Mathematical Journal of Okayama University

Volume 18, *Issue* 1 1975 *Article* 4

DECEMBER 1975

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ON SEPARABILITY OF GALOIS EXTENSIONS

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Througout, A will represent a ring with 1, and B a subring of A containing 1. Let V be the centralizer $V_A(B)$ of B in A, $H=V_A(V)$, and C the center of A.

The present note contains several results concerning the separability of Galois extensions of (Artinian) simple rings, where a simple ring extension A/B is called a [*finite*] *Galois extension* if [${}_BA$ is finitely generated and] V is a simple ring and B coincides with the fixring of the group G of all B-ring automorphisms in A. A Galois extension A/B of simple rings is said to be *outer Galois* or *inner Galois* according as $V=C$ or $H=B$. If A/B is finite outer Galois then it is known that A/B is *G-Galois* (cf. [8, Proposition 9.6]). As for the notation and terminology employed here, we follow [3] and [8].

At first, we shall prove the following which will enrich substantially the theory of separable extensions (cf. [1]):

Theorem 1. *If A/B is a finite Galois extension of simple rings then it is Frobenius and separable.*

Proof. Although it is known that A/B is a Frobenius extension (cf. [2, pp. 463–464]), we shall give here the proof. Let $A^*=\mathrm{Hom}\,({}_BA, {}_BB)$, and $(A\otimes_B A)^A=\{u\in A\otimes_B A \mid au=ua$ for all $a\in A\}$. As is well known, there holds the following:

$$ {}_AA\otimes_B A_A\cong {}_A\mathrm{Hom}\,(B_B, A_B)\otimes_B A_A\cong {}_A\mathrm{Hom}\,(A^*{}_B, A_B)_A $$

and

$$ {}_AA\otimes_B A_A\cong {}_A\mathrm{Hom}\,(A_A, A_A)\otimes_B A_A\cong {}_A\mathrm{Hom}\,((\mathrm{End}\,{}_BA)_A, A_A)_A. $$

Combining this with the fact that $\mathrm{End}\,{}_BA=GA_R=\bigoplus_{i=1}^n \sigma_i A_R$ with some $\sigma_1=1, \sigma_2, \cdots, \sigma_n\in G$, one will easily see

$$ 0\neq \mathrm{Hom}\,({}_A(\mathrm{End}\,{}_BA)_A, {}_AA_A)\cong (A\otimes_B A)^A\cong \mathrm{Hom}\,({}_AA^*{}_B, {}_AA_B). $$

Recalling that ${}_AA_B$ and ${}_BA_A$ are homogeneously completely reducible and their lengths coincide with the capacity of V ([8, Theorem 6.1]), we can easily see that there exists an epimorphism $h:\ {}_AA^*{}_B \longrightarrow {}_AA_B$. Since $[A:B]=[A^*:B]_B$, h is indeed an isomorphism. This proves that A/B is a Frobenius extension.

In particular, the finite inner Galois extension A/H is a Frobenius extension and End $A_H = A_L \cdot V_R$. Hence,

$$ {}_A A \otimes_H A_A \cong {}_A \mathrm{Hom}\,((\mathrm{Hom}\,({}_H A, {}_H H)_H, A_H)_A \cong {}_A(\mathrm{End}\, A_H)_A $$
$$ \cong {}_A(A \oplus \cdots \oplus A)_A, $$

and so ${}_A A \otimes_H A_A$ is homogeously completely reducible and the A-A-homomorphism $A \otimes_H A \longrightarrow A$ $(a \otimes a' \longmapsto aa')$ splits. On the other hand, H/B being finite outer Galois, by [1, Proposition 3. 3] the H-H-homomorphism $H \otimes_B H \longrightarrow H$ $(h \otimes h' \longmapsto hh')$ splits, whence it follows that the A-A-homomorphism $A \otimes_B A \longrightarrow A \otimes_H A$ $(a \otimes a' \longmapsto a \otimes a')$ splits. Combining those above, we readily see that the A-A-homomorphism $A \otimes_B A \longrightarrow A$ $(a \otimes a' \longmapsto aa')$ splits.

If A/B is *H-separable*, i. e., if ${}_A A \otimes_B A_A < \oplus\, {}_A(A \oplus \cdots \oplus A)_A$, then there exist some $v_i \in V$ and $\sum_j x_{ij} \otimes y_{ij} \in (A \otimes_B A)^A$ such that $\sum_{i,j} x_{ij} \otimes y_{ij} v_i = 1 \otimes 1$. Given $g \in \mathrm{End}\, A_B$, $\sum_{i,j} x_{ij} \otimes y_{ij} a v_i = a \otimes 1$ implies $\sum_{i,j} g(x_{ij}) \otimes y_{ij} a v_i = g(a) \otimes 1$, whence it follows $\sum_{i,j} g(x_{ij}) y_{ij} a v_i = g(a)$ $(a \in A)$. Especially, we have

Lemma 1. *If A/B is H-separable then* End $A_B = A_L \cdot V_R$.

As was shown in [9, Proposition 1. 1], if A is a separable R-algebra and a projective R-module then A is a finitely generated R-module. The next is an analogue of the above for H-separable extensions ([7]).

Lemma 2. *If A/B is H-separable and A_B is projective then A_B is finitely generated.*

Proof. For the sake of completeness, we shall give the proof. Let $\{a_\kappa ; f_\kappa\}_{\kappa \in K}$ $(a_\kappa \in A,\ f_\kappa \in \mathrm{Hom}\,(A_B, B_B))$ be a projective coordinate system for A_B. Then, f_κ extends naturally to $f_\kappa^* \in \mathrm{Hom}(A \otimes_B A_A, A_A)$ and $\{a_\kappa \otimes 1, f_\kappa^*\}_{\kappa \in K}$ is a projective coordinate system for $A \otimes_B A_A$. Since $A \otimes_B A_A$ is finitely generated, we can find a finite subset K' of K such that $A \otimes_B A = \sum_{\kappa \in K'} (a_\kappa \otimes 1)A$. We consider here the set $K'' = \{\kappa \in K \mid f_\kappa(a_{\kappa'}) \neq 0$ for some $\kappa' \in K'\}$, which is obviously a finite subset of K. If a is an arbitrary element of A then $\{\kappa \in K \mid f_\kappa^*(a \otimes 1) \neq 0\} \subset K''$ and we have

$$ a \otimes 1 = \textstyle\sum_{\kappa \in K} (a_\kappa \otimes 1) f_\kappa^*(a \otimes 1) = \sum_{\kappa \in K''} (a_\kappa \otimes 1) f_\kappa^*(a \otimes 1) $$
$$ = \textstyle\sum_{\kappa \in K''} a_\kappa f_\kappa(a) \otimes 1, $$

which implies $a \in \sum_{\kappa \in K''} a_\kappa A$.

The next is only a combination of [5, Theorem 1. 5] and [5, Theorem 2. 1]. However, Lemmas 1, 2 and the proof of Theorem 1 enable us to obtain a shorter proof.

Theorem 2. *If B is simple, then the following conditions are equivalent*:

(1) *A is simple and A/B is finite inner Galois.*

(2) ${}_AA\otimes_B A_A \cong {}_A(A\oplus\cdots\oplus A)_A$.

(3) *A/B is H-separable.*

Proof. As (1)$\Longrightarrow$(2) is obvious by the proof of Theorem 1 and (2)$\Longrightarrow$(3) is trivial, it remains only to prove (3)$\Longrightarrow$(1). By Lemma 2, A_B is finitely generated projective. Hence, there exist $a_1, \cdots, a_n \in A$ and $f_1, \cdots, f_n \in \mathrm{Hom}\,(A_B, B_B)$ End $A_B = A_L\cdot V_R$ (Lemma 1) such that $a=\sum_i a_i f_i(a)$ for all $a\in A$. If I is an arbitrary non-zero ideal of A then $f_i(I)\subseteq AIV\cap B = I\cap B$, and so $I\subseteq \sum_i Af_i(I)\subseteq A(I\cap B)$, namely, $I=A(I\cap B)=AB=A$. This means that A is simple. Moreover, the simplicity of End $A_B = A_L\cdot V_R\ (\cong A\otimes_C V^\circ)$ yields the simplicity of V and $[V:C]=[A:B]_R$. Hence, A/B is finite inner Galois.

Finally, we shall prove a slight improvement of [3, Theorem 3].

Theorem 3. *Assume that A/B is an algebraic Galois extension of simple rings. If $[V:C]<\infty$ then the following conditions are equivalent*:

(1) *${}_BA_B$ is completely reducible.*

(2) *${}_HA_H$ and ${}_BH_B$ are completely reducible.*

(3) *${}_HH_H <\oplus\, {}_HA_H$ and ${}_BB_B <\oplus\, {}_BH_B$.*

(4) *V/C is separable and ${}_BB_B <\oplus\, {}_BH_B$.*

Proof. The equivalence of (1), (2) and (4) has been proved in [3, Theorem 3]. It suffices therefore to show that if ${}_HH_H <\oplus\, {}_HA_H$ then V/C is separable. To be easily seen, End ${}_HA_H = V_L\cdot V_R$ is canonically V-V-isomorphic to $V\otimes_C V$. Hence, there exists an element $\sum_i v_i\otimes v_i'\in V\otimes_C V$ such that $\sum_i v_iv_i'=1$ and $\sum_i v_iav_i'\in H$ for all $a\in A$. If v is in V then $\sum_i vv_iav_i' = \sum_i v_iav_i'v$, which means that $\sum_i v_i\otimes v_i'\in (V\otimes_C V)^V$. Accordingly, V/C is separable.

As a special case of Theorem 3, we have the following which contains [6, Corollary 2 (2)] :

Corollary 1. *Assume that A/B is a finite inner Galois extension of simple rings. If B' is a simple intermediate ring of A/B, then the* *following conditions are equivalent*:

(1) ${}_{B'}A_{B'}$ *is completely reducible.*

(2) ${}_{B'}B'_{B'} < \oplus {}_{B'}A_{B'}$.

(3) $V_A(B')/C$ *is separable.*

Remarks. (1) Let A be the 2×2-matrix ring over a division ring D; $A = \sum_{i,j=1}^{2} De_{ij}$, and $B = De_{11} \oplus De_{22}$. Then, A/B is a free H-separable extension (but A is not *centrally projective* over B, i. e., ${}_BA_B$ cannot be a direct summand of ${}_B(B \oplus \cdots \oplus B)_B$). In fact, $\{1, t = e_{12} + e_{21}\}$ is a right [left] free B-basis of A and $\{1 \otimes e_{11} + t \otimes e_{12}, 1 \otimes e_{22} + t \otimes e_{21}\}$ is a free A-basis of $A \otimes_B A$ contained in $(A \otimes_B A)^A$. This will show that for a projective H-separable extension A/B the simplicity of A need not imply that of B (cf. Theorem 2).

(2) Let P/Φ be a purely inseparable field extension with $[P:\Phi] = 2$. If we set $A = (\Phi)_2$ and $B = \Phi$, then A/B is finite inner Galois and there exists an intermediate ring B' of A/B which is B-ring isomorphic to P. By Corollary 1, ${}_{B'}A_{B'}$ is not completely reducible, but ${}_BA_B$ is obviously completely reducible.

REFERENCES

[1] K. HIRATA and K. SUGANO: Qn semisimple extensions and separable extensions over non commutative rings, J. Math. Soc. Japan **18** (1966), 360—373.

[2] F. KASCH: Grundlagen einer Theorie der Frobeniuserweiterungen, Math. Ann. **127** (1954), 453—474.

[3] A. KAYA and H. TOMINAGA: Some. remarks on the bimodule structure of Galois extensions, Math. J. Okayama Univ. **15** (1972), 129—135.

[4] T. NAKAMOTO and K. SUGANO: Note on H-separable extensions, Hokkaido Math. J. **4** (1975), 295—299.

[5] K. SUGANO: On some commutor theorems of rings, Hokkaido Math. J. **1** (1972), 242—249.

[6] K. SUGANO: On projective H-separable extensions, Hokkaido Math. J. (to appear).

[7] H. TOMINAGA: A note on H-separable extensions, Proc. Japan Acad. **50** (1974), 446—447.

[8] H. TOMINAGA and T. NAGAHARA: Galois Theory of Simple Rings, Okayama Math. Lectures, 1970.

[9] O. E. VILLAMAYOR and D. ZELINSKY: Galois theory for rings with finitely many idempotents, Nagoya Math. J. **27** (1966), 721—731.

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(*Received April 28, 1975*)