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On s-unital rings

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ON s-UNITAL RINGS

Dedicated to Professor Mikao Moriya on his 70th birthday

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The present paper attempts to generalize several results in [10], [21], [22] and [24] obtained for rings with identity. In fact, we can prove similar ones for left (and right) s-unital rings, where a ring R ($\neq 0$) is called a left s-unital ring if $Ra \ni a$ for any $a \in R$. Needless to say, the class of left s-unital rings includes those of rings with identity and of regular rings. In [6], [18] and [23] we treated with left s-unital rings in the connection with regular rings. In the present paper, our attention will be directed towards the classes of left V-rings, fully left idempotent rings, and of almost commutative rings, those which are closely related to the class of regular rings. § 1 contains a fundamental proposition, a characterization of prime ideals of a left s-unital ring in terms of its right modules as in Beachy [3], and a slight generalization of a result of Hansen [13]. The material of § 2 comes from Fisher [10], Michler-Villamayor [21], Ramamurthi [22] and Yue Chi Ming [25], and left V-rings will be concerned in regular rings, left p-V-rings and fully left idempotent rings. In §§ 3 and 4, almost all results of Wong [24] will be carried over to s-unital rings.

For future reference, R ($\neq 0$) will represent always a ring (with or without identity), and C the center of R. The Jacobson radical and the prime radical of R will be denoted by $J(R)$ and $P(R)$, respectively. As for other notations, we follow [18] and [23].

1. s-unital rings. A left R-module $M \neq 0$ is defined to be *s-unital* if $Ru \ni u$ for any $u \in M$. For instance, every irreducible left R-module is s-unital. Needless to say, if ${}_RM$ is s-unital then it is unital, and in case R contains 1 these notions are identical. We can define similarly an s-unital right R-module.

Theorem 1. *If M ($\neq 0$) is a left R-module then the following are equivalent:*

1) *${}_RM$ is s-unital.*

2) *For any $u_1, \cdots, u_n \in M$ there exists an element $e \in R$ such that $eu_i = u_i$ ($i = 1, \cdots, n$).*

3) *For any positive integer* n, *every* $(R)_n$*-submodule of the direct sum* ${}^{(n)}M$ *of* n *copies of* M *is of the form* ${}^{(n)}N$ *with some* ${}_RN \subseteq {}_RM$, *where* $(R)_n$ *denotes the* $n \times n$ *matrix ring over* R.

Proof. 1) $\Longleftrightarrow$ 2). Assume that ${}_RM$ is s-unital. Choose an element $e_n \in R$ such that $e_n u_n = u_n$, and set $v_i = u_i - e_n u_i$ $(i=1, \cdots, n-1)$. By induction method, there exists an element $e' \in R$ such that $e'v_i = v_i$ $(i=1, \cdots, n-1)$. Then, one will easily see that $e = e' + e_n - e'e_n$ is an element with the property requested in 2). The converse is trivial.

1) $\Longleftrightarrow$ 3). Given $a \in R$, $E_{ij}(a)$ will denote the element of $(R)_n$ with a in the (i, j)-position and zeros elsewhere. If $u_1, \cdots, u_n \in M$ then

$$E_{1i}(a)\begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} = \begin{pmatrix} au_i \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix},$$

whence we can easily see that 1) implies 3). The converse is also easy by the fact that $\begin{pmatrix} Ru + Zu \\ Ru \\ \cdot \\ \cdot \\ Ru \end{pmatrix}$ is an $(R)_n$-submodule of ${}^{(n)}M$ for any $u \in M$.

If ${}_RR$ (resp. R_R) is s-unital, R is said to be *left* (resp. *right*) *s-unital.* To be easily seen, every (non-zero) homomorphic image of a left s-unital ring is left s-unital, and any regular ring is left and right s-unital. (In Ramamurthi [22], a left s-unital ring is cited as a *left D-regular ring.*)

Corollary 1. *If* R *is left s-unital then so is* $(R)_n$, *and conversely.*

Proof. If $A = (a_{ij})$ is an arbitrary element of $(R)_n$, then by Theorem 1 there exists an element $e \in R$ such that $ea_{ij} = a_{ij}$ $(i, j = 1, \cdots, n)$, whence it follows $\mathrm{diag}\{e, \cdots, e\} \cdot A = A$. Conversely, if $Ra \not\ni a$ then $(R)_n \cdot \mathrm{diag}\{a, \cdots, a\}$ does not contain $\mathrm{diag}\{a, \cdots, a\}$.

Proposition 1 (cf. [2, Proposition 5]). *Let* $\mathfrak{r}$ *be a non-zero right ideal of* R. *Then the following are equivalent*:

1) $\mathfrak{r}$ *is a left s-unital ring.*
2) $\mathfrak{r} \cap \mathfrak{l} = \mathfrak{r}\mathfrak{l}$ *for any left ideal* $\mathfrak{l}$ *of* R.

If R *is right s-unital then* 1) *is also equivalent to the following*:

3) *$\mathfrak{r}M\cap N=\mathfrak{r}N$ for any left R-modules ${}_RN\subseteq{}_RM$.*
(In case R contains 1, it is known that 1) is nothing but to say that $(R/\mathfrak{r})_R$ is flat (see for instance [19, Proposition 3, p. 133]).)

Proof. 1) $\Longleftrightarrow$ 2) is easy, and in case R is right s-unital 2) is obviously a special case of 3).

1) $\Longrightarrow$ 3). Let $u=a_1u_1+\cdots+a_nu_n$ ($a_i\in\mathfrak{r}$, $u_i\in M$) be an arbitrary element of $\mathfrak{r}M\cap N$, and choose $e\in\mathfrak{r}$ with $ea_i=a_i$ for all i (Theorem 1). Then $u=ea_1u_1+\cdots+ea_nu_n=eu\in\mathfrak{r}N$.

The next will play occasionally an important role in our subsequent study.

Proposition 2. *Let R be a left (resp. right) s-unital ring.*

(1) *If $\mathfrak{a}$ is a proper ideal of R then $\mathfrak{a}$ is contained in a proper prime ideal.*

(2) *Let R'/R be a ring extension. If $\mathfrak{a}'$ is an ideal of R' and $\mathfrak{a}'\cap R\neq R$ then there exists a maximal left (resp. right) ideal $\mathfrak{m}'$ of R' such that $\mathfrak{m}'\supseteq\mathfrak{a}$ and $\mathfrak{m}'\cap R\neq R$. Especially, if $\mathfrak{a}$ is a proper ideal of R then $\mathfrak{a}$ is contained in a maximal left (resp. right) ideal of R* (cf. [23, Lemma 1 (a)]).

Proof. (1) Let $r\in R\backslash\mathfrak{a}$, and choose $e\in R$ such that $r=er$. Then $E=\{e^i \mid i=1,2,\cdots\}$ is an m-system excluding $\mathfrak{a}$. If $\mathfrak{p}\supseteq\mathfrak{a}$ is an ideal of R which is maximal with respect to excluding E, then $\mathfrak{p}$ is a proper prime ideal.

(2) Let $r\in R\backslash(\mathfrak{a}'\cap R)$, and choose $e\in R$ such that $r=er$. By Zorn's lemma, there exists a maximal member $\mathfrak{m}'$ in the family of left ideals $\mathfrak{b}'$ of R' with $\mathfrak{b}'\supseteq\{x'\in R' \mid x'r\in\mathfrak{a}'\}(\supseteq\mathfrak{a}')$ and $\mathfrak{b}'\not\ni e$. Obviously $\mathfrak{m}'\cap R\neq R$, and one will easily see that $\mathfrak{m}'$ is a maximal left ideal of R'.

For a right R-module M_R, we set $\tau(M_R)=\sum_f fM$ ($f\in\mathrm{Hom}(M_R, R_R)$) and $\mathrm{Ann}(M_R)=\{x\in R \mid Mx=0\}$. To be easily seen, $\tau(M_R)$ is an ideal of R and $\mathrm{Ann}(M_R)\subseteq\mathrm{Ann}(\tau(M_R)_R)$.

Now, let M_R and M'_R be non-zero right R-modules. If for each $u\neq 0$ in M there exists $f\in\mathrm{Hom}(M_R, M'_R)$ such that $fu\neq 0$, then we write $M_R>M'_R$. If $M_R>M'_R$ and $M'_R>M_R$, then we write $M_R\sim M'_R$. It is easy to see that the relations $>$ and $\sim$ are transitive. Obviously, $M_R>R_R$ is nothing but to say that M_R is torsionless, and then we have $\mathrm{Ann}(M_R)=\mathrm{Ann}(\tau(M_R)_R)$. If M_R is faithful then $R_R>M_R$, and in case R is left s-unital the converse is also true.

In what follows, we shall present a characterization of proper prime ideal of a left s-unital ring in terms of its right modules. If R is a prime ring and $M_R > R_R$ then $\tau(M_R)$ is non-zero and $\mathrm{Ann}(M_R) = \mathrm{Ann}(\tau(M_R)_R) = 0$, namely, M_R is faithful. Conversely, if every torsionless right R-module is faithful then R is seen to be prime. Hence, for a left s-unital ring R, we see that R is prime if and only if $M_R > R_R$ implies always $M_R \sim R_R$.

Theorem 2 (cf. [3, Theorem 2]). *If $\mathfrak{p}$ is a proper ideal of a left s-unital ring R then the following are equivalent:*

1) *$\mathfrak{p}$ is a prime ideal.*
2) *$M_R > (R/\mathfrak{p})_R$ implies always $M_R \sim (R/\mathfrak{p})_R$.*

Proof. If $M_R > (R/\mathfrak{p})_R$ then $\mathrm{Ann}(M_R) \supseteq \mathrm{Ann}((R/\mathfrak{p})_R) = \mathfrak{p}$, and so M_R may be regarded as $M_{R/\mathfrak{p}}$. Hence, $R/\mathfrak{p}$ is a prime ring if and only if $M_R \sim (R/\mathfrak{p})_R$ for any $M_R > (R/\mathfrak{p})_R$.

Corollary 2 (cf. [3, Theorem 3]). *Let R be a left s-unital ring. If N_R ($\neq 0$) is a unital module then the following are equivalent:*

1) *$M_R > N_R$ implies always $M_R \sim N_R$.*
2) *$N_R \sim (R/\mathfrak{p})_R$ for a proper prime ideal $\mathfrak{p}$.*

Proof. 1) $\Rightarrow$ 2). Let $\mathfrak{p} = \mathrm{Ann}(N_R)$ ($\neq R$). Since $N_{R/\mathfrak{p}}$ is faithful, we have $(R/\mathfrak{p})_{R/\mathfrak{p}} > N_{R/\mathfrak{p}}$, and hence $(R/\mathfrak{p})_R \sim N_R$. If $M_R > (R/\mathfrak{p})_R$ then $M_R > N_R$, and $M_R \sim N_R \sim (R/\mathfrak{p})_R$, whence it follows that $\mathfrak{p}$ is a prime ideal (Theorem 2).

2) $\Rightarrow$ 1). Since $M_R > N_R \sim (R/\mathfrak{p})_R$ and $\mathfrak{p}$ is prime, Theorem 2 shows that $M_R \sim (R/\mathfrak{p})_R \sim N_R$.

As was shown in [13], every left Noetherian, left s-unital ring has a left identity. The next is a slight generalization of the result.

Theorem 3. *If a left Goldie ring R is left s-unital then R contains a left identity.*

Proof. To be easily seen, the left singular ideal $Z_l(R)$ is contained in $P(R)$ that is nilpotent by Lanski's theorem (cf. [16, p. 24]). By [9, Theorem 1.3], $R/Z_l(R)$ satisfies the maximum condition for right annihilators. Then $R/Z_l(R)$ has a left identity by [14, Proposition 2.1], and hence the semi-prime ring $R/P(R)$ has the identity. Now, we shall proceed by the induction with respect to the nilpotency index n of $P(R)$. The case $n=1$ is obvious by the above. Assume $n > 1$. Since $R/P(R)^{n-1}$ has a left identity by the induction hypothesis and $R/P(R)$ has the identity, a result of Herstein (cf. [15, p. 31]) shows that R has a left identity.

Corollary 3. *If R is left s-unital then the following are equivalent:*
1) *R is a left Artinian ring.*
2) *R is a left Noetherian π-regular ring.*
3) *R is a fully left Goldie π-regular ring.*

Proof. If R is left Artinian then R is left Noetherian by Hopkins' theorem (cf. [17, Theorem 34, p. 134]). Moreover, R being of bounded index, R is π-regular by [1, Theorem 5]. Since 2) implies 3) obviously, it remains only to prove that 3) implies 1). As was claimed in the proof of Theorem 3, $P(R)$ is nilpotent and $\bar{R}=R/P(R)$ has the identity. Now, let $\bar{a}$ be an arbitrary regular element of $\bar{R}$, and $\bar{a}^n\bar{x}\bar{a}^n=\bar{a}^n$. Then, $\bar{a}^n(1-\bar{x}\bar{a}^n)=0$ implies $\bar{x}\bar{a}^n=1$, and similarly $\bar{a}^n\bar{x}=1$. Hence, every regular element of $\bar{R}$ is a unit, which means that $\bar{R}$ coincides with its left quotient ring that is Artinian semiprimitive. Recalling here that $R/P(R)^{k+1}$ is a left s-unital, left Goldie ring, one will easily see that ${}_{\bar{R}}(P(R)^k/P(R)^{k+1})$ is completely reducible and of finite length. It follows therefore that ${}_RR$ has a composition series.

Corollary 4. *Let R be a left s-unital, fully left Goldie ring whose prime factor rings are π-regular. If $\mathfrak{a}$ is an ideal of R and ${}_R\mathfrak{a}_R$ is of finite length, then ${}_R\mathfrak{a}$ is of finite length.*

Proof. To our end, it suffices to prove the assertion for a minimal ideal $\mathfrak{a}$. Obviously, $l(\mathfrak{a})$ is a prime ideal of R and $S=R/l(\mathfrak{a})$ is Artinian simple by Corollary 3. Since R is left Goldie and ${}_S\mathfrak{a}$ is completely reducible, ${}_R\mathfrak{a}$ is of finite length.

The next is perhaps in the same vein as Corollary 4, and can be proved in the same way as in the proof of [20, Proposition].

Corollary 5. *Let R be a left s-unital, left Noetherian ring. If $\mathfrak{a}$ is an ideal of R and $\mathfrak{a}_R$ is of finite length, then ${}_R\mathfrak{a}$ is of finite length, too.*

Remarks. (1) Every s-unital left R-module is a homomorphic image of a direct sum of copies of ${}_RR$.

(2) Let M be an s-unital left R-module over a left s-unital ring R. We consider the map $f: M \longrightarrow R\otimes_R M$ defined by $u \longmapsto e'\otimes u$, where $e'u=u$. If $e''u=u$ $(e''\in R)$ then there exists an element $e\in R$ such that $ee'=e'$ and $ee''=e''$ (Theorem 1) and we have $e'\otimes u=ee'\otimes u=e\otimes e'u$

$=e\otimes e''u=e''\otimes u$. Hence, f is well-defined and is an R-homomorphism. Now, let $\sum_i a_i\otimes u_i$ be an arbitrary element of $R\otimes_R M$. Again by Theorem 1, we can find an element $a\in R$ such that $aa_i=a_i$ for all i. We have then $(\sum_i a_iu_i)f=a\otimes\sum_i a_iu_i=\sum_i aa_i\otimes u_i=\sum_i a_i\otimes u_i$. This proves that ${}_RR\otimes_R M$ is canonically isomorphic to ${}_RM$. Similarly, if R is commutative then we can prove the same for any s-unital module ${}_RM$.

(3) An s-unital module M_R will be defined to be *s-flat* if for each pair of s-unital left R-modules $A\subseteq B$ (with the inclusion map ι) $1\otimes\iota$: $M\otimes_R A\longrightarrow M\otimes_R B$ is a monomorphism. As a consequence of (2), one will easily see that if R is left and right s-unital then R_R is s-flat. Moreover, we can prove the following: Let R be a left and right s-unital ring, and $\mathfrak{l}$ a left ideal of R. If M_R is s-flat then $M\otimes_R\mathfrak{l}$ is canonically isomorphic to $M\mathfrak{l}$.

2. V-rings. An s-unital left R-module M is defined to be *s-injective* if M has the property that for each pair of s-unital left R-modules $A\subseteq B$ each $f\in$ Hom $({}_RA, {}_RM)$ can be extended to an element of Hom $({}_RB, {}_RM)$. If ${}_RM$ is s-injective then ${}_RM<\oplus\ {}_RM'$ for any s-unital ${}_RM'\supseteq{}_RM$. Moreover, the proof of [8, Theorem 1. 6] enables us to obtain the following:

Proposition 3 (Baer Criterion). *Let R be a left s-unital ring, and M an s-unital left R-module. Then ${}_RM$ is s-injective if and only if for each left ideal $\mathfrak{l}$ of R each $f\in$ Hom $({}_R\mathfrak{l}, {}_RM)$ can be extended to an element of* Hom$({}_RR, {}_RM)$.

An s-unital left (resp. right) R-module M is called a *V-module* if every R-submodule of M is an intersection of maximal R-submodules. If ${}_RR$ (resp. R_R) is a V-module, R is called a *left* (resp. *right*) *V-ring* (cf. [5]). As was mentioned in [18, Remark], we obtain the following which corresponds to [21, Theorem 2. 1]:

Theorem 4. *The following are equivalent*:

1) *R is a left V-ring.*

2) *R is left s-unital and every irreducible left R-module is s-injective.*

3) *R is left s-unital and every s-unital left R-module is a V-module.*

4) *R is left s-unital, and for any s-unital left R-module M the intersection of all maximal R-submodules is* 0; rad ${}_RM=0$.

5) *For any positive integer n, $(R)_n$ is a left V-ring.*

Proof. First, we shall prove the equivalence of 1)—4). Obviously, 4) $\Longleftrightarrow$ 3) $\Longrightarrow$ 1).

2) $\Longrightarrow$ 4). Let M be an arbitrary s-unital left R-module. If $0\neq u\in$

M, then there exists an R-submodule Y of M which is maximal with respect to $Y \not\ni u$. Let S be the set of R-submodules of M properly containing Y, and $D = \cap_{X \in S} X$ $(\ni u)$. Since D/Y is an irreducible R-module, by 2) there exists an R-submodule K of M containing Y such that $M/Y = D/Y \oplus K/Y$. Then $u \notin K$, and hence $Y = K$, namely, $M = D$. This means that Y is a maximal R-submodule of M and rad ${}_R M = 0$.

1) $\Longrightarrow$ 2). Let M be an irreducible left R-module, and $\mathfrak{l}$ a left ideal of R. If f is a non-zero element of Hom$({}_R\mathfrak{l}, {}_RM)$, then $\mathfrak{l}' = \mathrm{Ker}\, f \subset \mathfrak{l}$. By 1), there exists a maximal left ideal $\mathfrak{m}$ such that $\mathfrak{m} \supseteq \mathfrak{l}'$ and $\mathfrak{m} \not\supseteq \mathfrak{l}$. Since ${}_RM \cong {}_R(\mathfrak{l}/\mathfrak{l}')$ is irreducible and $\mathfrak{l} \supset \mathfrak{m} \cap \mathfrak{l} \supseteq \mathfrak{l}'$, we have $\mathfrak{m} \cap \mathfrak{l} = \mathfrak{l}'$. Now, taking this into mind, we can well-define an extension $g \in$ Hom $({}_RR, {}_RM)$ of f by $l + m \longmapsto lf$ $(l \in \mathfrak{l},\ m \in \mathfrak{m})$. Hence ${}_RM$ is s-injective by Proposition 3.

Next, we shall prove 1) $\Longrightarrow$ 5) $\Longrightarrow$ 4).

1) $\Longrightarrow$ 5). The direct sum $R^{(n)}$ of n copies of R is an s-unital left R-module (Theorem 1), and we have seen that ${}_RR^{(n)}$ is a V-module. Again by Theorem 1, every $(R)_n$-submodule of $(R)_n = {}^{(n)}(R^{(n)})$ is of the form ${}^{(n)}N$ with some ${}_RN \subseteq {}_RR^{(n)}$. Since ${}_RR^{(n)}$ is a V-module, $N = \cap_\alpha M_\alpha$ with maximal submodules ${}_RM_\alpha \subseteq {}_RR^{(n)}$. Hence ${}^{(n)}N = \cap_\alpha {}^{(n)}M_\alpha$, and $(R)_n$ is a left V-ring.

5) $\Longrightarrow$ 4). Again by Theorem 1, given an s-unital ${}_RM$, the left $(R)_n$-module ${}^{(n)}M$ is s-unital and rad ${}_{(R)_n}{}^{(n)}M = 0$, whence it follows rad ${}_RM = 0$.

A left R-module M is said to be *p-injective* if for any principal left ideal $(a|$ of R and $f \in$ Hom$({}_R(a|, {}_RM)$ there exists an element $u \in M$ such that $xf = xu$ for all $x \in (a|$. As was noted in [6], R is regular if and only if every left R-module is p-injective (cf. also [25]). In connection with Theorem 3, a left s-unital ring R is defined to be a *left p-V-ring* if every irreducible left R-module is p-injective. We can define a *right p-V-ring* in an obvious way. In case R contains 1, a left V-ring is a left p-V-ring. More generally we have

Proposition 4. *If R is a right s-unital, left V-ring then it is a left p-V-ring.*

Proof. Let ${}_RM$ be irreducible, and $(a|\,(=Ra)$ an arbitrary principal left ideal of R. Choose an element $e \in R$ with $ae = a$. If $f \in$ Hom$({}_R(a|, {}_RM)$ and $g \in$ Hom$({}_RR, {}_RM)$ is an extension of f, then for any $x \in R$ there holds $(xa)f = (xa)g = (xae)g = xa \cdot eg$.

If every left (resp. right) ideal of R is idempotent, R is said to be *fully left* (resp. *right*) *idempotent*. (In [22], a fully left idempotent ring is cited as a *left weakly regular ring.*) On the other hand, R is said to be *fully idempotent* if every ideal of R is idempotent.

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Proposition 5. (1) *The following are equivalent*:

1) *R is fully left idempotent.*
2) *$(Ra)^2 \ni a$ for any $a \in R$.*
3) *For each pair of left ideals $\mathfrak{l} \subseteq \mathfrak{l}'$ of R, there holds $\mathfrak{l}'\mathfrak{l} = \mathfrak{l}$.*
4) *For any positive integer n, $(R)_n$ is fully left idempotent.*

If R is right s-unital then 1) *is also equivalent to each of the following*:

5) *For each ideal $\mathfrak{a}$ and each left ideal $\mathfrak{l}$ of R there holds $\mathfrak{a} \cap \mathfrak{l} = \mathfrak{a}\mathfrak{l}$.*
6) *For each ideal $\mathfrak{a}$ of R and each pair of left R-modules ${}_RN \subseteq {}_RM$ there holds $\mathfrak{a}M \cap N = \mathfrak{a}N$.*

(2) *The following are equivalent*:

1) *R is fully idempotent.*
2) *$(RaR)^2 \ni a$ for any $a \in R$.*
3) *Every ideal of R is semiprime.*
4) *For each pair of ideals $\mathfrak{a}$, $\mathfrak{a}'$ of R there holds $\mathfrak{a} \cap \mathfrak{a}' = \mathfrak{a}\mathfrak{a}'$.*
5) *For any positive integer n, $(R)_n$ is fully idempotent.*

Proof. The assertion (2) is given in [7]. Concerning (1), the equivalence of 1)−3) is given in [22, Proposition 1] and 4) $\Longrightarrow$ 1) is trivial. Moreover, the latter part will be obvious by Proposition 1.

1) $\Longrightarrow$ 4). We shall modify slightly the proof of [12, Theorem 4]. At first, we consider the case $n = 2$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary element of $\mathfrak{R} = (R)_2$. If $a = \sum_i w_i a w_i' a$ (w_i, $w_i' \in R$), then $A - X = \begin{pmatrix} 0 & b' \\ c & d \end{pmatrix}$ for $X = \sum_i \begin{pmatrix} w_i & 0 \\ 0 & 0 \end{pmatrix} A \begin{pmatrix} w_i' & 0 \\ 0 & 0 \end{pmatrix} A$. Next, if $d = \sum_j x_j d x_j' d$ (x_j, $x_j' \in R$), then $A - X - Y = \begin{pmatrix} 0 & b' \\ c' & 0 \end{pmatrix}$ for $Y = \sum_j \begin{pmatrix} 0 & 0 \\ 0 & x_j \end{pmatrix} (A - X) \begin{pmatrix} 0 & 0 \\ 0 & x_j' \end{pmatrix} (A - X)$. Finally, if $b' = \sum_k y_k b' y_k' b'$ and $c' = \sum_k z_k c' z_k' c'$ (y_k, y_k', z_k, $z_k' \in R$) then $A - X - Y = \sum_k \begin{pmatrix} y_k & 0 \\ 0 & z_k \end{pmatrix} (A - X - Y) \begin{pmatrix} 0 & z_k' \\ y_k' & 0 \end{pmatrix} (A - X - Y)$. We obtain therefore $A = (A - X - Y) + Y + X \in (\mathfrak{R}(A - X - Y))^2 + (\mathfrak{R}(A - X))^2 + (\mathfrak{R}A)^2 = (\mathfrak{R}A)^2$, namely, $\mathfrak{R}$ is fully left idempotent.

Since $(R)_{2^k} \cong (R_{2^{k-1}})_2$, one will easily see that $(R)_{2^k}$ is fully left idempotent. Given arbitrary n, we choose k so that $2^k \geq n$. If $A \in (R)_n$, we choose $A' \in (R)_{2^k}$ with A in the upper left-hand corner and zeros elsewhere. Now, $A' \in ((R)_{2^k} A')^2$ and a brief computation gives $A \in ((R)_n A)^2$.

Proposition 6. *Every left p-V-ring is fully left idempotent, and so every right s-unital, left V-ring is fully left idempotent.*

Proof. If not, there exists a non-zero element $a\in R$ such that $(a|^2 \neq (a|$ $(=Ra)$. Let $\mathfrak{m}$ be a maximal member in the family of left ideals $\mathfrak{l}$ of R such that $(a|^2\subseteq\mathfrak{l}\subset(a|$. Since the irreducible left R-module $(a|/\mathfrak{m}$ is p-injective, there exists an element $b\in(a|$ such that $x+\mathfrak{m}=xb+\mathfrak{m}$ for all $x\in(a|$. But this implies a contradiction $(a|=\mathfrak{m}$. The latter part is evident by Proposition 4.

As a direct consequence of Proposition 6 and [11, Theorem 1.1], we obtain the following:

Corollary 6 (cf. [10, Theorem 13]). *If R is a left V-ring then the following are equivalent:*

1) *R is a regular ring.*

2) *R is right s-unital and every prime factor ring of R is a regular ring.*

Proposition 7 (cf. [21], [22]). *Let R be fully left idempotent.*

(1) *R is right non-singular*; $Z_r(R)=0$.

(2) *R is semiprimitive*; $J(R)=0$.

(3) *If $a\in R$ is left regular then $R=RaR$.*

(4) *C is a regular ring.* [1)]

Proof. (1) Let $z\in Z_r(R)$, and choose an element $y\in Z_r(R)$ such that $z=yz$ (Proposition 5 (1)). If $nz+zx$ (n an integer and $x\in R$) is an arbitrary element of $|z)\cap r(y)$, then $0=y(nz+zx)=nyz+yzx=nz+zx$. Hence, $|z)\cap r(y)=0$, which means $z=0$.

(2) Let $z\in J(R)$, and choose $y\in J(R)$ such that $z=yz$. Since $\{xy-x\,|\,x\in R\}=R$, it follows $Rz=0$, namely, $z=0$.

(3) This is evident by $Ra=RaRa$.

(4) If c is an arbitrary element of C then $c\in(Rc)^2=Rc^2$ by Proposition 5 (1). Hence, C is regular by [1, Lemma 1].

Theorem 5 (cf. [10, Theorem 14]). *If R is right s-unital then the following are equivalent:*

1) *R is a left V-ring.*

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1) If R is fully idempotent then it is almost evident that C is still regular and the centroid $\mathfrak{C}$ of R is commutative. Moreover, as was shown by R. Courter [Proc. Amer. Math. Soc. 43 (1974), 293—295], $\mathfrak{C}$ is a regular ring. In fact, given an arbitrary element γ of $\mathfrak{C}$, one will easily see that $R\gamma^2=(R^2)\gamma^2=(R\gamma)^2=R\gamma$ and $R\gamma\cap\operatorname{Ker}\gamma=(R\gamma\cap\operatorname{Ker}\gamma)^2=0$. Hence, $R=R\gamma\oplus\operatorname{Ker}\gamma$ and γ induces an automorphism of ${}_R R\gamma_R$. We can find then an element γ' of $\mathfrak{C}$ such that $\gamma=\gamma^2\gamma'$.

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2) *R is fully left idempotent and every left primitive factor ring of R is a left V-ring.*

Proof. 1) $\Longrightarrow$ 2). This is a consequence of Proposition 6.

2) $\Longrightarrow$ 1). Let M be an irreducible left R-module, and $\mathfrak{l}$ a left ideal of R. Let f be a non-zero element of $\mathrm{Hom}({}_R\mathfrak{l}, {}_RM)$. Obviously $\mathfrak{a}=\mathrm{Ann}({}_RM)$ is a left primitive ideal of R. Noting that $\mathfrak{a}\cap\mathfrak{l}=\mathfrak{a}\mathfrak{l}$ (Proposition 5 (1)), one will easily see the map defined by $l+a \longmapsto lf$ ($l\in\mathfrak{l}$, $a\in\mathfrak{a}$) is an extension of f in $\mathrm{Hom}({}_R(\mathfrak{l}+\mathfrak{a}), {}_RM)$. Now, the rest of the proof proceeds in the same way as for 1) $\Longrightarrow$ 2) of Theorem 4.

Corollary 7 (cf. [10, Corollary 15]). *If R is a regular ring then the following are equivalent*:

1) *R is a left V-ring.*
2) *Every left primitive factor ring of R is a left V-ring.*

A left (resp. right) s-unital ring is said to be *left* (resp. *right*) *semi-artinian* if every s-unital left (resp. right) R-module contains an irreducible R-submodule.

Theorem 6 (cf. [10, Theorem 17]). *If R is left semiartinian then the following are equivalent*:

1) *R is a regular ring.*
2) *R is fully idempotent.*
3) *R is fully left idempotent.*
3′) *R is fully right idempotent.*
4) *R is a left p-V-ring.*
4′) *R is a right p-V-ring.*

When this is the case, R is right semiartinian.

Proof. 1) $\Longrightarrow$ 4) (resp. 4′)) $\Longrightarrow$ 3) (resp. 3′)) $\Longrightarrow$ 2). These are obvious by the remark mentioned before Proposition 4 and Proposition 6.

2) $\Longrightarrow$ 1). Let S ($\neq 0$) be the left socle of R. If $\mathfrak{l}$ is a left ideal of S then it is easy to see that $R\mathfrak{l}=R\mathfrak{l}\cdot R\mathfrak{l}\subseteq S\mathfrak{l}\subseteq\mathfrak{l}$, namely, $\mathfrak{l}$ is a left ideal of R. (Note that R is semiprime.) Hence ${}_SS$ is completely reducible. Since S is also semiprime, each homogeneous component of ${}_SS$ is (non-trivial) simple and regular. Hence S is regular. Now, let $\mathfrak{m}$ ($\supseteq S$) be the maximal regular ideal of R (cf. [4]). Suppose $\mathfrak{m}\neq R$. Then $R/\mathfrak{m}$ is fully idempotent and has non-zero left socle. By the above argument, we see that the maximal regular ideal of $R/\mathfrak{m}$ is non-zero, which contradicts the maximality of $\mathfrak{m}$. We have seen thus $R=\mathfrak{m}$. Finally, noting that S coincides with the right socle of R, one will easily see that R has a right

socle sequence, namely, R is right semiartinian.

3. *AC*-rings. R is called an *AC-ring* (almost commutative ring) if for any proper prime ideal $\mathfrak{p}$ of R and $a \notin \mathfrak{p}$ there exists x such that $ax \in C \backslash \mathfrak{p}$. Any P_1-ring is obviously an AC-ring (cf. [6]), and the next will be easily seen (cf. [24, Theorem 1]).

Proposition 8. *Let R be an AC-ring.*

(1) *Every homomorphic image of R is an AC-ring.*

(2) *Every prime ideal of R is completely prime. In particular, R is a prime ring if and only if it is a domain.*

(3) *Every semiprime ideal of R is completely semiprime. In particular, R is a semiprime ring if and only if it is a reduced ring.*

(4) *For any proper prime ideal* $\mathfrak{p}$ *of R and* $a \notin \mathfrak{p}$ *there exists y such that* $ya \in C \backslash \mathfrak{p}$. (The notion of AC is right-left symmetric.)

By Proposition 8 (3), the prime radical of an AC-ring coincides with the set of all nilpotent elements. If R is an AC-ring and $R \neq P(R)$ then $(R)_n$ cannot be an AC-ring for $n > 1$.

Proposition 9. *The following are equivalent*:

1) *R is a division ring.*

2) $aR = R$ *for any* $a \neq 0$ *in R.*

2′) $Ra = R$ *for any* $a \neq 0$ *in R.*

3) *R is a (non-trivial) simple AC-ring.*

4) *R is a prime AC-ring with minimum condition on ideals.*

5) *R is a fully idempotent, prime AC-ring.*

6) *R is a regular, prime AC-ring.*

Proof. Obviously, 1) implies each of 2)—6) and 6) implies 5). Next, assume 2). Since R is strongly regular, there exists x such that $axa = a$ and $ax = xa$. By $axR = aR = R$, we see that the central idempotent ax is the identity of R, and 1) is obvious. Similarly, 1) $\Longrightarrow$ 2′) $\Longrightarrow$ 1). Finally, assume one of 3)—5). For any $a \neq 0$ there exists x such that ax is a non-zero central element. Since R is a domain (Proposition 8), one will easily see $R = axR = aR$.

Corollary 9 (cf. [24, Theorem 3]). *The following are equivalent*:

1) *R is a finite direct sum of division rings.*

2) *R is a semiprime AC-ring with minimum condition on ideals.*

3) *R is a semiprimitive AC-ring with minimum condition on ideals.*

Proof. It suffices to prove 2) $\Longrightarrow$ 1). For any proper prime ideal $\mathfrak{p}$, $R/\mathfrak{p}$ is a division ring (Proposition 9). Hence, R is a subdirect sum of division rings. As is well known, by the minimum condition on ideals, R is then a finite direct sum of division rings.

Theorem 7 (cf. [24, Theorem 2]). *If R is a left (resp. right) s-unital AC-ring and $\mathfrak{n}$ is a submodule of R, then the following are equivalent:*

1) *$\mathfrak{n}$ is a maximal right (resp. left) ideal.*
2) *$\mathfrak{n}$ is a maximal ideal.*
3) *$\mathfrak{n}$ is a right (resp. left) primitive ideal.*

Proof. Since R is left s-unital, $R^2 = R$ and any maximal ideal of R is a prime ideal. Moreover, if $\mathfrak{n}$ is a right ideal of R then $(\mathfrak{n}: R) = \{x \in R \mid Rx \subseteq \mathfrak{n}\}$ coincides with the largest ideal contained in $\mathfrak{n}$.

1) $\Longrightarrow$ 2). If $\mathfrak{a} = (\mathfrak{n}: R)$ $(\subseteq \mathfrak{n} \neq R)$ is not maximal, then $\mathfrak{a}$ is properly contained in a proper prime ideal $\mathfrak{p}$ (Proposition 2 (1)). Evidently, there exists an element $a \in \mathfrak{n} \backslash \mathfrak{p}$, and $ax \in (C \cap \mathfrak{n}) \backslash \mathfrak{p}$ for some x. But this is impossible by $ax \in \mathfrak{a} \subseteq \mathfrak{p}$. This proves that $\mathfrak{a}$ is a maximal ideal. Hence, $R/\mathfrak{a}$ is a division ring (Proposition 9), and $\mathfrak{n} = \mathfrak{a}$.

2) $\Longrightarrow$ 3). Since $R/\mathfrak{n}$ is a division ring (Proposition 9), $\mathfrak{n}$ is primitive.

3) $\Longrightarrow$ 1). There exists a maximal right ideal $\mathfrak{m}$ of R such that $\mathfrak{m} \supseteq \mathfrak{n}$ and $(\mathfrak{m}/\mathfrak{n}: R/\mathfrak{n}) = 0$. Since $R/\mathfrak{n}$ is a left s-unital AC-ring, as was shown in 1) $\Longrightarrow$ 2), we obtain $\mathfrak{m}/\mathfrak{n} = (\mathfrak{m}/\mathfrak{n}: R/\mathfrak{n}) = 0$, i. e., $\mathfrak{m} = \mathfrak{n}$.

By Theorem 7, if R is a left (resp. right) s-unital AC-ring then $J(R)$ is the intersection of maximal ideals, and so a left (resp. right) s-unital semiprimitive AC-ring is a subdirect sum of division rings.

Theorem 8. *The following are equivalent:*

1) *R is a strongly regular ring.*
2) *R is a regular AC-ring.*
3) *R is an AC-ring and a left (or right) p-V-ring.*
4) *R is a fully idempotent AC-ring.*
5) *R is an AC-ring whose ideals are semiprime.*
6) *R is a reduced ring such that $R/\mathfrak{p}$ is regular (in fact a division ring) for any proper prime ideal $\mathfrak{p}$.*
7) *R is a reduced ring whose proper completely prime ideals are maximal left ideal.*

Proof. 1) $\Longleftrightarrow$ 6) $\Longleftrightarrow$ 7) are given in [5] (and also in [11]), 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3) are trivial, 3) $\Longrightarrow$ 4) by Proposition 6, and 4) $\Longrightarrow$ 6) is a consequence of Propositions 8 and 9. Finally, 4) $\Longleftrightarrow$ 5) is contained in Proposition 5 (2).

Following [24], R is *primary* if every zero-divisor is nilpotent, and is *local* if it has exactly one maximal ideal.

Theorem 9 (cf. [24, Theorem 5]). (1) *If R is an AC-ring then the following are equivalent*:

1) *R is primary.*
2) *Every right zero-divisor is nilpotent.*
3) *Every left zero-divisor is nilpotent.*
4) *There exists a minimal prime ideal $\mathfrak{p}$ of R which contains all zero-divisors.*

(2) *If R is a left s-unital AC-ring then the following are equivalent*:

1) *R has a unique prime ideal $\mathfrak{p} \neq R$.*
2) *R is local and $P(R) = J(R)$.*
3) *$R/P(R)$ is a division ring.*

Proof. (1) 2)$\Longrightarrow$3). Let $xy=0$, $y \neq 0$. If $x \notin P(R)$ then $x \notin \mathfrak{p}_0$ for some prime ideal $\mathfrak{p}_0$. Choose $u \in R$ such that $ux \in C \backslash \mathfrak{p}_0$ (cf. Proposition 8). But, by 2), $0 = uxy = yux$ yields a contradiction $ux \in P(R)$. Similarly, we have 3)$\Longrightarrow$2). Obviously, $P(R)$ is a prime ideal.

1)$\Longrightarrow$2)$\Longrightarrow$4). Trivial.

4)$\Longrightarrow$1). It suffices to show that if x is non-nilpotent then $x \notin \mathfrak{p}$. To be easily seen, $T = \{x^k s \mid k \geq 0,\ s \in R \backslash \mathfrak{p}\} \cup \{x^k \mid k > 0\}$ is an m-system such that $x \in T$ and $0 \notin T$. Then there exists a prime ideal $\mathfrak{p}_0$ such that $\mathfrak{p}_0 \cap T = \emptyset$. Since $\mathfrak{p}$ is a minimal prime ideal and $\mathfrak{p}_0 \subseteq R \backslash T \subseteq \mathfrak{p}$, we have $\mathfrak{p} = \mathfrak{p}_0 \not\ni x$.

(2) 1)$\Longrightarrow$2). Every maximal ideal of R is a prime ideal. If $\mathfrak{p}$ is not maximal then it is properly contained in a proper prime ideal (Proposition 2 (1)), a contradiction.

2)$\Longrightarrow$3). Since $P(R) = J(R)$ is a unique maximal ideal, $R/P(R)$ is a division ring (Proposition 9).

3)$\Longrightarrow$1). Trivial.

4. Integral extensions of s-AC-rings. In [24], R with 1 is called an *SAC-ring* if for any proper ideal $\mathfrak{a}$ and $x \notin \mathfrak{a}$ there exists y such that $xy \in C \backslash \mathfrak{a}$. However, in our present study, we shall employ a somewhat weaker (but right-left symmetric) definition: An AC-ring is called an *s-AC-ring* if for any non-prime ideal $\mathfrak{a}$ and $x \notin \mathfrak{a}$ there holds $RxR \cap C \not\subseteq \mathfrak{a}$. To be easily seen, every s-AC-ring has the following property:

(*) For any proper ideal $\mathfrak{a}$ and $x \notin \mathfrak{a}$ there holds $RxR \cap C \not\subseteq \mathfrak{a}$.

Any strongly regular ring is s-AC, and conversely any P_1-ring with the property (*) is strongly regular.

For a while, we assume that R is a ring with the property (*). By a routine manner, we can show that an ideal $\mathfrak{a}$ is prime (resp. semiprime) if and only if $\mathfrak{a}\cap C$ is prime (resp. semiprime) in C. Accordingly, a ring is strongly regular if and only if it is an s-AC-ring whose center is regular (Theorem 8). We assume further that R'/R is a ring extension such that C is contained in the center C' of R'. Then we can easily see that if $\mathfrak{a}'$ is a prime (resp. semiprime) ideal of R' then $\mathfrak{a}'\cap R$ is a prime (resp. semiprime) ideal of R.

In what follows, R'/R will mean a ring extension, and C' the center of R'. R'/R is called a *left integral extension* if $C\subseteq C'$ and for each $x\in R'$ there exist $a_0, \cdots, a_{n-1}$ in R such that $x^n+a_{n-1}x^{n-1}+\cdots+a_0=0$.

Concerning "going up" we have the following:

Theorem 10 (cf. [24, Theorem 7 and Corollary 2]). *Let R be an s-AC-ring, R' a left (or right) s-unital ring, and let R'/R be a left integral extension. If $\mathfrak{a}'$ is an ideal of R' and $\mathfrak{p}$ is a proper prime ideal of R containing $\mathfrak{a}'\cap R$, then there exists a proper prime ideal $\mathfrak{p}'$ of R' such that $\mathfrak{p}'\supseteq\mathfrak{a}'$ and $\mathfrak{p}'\cap R=\mathfrak{p}$.*

Proof. Let M be the non-empty m-system $R\backslash\mathfrak{p}$, and $\mathfrak{p}'\supseteq\mathfrak{a}'$ an ideal of R' which is maximal with respect to excluding M. Then $\mathfrak{p}'$ is a proper prime ideal and $\mathfrak{p}'\cap R\subseteq\mathfrak{p}$. If $\mathfrak{p}'\cap R\subset\mathfrak{p}$ then there exists $c\in(C\cap\mathfrak{p})\backslash(\mathfrak{p}'\cap R)$. Since $(cR'+\mathfrak{p}')\cap M\neq\emptyset$, $cx+p'=m$ with some $x\in R'$, $p'\in\mathfrak{p}'$, $m\in M$. Suppose $x^n+a_{n-1}x^{n-1}+\cdots+a_0=0$ $(a_i\in R)$. Then $0=x^nc^n+a_{n-1}x^{n-1}c^n+\cdots+a_0c^n=(m-p')^n+a_{n-1}(m-p')^{n-1}c+\cdots+a_0c^n$. There exist therefore $r\in R$ and $q'\in\mathfrak{p}'$ such that $m^n+rc+q'=0$. This shows $q'\in\mathfrak{p}'\cap R\subseteq\mathfrak{p}$, and hence $m^n\in\mathfrak{p}$, whence it follows a contradiction $m\in\mathfrak{p}$ (Proposition 8).

Corollary 9. *Let R be a left s-unital s-AC-ring, R' a left s-unital ring, and let R'/R be a left integral extension. If $\mathfrak{a}$ is a proper ideal of R then $\mathfrak{a}R'$ is a proper ideal of R'.*

Proof. By Proposition 2 (1), $\mathfrak{a}$ is contained in a proper prime ideal $\mathfrak{p}$ of R, and then there exists a proper prime ideal $\mathfrak{p}'$ of R' such that $\mathfrak{p}'\cap R=\mathfrak{p}$ (Theorem 10). Hence $\mathfrak{a}R'\subseteq\mathfrak{p}'\neq R'$. Next, to be easily seen, $R(\mathfrak{a}\cap C)=\mathfrak{a}$. It follows therefore $R'(\mathfrak{a}R')=R'R(\mathfrak{a}\cap C)R'\subseteq\mathfrak{a}R'$.

Lemma 1. *Let R'/R be a left integral extension. If a completely prime ideal $\mathfrak{p}'$ of R' is contained in a left ideal $\mathfrak{n}'$ and $\mathfrak{n}'\cap R=\mathfrak{p}'\cap R$, then*

$\mathfrak{n}'=\mathfrak{p}'$.

Proof. Suppose there exists $x\in\mathfrak{n}'\backslash\mathfrak{p}'$. Let n be the smallest integer such that $x^n+a_{n-1}x^{n-1}+\cdots+a_0=p'\in\mathfrak{p}'$ $(a_i\in R)$. This implies $a_0\in\mathfrak{n}'\cap R=\mathfrak{p}'\cap R$ and $n>1$. Since $\mathfrak{p}'$ is completely prime, $(x^{n-1}+a_{n-1}x^{n-2}+\cdots+a_1)x=p'-a_0\in\mathfrak{p}'$ yields a contradiction $x^{n-1}+\cdots+a_1\in\mathfrak{p}'$.

Theorem 11 (cf. [24, Corollary 4]). *Let R be a right and left s-unital s-AC-ring, R' a left s-unital ring, and let R'/R be a left integral extension. Let $\mathfrak{p}'$ be a completely prime ideal of R'. Then $\mathfrak{p}'$ is a maximal left ideal if and only if $\mathfrak{p}'\cap R$ is a maximal ideal of R.*

Proof. Suppose $\mathfrak{p}'\cap R$ is a maximal ideal. By Proposition 2 (2), there exists a maximal left ideal $\mathfrak{m}'$ of R' such that $\mathfrak{m}'\supseteq\mathfrak{p}'$ and $\mathfrak{m}'\cap R\neq R$. Since $\mathfrak{p}'\cap R$ is a maximal left ideal by Theorem 7, we have $\mathfrak{m}'\cap R=\mathfrak{p}'\cap R$, whence it follows $\mathfrak{m}'=\mathfrak{p}'$ (Lemma 1). Conversely, suppose $\mathfrak{p}'$ is a maximal left ideal. We claim here $\mathfrak{p}'\cap R\neq R$. In fact, if $x\in R'\backslash\mathfrak{p}'$ and $x^n+a_{n-1}x^{n-1}+\cdots+a_0=0$ $(a_i\in R)$ then $\mathfrak{p}'\supseteq R$ gives a contradiction $x^n\in\mathfrak{p}'$. Now, suppose $\mathfrak{p}'\cap R$ is not maximal. Then $\mathfrak{p}'\cap R$ is properly contained in a proper prime ideal $\mathfrak{p}_0$ of R (Proposition 2 (1)), and we can find a proper prime ideal $\mathfrak{p}_0'\supseteq\mathfrak{p}'$ of R' such that $\mathfrak{p}_0'\cap R=\mathfrak{p}_0$ (Theorem 10). Since $\mathfrak{p}_0'$ has to be equal to $\mathfrak{p}'$, we have a contradiction $\mathfrak{p}'\cap R=\mathfrak{p}_0'\cap R=\mathfrak{p}_0$.

Theorem 12 (cf. [24, Theorem 9]). *Let R be a left and right s-unital s-AC-ring, R' a left and right s-unital reduced ring, and let R'/R be a left intergral extension. Then, R is regular if and only if so is R'.*

Proof. If R is (strongly) regular then every proper prime ideal of R is a maximal left ideal (Theorem 8). By the proof of Theorem 11, for any proper completely prime ideal $\mathfrak{p}'$ of R', $\mathfrak{p}'\cap R$ is a proper prime ideal of R, and so it is a maximal left ideal. Hence $\mathfrak{p}'$ is a maximal left ideal by Theorem 11, and again by Theorem 8 R' is a regular ring. Conversely, if R' is a regular ring, then for any proper prime ideal $\mathfrak{p}$ of R there exists a proper prime ideal $\mathfrak{p}'$ of R' such that $\mathfrak{p}'\cap R=\mathfrak{p}$ (Theorem 10) and $\mathfrak{p}'$ is a maximal left ideal (Theorem 8). Hence, $\mathfrak{p}$ is a maximal ideal by Theorem 11, and so it is a maximal left ideal (Theorem 7). Theorem 8 proves therefore that R is regular.

Theorem 13 (cf. [24, Theorem 10]). *Let R'/R be a left integral extension. If R is strongly regular then for each $x\in R'$ there exist $y\in R'$ which can be expressed as a (left) polynomial in x over $R^1=R+Z$ and a natural number n such that $yx^{n+1}=x^n$.*

Proof. Let $A(x)=\{p(x) \mid p(x)$ is a monic polynomial of positive degree in x over R such that $p(x)x^m=0$ for some $m\}$. In $A(x)$ we choose $p(x)=x^k+a_{k-1}x^{k-1}+\cdots+a_1x+a_0$ of the least degree; $p(x)x^{n-1}=0$ $(n>1)$. By [1, Lemma 1], there exists (uniquely) an element $a\in R$ such that $a_0a=aa_0$, $a_0^2a=a_0$ and $a^2a_0=a$. Obviously, $e=a_0a$ is a central idempotent with $ea_0=a_0$ and $ea=a$. If $k=1$ then $x^n+a_0x^{n-1}=0$ implies $0=x^n+a_0x^{n-1}-e(x^n+a_0x^{n-1})=x^n-ex^n$, i.e., $ex^n=x^n$. Hence, $0=a(x^n+a_0x^{n-1})x=ax^{n+1}+ex^n=ax^{n+1}+x^n$, whence it follows $-ax^{n+1}=x^n$. Next, we shall consider the case $k>1$, and set $x_0=x-ex$. Since $0=p(x)x^{n-1}-ep(x)x^{n-1}=(x^k+a_{k-1}x^{k-1}+\cdots+a_1x)x^{n-1}-e(x^k+a_{k-1}x^{k-1}+\cdots+a_1x)x^{n-1}=(x_0^k+a_{k-1}x_0^{k-1}+\cdots+a_1x_0)x^{n-1}=(x_0^{k-1}+a_{k-1}x_0^{k-2}+\cdots+a_1)x_0^n$, $A(x_0)$ contains a polynomial of degree $k-1$. By induction method, there exists a polynomial $f(x_0)$ over R^1 such that $f(x_0)x_0^{m+1}=x_0^m$ for some m. Since $x^{n-1}=(-a)p(x)x^{n-1}+x^{n-1}=(-a)(x^{k-1}+a_{k-1}x^{k-2}+\cdots+a_1)x^n+x_0^{n-1}$, we obtain $x^{n+m}=(-a)(x^{k-1}+a_{k-1}x^{k-2}+\cdots+a_1)x^{n+m+1}+x_0^{n+m}$, whence it follows $x^{n+m}=\{(-a)(x^{k-1}+a_{k-1}x^{k-2}+\cdots+a_1)+f(x_0)-ef(x_0)\}x^{n+m+1}$.

Finally, we shall prove the following, which will enable us to readily obtain [24, Theorem 11].

Corollary 10. *Let R'/R be a left and right integral extension. If R is strongly regular then for any $x\in R'$ there exists a quasi-regular element u in $R[x]$ such that $x^{2n}-ux^{2n}=x^n$* and *$ux^n=x^nu$ for some n.*

Proof. By Theorem 13, there exist s and t in $R[x]$ such that $sx^{n+1}=x^n=x^{n+1}t$ for some n. To be easily seen, $s^nx^{2n}=x^n=x^{2n}t^n$. If we set $a=x^n$ and $b=s^{2n}x^n$, then it is known that $ab=ba$, $a^2b=a$ and $ab^2=b$ (cf. the proof of [1, Lemma 1]). Obviously, $e=ab$ is an idempotent and $u=e-b$ is a quasi-regular element in $R[x]$ with quasi-inverse $e-a$. Now, it is easy to see that $a^2-ua^2=a$.

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Added in proof. Recently, Theorem 3 has been proved also by F. Hansen [Proc. Amer. Math. Soc. 55 (1976), 281—286].