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REGULAR MODULES AND V-MODULES. II

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This is a natural sequel to [1]. The notation and terminology employed there will be used here.

Let M_R be a module, and $S = \mathrm{End}_R(M)$. An element m of M is called *regular* (in M_R) if there exists an element f of $M^* = (M_R)^* = \mathrm{Hom}_R(M,R)$ such that $mf(m) = m$. A submodule N of M_R is called a *regular submodule* of M_R if every element of N is regular in M_R. Carefully examining the proof of [2, Theorem 2.2], we have the following proposition.

Proposition 1. (1) *Let m be an element of a module M_R. Then the following conditions are equivalent:*

1) *m is regular in M_R.*
2) *mR is projective and is a direct summand of M_R.*
3) *mR is projective and the restriction map $M^* \to (mR_R)^*$ is epic.*

(2) *If N is a regular submodule of M_R, then for every $m_1, \cdots, m_t \in N$, $m_1R + \cdots + m_tR$ is projective and is a direct summand of M_R.*

We call a module *finite dimensional* if it contains no infinite direct sums of submodules.

Theorem 1. *Let M_R be a finite dimensional module.*

(1) *There exists a decomposition $M = N \oplus P$ where N is a regular sudmodule of M_R and P is an S-R-submodule which has no nonzero regular submodules. Such a P is uniquely determined.*

(2) *There exists an S-R-decomposition $M = A \oplus B$ where A_R* is a *completely reducible, artinian, projective module and B has no nonzero S-admissible regular submodules.*

Proof. By [2, Theorem 1.8] every regular submodule is isomorphic to a finite direct sum of minimal right ideals generated by idempotents. Hence there exists a maximal regular submodule N of M_R. By Proposition 1 (2), $M = N \oplus P$ for some submodule P of M_R. Let $p: M \to N$ be the natural projection. If $s(P) \cap N \neq 0$ for some $s \in S$, then $0 \neq ps(P) \cap N$. Since $ps(P)$ is projective (Proposition 1 (2)), we have a decomposition $P = P' \oplus P''$ with $ps(P) \simeq P'$. Then $N \oplus P'$ is a regular submodule of M_R.

This contradicts the maximality of N. Therefore P is S-admissible. Now, let $M = N_1 \oplus P_1$ be another such decomposition with a maximal regular submodule N. Let $p_1 : M \to N_1$ be the natural projection. If $p_1(P) \neq 0$, then by the same argument as the above we have a contradiction. Thus we have $P \subseteq P_1$. Similarly, we have $P_1 \subseteq P$, and hence $P = P_1$.

For the proof of (2), let A be a maximal S-admissible regular submodule of M_R. Then we have a decomposition $M = A \oplus B$ with some submodule B. It remains only to show that B is S-admissible. If $SB \cap A \neq 0$, then there exists an $s \in S$ such that $0 \neq s(B) \subseteq A$. Since $s(B)$ is projective, there exists a decompsition $B = B' \oplus B''$ with $s(B) \simeq B'$. Since the isomorphism $s(B) \simeq B'$ can be extended to an element t of S, we have $B' = ts(B) \subseteq A$, a contradiction.

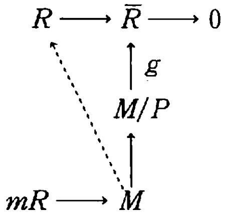
Remarks. (1) Let K be a field and $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$. Then, $I = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$ is a maximal regular submodule of R_R, but I is not S-admissible.

(2) Needless to say, Theorem 1 (2) is an extension of [2, Corollary 1.10] to modules.

Theorem 2. *Let M_R be a locally projective module. Let P be an S-R-submodule of M, N a submodule of M_R containing P, and $\bar{R} = R/\mathrm{Ann}_R(M/P)$. Then N is a regular submodule of M_R if and only if P is a regular submodule of M_R and N/P is a regular submodule of $M/P_{\bar{R}}$.*

Proof. Assume that N is a regular submodule of M_R. Let $\bar{m} = m + P$ be an element of N/P. By hypothesis there exists an $f \in M^*$ such that $mf(m) = m$. Since P is an S-R-submodule of M, f induces $\bar{f} \in (M/P_{\bar{R}})^*$ with $\bar{m}\bar{f}(\bar{m}) = \bar{m}$. Hence N/P is a regular submodule of $M/P_{\bar{R}}$.

Conversely, assume that P is a regular submodule of M_R and N/P is a regular submodule of $M/P_{\bar{R}}$. Let m be an element of N, and $\bar{m} = m + P$. Then there exists a $g \in (M/P_{\bar{R}})^*$ such that $\bar{m}g(\bar{m}) = \bar{m}$. Consider the following diagram:



Since M_R is locally projective, there exists a $g' \in M^*$ such that $pg'(m) = g(\overline{m})$. Hence we have $n = mg'(m) - m \in P$. Since P is a regular submodule of M_R, there exists an $h \in M^*$ with $nh(n) = n$. Hence we have

$$m = m(g' - h - g'(m)h(m)g' + g'(m)h + h(m)g')m,$$

that is, m is regular in M_R. Since $m \in N$ is arbitrary, we conclude that N is a regular submodule of M_R.

It is well known that every ring has a unique maximal regular ideal. For locally projective modules, we have

Theorem 3. *Let M_R be a locally projective module. Then there exists a unique maximal S-admissible regular submodule N,* and $M/N_{\bar{R}}$ has no *nonzero S-admissible regular submodule, where $\bar{R} = R/\mathrm{Ann}_R(M/N)$.*

Proof. Let N_1 and N_2 be S-admissible regular submodules of M. Then $(N_1+N_2)/N_1$ is clearly a regular submodule of $(M/N_1)_{R'}$, where $R' = R/\mathrm{Ann}_R(M/N_1)$. Thus, by Theorem 2, N_1+N_2 is a regular submodule of M_R. And hence the sum of all S-admissible regular submobules of M_R is the unique largest S-admissible regular submodule of M_R. The second assertion is also clear by Theorem 2.

A module M_R is said to be *semi-artinian* if every nonzero homomorphic image of M_R has the nonzero socle. We call a module M_R *a fully idempotent module*, if for each $m \in M$, there are $s_1, \cdots, s_n, \in S$, $f_1, \cdots, f_n \in M^*$ and $r_1, \cdots, r_n \in R$ such that $m = \sum_{i=1}^{n} s_i(m)f_i(m)r_i$. The following theorem is a generalization of [1, Proposition 4.5].

Theorem 4. *If M_R is semi-artinian, then the following conditions are equivalent*:

1) *M_R is a regular module.*
2) *M_R is a locally projective, fully idempotent module.*

Proof. It is enough to prove that 2) implies 1) (Proposition 1 (2)). Let N be as in Theorem 3. If $N \neq M$, then by hypothesis $X = \mathrm{Soc}(M/N_R)$ is nonzero. We shall show that X is a regular submodule of $\overline{M}_{\bar{R}}$, where $\overline{M} = M/N$ and $\bar{R} = R/\mathrm{Ann}_R(\overline{M})$. Now, let Y be a simple submodule of $\overline{M}$. Since M_R is semiprime by [1, Proposition 2.2], there exists an $f \in (\overline{M}_{\bar{R}})^*$ such that $Yf(Y) \neq 0$. Then $f(Y)$ is a non-nilpotent minimal right ideal of $\bar{R}$, and so $f(Y) = e\bar{R}$ with some idempotent e in $\bar{R}$. Let y be a nonzero element of Y. Since $f(Y) = e\bar{R}$ is a minimal right ideal, there exists an $r \in \bar{R}$ such that $f(y)r = e$. Let g be the element of $(\bar{R}_{\bar{R}})^*$ induced by the left multiplication by the element r. Then we obtain $f(y) = f(ygf(y))$. Since $f|Y$ is monic, there holds $y = ygf(y)$. Hence ygf is an idempotent of $\mathrm{End}_{\bar{R}}(\bar{M})$, and therefore Y is a direct summand of $M_{\bar{R}}$. We show by induction that any finite direct sum of simple submodules Y_i is a direct summand of $\bar{M}_{\bar{R}}$. Assume $\bar{M} = Y_1 \oplus \cdots \oplus Y_{n-1} \oplus K$ with some submodule K. Let p be the natural projection $\bar{M} \rightarrow K$. Then we can easily see that $Y_1 \oplus \cdots \oplus Y_n = Y_1 \oplus \cdots \oplus Y_{n-1} \oplus p(Y_n)$ and $p(Y_n)$ is a direct summand of M, which completes the induction. Now, let m be an element of X. Then $m\bar{R}$ is a finite direct sum of simple submodules of $\bar{M}$, and hence by the above $m\bar{R}$ is projective and is a direct summand of $\bar{M}_{\bar{R}}$. Thus, by Proposition 1 (1) m is regular in $\bar{M}$, namely X is a nonzero S-admissible regular submodule of $\bar{M}_{\bar{R}}$. This contradicts the choice of N (Theorem 3).

We call a module M_R a *V-module*, if every submodule is an intersection of maximal submodules of M_R. Since every locally projective V-module is fully idempotent by [1, Proposition 3.7], we readily obtain the following corollary.

Corollary 1. *If M_R is a locally projective, semi-artinian V-module, then M_R is a regular module.*

In the previous paper [1], we proved that a module M over a *P.I.*-ring R is a regular module if and only if it is a locally projective V-module. If M_R is a V-module, then every simple module is M-injective, and conversely ([1, Proposition 3.1]). As an application of these results, we have

Theorem 5. *Let R be a P.I.-ring. Then a locally projective module M_R is completely reducible if (and only if) every completely reducible module is M-injective.*

Proof. Let m be an arbitrary element of M. Since M_R is regular by [1, Theorem 4.4], mR_R is a regular module and every completely reducible module is mR-injective. Now, we shall show that mR is finite dimensional. Assume, to the contrary, that mR contains an infinite direct sum $N = \oplus_{\alpha \in A} M$ with $M_\alpha \neq 0$. Since each M_α is a V-module ([1, Proposition 3.1]), it contains a maximal submodule M'_α. Then $N' = \oplus_{\alpha \in A} M_\alpha / M'_\alpha$ is completely reducible, and hence by hypothesis mR-injective. Thus the canonical homomorphism $N \rightarrow N'$ can be extended to a homomorphism

$f: mR \to N'$. Noting that $f(m) \in \bigoplus_{\alpha \in A'} M_\alpha / M'_\alpha$ with a finite subset A' of A, we obtain $N' = f(N) \subseteq f(mR) \subseteq \bigoplus_{\alpha \in A'} M_\alpha / M'_\alpha$, which is a contradiction. Thus we see that mR is isomorphic to a finite direct sum of minimal right ideals by [2, Theorem 1.8], concluding that M_R is a sum of simple submodules.

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