Mathematical Journal of Okayama University

Volume 18, *Issue* 1 1975 *Article* 11

DECEMBER 1975

Remarks to some Tauberian conditions of mean type

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REMARKS TO SOME TAUBERIAN CONDITIONS OF MEAN TYPE[0]

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1. We shall consider in this paper a complex series

$$\sum_{n=1}^{\infty} a_n . \tag{1}$$

If we know that it is Abel or Cesàro summable to a finite sum A, and if in addition

$$a_n = o\left(\frac{1}{n}\right) \qquad (n \to \infty) \tag{2}$$

holds, then we can assert that (1) is necessarily convergent to the same A. This simple but remarkable theorem was first obtained by A. Tauber [17], who also showed that (2) is replaced by its mean type condition

$$\sum_{k=1}^{n} k\, a_k = o(n) . \tag{3}$$

It was Hardy and Littlewood [4] who extensively studied with particular interest such Tauberian theorems for various summability methods and applied them to many problems. Littewood [9] proved that (2), when a_n is real, is improved to

$$a_n > -\frac{K}{n} \qquad (K \text{ some real constant}), \tag{4}$$

which is best possible in this direction.

Since then Tauberian theorems have brought many important results into various branches of mathematics, and in particular N. Wiener [19] was led to his general Tauberian theorems. On the other hand, Tauberian theorems of (discrete) mean type, that is, those with Tauberian condions of mean type like (3) instead of (2) or of (4), seem to have been less noticed almost until the appearance of a series of research papers by D. Gaier [2], W. Meyer-König and H. Tietz [12, 13, 14]. In this paper we shall give new proofs of some Tauberian theorems of mean type and also some counter-examples to prove such Tauberian theorems

0) The content of this paper have been partly reported in [7].

being best possible of their kind.

2. It is known [4] that (3) is necessary and sufficient for (1) to be convergent to a finite sum A if (1) is summable Abel or Cesàro to A, so there is nothing to do in this form. However, it will be observed that we may substitute the last 'half sum' of the left of (3) for the original 'full sum' in either case. In fact we prove

Theorem 1. *If* (1) *is summable Abel or* (C, k) *for some* $k>-1$ *to a finite sum A, it is convergent to A if and only if* $a_n\to 0$ *and*

$$\sum_{\frac{n}{2}<k\leq n} k\,a_k = o(n). \tag{5}$$

Proof. Necessity part is evident because (3) is necessary and

$$\sum_{\frac{n}{2}<k\leq n} k\,a_k = \sum_{1\leq k\leq n} k\,a_k - \sum_{1\leq k\leq \frac{n}{2}} k\,a_k.$$

To prove that (5) is sufficient, set

$$A(x) = \sum_{1\leq k\leq x} k\,a_k$$

for positive real numbers x. Then

$$A(x) - A\left(\frac{x}{2}\right) = \sum_{\frac{x}{2}<k\leq x} k\,a_k.$$

Observing that $a_n\to 0$, it follows that (5) is equivalent to

$$A(x) - A\left(\frac{x}{2}\right) = o(x) \qquad (x\to\infty). \tag{6}$$

Let us here substitute $\frac{x}{2^l}$ for x, where $l=0, 1, 2, \cdots, m-1$, and $m=[\log_2\sqrt{x}]$, $x\geq x_0$. Then on adding, we obtain

$$A(x) - A\left(\frac{x}{2^m}\right) = o\left(x\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)\right) = o(x). \tag{7}$$

But we have from $a_n = o(1)$,

$$A(x) = o(x^2),$$

so we find from (7) that

$$A(x) = o\left(\frac{x^2}{2^{2m}}\right) + o(x) = o\left(\frac{x^2}{2^{\log x}}\right) + o(x) = o(x).$$

This is (3), and our result follows.

3. Next we shall consider the Tauberian theorems of mean type for Borel summability method. Hardy and Littlewood [4] showed that

$$a_n = o\left(\frac{1}{\sqrt{n}}\right)^{1)} \tag{8}$$

is a Tauberian condition, while D. Gaier [2] proved that its mean type condition

$$\sum_{k=1}^{n} \sqrt{k}\, a_k = o(n) \tag{9}$$

is not sufficient to ensure the convergence of (1) when it is summable Borel, that is, that (9) is not the Tauberian condition of mean type for Borel summability method. His proof depends on the theorem of C. N. Moore [14] concerning the Borel summability of Fourier series. We prove this by another method which leads us to the following theorem.

Theorem 2. *If* (1) *is summable Borel and in addition*

$$a_n > -K, \qquad (a_n \text{ real, } K \text{ a real constant}) \tag{10}$$

then

$$\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} \tag{11}$$

is convergent, and so (9) *holds accordingly. Moreover, we cannot replace the convergence of* (11) *by that of*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{\sqrt{n}} \tag{12}$$

or

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{\frac{1}{2}-\varepsilon}} \qquad (\forall \varepsilon > 0). \tag{13}$$

Corollary 1. (9) *or, in fact, even the convergence of* (11) *is not sufficient to ensure the convergence of* (1) *when it is summable Borel.*

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1) They could so far prove that $a_n = O\left(\frac{1}{\sqrt{n}}\right)$ is a Tauberian condition for Borel summability method.

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Proof of Theorem 2. We need the following theorem of R. D. Lord [10][2].

Theorem A. *If* (1) *is summable Borel and*

$$a_n > -K n^{p-1}$$

for some constants $K>0$ *and* $p \geq \frac{1}{2}$, *then* (1) *is summable* (C, $2p-1$) *to the same sum.*

We know from this theorem that (1) is summable (C, 1), and so we obtain from the theorem below due to L. J. Mordell [15],

$$S_n = \sum_{k=1}^{n} a_k = o(\sqrt{n}). \tag{14}$$

Theorem B. *If* (1) *is summable* (C, 1) *to* 0 [3] *and*

$$a_n > -Kn^{-c}$$

for some constants $K>0$ *and* $c \in (-1, 1]$, *then*

$$S_n = \sum_{k=1}^{n} a_k = o(n^{\frac{1-c}{2}}).$$

Hence, by partial summation,

$$\sum_{k=1}^{n} \sqrt{k}\, a_k = \sum_{k=1}^{n-1} (\sqrt{k} - \sqrt{k+1})\, S_k + \sqrt{n}\; S_n = o\left(\sum^{n} \frac{1}{\sqrt{k}} \sqrt{k}\right) + o(n)$$
$$= o(n) + o(n) = o(n),$$

which proves (9). For the proof of the convergence of (11), we further need a simple lemma.

Lemma 1. (cf. [1] p. 454) *If* (1) *is summable* (C, 1) *and*

$$S_n = \sum_{k=1}^{n} a_k = o\left(\frac{1}{\lambda_n}\right), \qquad (n \to \infty)$$

where λ_n *is a convex null-sequence, then*

$$\sum_{n=1}^{\infty} \lambda_n a_n$$

is convergent.

2) He obtained a much more general theorem ([10] Theorem 5) than Theorem A.

3) This is merely a formal condition so that we can obtain a single form of the estimate for S_n (consider the case $c = 1$).

Now we take $\lambda_n = n^{-\frac{1}{2}}$ in this lemma, so that we can assert from (14) that (11) is convergent [4]. To prove the rest of our theorem let us consider e. g.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \text{Re } n^{-s} \exp(ixn^{\alpha}), \tag{15}$$

where $s = \sigma + it$, $x > 0$ and $0 < \alpha < 1$. Then it is known that (15) is summable Borel for all complex s provided $\frac{1}{2} < \alpha < 1$ (cf. [3] p. 411 and [4] p. 225), while

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{\frac{1}{2}-\varepsilon}} = \sum_{n=1}^{\infty} \text{Re } n^{-(s+\frac{1}{2}-\varepsilon)} \exp(ixn^{\alpha}) \tag{16}$$

is divergent when the condition

$$\alpha + (\sigma + \frac{1}{2} - \varepsilon) \leq 1 \tag{17}$$

is satisfied (cf. [4] p. 141). Hence, if we take $\sigma = 0$ and $\alpha = \frac{1}{2} + \delta$ $(0 < \delta \leq \varepsilon)$, then (17) holds good and (16) diverges. This means that for any $\varepsilon > 0$ there exists a divergent series (15) which is summable Borel, when (16) is divergent. That the convergence of (12) is not necessary at all is trivially explained by the example

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}},$$

which is a convergent series but

$$\sum_{n=1}^{\infty} \frac{|a_n|}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty .$$

4. We have seen in the preceding section that (9) is not a Tauberian condition for Borel summability method. Thus we naturally ask for what $f(n)$

$$\sum_{k=1}^{n} \sqrt{k}\, a_k = o\,(f(n)), \qquad (n \to \infty)$$

becomes a Tauberian condition for Borel summability method.

According to a general theorem of Meyer-König and Tietz [12, 13],

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4) (9) will follow from this fact also by Lemma 3.

we can take $f(n) = \sqrt{n}$, so that

$$\sum_{k=1}^{n} \sqrt{k}\, a_k = o(\sqrt{n}) \tag{18}$$

is a Tauberian condition of mean type for Borel summability method.

Furthermore it is shown by a general theorem of H. Tietz [18] that (18) is best possible in the sense that

$$\sum_{k=1}^{n} \sqrt{k}\, a_k = O(\sqrt{n}) \tag{19}$$

is not a Tauberian condition [5]. Firstly, we shall give a simple proof of the following special case of the theorem due to Meyer-König and Tietz.

Theorem 3. *If* (1) *is summable Borel to a finite sum A and* (18) *holds, then* (1) *is necessarily convergent to A.*

Proof. We have by partial summation

$$\sum_{k=1}^{n} k\, a_k = \sum_{k=1}^{n} \sqrt{k}\, (\sqrt{k}\, a_k) = \sum_{k=1}^{n-1} (\sqrt{k} - \sqrt{k+1})\, T_k + \sqrt{n}\, T_n,$$

with

$$T_n = \sum_{k=1}^{n} \sqrt{k}\, a_k .$$

Hence we obtain from (18),

$$\sum_{k=1}^{n} k a_k = o\Big(\sum^{n} \frac{1}{\sqrt{k}} \sqrt{k}\Big) + o(n) = o(n) + o(n) = o(n). \tag{20}$$

On noticing that (20) implies $a_n = o(1)$, we find that (1) is summable (C, 1) to A by the same reasoning as in the proof of Theorem 2. Thus we conclude that (18) is the Tauberian condition of mean type for (C, 1) method. Next we shall show that (19) is not a Tauberian condition. More precisely, we can prove the theorem below.

Theorem 4. *There exists a series* (1) *such that*

(a) *it is not convergent but is summable Borel,*

(b) *there holds the relation* (19) [6], *and*

5) The present writer owes this point to Prof. W. Meyer-König who informed him the work of H. Tietz.

6) It is clear that (19) follows from the boundedness of $\sum_{k=1}^{n} a_k$.

(c) $$a_n = O(n^{-\frac{1}{2}+\varepsilon}) \qquad \textit{for any } \varepsilon > 0.$$

Proof. We shall show that

$$a_n = \sin(x\sqrt{n}\log n) - \sin(x\sqrt{n+1}\log(n+1)), \qquad (x > 0)$$

is such an example. It is obvious in this case that (1) is not convergent and

$$a_n = O(x|\Delta(\sqrt{n}\log n)|)^{7)} = O\left(x\frac{\log n}{\sqrt{n}}\right) = O(n^{-\frac{1}{2}+\varepsilon}).$$

So, it needs only to show that (1) is summable Borel. To do this, we shall apply the following theorem due to G. H. Hardy ([4] p. 213 and [3]: See also [6] which is closely related to Theorem 4.).

Theorem C. *If* σ_n, *the* n_{th} $(C, 1)$ *mean of the partial sum of* (1), *satisfies*

$$\sigma_n = A + o(n^{-\frac{1}{2}}), \qquad (n \to \infty)$$

then (1) *is summable Borel to* A.

Thus it will suffice to prove that

$$T_n = \sum_{k=1}^{n} \sin(x\sqrt{k}\log k) = o(\sqrt{n}). \tag{21}$$

We estimate T_n by the following well-known lemma of van der Corput (cf. [20] p. 198).

Lemma 2. *Let* $f(t)$ *be defined in* $(a, b]$ *and* $f'(t)$ *be monotone there. Then*

(a) *if* $f'(t) \geq m > 0$ *(or* $f'(t) \leq -m < 0$*) in* (a, b),

$$\left|\int_a^b \exp(if(t))\,dt\right| \leq \frac{4}{m},$$

(b) *if* $|f'(t)| < 1$ *in* (a, b),

$$\sum_{a<n\leq b} \exp(2\pi\, if(n)) = \int_a^b \exp(2\pi\, if(t))\,dt + O(1),$$

where the constant implied by the O *is absolute.*

We obtain, applying (b) of this lemma,

7) $\Delta c_n = c_n - c_{n+1}$.

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$$T_n = \int_1^n \sin(x\sqrt{t}\ \log t)\ dt + O(1),$$

provided $0 < x < 2\pi$. Also from (a) it follows that

$$\int_1^n \sin(x\sqrt{t}\ \log t)\ dt = O\left(\frac{1}{x}\frac{\sqrt{n}}{\log n}\right).$$

Hence we get

$$T_n = O\left(\frac{\sqrt{n}}{\log n}\right), \qquad (n \to \infty)$$

which implies (21), whence our result.

It will be noteworthy that our method of the proof can substantially show that condition (c) of Theorem 4 is replaced by

(d) $$a_n = O(n^{-\frac{1}{2}} f(n)), \qquad (n \to \infty)$$

where $f(n)$ is a positive function which diverges to infinity arbitrarily slowly in a monotonic way. This shows in its turn the best possibility of the Tauberian condition of Hardy and Littlewood previously mentioned in section 3 in the footnote.

5. Finally we shall consider Tauberian theorems for the logarithmic summability method which was first introduced by M. Riesz in his investigation of the convergence domain of $1/\zeta(s)$, $\zeta(s)$ being the Riemann zeta-function. He defined, as a matter of fact, more general notion of the means which is now called "Riesz typical means". It has already been obtained by him that a Tauberian condition for the logarithmic mean is

(22) $$a_n = O\left(\frac{1}{n \log n}\right).$$

It is known [5, 8, 16] that there is a certain summability method called (L) summability which includes the logarithmic summability :

We say that (1) is summable (L) if a finite limit

$$\lim_{x \to 1^-} \frac{-1}{\log(1-x)} \sum_{n=0}^{\infty} S_n \frac{x^{n+1}}{n+1} = A$$

exists, where S_n is the nth partial sum of (1) [8].

8) This notion of definition can already be found in [4] p. 81.

There are also some Tauberian conditions of mean type for (L) summability ;

$$\sum_{k=1}^{n} a_k\, k \log k = O(n), \tag{23}$$

$$\sum_{k=1}^{n} k\, a_k = O\left(\frac{n}{\log n,}\right), \tag{24}$$

$$\sum_{k=1}^{n} a_k \log k = o(\log n). \tag{25}$$

Of these, it is easily seen that (23) and (24) are equivalent and the o in (25) cannot be replaced by O. It should be remarked that we have the O-estimate in (23), which is impossible in all the cases of Abel, Cesàro and Borel summabilities. We shall first prove a theorem below similar to Theorem 4.

Theorem 5. *There exists a series* (1) *such that*

(a) *it is not convergent but is summable* (L),

(b) *relation* (23) *holds true, and*

(c) $a_n = O(\text{loglog}\, n / n \log n), \qquad (n \to \infty)$.

Proof. We shall show that

$a_1 = 0,$

$a_n = \sin(x(\text{loglog}\, n)^2) - \sin(x(\text{loglog}(n+1))^2), \qquad (n \geq 2,\ x > 0)$

is such an example. As before, we know that it will suffice to show its logarithmic summability, i. e.

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=2}^{n} \frac{\sin(x(\text{loglog}\, k)^2)}{k} = A. \tag{26}$$

We find from Lemma 2 (b),

$$\sum_{k=2}^{n} \frac{\sin(x(\text{loglog}\, k)^2)}{k} = \int_2^n \frac{\sin(x(\text{loglog}\, t)^2)}{t}\, dt + O(1),$$

when $0 < x < \frac{\pi}{2}$. On the other hand it follows from Lemma 2 (a) that

$$\int_2^n \frac{\sin(x(\text{loglog}\, t)^2)}{t}\, dt = O\left(\frac{\log n}{\text{loglog}\, n}\right), \qquad (n \geq n_0).$$

Hence we can assert that $A = 0$ in (26), which proves our theorem.

We may prove in like manner, as in Theorem 4, that condition (c) of Theorem 5 can be replaced by

(d) $$a_n = O\left(\frac{f(n)}{n \log n}\right), \qquad (n \to \infty)$$

where $f(n) > 0$ is a similar function to that of (d) in Theorem 4, which shows on the one hand that (22) is best possible in this sense.

As an interesting application of the Tauberian theorem for the logarithmic summability, we shall prove the following theorem which furnishes everywhere non-summable (C, k) for any $k > 0$ trigonometric series with coefficients tending to zero [9].

Theorem 6. *The exponential series*

(27) $$\sum_{n=2}^{\infty} \frac{\exp(ix\lambda_n)}{n \log n}$$

is not summable (L) *for any* $x > 0$, *so in particular is divergent for all* $x > 0$ *provided*

(28) $$\Delta\lambda_n = O\left(\frac{1}{n \log n}\right).$$

Proof. Suppose on the contrary that (27), with (28), is summable (L). Then it is convergent according to (22). Hence, from a known lemma below we have

(29) $$\sum_{n=2}^{N} \frac{\exp(ix\lambda_n)}{n} = o(\log N), \qquad (N \to \infty).$$

Lemma 3 (cf. [4] p. 73). *If* (1) *is convergent, then for any positive* $c_n \uparrow \infty$

$$\sum_{n=1}^{N} a_n c_n = o(c_N), \qquad (N \to \infty).$$

Relation (29) implies that $\exp(ix\lambda_n)$ is logarithmically summable to 0. On the other hand from (28),

$$\Delta(\exp(ix\lambda_n)) = O(x|\Delta\lambda_n|) = O\left(\frac{x}{n \log n}\right), \qquad (n \to \infty)$$

which implies by virtue of (22) together with (29) that

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9) It is known that if a series summable (C, k) for some $k \geq 0$ then it is summable both by Abel's method and by (L) method.

$$\lim_{n\to\infty} \exp(ix\,\lambda_n) = 0.$$

This absurdity proves the truth of our theorem.

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(Received July 9, 1975)