On images of topological ordered spaces under some quotient mappings

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ON IMAGES OF TOPOLOGICAL ORDERED SPACES UNDER SOME QUOTIENT MAPPINGS

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A topological ordered space is a topological space equipped with a partial order. Since a topological space may be regarded as a topological ordered space equipped with the discrete order, i. e., $a\rho b$ if and only if $a=b$, the study of topological ordered spaces not only includes that of topological spaces but also reveals many generalizations of well-known results concerning topological spaces. From this point of view, the study of topological ordered spaces was first taken up by L. Nachbin [12]. In this paper, at first we survey some results obtained hitherto concerning the images of T_i-ordered spaces ($i=2, 3, 4$) under some quotient mappings[1], and then we establish the principal theorem which asserts that the image of a T_i-ordered space under a proper mapping is a T_i-ordered space ($i=2, 3$) and the image of a normally ordered space under a closed mapping is a normally ordered space. Needless to say, the theorem reduces to the known facts of topological spaces when the partial order concerned is the discrete order (cf. [5, §10, Corollaire 2 à Proposition 5 and Exercice 5] and [6, §4, Exercice 15]). Finally, we shall say a few words concerning the images of T_1-ordered spaces under quotient mappings.

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1. Throughout this paper, $\mathscr{U}$ and $\mathscr{V}$ denote topologies, and ρ and τ partial orders. The notation $(X, \mathscr{U}, \rho)$ is used to denote a set X endowed with a topology $\mathscr{U}$ and a partial order ρ. The notations $(X, \mathscr{U})$ and (X, ρ) are to be understood similarly. All mappings are assumed to be continuous.

Notation. In (X, ρ), for x, $y \in X$, $x \parallel y$ means that neither $x\rho y$ nor

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$y\rho x$. $x\bar{\rho}y$ if and only if $x \neq y$, $x\rho y$ or $x \parallel y$, and $x\rho'y$ stands for $x \neq y$, $y\rho x$ or $x \parallel y$. For $a \in X$ and $U, V \subset X$, $a\bar{\rho}U$ means that $a\bar{\rho}x$ for all $x \in U$, similarly $U\bar{\rho}V$ means that $x\bar{\rho}y$ for all $x \in U$ and all $y \in V$. (Note that these notations are different from those in [2].)

Definition 1. In (X, ρ), $[x, \longrightarrow]$ and $[\longleftarrow, x]$ denote the sets $\{y \in X : x\rho y\}$ and $\{y \in X : y\rho x\}$ respectively. In case $A \subset Y \subset X$, we put $i_Y(A) = \{\cup \{[a, \longrightarrow] : a \in A\}\} \cap Y$, $d_Y(A) = \{\cup \{[\longleftarrow, a] : a \in A\}\} \cap Y$, and A is said to be *increasing* (resp. *decreasing*) in Y if and only if $A = i_Y(A)$ (resp. $A = d_Y(A)$).

Definition 2. A space $(X, \mathcal{U}, \rho)$ is called a T_1-*ordered* (resp. T_2-*ordered*) *space* if for each pair $a, b \in X$ such that $a\rho'b$, there exist an increasing neighborhood U of a and a decreasing neighborhood V of b such that $b \notin U$ and $a \notin V$ (resp. $U \cap V = \emptyset$) (see [9]).

In these connections, the term O_i-space $(i = 1, 2)$ is used in [2] and [3] instead.

Definition 3. A space $(X, \mathcal{U}, \rho)$ is called a *regularly ordered space* if for each decreasing (resp. increasing) closed set $F \subset X$ and each element $a \notin F$, there exist disjoint neighborhoods U of a and V of F such that U is increasing (resp. decreasing) and V is decreasing (resp. increasing) in X. A space $(X, \mathcal{U}, \rho)$ is a T_3-*ordered space* if and only if $(X, \mathcal{U}, \rho)$ is both T_1-ordered and regularly ordered (cf. [9]).

Definition 4. A space $(X, \mathcal{U}, \rho)$ is called a *normally ordered space* if for each pair of disjoint closed sets $F_1, F_2 \subset X$ where F_1 is increasing and F_2 is decreasing in X, there exist disjoint neighborhoods U_1 of F_1 and U_2 of F_2 such that U_1 is increasing and U_2 is decreasing in X. A space $(X, \mathcal{U}, \rho)$ is a T_4-*ordered space* if and only if $(X, \mathcal{U}, \rho)$ is both T_1-ordered and normally ordered (cf. [9]).

In D. Adnadjević [1], $(X, \mathcal{U}, \rho)$ is said to be T_3-ordered if $(X, \mathcal{U})$ is a T_3 space and for each closed set F and each point a such that $a\bar{\rho}F$ (resp. $F\bar{\rho}a$) there exist neighborhoods U of a and V of F such that $U\bar{\rho}V$ (resp. $V\bar{\rho}U$), and is said to be T_4-ordered if $(X, \mathcal{U})$ is a T_4 space and for each pair of closed sets F_1, F_2 such that $F_1\bar{\rho}F_2$ there exist neighborhoods U of F_1 and V of F_2 such that $U\bar{\rho}V$. (Note that $a\bar{\rho}F$ (resp. $U\bar{\rho}V$) implies $a \notin F$ (resp. $U \cap V = \emptyset$).) In these connections, the notion of "T_i-ordered in Adnadjević' sense" is properly stronger than ours $(i = 3, 4)$ (cf. [9, Example 3] and [10, Example 2]).

Definition 5. Let f be a mapping of (X, ρ) onto (Y, τ). Then τ is called a *quotient order of ρ induced by f* if $x\tau y$ for $x, y \in Y$ if and only if there exist $u \in f^{-1}(x)$, $v \in f^{-1}(y)$ such that $u\rho v$.

Definition 6. A mapping f of (X, ρ) onto (Y, τ) is said to be *isotonic* (resp. *dually isotonic*) if $x\rho y$ (resp. $f(x)\tau f(y)$) implies $f(x)\tau f(y)$ (resp. $x\rho y$) (cf. [1], [2]). In [12, p. 21], an isotonic mapping is cited as an increasing mapping.

Remark 1. In Definition 5, let R be the equivalence relation on X agreeing that x and y are equivalent if and only if $f(x)=f(y)$, identify Y with X/R, and regard f as the projection of X onto X/R. Then the order τ on Y is viewed as the order induced on X/R as follows: for $A, B \in X/R$, $A\tau B$ if and only if there exist $a\in A$, $b\in B$ such that $a\rho b$ (see [13, § 4]). Different orders on X/R were also considered. For instance, in [4, § 1, Exercice 2] $A\tau_1 B$ for $A, B\in X/R$ if and only if there exists $b\in B$ such that $a\rho b$ for all $a\in A$, and in [8] $A\tau_2 B$ for $A, B\in X/R$ if and only if $a\rho b$ for each $a\in A$ and each $b\in B$. In the latter case, the equivalence relation considered in [8][2] is a very special one and τ_2 is then the quotient order of Definition 5. While, if f is dually isotonic then τ coincides with τ_2.

2. Suppose f is an open mapping of a T_2-ordered space $(X, \mathscr{U}, \rho)$ onto $(Y, \mathscr{V}, \tau)$ where $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ are T_2 spaces. As was shown in [2, Proposition 5], if f is isotonic and dually isotonic then $(Y, \mathscr{V}, \tau)$ is a T_2-ordered space. However, as the next example shows, if τ is the quotient order of ρ induced by f then the above does not hold generally, namely, the hypothesis that f is dually isotonic is indispensable.

Example 1. Let X be the set $\{(a, x, y): a=0 \text{ or } 1, x\in[0, \infty) \text{ and } y\in(-\infty, \infty)\}$. We define an equivalence relation R on X as follows: $(a, x, y) R (b, u, v)$ if and only if $a=b$, $x=u$. The topology $\mathscr{U}$ on X is the usual one. We define a partial order ρ in X as follows: $(a, x, y) \rho (b, u, v)$ if and only if $a=0$, $b=1$, $x=u\neq 0$, $y=v=\frac{1}{x}$; or $a=b$, $x=u$, $y=v$. Let $Y=X/R$, and f the projection of X onto Y. If we take the identification topology determined by f as the topology $\mathscr{V}$ of Y and the quotient order of ρ induced by f as the partial order τ in Y, then the mapping f is isotonic but not dually isotonic. And $(Y, \mathscr{V}, \tau)$ is not T_2-ordered. This is

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[2] In (X, ρ), $(x, \rightarrow]$ and $[\leftarrow, x)$ denote the sets $\{y\in X: x\rho y \text{ and } x\neq y\}$ and $\{y\in X: y\rho x \text{ and } x\neq y\}$ respectively. Then the equivalence relation D on X used in [8] is defined by agreeing that for $x, y\in X$, $(x, y)\in D$ if and only if $(x, \rightarrow]=(y, \rightarrow]$ and $[\leftarrow, x)=[\leftarrow, y)$.

because $(0, 0)^* \| (1, 0)^*$ in Y where $(a, x)^* = f((a, x, y))$, but there do not exist an increasing neighborhood U of $(0, 0)^*$ and a decreasing neighborhood V of $(1, 0)^*$ such that $U \cap V = \varnothing$.

In $(X, \mathscr{U}, \rho)$, let R be an equivalence relation on X, and f the projection of X onto X/R. In X/R, suppose that $\mathscr{V}$ is the identification topology determined by f and τ is the quotient order of ρ induced by f. As a generalization of a well-known result in topological space (cf. [5, §8, Proposition 8]), Theorem 4 of [11] asserts that if $(X/R, \mathscr{V}, \tau)$ is T_2-ordered then the graph $G(R)$ is saturated order closed (s. o. closed) in X^2, namely, for each $(x, y) \notin G(R)$ with $f(x) \tau' f(y)$ there exist a saturated increasing neighborhood U of x and a saturated decreasing neighborhood V of y such that $(U \times V) \cap G(R) = \varnothing$, and conversely if f is open and $G(R)$ is s. o. closed in X^2 then $(X/R, \mathscr{V}, \tau)$ is T_2-ordered. By Example 1, we see that, in the latter half of the above assertion, the hypothesis that $G(R)$ is s. o. closed in X^2 is indispensable.

Now, we shall prove our principal theorem which includes Theorems 2.4 and 2.5 of [1].

Theorem. *Suppose f is a mapping of $(X, \mathscr{U}, \rho)$ onto $(Y, \mathscr{V}, \tau)$ where τ is the quotient order of ρ induced by f.*

(1) *If f is a proper mapping and $(X, \mathscr{U}, \rho)$ is a T_2-ordered space, then $(Y, \mathscr{V}, \tau)$ is a T_2-ordered space.*

(2) *If f is a proper mapping and $(X, \mathscr{U}, \rho)$ is a T_3-ordered space, then $(Y, \mathscr{V}, \tau)$ is a T_3-ordered space.*

(3) *If f is a closed mapping and $(X, \mathscr{U}, \rho)$ is a normally ordered space, then $(Y, \mathscr{V}, \tau)$ is a normally ordered space.*

Proof. (1) If $G(\rho)$ and $G(\tau)$ are the graphs of ρ and τ respectively, then $G(\rho)$ is closed in X^2 since $(X, \mathscr{U}, \rho)$ is T_2-ordered ([12, p. 26, Proposition 1]). Let g be a mapping of X^2 onto Y^2 defined by $g((x, y)) = (f(x), f(y))$. Then g is proper by [5, §10, Proposition 4]. Further $g(G(\rho)) = G(\tau)$. Therefore $G(\tau)$ is closed in Y^2. Thus $(Y, \mathscr{V}, \tau)$ is T_2-ordered by [12, p. 26, Proposition 1].

(2) Let F be an increasing closed set of Y, and a any element of Y not contained in F. (In case F is a decreasing set of Y and $a \notin F$, the proof will go as well.) Then $f^{-1}(F)$ is an increasing closed set of X, $f^{-1}(a)$ a compact set, and $f^{-1}(a) \cap f^{-1}(F) = \varnothing$. Hence for each $x \in f^{-1}(a)$, there exist a decreasing neighborhood $U(x)$ of x and an increasing neighborhood $V(x)$ of $f^{-1}(F)$ such that $U(x) \cap V(x) = \varnothing$. Since $f^{-1}(a)$ is compact, there exists a finite set $\{x_1, \cdots, x_n\} \subset f^{-1}(a)$ such that $f^{-1}(a) \subset \bigcup_{i=1}^n U(x_i) = U_1$ and U_1 is a neighborhood of $f^{-1}(a)$. Let $V_1 = \bigcap_{i=1}^n V(x_i)$, then V_1 is an

increasing neighborhood of $f^{-1}(F)$ and satisfies $U_1 \cap V_1 = \emptyset$. Let $U_2 = Y - f(X - U_1)$, $V_2 = Y - f(X - V_1)$, then U_2 and V_2 are disjoint neighborhood of a and F respectively such that $U_2 \bar{\mp} V_2$. Therefore $U = d_r(U_2)$ is a decreasing neighborhood of a and $V = i_r(V_2)$ is an increasing neighborhood of F such that $U \cap V = \emptyset$. Thus $(Y, \mathscr{V}, \tau)$ is T_3-ordered.

(3) Let F_1 and F_2 be disjoint closed sets of Y such that F_1 is decreasing and F_2 is increasing. Then $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint closed sets of X such that $f^{-1}(F_1)$ is decreasing and $f^{-1}(F_2)$ is increasing. Therefore there exist a decreasing neighborhood U_1 of $f^{-1}(F_1)$ and an increasing neighborhood V_1 of $f^{-1}(F_2)$ such that $U_1 \cap V_1 = \emptyset$. Let $U_2 = Y - f(X - U_1)$, $V_2 = Y - f(X - V_1)$. Then U_2 and V_2 are disjoint neighborhoods of F_1 and F_2 respectively such that $U_2 \bar{\mp} V_2$. Hence $U = d_r(U_2)$ is a decreasing neighborhood of F_1 and $V = i_r(V_2)$ is an increasing neighborhood of F_2 such that $U \cap V = \emptyset$. Thus $(Y, \mathscr{V}, \tau)$ is a normally ordered space. Q. E. D.

Remark 2. If $(X, \mathscr{U}, \rho)$ is a compact space $(X, \mathscr{U})$ equipped with a closed order ρ (i. e., the graph of ρ is closed in X^2), then $(X, \mathscr{U}, \rho)$ is called a compact ordered space. Therefore a compact ordered space is just the same as a compact T_2-ordered space (cf. [12, pp. 25, 44]). Now, suppose that f is a mapping of $(X, \mathscr{U}, \rho)$ onto $(Y, \mathscr{V}, \tau)$ where τ is the quotient order of ρ induced by f. Then the following results were obtained:

(1) In case $\mathscr{V}$ is the identification topology determined by f and $(X, \mathscr{U}, \rho)$ is a compact ordered space, $(Y, \mathscr{V}, \tau)$ is a compact ordered space if and only if $(Y, \mathscr{V})$ is a T_2 space ([13, Proposition 9]).

(2) If f is a proper mapping and $(X, \mathscr{U}, \rho)$ is a locally compact T_2-ordered space, then $(Y, \mathscr{V}, \tau)$ is a locally compact T_2-ordered space ([10, Theorem 1]).

Remark 3. The assertion (2) (resp. (3)) of the Theorem is still valid if "T_3-ordered" (resp. "normally ordered") is replaced by "T_3-ordered in Adnadjević' sense" (resp. "T_4-ordered in Adnadjević' sense").

In topological spaces, the closed image of a T_1 space is also T_1. However, in topological ordered spaces, even the proper image of T_1-ordered space is not necessarily T_1-ordered. We shall conclude our study with exhibiting an example for this.

Example 2. Let X be the set $\{(a, x) : a = 0 \text{ or } 1, x \text{ is a real number}\}$. The topology $\mathscr{U}$ on X is the usual one, and the partial order ρ on X is defined as follows: $(a, x) \rho (b, y)$ if and only if $a = 0$, $b = 1$, $x = y$ and x is a rational number; or $a = b$, $x = y$. Then $(X, \mathscr{U}, \rho)$ is T_1-ordered but not

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T_2-ordered. Let $A = \{(0, x): x \in [0, 1]\}$. We introduce an equivalence relation R on X as follows: $(r, s) \in R$ if and only if $r, s \in A$ or $r = s$. Let $Y = X/R$, and f the projection of X onto Y. If $\mathscr{V}$ is the identification topology determined by f and τ is the quotient order of ρ induced by f then $(Y, \mathscr{V}, \tau)$ is a topological ordered space and f is a proper mapping. But $(Y, \mathscr{V}, \tau)$ is not T_1-ordered. This is because, for $a^* = f(A) \in Y$, $b^* = \{(1, b)\} \in Y$ where b is an irrational number contained in $[0, 1]$, $a^* \parallel b^*$, and every decreasing neighborhood of b^* should necessarily contain a^*.

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