Mathematical Journal of Okayama University

Volume 23, *Issue* 2 1981 *Article* 17

DECEMBER 1981

Counterexamples in the individual ergodic theory of pseudo-resolvents

Ryotaro Sato*

*Okayama University

Math. J. Okayama Univ. 23 (1981), 207—211

COUNTEREXAMPLES IN THE INDIVIDAL ERGODIC THEORY OF PSEUDO-RESOLVENTS

RYOTARO SATO

1. Introduction. Let D denote the set of all complex numbers λ with Re $(\lambda) > 0$, and D_+ the set of all positive reals. Let $\boldsymbol{J} = (J_\lambda : \lambda \in D)$ be a pseudo-resolvent of bounded linear operators on L_1 of a σ-finite measure space. Thus $J_\lambda - J_\nu = (\nu - \lambda) J_\lambda J_\nu$ for all λ and ν in D. Previously it was proved (cf. [5]) that if $\boldsymbol{J}$ satisfies that

$$\|\lambda J_\lambda\|_1 \leq 1 \text{ for all } \lambda \in D_+$$

and also that for some constant $M \geq 1$

$$\|\lambda J_\lambda f\|_\infty \leq M \|f\|_\infty \text{ for all } \lambda \in D_+ \text{ and } f \in L_1 \cap L_\infty,$$

then the following individual ergodic limits

$$\lim_{\substack{\lambda \to 0 \\ \lambda \in D_+}} \lambda J_\lambda f(\omega) \quad \text{and} \quad \lim_{\substack{\lambda \to \infty \\ \lambda \in D_+}} \lambda J_\lambda f(\omega)$$

exist almost everywhere, whenever $f \in L_p$ with $1 \leq p < \infty$.

The purpose of this note is to examine the necessity of the above norm conditions on $\boldsymbol{J}$ and to show, by examples, that these conditions can not be weakened without failing to hold the individual ergodic theorem.

2. Preliminary lemmas.

Lemma 1. *If $(a_n : n \geq 0)$ is a sequence of nonnegative reals, then*

$$\sup_{n \geq 1} \frac{1}{n} \sum_{i=0}^{n-1} a_i \leq e \cdot \sup_{0<r<1} (1-r) \sum_{i=0}^{\infty} r^i a_i$$

and

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} a_i \leq e \cdot \limsup_{r \to 1-0} (1-r) \sum_{i=0}^{\infty} r^i a_i.$$

Proof. Putting

$$C_n = \frac{1}{n} \sum_{i=0}^{n-1} a_i \quad (n \geq 1) \quad \text{and} \quad A_r = (1-r) \sum_{i=0}^{\infty} r^i a_i \quad (0 < r < 1),$$

we see that

$$A_r \geq (1-r) \sum_{i=0}^{n-1} r^i a_i \geq (1-r) r^{n-1} \sum_{i=0}^{n-1} a_i = n(1-r) r^{n-1} C_n.$$

Mathematical Journal of Okayama University, Vol. 23 [1981], Iss. 2, Art. 17

On the other hand, for each $n \geq 1$

$$\sup_{0<r<1} n(1-r)r^{n-1} = n(1-[1-\frac{1}{n}])(1-\frac{1}{n})^{n-1} > e^{-1};$$

therefore the lemma follows.

Lemma 2. *If $a(t)$ is a nonnegative Lebesgue measurable function on the interval $(0, \infty)$, then*

$$\sup_{b>0} \frac{1}{b}\int_0^b a(t)\ dt \leq e\cdot\sup_{\lambda>0} \lambda\int_0^\infty e^{-\lambda t}a(t)\ dt,$$

$$\limsup_{b\to+\infty} \frac{1}{b}\int_0^b a(t)\ dt \leq e\cdot\limsup_{\lambda\to+0} \lambda\int_0^\infty e^{-\lambda t}a(t)\ dt$$

and

$$\limsup_{b\to+0} \frac{1}{b}\int_0^b a(t)\ dt \leq e\cdot\limsup_{\lambda\to+\infty} \lambda\int_0^\infty e^{-\lambda t}a(t)\ dt.$$

Proof. Since

$$\lambda\int_0^\infty e^{-\lambda t}a(t)\ dt \geq \lambda e^{-\lambda b}\int_0^b a(t)\ dt$$
$$= (\lambda b)e^{-\lambda b}\cdot\frac{1}{b}\int_0^b a(t)\ dt,$$

and since for each $b > 0$

$$\sup_{\lambda>0} (\lambda b)e^{-\lambda b} = e^{-1},$$

the lemma follows immediately.

3. Counterexamples.

Example 1 (cf. [7]). *There exists a pseudo-resolvent $\boldsymbol{J} = (J_\lambda : \lambda \in D)$ on $L_1(0, 1)$ such that*

(i) *for all $\lambda \in D_+$, $J_\lambda \geq 0$ and $\|\lambda J_\lambda\|_1 = 1$,*

(ii) *for some $f \in L_1(0, 1)$ the limit*

$$\lim_{\substack{\lambda\to 0 \\ \lambda\in D_+}} \lambda J_\lambda f(\omega)$$

does not exist almost everywhere on the whole interval $(0, 1)$.

To see this, let T be a positive isometry on $L_1(0,1)$ such that for some $0 \leq f \in L_1(0, 1)$, $\limsup_n \frac{1}{n}\sum_{i=0}^{n-1} T^i f(\omega) = \infty$ almost everywhere on $(0, 1)$ (cf. [1]). Then, since $\|T\|_1 = 1$, we may define

$$J_\lambda = (\lambda+1-T)^{-1} = \frac{1}{\lambda+1}\sum_{i=0}^{\infty}\frac{T^i}{(\lambda+1)^i} \quad (\lambda \in D).$$

Clearly, $\boldsymbol{J} = (J_\lambda : \lambda \in D)$ is a pseudo-resolvent on $L_1(0, 1)$ such that

$$J_\lambda \geq 0 \text{ and } \|\lambda J_\lambda\|_1 = 1 \text{ for all } \lambda \in D_+.$$

Furthermore, by Lemma 1, we have

$$\limsup_{\substack{\lambda\to 0\\ \lambda\in D_+}} \lambda J_\lambda f(\omega) = \infty$$

almost everywhere on (0, 1). On the other hand, by Fatou's lemma, if we set

$$h(\omega) = \liminf_{\substack{\lambda\to 0\\ \lambda\in D_+}} \lambda J_\lambda f(\omega) \quad \text{for all } \omega \in (0, 1)$$

then $0 \leq h \in L_1(0, 1)$. Therefore $h(\omega) < \infty$ almost everywhere on (0, 1), and this completes the proof.

Example 2. *Given an $\varepsilon > 0$ there exists a pseude-resolvent $\boldsymbol{J} = (J_\lambda : \lambda \in D)$ on L_1 of a finite measure space such that*

(i) *for all $\lambda \in D_+$*

$$J_\lambda \geq 0, \ \lambda J_\lambda 1 = 1 \ \textit{and} \ \|\lambda J_\lambda\|_1 \leq 1+\varepsilon,$$

(ii) *for some $f \in L_1$ the limit*

$$\lim_{\substack{\lambda\to 0\\ \lambda\in D_+}} \lambda J_\lambda f(\omega)$$

does not exist almost everywhere on a certain measurable subset of positive measure.

To see this, let S be an ergodic and invertible measure preserving point transformation on the interval (0, 1] and define also $Sg(\omega) = g(S\omega)$ for $g \in L_1(0, 1]$. Take $0 \leq f \in L_1(0, 1]$ such that $f \log^+ f \notin L_1(0, 1]$. Then, by [3] and Lemma 1, we see that

$$\sup_{0<r<1} (1-r)\sum_{i=0}^{\infty} r^i S^i f(\omega) \notin L_1(0, 1].$$

Thus, as in Derriennic and Lin [2], there exists a sub-σ-field $\mathcal{B}$ of the Lebesgue measurable subsets of (0, 1] such that the limit

$$\lim_{r\to 1-0} (1-r) \sum_{i=0}^{\infty} r^i E[S^i f \mid \mathcal{B}](\omega)$$

does not exist almost everywhere on (0, 1], where $E\,[\cdot\,|\mathcal{B}]$ stands for the conditional expectation operator with respect to $\mathcal{B}$. Define a positive linear operator T on $L_1(0, 1+\varepsilon]$ by

$$Tg(\omega) = \begin{cases} S(g1_{(0,1]})(\omega) & \text{if } 0 < \omega \leq 1, \\ E[S(g1_{(0,1]}) \mid \mathcal{B}](\frac{\omega-1}{\varepsilon}) & \text{if } 1 < \omega \leq 1+\varepsilon. \end{cases}$$

It is easily seen that $T1 = 1$ and $\|T^n\|_1 = 1+\varepsilon$ $(n \geq 1)$. Thus if we set

$$J_\lambda = (\lambda+1-T)^{-1} = \frac{1}{\lambda+1}\sum_{i=0}^{\infty}\frac{T^i}{(\lambda+1)^i} \quad (\lambda \in D)$$

then for all $\lambda \in D_+$ we have

$$J_\lambda \geq 0, \ \lambda J_\lambda 1 = 1 \text{ and } \|\lambda J_\lambda\|_1 \leq 1+\varepsilon;$$

furthermore, $\lim_{\substack{\lambda\to 0\\ \lambda\in D_+}} \lambda J_\lambda f(\omega)$ does not exist almost everywhere on $(1, 1+\varepsilon]$. Hence the proof is completed.

Example 3. *Given an $\varepsilon > 0$ there exists a pseudo-resolvent $\boldsymbol{J} = (J_\lambda : \lambda \in D)$ on L_1 of a finite measure space such that*

(i) *for all $\lambda \in D_+$*

$$J_\lambda \geq 0, \ \lambda J_\lambda 1 = 1 \text{ and } \|\lambda J_\lambda\|_1 \leq 1+\varepsilon,$$

(ii) *for some $f \in L_1$ the limit*

$$\lim_{\substack{\lambda\to\infty\\ \lambda\in D_+}} \lambda J_\lambda f(\omega)$$

does not exist almost everywhere on a certain measurable subset of positive measure.

To see this, let $(S_t : t \geq 0)$ be the strongly continuous semigroup of positive isometries on $L_1(0, 1]$ defined by

$$S_t g(\omega) = g(\omega \dot{+} t) \qquad (g \in L_1(0, 1], \ \omega \in (0, 1]),$$

where $\omega \dot{+} t = \omega + t$ if $\omega + t \leq 1$ and $\omega \dot{+} t = \omega + t - n$ if $n < \omega + t \leq n+1$. By [4], for some $0 \leq f \in L_1(0, 1]$ and some sequence (b_n) of positive reals with $\lim_n b_n = 0$ we have

$$\sup_n \frac{1}{b_n}\int_0^{b_n} S_t f(\omega)\, dt \notin L_1(0, 1).$$

Thus, by Lemma 2 and the argument given in Example 2 (cf. also [4]), there exists a strongly continuous semigroup $\Gamma = (T_t : t \geq 0)$ of positive linear operators on $L_1(0, 1+\varepsilon]$ such that for all $t \geq 0$

$$T_t 1 = 1 \text{ and } \|T_t\|_1 = 1+\varepsilon,$$

and also such that the limit

$$\lim_{\substack{\lambda\to\infty \\ \lambda\in D_+}} \lambda\int_0^\infty e^{-\lambda t}\, T_t f(\omega)\, dt$$

does not exist almost everywhere on $(1, 1+\varepsilon]$. For $\lambda\in D$ define

$$J_\lambda g = \int_0^\infty e^{-\lambda t}\, T_t g\, dt \quad (g\in L_1(0, 1+\varepsilon]).$$

Obviously J_λ is a bounded linear operator on $L_1(0, 1+\varepsilon]$ satisfying $J_\lambda T_0 = J_\lambda$, and if $\lambda\in D_+$ then $J_\lambda\ge 0$ and $\|\lambda J_\lambda\|_1 = 1+\varepsilon$. Thus, to complete the proof it is now enough to check that $J=(J_\lambda:\lambda\in D)$ is a pseudo-resolvent. To this end, put $M=T_0L_1(0, 1+\varepsilon]$. Then M is a closed subspace of $L_1(0, 1+\varepsilon]$, $T_tM\subset M$ for all $t\ge 0$, and $T_0=I$ on M. Thus, by restricting $\Gamma=(T_t:t\ge 0)$ to M and applying Corollary IX.4.1 and Theorem VIII.2.2 in [6], we see that $J_\lambda - J_\nu = (\nu-\lambda)J_\lambda J_\nu$ on M. Hence for every $g\in L_1(0, 1+\varepsilon]$

$$\begin{aligned} J_\lambda g - J_\nu g &= J_\lambda T_0 g - J_\nu T_0 g \\ &= (\nu-\lambda)J_\lambda J_\nu T_0 g = (\nu-\lambda)J_\lambda J_\nu g, \end{aligned}$$

completing the proof.

References

[1] R. V. Chacon: A class of linear transformations, Proc. Amer. Math. Soc. **15** (1964), 560—564.

[2] Y. Derriennic and M. Lin: On invariant measures and ergodic theorems for positive operators, J. Functional Analysis **13** (1973), 252—267.

[3] D. Ornstein: A remark on the Birkhoff ergodic theorem, Illinois J. Math. **15** (1971), 77—79.

[4] R. Sato: On local ergodic theorems for positive semigroups, Studia Math. **63** (1978), 45—55.

[5] R. Sato: Individual ergodic theorems for pseudo-resolvents, Math. J. Okayama Univ. **23** (1981), 59—67.

[6] K. Yosida: Functional Analysis, Springer, Berlin, 1965.

[7] Chr. Grillenberger and U. Krengel: On matrix summation and the pointwise ergodic theorem, Théorie Ergodique, Springer Lecture Notes in Math. **532**, pp. 113—124, Springer, Berlin, 1976.

Department of Mathematics
Okayama University

(*Received January 12, 1981*)