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Ryotaro Sato*

*Okayama University

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ON ABSTRACT MEAN ERGODIC THEOREMS. II

RYOTARO SATO

1. Introduction

This is a continuation of [7]. In [7] we proved abstract mean ergodic theorems for weakly right ergodic semigroups $\mathfrak{S}$ of continuous linear operators on a *complete* locally convex topological vector space E. Among other things it was proved that the fixed points of $\mathfrak{S}$ separate the fixed points of the adjoint semigroup $\mathfrak{S}^*=\{T^*: T\in\mathfrak{S}\}$ if and only if E is the direct sum of the fixed points of $\mathfrak{S}$ and the closed linear subspace of E determined by the set $\{x-Tx: x\in E \text{ and } T\in\mathfrak{S}\}$. This generalizes Sine's mean ergodic theorem for a single Banach space contraction operator ([5], [6], [9]). In the present paper we shall first derive a criterion, called the finite dimension criterion, for the validity of a mean ergodic theorem for a weakly right ergodic semigroup which has as a special case Sine's and Atalla's criterion for a single Banach space contraction operator ([2], [8], [11]). We shall next study ergodic properties of right ergodic semigroups of Markov operators on $C(X)$, $C(X)$ being the Banach space of all (real or complex) continuous functions on a compact Hausdorff space X with the supremum norm. Sine's results in [10] will be generalized.

2. Definitions and the finite dimension criterion

Throughout this section, E will be a *complete* locally convex topological vector space (t. v. s.) and $\mathfrak{S}$ a semigroup of continuous linear operators on E. For an $x\in E$ we denote by $A(x)$ the affine subspace of E determined by the set $\{Tx: T\in\mathfrak{S}\}$, i. e.

$$A(x)=\{y: y=\sum_{i=1}^{k} a_i T_i x,\ \sum_{i=1}^{k} a_i=1,\ T_i\in\mathfrak{S},\ 1\leq k<\infty\},$$

and by $\bar{A}(x)$ the closure of $A(x)$ in E. A net $(T_n,\ n\in\Delta)$ of linear operators on E is said to be (*weakly*) *right* [resp. (*weakly*) *left*] $\mathfrak{S}$-*ergodic* if it satisfies:

(I) For every $x\in E$ and all $n\in\Delta$, $T_n x\in\bar{A}(x)$.

(II) The transformations T_n are equicontinuous.

(III) For every $x\in E$ and all $T\in\mathfrak{S}$,

(weak-) $\lim_n T_n Tx-T_n x=0$ [resp. (weak-) $\lim_n TT_n x-T_n x=0$].

The semigroup $\mathfrak{S}$ is said to be (*weakly*) *right* [resp. (*weakly*) *left*]

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ergodic if it possesses at least one (weakly) right [resp. (weakly) left] $\mathfrak{S}$-ergodic net $(T_n,\ n\in J)$. Whenever $(T_n,\ n\in J)$ is a both (weakly) right and left $\mathfrak{S}$-ergodic net, we call it simply (weakly) $\mathfrak{S}$-*ergodic*. And if $\mathfrak{S}$ possesses at least one (weakly) $\mathfrak{S}$-ergodic net, $\mathfrak{S}$ is said to be (*weakly*) *ergodic*. (See [4] and [7].)

The adjoint semigroup of $\mathfrak{S}$ is the semigroup $\mathfrak{S}^*=\{T^*: T\in\mathfrak{S}\}$, where T^* is the adjoint operator of T defined by $\langle x,\ T^*x^*\rangle=\langle Tx,\ x^*\rangle$ for all $x\in E$ and all $x^*\in E^*$, E^* being the topological dual of E. We let

$$F=\{x\in E:\ Tx=x \text{ for all } T\in\mathfrak{S}\}$$

and

$$F^*=\{x^*\in E^*:\ T^*x^*=x^* \text{ for all } T^*\in\mathfrak{S}^*\}.$$

Lemma. *Let $\mathfrak{S}$ be a weakly right ergodic semigroup. If* $\dim F<\infty$, *then* $\dim F^*\geq\dim F$.

Proof. Let f be any linear functional on F. Then f is continuous on F, as F is finite dimensional. Therefore by the Hahn-Banach theorem there exists an $f^*\in E^*$ such that

$$f^*=f \quad\text{on}\quad F.$$

Write

$$U=\{x\in E:\ |\langle x,\ f^*\rangle|<1\}.$$

Now if $(T_n,\ n\in J)$ is a weakly right $\mathfrak{S}$-ergodic net, then by the equicontinuity of the operators T_n we can choose a neighborhood W of the origin of E such that $W\subset U$ and also such that

$$T_nW\subset U \quad\text{for all } n\in J.$$

Let

$$A^*=\{x^*\in E^*:\ x^*=f \text{ on } F \text{ and } |\langle x,\ x^*\rangle|\leq 1 \text{ for all } x\in U\}$$

and

$$B^*=\{x^*\in E^*:\ x^*=f \text{ on } F \text{ and } |\langle x,\ x^*\rangle|\leq 1 \text{ for all } x\in W\}.$$

It is easily seen that $f^*\in A^*\subset B^*$ and that

$$x^*\in A^* \text{ implies } T_n^*x^*\in B^* \text{ for all } n\in J.$$

The Banach-Alaoglu theorem shows that B^* is weak*-compact, thus there exists a subnet $(T_{n'}^*f^*,\ n'\in J')$ of the net $(T_n^*f^*,\ n\in J)$ which converges in the weak*-topology to a point g^* in B^*. Hence for every $T^*\in\mathfrak{S}^*$ and all $x\in E$

$$\langle x,\ T^*g^*\rangle=\lim_{n'}\langle x,\ T^*T_{n'}^*f^*\rangle=\lim_{n'}\langle T_{n'}Tx,\ f^*\rangle$$

$$=\lim_{n'}\langle T_{n'}x,\ f^*\rangle=\lim_{n'}\langle x,\ T_{n'}{}^*f^*\rangle$$
$$=\langle x,\ g^*\rangle.$$

It follows that $g^*\in F^*$, and since $g^*=f$ on F, we immediately conclude that $\dim F^*\geq\dim F$. The proof is complete.

Remark 1. The above-given argument can easily be modified to show that if $\mathfrak{S}$ is a weakly right ergodic semigroup, then $\dim F^*<\infty$ implies $\dim F\leq\dim F^*$. Any continuous linear functional on F can be extended to a continuous linear functional on E belonging to F^*.

Theorem 1. *Let $\mathfrak{S}$ be a weakly right ergodic semigroup of continuous linear operators on a complete locally convex t. v. s. E. If either* $\dim F<\infty$ *or* $\dim F^*<\infty$ *then the following conditions are equivalent*:

(a) $\dim F=\dim F^*$.

(b) *E is the direct sum of F and N, where N is the closed linear subspace of E determined by the set* $\{x-Tx:\ x\in E$ *and* $T\in\mathfrak{S}\}$.

Proof. By the previous lemma and remark, we see that (a) is equivalent to the following: For any nonzero $x^*\in F^*$ there exists an $x\in F$ satisfying $\langle x, x^*\rangle\neq 0$, i. e. F separates F^*. And this condition is equivalent to (b), as is stated in Introduction. The proof is complete.

Remark 2. In the above theorem, the hypothesis that F is finite dimensional is not omitted. In fact there are many spaces E such that $\dim E<\dim E^*$. If we let $S=\{I\}$, where I denotes the identity operator on such a space E, then clearly (b) holds but (a) does not.

3. Frgodic properties of Markov operator semigroups

Let X be a compact Hausdorff space and $C(X)$ the Banach space of all (real or complex) continuous functions on X with the supremum norm. A linear operator T on $C(X)$ is said to be a *Markov operator* if $T1=1$ and if $f\geq 0$ implies $Tf\geq 0$. Let $\mathfrak{S}$ be a fixed semigroup of Markov operators on $C(X)$, and put

$$C_i(X)=\{f\in C(X):\ Tf=f\ \text{ for all }\ T\in\mathfrak{S}\}.$$

It is well-known ([3], p. 265) that the topological dual space $C^*(X)$ of $C(X)$ is identified with the space of all regular finite (countably additive) measures on the σ-field Σ of Borel subsets of X. Denote by $\mathscr{P}(X)$ the regular probability measures on Σ, and put

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$$\mathscr{P}_i(X)=\{\mu\in\mathscr{P}(X):\ T^*\mu=\mu \text{ for all } T^*\in\mathfrak{S}^*\}.$$

We define, as in Sine [10], the center M of $\mathfrak{S}$ by

$$M=\text{closure}\cup\{\text{supp}\ \mu:\ \mu\in\mathscr{P}_i(X)\}.$$

A closed subset K of X is said to be $\mathfrak{S}$-*invariant* if supp $T^*e_x\subset K$ for every $x\in K$, where e_x denotes the unit mass concentrated at x. It is easily seen from Sine [8] that M is $\mathfrak{S}$-invariant.

Proposition. *Let $\mathfrak{S}$ be a weakly right ergodic semigroup of Markov operators on $C(X)$. Then any $g\in C(X)$ with $g=0$ on M is in the closed linear subspace N of $C(X)$ determined by the set $\{f-Tf:\ f\in C(X)$ and $T\in\mathfrak{S}\}$.*

Proof. Let $(T_n,\ n\in\Delta)$ be a weakly right $\mathfrak{S}$-ergodic net. If $\mu\in\mathscr{P}(X)$, then as in the proof of the lemma there exists a subnet $(T^*_{n'}\mu,\ n'\in\Delta')$ of the net $(T^*_n\mu,\ n\in\Delta)$ and an element $\tilde{\mu}\in C^*(X)$ such that

$$\text{weak*-}\lim_{n'} T^*_{n'}\mu=\tilde{\mu}.$$

It follows that $T^*\tilde{\mu}=\tilde{\mu}$ for all $T^*\in\mathfrak{S}^*$. Since $\|T^*\|=1$ for all $T^*\in\mathfrak{S}^*$, it follows that $\tilde{\mu}$ is a finite linear combination of elements of $\mathscr{P}_i(X)$. Hence supp $\tilde{\mu}\subset M$, and so we have

$$\lim_{n'}\langle T_{n'}g,\ \mu\rangle=\lim_{n'}\langle g,\ T^*_{n'}\mu\rangle=\langle g,\tilde{\mu}\rangle=0.$$

By this and an easy induction argument, the zero function 0 is a weak cluster element of the net $(T_n g,\ n\in\Delta)$, and thus we have $0\in\overline{A}(g)$. Therefore given an $\varepsilon>0$ there exists an $h=\sum_{i=1}^{k}a_iT_ig$ with $\|h\|<\varepsilon$, where $\sum_{i=1}^{k}a_i=1$ and $T_i\in\mathfrak{S}$ for each i. Consequently

$$g=h+\sum_{i=1}^{k}a_i(g-T_ig),$$

and this proves the proposition.

In Theorem 2 below we study ergodic properties of $\mathfrak{S}$ restricted to the center M. A semigroup $\mathfrak{S}$ is said to be *continuously scattered* if there exists a family of functions in $C(X)$ so that each function in the family is constant on the support of each extreme measure of $\mathscr{P}_i(X)$ and the family separates the extreme measures of $\mathscr{P}_i(X)$.

Theorem 2. *Let $\mathfrak{S}$ be a semigroup of Markov operators on $C(X)$ and*

$(T_n, n\in\Delta)$ a right $\mathfrak{S}$-ergodic net of linear operators on $C(X)$. Then the following conditions are equivalent:

(a) *$\mathfrak{S}$ is continuously scattered.*

(b) *For any $f\in C(X)$ the net $(T_nf,\ n\in\Delta)$ converges uniformly on the center M, and further $\lim_n TT_nf-T_nf=0$ uniformly on M for all $T\in\mathfrak{S}$.*

Proof. We proceed partly as in Sine [10]. Since M is $\mathfrak{S}$-invariant, we may and will assume without loss of generality that M equals the whole space X.

(a) $\Longrightarrow$ (b). Let $\mathscr{A}$ be the family of all $f\in C(X)$ that are constant on the support of each extreme measure of $\mathscr{P}_i(X)$. Then we see that $\mathscr{A}$ is a norm closed algebra and that if $f\in\mathscr{A}$ and if μ is an extreme measure of $\mathscr{P}_i(X)$ then $Tf=f$ on supp μ for all $T\in\mathfrak{S}$, because supp μ is $\mathfrak{S}$-invarinat. By the Krein-Milman theorem, the union

$$\bigcup\{\text{supp } \mu:\ \mu \text{ is an extreme measure of } \mathscr{P}_i(X)\}$$

is dense in $X(=M)$, and thus the continuity of f implies that $Tf=f$ on X for all $T\in\mathfrak{S}$. Let Y be the quotient topological space $X/\mathscr{A}$. The quotient map q is defined by

$$q(x)=\{z\in X: f(z)=f(x) \text{ for all } f\in\mathscr{A}\}\ (\in Y)$$

for all $x\in X$. Y is then a compact Hausdorff space and q is continuous. The Stone-Weierstrass theorem implies that $\mathscr{A}$ can be identified with the Banach space $C(Y)$, and from this it may be readily seen that for any $y\in Y$ the set $q^{-1}(y)$ is $\mathfrak{S}$-invariant. Since by assumption $\mathfrak{S}$ is continuously scattered, there exists a unique measure μ in $\mathscr{P}_i(X)$ such that supp $\mu\subset q^{-1}(y)$. It follows from Corollary 1 of [7] and the results of the preceding section that $(T_nf,\ n\in\Delta)$ converges uniformly on $q^{-1}(y)$ to a constant function for each $f\in C(X)$.

Let

$$F(x)=\lim_n T_nf(x)\quad (x\in X).$$

To prove the uniform convergence of $(T_nf,\ n\in\Delta)$ to F, let $x\in q^{-1}(y)$. Since $(T_nf,\ n\in\Delta)$ converges uniformly on $q^{-1}(y)$, given an $\varepsilon>0$ there exists an $N\in\Delta$ such that

$$q^{-1}(y)\subset\{z\in X: |T_Nf(z)-F(x)|<\varepsilon\}.$$

The latter set is open, and hence there exists an open set U in Y so that

$$q^{-1}(y)\subset q^{-1}(U)\subset\{z\in X: |T_Nf(z)-F(x)|<\varepsilon\}.$$

Since $T_Nf\in\overline{A}(f)$, we can choose $\sum_{i=1}^k a_iT_if\in A(f)$ so that

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$$\| T_N f - \sum_{i=1}^{k} a_i T_i f \| < \varepsilon .$$

It then follows that

$$| \sum_{i=1}^{k} a_i T_i f - F(x) | < 2\varepsilon \text{ on } q^{-1}(U)$$

and that

$$T_n f = T_n(\sum_{i=1}^{k} a_i (f - T_i f)) + T_n(\sum_{i=1}^{k} a_i T_i f - F(x)) + T_n F(x).$$

Since for every $z \in q^{-1}(U)$ and all $n \in \Delta$, supp $T_n^* e_z \subset q^{-1}(U)$, we have that

$$\begin{aligned} | T_n f - F(x) | &= | T_n f - T_n F(x) | \\ &\leq \sum_{i=1}^{k} | a_i | \| T_n f - T_n T_i f \| + 2\Lambda\varepsilon \text{ on } q^{-1}(U) \end{aligned}$$

where $\Lambda = \sup_n \| T_n \|$, and that

$$\lim_n \| T_n f - T_n T_i f \| = 0 \ (i = 1, \cdots, k).$$

Hence we see that $| F(z) - F(x) | \leq 2\Lambda\varepsilon$ for all $z \in q^{-1}(U)$, and furthermore that there exists an $N(x) \in \Delta$ such that if $n \geq N(x)$ then

$$| T_n f - F | < 5\Lambda\varepsilon \text{ on } q^{-1}(U).$$

Since X is compact, the uniform convergence of $(T_n f,\ n \in \Delta)$ to F on X follows. Since $F \in \mathscr{A} \subset C_i(X)$, we also have

$$\lim_n \| T T_n f - T_n f \| = \| TF - F \| = 0$$

for all $T \in \mathfrak{S}$.

(b) $\Longrightarrow$ (a). If (b) holds then by Corollary 1 of [7] $C_i(X)$ separates the extreme measures of $\mathscr{P}_i(X)$. On the other hand, every $f \in C_i(X)$ is constant on the support of each extreme measure of $\mathscr{P}_i(X)$ ([8]). Therefore $\mathfrak{S}$ is continuously scattered.

The following theorem may be regarded as a generalization of Theorem 3.2 of Atalla [1].

Theorem 3. *Let $\mathfrak{S}$ be a continuously scattered semigroup of Markov operators on $C(X)$ and $(T_n,\ n \in \Delta)$ a right $\mathfrak{S}$-ergodic net of linear operators on $C(X)$. Then the following conditions are equivalent:*

(a) *For any $f \in C(X)$ the net $(T_n f,\ n \in \Delta)$ converges uniformly on X.*

(b) *There exists a continuous linear operator S on $C(X)$ such that for every $f \in C(X)$ $\lim_n \| S T_n f - T_n f \| = 0$ and such that, for each $f \in C(X)$ with $f = 0$ on M, $Sf = 0$ on X.*

Proof. (a)$\Longrightarrow$(b). Let $Sf=\lim_n T_n f$ for all $f\in C(X)$. Since $(T_n, n\in \varDelta)$ is right $\mathfrak{S}$-ergodic, if $f\in C(X)$ satisfies $f=0$ on M then by the Proposition $Sf=0$ on X. Furthermore for every $T\in\mathfrak{S}$ and all $f\in C(X)$,

$$STf-Sf=\lim_n T_n Tf-T_n f=0.$$

Hence, immediately, $ST_n=S$ for all $n\in \varDelta$, and so (b) follows.

(b)$\Longrightarrow$(a). Since $\mathfrak{S}$ is continuously scattered by hypothesis, Theorem 2 shows that the net $(T_n f,\ n\in \varDelta)$ converges uniformly on the center M for every $f\in C(X)$. Choose an $F\in C(X)$ so that

$$F(x)=\lim_n T_n f(x) \quad (x\in M).$$

(b) implies that supp $S^*e_x\subset M$ for all $x\in X$, and hence we have

$$\lim_n \|SF-ST_n f\| \leq \|S\| \lim_n (\sup \{|F(z)-T_n f(z)| : z\in M\})=0.$$

Therefore, by (b) again, we have

$$\lim_n \|SF-T_n f\| \leq \lim_n \|SF-ST_n f\| + \lim_n \|ST_n f-T_n f\| =0,$$

completing the proof.

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DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY

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