} R[T(i_1 t_1, \dots, i_k t_k) - I]. \end{aligned}$$

Doing this process successively and applying an approximation argument, we finally observe that

$$\begin{aligned} & \overline{\text{sp}} \bigcup_{s_1, \dots, s_k > 0} R[T(s_1, \dots, s_k) - I] \\ & \subset \overline{\text{sp}} \bigcup_{i_1, \dots, i_k=1}^{\infty} R[T(i_1 t_1, \dots, i_k t_k) - I]. \end{aligned}$$

Since P_0 (resp. P_{t_1, \dots, t_k}) is a projection onto $\bigcap_{s_1, \dots, s_k > 0} N[T(s_1, \dots, s_k) - I]$ (resp. $\bigcap_{i_1, \dots, i_k=1}^{\infty} N[T(i_1 t_1, \dots, i_k t_k) - I]$), the proof is completed.

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DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY

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