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ON TORSION FREE MODULES OVER REGULAR RINGS III

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Throughout this paper, all rings are commutative rings with identity and all modules are unital. Let R be a ring. An R-module A is said to be torsion free if $\{x \in A \mid \mathrm{Hom}_R(Rx, I(R)) = 0\} = 0$, where $I(R)$ is the injective hull of R as an R-module.

This paper is a continuous work of [5] and [6]. For a given regular ring R, in § 2, we construct the Baer hull of R, in the sense of Mewborn [3], by a sheaf theoretic method.

In § 3, we give several characterizations of those regular rings whose corresponding Boolean spaces contain no n-points.

Using results of §§ 2, 3, in the final § 4, we study the converse of the following Pierce's theorem: If R is a regular ring and every finitely generated torsion free R-module is a direct sum of cyclic R-modules, then its corresponding Boolean space contains no 3-points ([7]). Our main theorem of this paper gives some equivalent conditions that, for a regular ring R, every finitely generated torsion free R-module to be expressed as a direct sum of cyclic R-modules.

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1. Preliminaries. For a ringed space (X, R) and a sheaf A of R-modules over X, we denote the ring of all global sections of R over X by $\Gamma(X, R)$, and the $\Gamma(X, R)$-module of all global sections of A over X by $\Gamma(X, A)$.

Let R be a ring. $B(R)$ will represent the Boolean ring consisting of all idempotents of R, and $X(R)$ the spectrum of $B(R)$ consisting of all prime ideals of $B(R)$. Let x be a point of $X(R)$. For e in x, we call $U_e^{B(R)} = \{y \in X(R) \mid e \in y\}$ a neighborhood of x. As is well known, these neighborhoods form a basis of open subsets in $X(R)$, and with this topology, $X(R)$ becomes a Boolean space, i. e., a compact, Hausdorff totally disconnected space.

Let A be an R-module and $x \in X(R)$. We denote the factor module A/Ax by A_x where $Ax = \{ae \mid a \in A,\ e \in x\}$ and the image of $a\,(\in A)$ under the natural homomorphism $A \longrightarrow A_x$ by a_x.

For a given ring R, we recall that Pierce sheaf $\boldsymbol{R}(R)$ of rings over $X(R)$ is one whose stalks are R_x for $x \in X(R)$, and that, for an R-module A, the sheaf $\boldsymbol{A}(A)$ of $\boldsymbol{R}(R)$-modules over $X(R)$ is one whose stalks are A_x for $x \in X(R)$. If $r \in R$ $(a \in A)$, σ_r (σ_a) denotes the section of $\boldsymbol{R}(R)$ $(\boldsymbol{A}(A))$ over $X(R)$ given by $x \longrightarrow r_x$ $(x \longrightarrow a_x)$. Then the mapping $\Phi : r \longrightarrow \sigma_r$ is a ring isomorphism of R onto $\Gamma(X(R), \boldsymbol{R}(R))$, and the mapping $\psi : a \longrightarrow \sigma_a$ is a group isomorphism of A onto $\Gamma(X(R), \boldsymbol{A}(A))$ satisfying $\psi(ar) = \psi(a)\Phi(r)$ for any $r \in R$ and $a \in A$. One may remark here, for x in $X(R)$, its corresponding prime ideal, say $m(X)$, of $B(\Gamma(X(R), \boldsymbol{R}(R)))$ is the set $\{\sigma \in B(\Gamma(X(R), \boldsymbol{R}(R))) \mid \sigma(x) = 0_x\}$ (see [7, Lemma 5.2]).

Let (X, R) be a ringed space and M an open-closed subset of X. σ_M denotes the section of R over X given by

$$\sigma_M(x) = \begin{cases} 1_x & (x \in M) \\ 0_x & (x \notin M) \end{cases}$$

where 0_x are 1_x the zero element and the identity of the stalk for x respectively. Note that if X is a Boolean space and R a reduced sheaf of rings over X, then

$$\begin{aligned} &B(\Gamma(X, R)) \\ &= \{\sigma \in \Gamma(X, R) \mid \sigma(x) = 0_x \text{ or } 1_x \text{ for all } x \in X\} \\ &= \{\sigma_M \mid M \text{ is an open-closed subset of } X\}. \end{aligned}$$

(see [7, p. 15]).

2. Baer hulls. Let R be a ring. Let us denote by $Y(R)$ the spectrum of the maximal ring of quotients $Q(B(R))$ of $B(R)$ (note that $Q(B(R))$ is also a Boolean ring (see [2, p. 44])), and by λ the natural (closed) mapping from $Y(R)$ onto $X(R)$ given by $y \longrightarrow y \cap R$. For an element y of $Y(R)$, we denote $R_{\lambda(y)}$ by $\boldsymbol{R}_y$ and define the disjoint union

$$\boldsymbol{R}^*(R) = \bigcup_{y \in Y(R)} \boldsymbol{R}_y.$$

First we show that the pair $(Y(R), \boldsymbol{R}^*(R))$ forms a ringed space. For $r \in R$, $\bar{\sigma}_r$ denotes the mapping from $Y(R)$ into $\boldsymbol{R}^*(R)$ given by $y \longrightarrow \sigma_r(\lambda(y))$. For r and s in R and y in $Y(R)$, assume that $\bar{\sigma}_r(y) = \bar{\sigma}_s(y)$. Then by [7, Lemma 4.3] there exists e in $B(R)$ with $\lambda(y) \in U_e^{B(R)}$ such that $\sigma_r(x) = \sigma_s(x)$ for all $x \in U_e^{B(R)}$. So e is in $Q(B(R))$ such that $y \in U_e^{Q(B(R))}$ and that $\bar{\sigma}_r(z) = \bar{\sigma}_s(z)$ for all $z \in U_e^{Q(B(R))}$ since $\lambda(U_e^{Q(B(R))}) = U_e^{B(R)}$. From this observation, it is easy to verify that $\boldsymbol{R}^*(R)$ becomes a topological space with an open basis consisting of the set $\{\bar{\sigma}_r(U_e^{Q(B(R))}) \mid r \in R,\ e \in Q(B(R))\}$. Furthermore we can verify that $\boldsymbol{R}^*(R)$ becomes a reduced sheaf of rings over $Y(R)$ whose stalks are $\boldsymbol{R}_y = R_{\lambda(y)}$ for $y \in Y(R)$ (cf. [7, Theorem 4.4]).

Concerning the ringed space $(Y(R),\ \boldsymbol{R}^*(R))$, we have

Proposition 2.1. (cf. [6, Proposition 2.3]). *$\Gamma(Y(R),\ \boldsymbol{R}^*(R))$ becomes a ring extension of $\Gamma(X(R),\ \boldsymbol{R}(R))$ with the same identity, and moreover coincides with the ring generated by the set of all idempotents of $\Gamma(Y(R),\ \boldsymbol{R}^*(R))$ over $\Gamma(X(R),\ \boldsymbol{R}(R))$.*

Proof. Let $\sigma \in \Gamma(X(R),\ \boldsymbol{R}(R))$ and $y \in Y(R)$. Then there is r_y in R such that $\sigma(\lambda(y)) = r^y_{\lambda(y)}$. We claim that the mapping σ^* given by $y \longrightarrow r^y_{\lambda(y)}$ is a section of $\boldsymbol{R}^*(R)$ over $Y(R)$. To see this, let $e \in Q(B(R))$ with $y \in U_e^{Q(B(R))}$. Since $\sigma(\lambda(y)) = r^y_{\lambda(y)} \in r^y_{\lambda(U_e^{Q(B(R))})} \subseteq r^y{}_{X(R)}$ (for s in R and a subset V of $X(R)$, s_V denots the set $\{s_x \mid x \in V\}$), there exists a neighborhood U of y such that $\sigma\lambda(U) \subseteq r^y_{X(R)}$. Pick an open subset U' in $Y(R)$ such that $y \in U' \subseteq U \cap U_e^{Q(B(R))}$. Then $\sigma\lambda(z) = r^z_{\lambda(z)} = r^y_{\lambda(z)} \in r^y_{\lambda(U_e^{Q(B(R))})}$ for all $z \in U'$. It follows that σ^* becomes a section of $\boldsymbol{R}^*(R)$. Identifying σ with σ^*, $\Gamma(X(R),\ \boldsymbol{R}(R))$ can be regarded as a subring of $\Gamma(Y(R),\ \boldsymbol{R}^*(R))$.

Let $\sigma \in \Gamma(Y(R),\ \boldsymbol{R}^*(R))$ and $y \in Y(R)$. Then,

$$\sigma(y) = \bar{\sigma}_r(y) = \sigma_r^*(y)$$

for some r in R. Hence, by [7, Lemma 3.2], there is a neighborhood M of y in $Y(R)$ such that $\sigma(z) = \sigma_r^*(z)$ for all $z \in M$. Furthermore, by making use of the partition property (see [7, p. 12]), we obtain a finite family $\{M_1, \cdots, M_s\}$ of open-closed subsets of $Y(R)$ and a finite subset $\{r_1, \cdots, r_s\}$ of R such that

$$Y(R) = \bigcup_{i=1} M_i,$$
$$M_j \cap M_k = \phi \text{ if } j \neq k, \text{ and}$$
$$\sigma(y) = \sigma_{r_i}^*(y) \text{ for all } y \in M_i,\quad i = 1,\ 2, \cdots,\ s.$$

Then $\sigma = \sigma_{r_1}^* \sigma_{M_1} + \cdots + \sigma_{r_s}^* \sigma_{M_s}$, and $\{\sigma_{M_1}, \cdots, \sigma_{M_s}\}$ is a set of (orthogonal) idempotents of $\Gamma(Y(R),\ \boldsymbol{R}^*(R))$.

Proposition 2.2. *For a regular ring R, $\Gamma(Y(R),\ \boldsymbol{R}^*(R))$ is a ring of quotients of $\Gamma(X(R),\ \boldsymbol{R}(R))$.*

Proof. Let $\sigma \in \Gamma(Y(R),\ \boldsymbol{R}^*(R))$. Then, by the proof of the preceding proposition, σ is expressed in the form

$$\sigma = \sigma_1^* \sigma_{M_1} + \cdots + \sigma_s^* \sigma_{M_s}$$

for some $\sigma_1, \cdots, \sigma_s$ in $\Gamma(X(R), \boldsymbol{R}(R))$ and pairwise disjoint open-closed subsets $M_1, \cdots, M_s$ of $Y(R)$. Now suppose that $\sigma \neq 0$. Then $\sigma_i \sigma_{M_i} \neq 0$ for some i. Since $\Gamma(X(R), \boldsymbol{R}(R))$ is a regular ring, there exists σ' in $\Gamma(X(R), \boldsymbol{R}(R))$ such that $\sigma_i \sigma' \sigma_i = \sigma_i$. Since $\sigma' \sigma_i$ is idempotent, $\sigma' \sigma_i = \sigma_W$ for some open-closed subset W of $X(R)$. Then $0 \neq \sigma_W^* \sigma_{M_i} = \sigma_{\lambda^{-1}(W)} \sigma_{M_i} = \sigma_{\lambda^{-1}(W) \cap M_i}$, and it follows that $\lambda^{-1}(W) \cap M_i \neq \emptyset$. Hence, by [6, Lemma 1.2], there exists $e(\neq 1)$ in $B(R)$ such that $U_e^{Q(B(R))} \subseteq \lambda^{-1}(W) \cap M_i$. Here $0 \neq \sigma_{U_e^{Q(B(R))}} = \sigma_W^* \sigma_{M_i} \sigma_{U_e^{B(R)}}^*$ and hence $0 \neq \sigma \sigma_{U_e^{B(R)}}^* \in \Gamma(X(R), \boldsymbol{R}(R))$. This shows that $\Gamma(Y(R), \boldsymbol{R}^*(R))$ is an essential extension of $\Gamma(X(R), \boldsymbol{R}(R))$ as a $\Gamma(X(R), \boldsymbol{R}(R))$-module.

Corollary 2.3. *If R is a regular ring, then $\Gamma(Y(R), \boldsymbol{R}^*(R))$ is the Baer hull of $\Gamma(X(R), \boldsymbol{R}(R))$ in the sense of Mewborn* [3].

Proof. By Propositions 2.1, 2.2 and [3, Proposition 2.5], we may only show that $B(\Gamma(Y(R), \boldsymbol{R}^*(R)))$ is the maximal ring of quotients of $B(\Gamma(X(R), \boldsymbol{R}(R)))$. At any rate, it is easily seen from Proposition 2.2 that $B(\Gamma(Y(R), \boldsymbol{R}^*(R)))$ is a ring of quotients of $B(\Gamma(X(R),$ of $\boldsymbol{R}(R)))$. Since the ringed space $(Y(R), \boldsymbol{R}^*(R))$ is reduced, the spectrum $B(\Gamma(Y(R), \boldsymbol{R}^*(R)))$ is homeomorphic to the extremely disconnected space $Y(R)$ (see [7, Lemma 5.2]). As is well known (e.g. [7, Proposition 24.1]) a Boolean ring is self-injective if and only if its spectrum is an extremely disconnected space, so we conclude that $B(\Gamma(Y(R), \boldsymbol{R}^*(R)))$ is self-injevtive.

Henceforce for the sake of simplicity we denote, for a given ring R, $\Gamma(Y(R), \boldsymbol{R}^*(R))$ by $C(R)$. We identify R with $\Gamma(X(R), \boldsymbol{R}(R))$ by the isomorphism $r \longrightarrow \sigma_r$, and also identify an R-module A with $\Gamma(X(R), \boldsymbol{A}(A))$ by the isomorphism $a \longrightarrow \sigma_a$.

Remark. When we see $C(R)$ to be an R-module by identifying R with $\Gamma(X(R), \boldsymbol{R}(R))$, we ought to note the following notations:

(1) Let r in R, and M an open-closed subset of $Y(R)$. Then $r\sigma_M$ means $\sigma_r^* \sigma_M$.

(2) For a point x in $X(R)$, $C(R)_x$ means $C(R)_{m(x)}$, where $m(x) = \{\sigma \in B(\Gamma(X(R), \boldsymbol{R}(R))) \mid \sigma(x) = 0_x\}$.

Lemma 2.4. *Let R be a ring, M an open-closed subset of $Y(R)$ and*

$x \in X(R)$. *Then* $(\sigma_M)_x \neq 0_x$ if and only if $\lambda^{-1}(x) \cap M \neq \emptyset$.

Proof. *Let* $0_x \neq (\sigma_M)_x = (\sigma_M)_{m(x)}$ and suppose that $\lambda^{-1}(x) \cap M = \emptyset$. Then there exists an open-closed subset $U \subseteq X(R)$ with $x \in U$ such that $U \cap \lambda(M) = \emptyset$. Obviously $\sigma_M \sigma_U^* = 0$. Since $(\sigma_M)_x \neq 0_x$, it follows that $(\sigma_U^*)_x = 0_x$. This implies that $x \notin U$, a contradiction.

Conversely, let $\lambda^{-1}(x) \cap M \neq \emptyset$. If $(\sigma_M)_x = 0_x$, then $\sigma_M(1-\sigma_U^*) = 0$ for some open-closed subset $U \subseteq X(R)$ such that $\sigma_U \in m(x)$. Take $y \in M$ such that $\lambda(y) = x$. Then $1_{\lambda(y)} = \sigma_M(y) = \sigma_U(\lambda(y)) = 0_{\lambda(y)}$, a contradiction.

3. *n*-point. Let X be a topological space, x in X and ξ an arbitrary ordinal number. Following Pierce [7], x is called a ξ-point if there is a collection $\{U_\eta \mid \eta < \xi\}$ of pairwise disjoint open subsets of X such that $x \in U_\eta^- - U_\eta$ for all $\eta < \xi$. (U_η^- denotes the closure of U_η in X).

Remark. For a natural number n, a point x in a topological space X is an n-point if and only if there exist n pairwise disjoint open sets $U_1, \cdots, U_n$ of X such that each $U_i^- - U_i$ contains x.

Lemma 3.1. *Let R be a ring and x a non-isolated point in $X(R)$. Then, x is an n-point if and only if $n \leq |\lambda^{-1}(x)|$, the number of elements of $\lambda^{-1}(x)$.*

Proof. See [6, Proposition 3. 2].

It is well known that a countably infinite Boolean space is not an extremely disconnected space (e. g. [1, p. 98]). Hence from the above lemma we have immediately

Corollary 3.2. (cf. [5, Example A]). *A countably infinite Boolean space contains an n-point* ($n=1, 2, \cdots$).

Lemma 3.3. *Let R be a regular ring, $C(R) = Q(R)$ and $x \in X(R)$. Then $n \leq |\lambda^{-1}(x)|$ if and only if $n \leq [Q(R)_x : R_x]$, the rank of $Q(R)_x$ over R_x.* ($Q(R)$ denotes the maximal ring of quotients of R).

Proof. Let $x \in X(R)$ and $n \leq |\lambda^{-1}(x)|$. Then we can choose pairwise disjoint open-closed subsets $M_1, \cdots, M_n$ of $Y(R)$ such that $\lambda^{-1}(x) \cap M_i \neq \emptyset$ ($i = 1, 2, \cdots, n$). Since each $(\sigma_{M_i})_x \neq 0_x$ by Lemma 2. 4, it follows that $\{(\sigma_{M_1})_x, \cdots, (\sigma_{M_n})_x\}$ is a linearly independent subset of $Q(R)_x$. Hence we have $n \leq [Q(R)_x : R_x]$.

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Conversely, suppose that $n \leq [Q(R)_x : R_x]$. Then there are $q_1, \cdots, q_n$ in $Q(R)$ such that $\{(q_1)_x, \cdots, (q_n)_x\}$ is a linearly independent set. Since $C(R) = Q(R)$, there exist pairwise disjoint open-closed subsets $M_1, \cdots, M_s$ of $Y(R)$ such that $Rq_1 + \cdots + Rq_n \subseteq R\sigma_{M_1} + \cdots + R\sigma_{M_s}$ ([5, Lemma 3.2]). Clearly the number of non-zero elements of $\{(\sigma_{M_1})_x, \cdots, (\sigma_{M_s})_x\}$ is greater than n. Thus by Lemma 2.4 we have $n \leq |\lambda^{-1}(x)|$.

Remark. Let R be a ring and x an isolated point of $X(R)$. Since Rx is a direct summand of R ([4, Proposition 2.2]), it is easily seen that $|\lambda^{-1}(x)| = 1$, and hence when R is regular, $[Q(R)_x : R_x] = 1$.

From Lemmas 3.1, 3.3, the remark above and [7, Corollary 13.8], we have

Theorem 3.4. *Let R be a regular ring with $C(R) = Q(R)$. Then the following conditions are equivalent*:

(a) $|\lambda^{-1}(x)| \leq n$ *for all* x *in* $X(R)$.

(b) $X(R)$ *contains no* $(n+1)$*-points.*

(c) $[Q(R)_x : R_x] \leq n$ *for all* x *in* $X(R)$.

(d) *Every finitely generated R-submodule of* $Q(R)$ *is generated by* n *elements.*

From this theorem and [6, Example], there is a regular ring R such that every finitely generated R-submodule of $Q(R)$ is generated by n elements but not by $n+1$ elements $(n = 1, 2, \cdots)$.

4. Main theorem. Let R be a regular ring and A an R-module. Following Pierce [7, p. 68], for each $x \in X(R)$ and each open set $U \subseteq X(R)$, we define

$$E_A(x, U) = \{a_x \mid a \in A,\ a_y = 0_y \text{ for all } y \in U\}$$

and denote by $L_A(x)$ the sublattice generated by $\{E_A(x, U) \mid U$ is open in $X(R)\}$ of the lattice of all subspace of A_x over R_x

Lemma 4.1. ([7, Corollary 17.5]). *Let R be a regular ring and A a finitely generated R-module. If A is a direct sum of cyclic R-modules, then $L_A(x)$ is distributive for all $x \in X(R)$.*

In general, the converse of this lemma does not hold ([7, Example 17.7]). However, in the case R is the ring of all global sections of the simple F-sheaf over a Boolean space, where F is a finite field, the converse also holds ([7, Theorem 17.6]).

In the following we have

Proposition 4.2. *Let R be a regular ring. Then the following conditions are equivalent:*

(a) *R is a Baer ring.*

(b) *$L_A(x)=\{0_x, A_x\}$ for all finitely generated torsion free R-module A and all $x \in X(R)$.*

(c) *$L_A(x)=\{0_x, A_x\}$ for all finitely generated R-submodules A of $Q(R)$ and all $x \in X(R)$.*

Proof. Note that a regular ring R is a Baer ring if and only if $Y(R) \overset{\lambda}{\simeq} X(R)$ ([5]).

(a)$\Longrightarrow$(b) Let A be a finitely generated torsion free R-module. Then A can be embedded in a direct product of finite copies of $Q(R)$ ([8, Corollary 5]). So assume that $A \subseteq Q(R) \times \cdots \times Q(R)$ (n copies). Now, if U is an open subset of $X(R)$, by [7, Lemma 17.2],

$$E_A(x, U) = \begin{cases} 0_x & (x \in U) \\ A_x & (x \notin U^-). \end{cases}$$

When $x \in U^- - U$, we show that $E_A(x, U) = 0_x$. Let $(a_1)_x \times \cdots \times (a_n)_x \in E_A(x, U) \subseteq (Q(R) \times \cdots \times Q(R))_x = Q(R)_x \times \cdots \times Q(R)_x$. Since $Q(R)$ is a regular ring, $a_i a_i' a_i = a_i$ for some $a_i' \in Q(R)$ $(i=1, 2, \cdots, n)$. Since $a_i a_i'$ is an idempotent, $a_i a_i' = \sigma_{M_i}$ for some open-closed subset M_i of $Y(R) = X(R)$ $(i=1, 2, \cdots, n)$. Clearly $(\sigma_{M_i})_z = 0_z$ for all $z \in U$ $(i = 1, 2, \cdots, n)$ and hence $U \subseteq M_i^c$ and $U^- \subseteq M_i^c$. (M_i^c denotes the complement of M_i in $X(R)$). It follows that $(\sigma_{M_i})_x = 0_x$ and $(a_i)_x = 0_x$ $(i = 1, 2, \cdots, n)$. Thus $(a_1)_x \times \cdots \times (a_n)_x = 0_x$ and $E_A(x, U) = 0_x$.

The implication (b)$\Longrightarrow$(c) is trivial.

(c)$\Longrightarrow$(a) If R is not a Baer ring, then there exist x in $X(R)$ and an open-closed subset M in $Y(R)$ such that $\lambda^{-1}(x) \cap M \neq \emptyset$ and $\lambda^{-1}(x) \cap M^c \neq \emptyset$. Then $(\sigma_M)_x \neq 0_x$ and $(\sigma_{M^c})_x \neq 0_x$ (Lemma 2.4). If we put

$$W = \cup \{ U_e^{B(R)} \mid e \in B(R),\ U_e^{Q(B(R))} \subseteq M,\ x \notin U_e^{B(R)} \},$$

then W is an open subset of $X(R)$ and $\lambda^{-1}(W) \subseteq M$. Moreover it is easily seen from [6, Lemma 1.3] that $x \in W^- - W$. Putting $A = R\sigma_M + R\sigma_{M^c}$, we have $(\sigma_{M^c})_x \in E_A(x, W)$ and $(\sigma_M)_x \notin E_A(x, W)$. This contradicts to the fact that $E_A(x, A) = 0_x$ or A_x.

Combining this proposition with [5, Theorem 3.7], we have

Corollary 4.3. *If R is a regular Baer ring but not a self-injective*

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ring, then there is a finitely generated torsion free R-module A such that $L_A(x)$ is distributive for all x in $X(R)$ and that A is not contained in a direct sum of cyclic torsion free R-modules.

Therefore the converse of Lemma 4. 1 is not true even through A is torsion free.

Now, Pierce has shown that if R is a regular ring and every torsion free R-module with two generators is a direct sum of cyclic R-modules, then $X(R)$ contains no 3-points ([7, Proposition 20. 1]). In the following, however, we have more strictly

Proposition 4.4. *Let R be a regular ring. If $L_A(x)$ is distributive for all $x \in X(R)$ and all R-submodules A of $C(R)$ with two generators, then $X(R)$ contains no 3-points.*

Proof. Let us suppose that $X(R)$ contains a 3-point, say x. Then there exist pairwise disjoint open-closed subsets M_1, M_2, M_3 of $Y(R)$ such that $M_i \cap \lambda^{-1}(x) \neq \emptyset$ $(i = 1, 2, 3)$. Putting

$$W_i = \cup \{U_e^{B(R)} \mid e \in B(\mathrm{R}),\ U_e^{\mathfrak{X}(B(R))} \subseteq M_i,\ x \notin U_e^{B(R)}\},$$

W_i is a non-empty open subset of $X(R)$ such that $\lambda^{-1}(W_i) \subseteq M_i$ and $x \in W_i^- - W_i$ $(i = 1, 2, 3)$ (cf. [6, Lemma 1. 3]).

Here we denote $R(\sigma_{M_1} + \sigma_{M_2}) + R(\sigma_{M_2} + \sigma_{M_3})$ by A. We show that $E_A(x, W_1) = R_x(\sigma_{M_2} + \sigma_{M_3})_x$. Clearly $R_x(\sigma_{M_2} + \sigma_{M_3})_x \subseteq E_A(x, W_1)$. Hence $1 \leq [E_A(x, W_1) : R_x]$. On the other hand, $[E_A(x, W_1) : R_x] \leq [A_x : R_x] = 2$. So, to show that $E_A(x, W_1) = R_x(\sigma_{M_2} + \sigma_{M_3})_x$, it suffices to show that $(\sigma_{M_1} + \sigma_{M_2})_x \notin E_A(x, W_1)$. Suppose that $(\sigma_{M_1} + \sigma_{M_2})_x \in E_A(x, W_1)$. Then $(\sigma_{M_1} + \sigma_{M_2})_x = a_x$ for some a an A with $a_z = 0_z$ for all $z \in W_1$. Since $(\sigma_{M_1} + \sigma_{M_2})_x = a_x$, there exists an open-closed subset V of $X(R)$ with $x \in V$ such that $(\sigma_{M_1} + \sigma_{M_2})_z = a_z$ for all $z \in V$ by [7, Lemma 3. 2]. $V \cap W_1 \neq \emptyset$ since $x \in W_1^-$. Take $z \in V \cap W_1$. Then $0_z \neq (\sigma_{M_1})_z = (\sigma_{M_1} + \sigma_{M_2})_z = a_z$, a contradiction. Thus $E_A(x, W_1) = R_x(\sigma_{M_2} + \sigma_{M_3})_x$. Similarly we can prove that $E_A(x, W_3) = R_x(\sigma_{M_1} + \sigma_{M_2})_x$ and $E_A(x, W_2) = R_x(\sigma_{M_1} - \sigma_{M_3})_x$. But it follows that

$$E_A(x, W_2) \cap (E_A(x, W_1) + E_A(x, W_3)) = \mathrm{E}_A(x, W_2), \quad \text{and}$$
$$(E_A(x, W_2) \cap E_A(x, W_1)) + (E_A(x, W_2) \cap E_A(x, W_3)) = 0_x.$$

This contradicts to the fact that $L_A(x)$ is distributive.

Now let A be a two dimensional vector space over a field R, say

$A = R\alpha + R\beta$. For a positive integer n, consider the external direct sum $B = A \times \cdots \times A$ of n copies of A. Let $a_1, a_2, \cdots, a_n$ be in B and write them as

$$\begin{aligned} a_1 &= (r_{11}\alpha + s_{11}\beta) \times \cdots \times (r_{1n}\alpha + s_{1n}\beta), \\ a_2 &= (r_{21}\alpha + s_{21}\beta) \times \cdots \times (r_{2n}\alpha + s_{2n}\beta), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ a_n &= (r_{n1}\alpha + s_{n1}\beta) \times \cdots \times (r_{nn}\alpha + s_{nn}\beta), \end{aligned}$$

where each r_{ij}, s_{ij} in R. We shall then call

$$\begin{pmatrix} r_{11} \cdots r_{1n} \; s_{11} \cdots s_{1n} \\ r_{21} \cdots r_{2n} \; s_{21} \cdots s_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ r_{n1} \cdots r_{nn} \; s_{n1} \cdots s_{nn} \end{pmatrix}$$

the matrix for $\{a_1, a_2, \cdots, a_n\}$. Under this situation, we shall prove the following

Lemma 4.5. *If $a_1, \cdots, a_n$ are linearly independent over R, then we have*

(1) *there exist $b_1, \cdots, b_n$ in B such that*
$Ra_1 + \cdots + Ra_n = Rb_1 + \cdots + Rb_n$,

(2) *the matrix for $\{b_1, \cdots, b_n\}$ is of the form*

$$\begin{pmatrix} 1 & & & & & & & & t_{11} \cdots t_{1n} \\ 0 & \cdot & & & & * & & & \\ \cdot & & \cdot & & & & & \cdot & \cdot \\ \cdot & & & 1 & & & & & \\ 0 & \cdot & \cdot & 0 & 0 & \cdot & 0 & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot & & \\ \cdot & & & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & 0 & \cdot & 0 & & t_{n1} \cdots t_{nn} \end{pmatrix}$$

with the property that, if there exist i and k such that

$$t_{k1} = t_{k2} = \cdots = t_{k,i-1} = 0 \text{ and } t_{ki} \neq 0,$$

then

$$t_{1i} = t_{2i} = \cdots = t_{k-1,i} = 0.$$

Proof. We may assume that some r_{ij} is not zero.

First, by change of basis for $a_1, \cdots, a_n$ appropriately, we obtain

$b_1, \cdots, b_n$ in B such that

$$Ra_1 + \cdots + Ra_n = Rb_1 + \cdots + Rb_n$$

and for which the matrix is of the following form

$$\begin{pmatrix}
\overbrace{1 & & & }^{p} & & * & & t_{11} & \cdots & t_{1n}\\
0 & \ddots & & & & & & & & \\
\vdots & & \ddots & & & & & \vdots & & \vdots\\
\vdots & & & 1 & & & & & & \\
0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & \vdots & & \vdots\\
\vdots & & & \vdots & \vdots & & \vdots & & & \\
\vdots & & & \vdots & \vdots & & \vdots & \vdots & & \vdots\\
0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & t_{n1} & \cdots & t_{nn}
\end{pmatrix}.$$

When $t_{n1} = \cdots = t_{n,j-1} = 0$ and $t_{nj} \neq 0$ (note that such a j surely exists if $p \neq n$), put

$$b_n^n = b_n,$$
$$b_i^n = b_i - t_{ij}\, t_{nj}^{-1}\, b_n, \quad i = 1, 2, \cdots, n-1,$$

and denote by $((r_{ij}^n)\,(t_{ij}^n))$ the matrix for $\{b_1^n, \cdots, b_n^n\}$.

In the next, if $t_{n-1,1}^n = \cdots = t_{n-1,k-1}^n = 0$ and $t_{n-1,k}^n \neq 0$, we make $\{b_1^{n-1}, \cdots, b_n^{n-1}\}$ by putting

$$b_n^{n-1} = b_n^n,$$
$$b_{n-1}^{n-1} = b_{n-1}^n,$$
$$b_i^{n-1} = b_i^n - t_{ik}^n (t_{n-1,k}^n)^{-1}\, b_{n-1}^n, \quad i = 1, 2, \cdots, n-2,$$

and denote the matrix for $\{b_1^{n-1}, \cdots, b_n^{n-1}\}$ by $((r_{ij}^{n-1})(t_{ij}^{n-1}))$.

We can proceed in this fashion up to $p+1$ and obtain $\{b_1^{p+1}, \cdots, b_n^{p+1}\}$ with the matrix $((r_{ij}^{p+1})(t_{ij}^{p+1}))$. If there exists l such that $t_{p1}^{p+1} = \cdots = t_{p,l-1}^{p+1} = 0$ and $t_{pl}^{p+1} \neq 0$, we make $\{b_1^p, \cdots, b_n^p\}$ by the same argument as above. Otherwise, i. e., in case $t_{p1}^{p+1} = \cdots = t_{pn}^{p+1} = 0$, we simply put

$$b_1^p = b_1^{p+1},$$
$$b_2^p = b_2^{p+1},$$
$$\cdots\cdots$$
$$b_n^p = b_n^{p+1}.$$

Finally we obtain $\{b_1^1, \cdots, b_n^1\}$ with the matrix $((r_{ij}^1)\,(t_{ij}^1))$. These $b_1^1, \cdots, b_n^1$ are required elements in B.

Proposition 4.6. *For a regular ring R with $C(R) = Q(R)$, if $X(R)$*

contains no 3*-point, then every finitely generated torsion free R-module is a direct sum of cyclic R-modules.*

Proof. Let A be a finitely generated torsion free R-module. By [7, Lemmas 14.2, 15.1] and the partition property, to prove that A is a direct sum of cyclic R-modules, it suffices to show that, for any x in $X(R)$, there exist an open-closed subset N of $X(R)$ containing x and a finite subset $\{a_1, \cdots, a_n\}$ of A such that the set of all non-zero elements of $\{(a_1)_z, \cdots, (a_n)_z\}$ is a basis of A_z over R_z for all $z \in N$.

Let $x \in X(R)$. We choose $a_1, \cdots, a_n$ in A such that $\{(a_1)_x, \cdots, (a_n)_x\}$ is a basis of A_x over R_x.

Since $Ra_1 + \cdots + Ra_n$ is a torsion free R-module, we can assume that $Ra_1 + \cdots + Ra_n$ is an R-submodule of $Q(R) \times \cdots \times Q(R)$ (n copies) ([8, Corollary 5]).

Since $|\lambda^{-1}(x)| \leq 2$, there exists an open-closed subset M of $Y(R)$ such that

$$\lambda^{-1}(x) \cap M \neq \emptyset, \quad \lambda^{-1}(x) \cap M^c \neq \emptyset \quad \text{if } x \text{ is a 2-point,}$$

$$\lambda^{-1}(x) \subseteq M \quad \text{if } x \text{ is a 1-point or an isolated point.}$$

If N is an open-closed subset of $Y(R)$, then we can verify using Lemma 2.4 that

$$(\sigma_N)_x = \begin{cases} 0_x & \text{if } \lambda^{-1}(x) \cap N = \emptyset\,, \\ (\sigma_M)_x & \text{if } \lambda^{-1}(x) \cap N = \lambda^{-1}(x) \cap M\,, \\ (\sigma_{M^c})_x & \text{if } \lambda^{-1}(x) \cap N = \lambda^{-1}(x) \cap M^c\,, \\ 1_x & \text{if } \lambda^{-1}(x) \subseteq N\,. \end{cases}$$

From this fact and the condition $C(R) = Q(R)$, it follows that

$$Q(R)_x = R_x(\sigma_M)_x \quad \text{if } x \text{ is a 1-point or an isolated point,}$$

$$Q(R)_x = R_x(\sigma_M)_x + R_x(\sigma_{M^c})_x \quad \text{if } x \text{ is a 2-point.}$$

So, we devide into two cases:

Case 1. x is a 1-point or an isolated point. In this case

$$(a_1)_x = (r_{11})_x(\sigma_M)_x \times \cdots \times (r_{n1})_x(\sigma_M)_x\,,$$

$$(a_2)_x = (r_{21})_x(\sigma_M)_x \times \cdots \times (r_{2n})_x(\sigma_M)_x\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(a_n)_x = (r_{n1})_x(\sigma_M)_x \times \cdots \times (r_{nn})_x(\sigma_M)_x$$

for some $r_{ij} \in R$ ($i, j = 1, 2, \cdots, n$). Putting

$$b_1 = \sigma_M \times 0 \times \cdots \times 0\,,$$

$$b_2 = 0 \times \sigma_M \times 0 \cdots \times 0\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$b_n = 0 \times \cdots \times 0 \times \sigma_M ,$$

we have

$$(Ra_1 + \cdots + Ra_n)_x = (Rb_1 + \cdots + Rb_n)_x .$$

Take an open-closed subset N of $X(R)$ containing x and $c_1, \cdots, c_n$ in A such that

$$(Rb_i)_z = (Rc_i)_z$$

for all i and all z in N. Then clearly non-zero elements of $\{(c_1)_z, \cdots, (c_n)_z\}$ are linearly independent over R_z, and

$$(Ra_1 + \cdots + Ra_n)_z = (Rc_1 + \cdots + Rc_n)_z$$

for all z in N.

Case 2. x is a 2-point. Then let us denote

$$(a_1)_x = ((r_{11})_x(\sigma_M)_x + (s_{11})_x(\sigma_{M^c})_x) \times \cdots \times ((r_{1n})_x(\sigma_M)_x + (s_{1n})_x(\sigma_{M^c})_x),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(a_n)_x = ((r_{n1})_x (\sigma_M)_x + (s_{n1})_x(\sigma_{M^c})_x) \times \cdots \times ((r_{nn})_x(\sigma_M)_x + (s_{nn})_x(\sigma_{M^c})_x)$$

where $r_{ij}, s_{ij} \in R$ $(i, j = 1, 2, \cdots, n)$. Now note that for each r in R, both $\{z \in X(R) \mid r_z = 0_z\}$ and $\{z \in X(R) \mid r_z \neq 0_z\}$ are open-closed subsets of $X(R)$. Here combining this remark with Lemma 4.5, we can choose $b_1, \cdots, b_n$ in A and an open-closed subset N of $X(R)$ containing x such that for each z in N

$$(Ra_1 + \cdots + Ra_n)_z = (Rb_1 + \cdots + Rb_n)_z$$

and the matrix for $\{(b_1)_z, \cdots, (b_n)_z\}$ is of the following form

$$\left(\begin{array}{cccccccccc}
1_z & & \overbrace{\qquad}^{p} & & & * & & & (t_{11})_z & \cdots \ (t_{1n})_z \\
0_z & \cdot & & & & & & & & \\
 & & \cdot & & & & & & & \\
0_z & \cdot & 0_z & 1_z & & & & & \cdot & \cdot \\
0_z & \cdot & \cdot & & 0_z & \cdot & \cdot & 0_z & \cdot & \cdot \\
\vdots & \cdot & & & & \cdot & \cdot & \vdots & \cdot & \cdot \\
0_z & \cdot & \cdot & \cdot & \cdot & & & 0_z & (t_{n1})_z & \cdots \ (t_{nn})_z
\end{array}\right)$$

with the property that if there exist i and k such that

$$(t_{k1})_z = (t_{k2})_z = \cdots = (t_{k,i-1})_z = 0_z \text{ and } (t_{ki})_z \neq 0_z$$

then $$(t_{1i})_z = (t_{2i})_z = \cdots = (t_{k-1,i})_z = 0_z .$$

We claim that $\{(b_i)_z \mid (b_i)_z \neq 0_z\}$ are linearly independent over R_z for each z in N. To see this, let

$$(r_1)_z(b_1)_z + \cdots + (r_n)_z (b_n)_z = 0_z$$

where $r_i \in R$, $i = 1, 2, \cdots, n$. In case $\lambda^{-1}(z) \cap M \neq \emptyset$ and $\lambda^{-1}(z) \cap M^c \neq \emptyset$, each $(r_i)_z$ must be zero. In case $\lambda^{-1}(z) \cap M \neq \emptyset$ and $\lambda^{-1}(z) \cap M^c = \emptyset$, $(r_1)_z = \cdots = (r_p)_z = 0_z$ and $(b_{p+1})_z = \cdots = (b_n)_z = 0_z$. Finally in case $\lambda^{-1}(z) \cap M = \emptyset$ and $\lambda^{-1}(z) \cap M_c \neq \emptyset$, for each i, we have $(r_i)_z = 0_z$ if $(b_i)_z \neq 0_z$. Thus at any rate, for each i, we obtain $(r_i)_z = 0_z$ if $(b_i)_z \neq 0_z$ as required.

Remark. For the ring of all global sections of simple F-sheaf over a Boolean space, where F is a finite field, the proposition above has been shown by Pierce [7, Theorem 20. 4]. But such a ring is indeed a regular ring with $C(R) = Q(R)$ ([6, Theorem 2. 4]). Hence the proposition above can be seen as a generalization of the Pierce's theorem.

From Theorem 3. 4, Lemmas 2. 4 and 4. 1, Propositions 4. 4 and 4. 6, and [5, Lemma 3. 4], we have our main theorem of this paper :

Theorem 4.7. *Let R be a regular ring. Then the following conditions are equivalent :*

(a) *Every finitely generated torsion free R-module is a direct sum of cyclic R-modules.*

(b) *Every finitely generated R-submodule of $Q(R)$ is a direct sum of cyclic R-modules.*

(c) *Every R-submodule of $Q(R)$ with two generators is a direct sum of cyclic R-modules.*

(d) $C(R) = Q(R)$ *and* $|\lambda^{-1}(x)| \leq 2$ *for all* $x \in X(R)$.

(e) $C(R) = Q(R)$ *and* $X(R)$ *contains no* 3-*points.*

(f) $C(R) = Q(R)$ *and* $[Q(R)_x : R_x] \leq 2$ *for all* $x \in X(R)$.

(g) $C(R) = Q(R)$ *and every finitely generated R-submodule of* $Q(R)$ *is generated by two elements.*

(h) $C(R) = Q(R)$ *and* $L_A(x)$ *is distributive for all finitely generated torsion free R-module A and all* $x \in X(R)$.

(i) $C(R) = Q(R)$ *and* $L_A(x)$ *is distributive for all finitely generated R-submodule A of* $Q(R)$ *and all* $x \in X(R)$.

(j) $C(R) = Q(R)$ *and* $L_A(x)$ *is distributive for all R-submodules A of* $Q(R)$ *with two generators and all* $x \in X(R)$.

Remark. There exists a regular Baer ring R such that $Q(R)$ is

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generated by two elements as an R-module, but $C(R) \neq Q(R)$ (see [5, Example C]). Then, by [7, Corollary 13. 10], every finitely generated R-submodule of $Q(R)$ is also generated by two elements. Therefore the condition $C(R) = Q(R)$ in the theorem above can not be removed in (d) to (g) and also in (h) to (j) by Proposition 4. 2.

Corollary 4.8. (cf. [6, Example]). *Let R be a regular ring with $C(R) = Q(R)$. If $Q(R)$ is generated by two elements as an R-module, then every finitely generated torsion free R-module is a direct sum of cyclic R-modules.*

Corollary 4.9. *Let I be an arbitrary index set. Suppose that for each $i \in I$, R_i is a regular ring and every finitely generated torsion free R_i-module is a direct sum of cyclic R_i modules. Then every finitely generated torsionn free R-module is a direct sum of cyclic R-modules, where $R = \prod_{i \in I} R_i$.*

Proof. We have $C(R) = Q(R)$ since $C(R_i) = Q(R_i)$ for all $i \in I$ and $Q(R) = \prod_{i \in I} Q(R_i)$ (see [2, Proposition 2. 3. 8]). Furthermore it is easy to see that since every finitely generated R_i-submodule of $Q(R_i)$ is generated by two elements for all $i \in I$, so does every finitely generated R-submodule of $Q(R)$. Thus this corollary follows from Theorem 4. 7 (e).

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